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The Kharitonov theorem and robust stabilization via orthogonal polynomials

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Kharitonov's theorem for interval polynomials is given in terms of orthogonal polynomials on $[0, +\infty)$ and their second kind polynomials. A family of robust stabilizing controls for the canonical system is proposed.
Keywords: Kharitonov theorem; orthogonal polynomials; Hurwitz polynomials; stabilization of control systems.

Абдон Е. Чоке-Ріверо **Теорема Харитонова та робастна стабілізація, засновані на ортогональних поліномах.** Представлена теорема Харитонова для інтервальних поліномів у термінах ортогональних поліномів на $[0, +\infty)$ та їх поліномів другого роду. Запропонований клас керувань, які робастно стабілізують канонічну систему.
Ключові слова: теорема Харитонова; ортогональні поліноми; поліноми Гурвиця; стабілізація керованих систем.

Абдон Э. Чоке-Риверо. **Теорема Харитонова и робастная стабилизация, основанные на ортогональных полиномах.** Представлена теорема Харитонова для интервальных полиномов в терминах ортогональных полиномов на $[0, +\infty)$ и их полиномов второго рода. Предложено семейство управлений, робастно стабилизирующее каноническую систему.
Ключевые слова: теорема Харитонова; ортогональные полиномы; полиномы Гурвица; стабилизация управляемых систем.

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1 Introduction

Throughout this paper, let n and m be positive integers. We will use $\mathbb{C}$ and $\mathbb{R}$ to denote the set of all complex numbers and the set of all real numbers, respectively.

The aim of this work is to rewrite Kharitonov's well-known theorem [26] on the Hurwitness of interval polynomials through orthogonal polynomials $[0, \infty)$ and their second kind polynomials; see Proposition 2 and Theorem 2. We will also construct positional robust controls $u = u_n(x)$ for the Brunovsky system of degree n via two sets of Markov parameter sequences or equivalently by using two families of Hurwitz polynomials; see Definition 8 and Theorem 3.

The motivation for present work comes from two sources. One comes from the interrelations between the Markov parameters, orthogonal polynomials and Hurwitz polynomials and their practical application on control theory. The second comes from the generalization of the indicated results for the matrix case.

The present work is based on the Markov parameter approach which is thoroughly studied in [20, Chapter XV]. We decisively use the explicit interrelation between the coefficients of given polynomials and their Markov parameters; see remark 1 or [10, Lemma 3.1]. This interrelation together with the Hurwitness criteria in terms of the positive definiteness of two Hankel matrices; see lemma 1 or [10, Theorem 3.4]. The explicit representation of a Hurwitz polynomial through orthogonal polynomials, allows us to rewrite the Kharitonov theorem on interval polynomials with the help of orthogonal polynomials; see Proposition 1 or [9, Theorem 7.10].

In this sense, the following notions play a relevant role for the present paper:

- The truncated Stieltjes moment problem,
- Orthogonal polynomials,
- Hurwitz polynomials.

In contrast to Kharitonov's theorem, instead of verifying the Hurwitzness of four polynomials of degree $n = 2m$ (resp. $n = 2m + 1$), we propose checking four polynomials of the degree $[\frac{n}{2}]$ (resp. $[\frac{n+1}{2}]$). To this end, the notion of Kharitonov quadruples is introduced. Roughly speaking, this notion highlights the fact that every stable interval polynomial can be constructed by two ordered sequences of Markov parameters. The latter means that the corresponding orthogonal polynomials and their second kind polynomials satisfy a certain order; see Definition 8.

The paper contains three conjectures. The first one states that every stable interval polynomial generates four sequences of ordered Markov parameters. The second conjecture says that the ordering of the quadruple $(h_n^{(\max)}, g_n^{(\max)}, h_n^{(\min)}, g_n^{(\min)})$ can be written in terms of the degree of the corresponding interval polynomial p_n. Finally, the third conjecture states the necessary and sufficient conditions for an interval polynomial to be a stable interval polynomial in terms of the Kharitonov quadruples.

The construction of robust controls of control systems in terms of the coefficient of certain interval polynomials was considered in [1], [25], [19], and references therein. In contrast to these works, we apply the Markov parameter approach. The advantages of using Markov parameters are explained in [22]. These consist mainly of the fact that the stable region in the coefficient space of a given polynomial is not convex, while the stable region in terms of the Markov parameters s_j with positive definite Hankel matrices (2) is convex set [24].

Future work can be devoted to the comparison of the descending degree procedure of the interval polynomial proposed in the present work (as in example 1) with the Routh procedure considered in [3]. Furthermore, future research on the characterization of two Markov sequences to be ordered sequences which generate Kharitonov quadruples is relevant. Such characterization could notably improve Algorithm 3.1.

This work is organized as follows. A brief summary of the truncated Stieltjes moment problem, orthogonal polynomials and the Hurwitz polynomial are given in the Introduction. In section 2, the Kharitonov theorem is represented via orthogonal polynomials on $[0, +\infty)$ and their second kind polynomials. An example of constructing a stable interval polynomial of degree $n = 7$ starting from two sequences of Markov parameters is given. Additionally, in remark 4 an example of a family of interval polynomials is proposed. In section 3, a result on the construction of stable interval polynomials via orthogonal polynomials is given; see Theorem 3. In subsection 3.1, an algorithm for the construction of a robust control is suggested. Following this algorithm, a family of robust controls is written; see examples 2 and 3. Finally, in section 4, the conclusion and three conjectures what develop or complete some results of section 2 are presented.

In the subsequent three subsections, we recall the definitions and relevant results concerning the Stieltjes moment problem, orthogonal polynomials on $[0, +\infty)$ and Hurwitz polynomials.

Note that in [12] the stabilization of the canonical system through orthogonal polynomials on $[0, +\infty)$ is treated.

1.1 The truncated Stieltjes moment problem and extremal solutions

The truncated Stieltjes moment problem is stated as follows: Let n be greater than or equal to 2. Given a sequence $(s_j)_{j=0}^{n-1}$ of real numbers, find the set $\mathcal{M}$ of nondecreasing functions σ of bounded variation on $[0, \infty)$ such that

$$s_j = \int_0^\infty t^j d\sigma(t), \quad 0 \le j \le n-1. \tag{1}$$

This problem was considered in [29, Page 176 and Page 192].

In case of an infinite sequence $(s_k)_{k=0}^\infty$ with (1) for $j \ge 0$, the stated problem is called the classical Stieltjes moment problem.

Let

$$\mathbf{H}_{1,j} := \begin{pmatrix} s_0 & s_1 & \dots & s_j \\ s_1 & s_2 & \dots & s_{j+1} \\ \vdots & \vdots & \vdots & \vdots \\ s_j & s_{j+1} & \dots & s_{2j} \end{pmatrix}, \quad \mathbf{H}_{2,j} := \begin{pmatrix} s_1 & s_2 & \dots & s_{j+1} \\ s_2 & s_3 & \dots & s_{j+2} \\ \vdots & \vdots & \vdots & \vdots \\ s_{j+1} & s_{j+2} & \dots & s_{2j-1} \end{pmatrix}. \tag{2}$$

It is known [16], [17] that the truncated Stieltjes moment problem with given moments $(s_j)_{j=0}^{2m+1}$ (resp. $(s_j)_{j=0}^{2m}$) as a solution if and only if $\mathbf{H}_{1,m}$ and $\mathbf{H}_{2,m-1}$

(resp. $\mathbf{H}_{1,m-1}$ and $\mathbf{H}_{2,m-1}$) are positive semidefinite. In [16], [17], the complete set of solutions of the truncated Stieltjes moment problem when $\mathbf{H}_{1,m}$ and $\mathbf{H}_{2,m-1}$ (resp. $\mathbf{H}_{1,m-1}$ and $\mathbf{H}_{2,m-1}$) are positive definite was given.

With the help of the analytic function in $\mathbb{C} \setminus [0, \infty)$

$$s(z) := \int_0^\infty \frac{d\sigma(t)}{t-z},$$

called associated solution with $\sigma \in \mathcal{M}$, the truncated Stieltjes moment problem is reduced to finding a set of associated analytic functions $s \in \mathcal{Z}$ such that

$$s(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \ldots - \frac{s_{n-1}}{z^n} - \ldots .$$

Assume that σ is normalized as $\sigma(t) = \frac{\sigma(t+0)+\sigma(t-0)}{2}$, and $\sigma(0) = 0$. From the Stieltjes inverse formula [2, Page 631], one gets a corresponding measure by

$$\sigma(t) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_0^t \operatorname{Im} s(x + i\epsilon) dx.$$

1.2 Orthogonal polynomials on $[0, +\infty)$

Orthogonal polynomials [6], [39] play an important role in a number mathematical areas. On one hand, orthogonal polynomials have been extensively used in applications for solving practical problems, such as in signal processing [32] and in filter design [38], [30]. On the other hand, the zeros of a certain family of orthogonal polynomials can be interpreted as the electrostatic energy for a system of a finite number of charges; see [43].

In the present subsection, we focus on truncated families of orthogonal polynomials on $[0, +\infty)$.

Definition 1 *The sequence $(s_j)_{j=0}^{2m}$ (resp. $(s_j)_{j=0}^{2m-1}$) is called a Stieltjes positive definite sequence if $\mathbf{H}_{1,m}$ and $\mathbf{H}_{2,m-1}$ (resp. $\mathbf{H}_{1,m-1}$ and $\mathbf{H}_{2,m-1}$) are positive definite matrices.*

In the sequel, we consider only Stieltjes positive definite sequences.

Definition 2 *Let $(s_j)_{j=0}^{2m-1}$ and $(s_j)_{j=0}^{2m}$ be Stieltjes positive definite sequences. For $k = 1, 2$, let*

$$\mathbf{D}_{k,j}(z) := \begin{pmatrix} s_{k-1} & s_k & \ldots & s_{j+k-1} \\ s_k & s_{k+1} & \ldots & s_{j+k} \\ \ldots & \ldots & \ldots & \ldots \\ s_{j+k-2} & s_{j+k-1} & \ldots & s_{2j+k-2} \\ 1 & z & \ldots & z^j \end{pmatrix},$$

$$\mathbf{E}_{k,j}(z) := \begin{pmatrix} s_{k-1} & s_k & \ldots & s_{j+k-1} \\ s_k & s_{k+1} & \ldots & s_{j+k} \\ \ldots & \ldots & \ldots & \ldots \\ s_{j+k-2} & s_{j+k-1} & \ldots & s_{2j+k-2} \\ e_{k,0}(z) & e_{k,1}(z) & \ldots & e_{k,j}(z) \end{pmatrix},$$

where $(e_{1,0}(z),\, e_{1,1}(z), \ldots, e_{1,j}(z)) := (0, -s_0, -zs_0 - s_1, \ldots, -\sum_{l=0}^{j-1} z^{j-l-1} s_l)$ *and*
$(e_{2,0}(z),\, e_{2,1}(z), \ldots, e_{2,j}(z)) := (-s_0, -zs_0 - s_1, \ldots, -\sum_{l=0}^{j} z^{j-l} s_l)$.
Denote by $p_{1,0}(z) := 1$*,* $q_{1,0}(z) := 0$*,* $p_{2,0}(z) := 1$*, and* $q_{2,0}(z) := s_0$*. For* $j \geq 1$ *and* $k = 1, 2$*, let*

$$p_{k,j}(z) := \frac{\det \mathbf{D}_{k,j}(z)}{\det \mathbf{H}_{k,j-1}}, \quad q_{k,j}(z) := \frac{\det \mathbf{E}_{k,j}(z)}{\det \mathbf{H}_{k,j-1}}. \tag{3}$$

The polynomials $q_{k,j}$ *are called second kind polynomials.*

Note that in [9] a matrix version of $p_{k,j}$ and $q_{k,j}$ is considered. In the proof of [8, Remark 2.6], the transformation from the matrix form to the determinant form (3) is performed.

Definition 3 *Let* $n = 2m$ *(resp.* $n = 2m+1$*). Let* $\sigma(t)$ *be a positive distribution on* $[0, \infty)$ *such that all moments* $s_j := \int_0^\infty t^j d\sigma(t)$ *are finite for* $0 \leq j \leq n-1$*. The sequence of monic polynomials* $(p_{1,j})_{j=0}^m$

$$\int_0^\infty p_{1,j}(t) p_{1,k}(t) d\sigma(t) = \begin{cases} 0, & j \neq k, \\ c_j, & j = k, \end{cases} \qquad c_j > 0$$

and respectively

$$\int_0^\infty p_{2,j}(t) p_{2,k}(t) t d\sigma(t) = \begin{cases} 0, & j \neq k, \\ d_j, & j = k, \end{cases} \qquad d_j > 0$$

are called the sequences of monic orthogonal polynomials on $[0, \infty)$ *with respect to* $d\sigma(t)$ *(resp* $t d\sigma(t)$*).*

For completeness, we recall two special, associated solutions of the truncated Stieltjes moment problem for $n = 2m+1$ (resp. $n = 2m$) called extremal solutions:

$$s_M^{(2m)}(z) =: -\frac{q_{1,m}(z)}{p_{1,m}(z)},\ s_\mu^{(2m)}(z) =: -\frac{q_{2,m}(z)}{z\, p_{2,m}(z)}, \tag{4}$$

$$s_M^{(2m-1)}(z) =: -\frac{q_{1,m}(z)}{p_{1,m}(z)},\ s_\mu^{(2m-1)}(z) =: -\frac{q_{2,m-1}(z)}{z\, p_{2,m-1}(z)}. \tag{5}$$

These solutions, introduced by Yu. Dyukarev in [18], play a relevant role as proving Proposition 1.

1.3 Hurwitz polynomials and Markov parameters

The real polynomial of degree n

$$f_n(z) := a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n$$

can be written as with the help of two polynomials h_n and g_n such that

$$f_n(z) = h_n(z^2) + z g_n(z^2),$$

where

$$h_n(z) := \begin{cases} a_0 z^m + a_2 z^{m-1} + \ldots + a_{n-2} z + a_n, & n = 2m, \\ a_1 z^m + a_3 z^{m-1} + \ldots + a_{n-2} z + a_n, & n = 2m+1, \end{cases} \tag{6}$$

$$g_n(z) := \begin{cases} a_1 z^{m-1} + a_3 z^{m-2} + \ldots + a_{n-3} z + a_{n-1}, & n = 2m, \\ a_0 z^m + a_2 z^{m-1} + \ldots + a_{n-3} z + a_{n-1}, & n = 2m+1. \end{cases} \tag{7}$$

A polynomial f_n is called a Hurwitz polynomial if all its roots have negative real parts.

Definition 4 *The numbers $(s_j)_{j=0}^{2m-1}$ (resp. $(s_j)_{j=0}^{2m}$) appearing in the asymptotic expansions*

$$\frac{g_{2m}(-z)}{h_{2m}(-z)} = -\frac{s_0}{z} - \frac{s_1}{z^2} - \frac{s_2}{z^3} - \ldots - \frac{s_{2m-2}}{z^{2m-1}} - \frac{s_{2m-1}}{z^{2m}} - \ldots, \tag{8}$$

$$\frac{h_{2m+1}(-z)}{(-z)\, g_{2m+1}(-z)} = -\frac{s_0}{z} - \frac{s_1}{z^2} - \frac{s_2}{z^3} - \ldots - \frac{s_{2m-1}}{z^{2m}} - \frac{s_{2m}}{z^{2m+1}} + \ldots. \tag{9}$$

are called Markov parameters of the polynomials f_n

Note that the expansion (8) appears in [20, Chapter XV], meanwhile expansion (9) was first introduced in [9] in the matrix case.

Here we highlight two of the Hurwitzness criteria.

- The algebraic Routh-Hurwitz criterion [23], [34], [4], which is given in terms of the coefficients a_k, of the polynomial f_n. More precisely, one should verify whether the so-called Hurwitz matrix, constructed by the coefficients a_k has positive principal minors; see [23], [34], [4].

- The Markov parameter criterion [20, Chapter XV] given in terms of the Markov parameters s_k. This criteria consists of finding out whether two Hankel matrices of the form (2) are positive definite; see lemma 1.

Lemma 1 *[10, Theorem 3.4] Let n be greater than or equal to 2. The polynomial f_{2m+1} (resp. f_{2m}) is a Hurwitz polynomial if and only if the associated Hankel matrices $\mathbf{H}_{1,m}$ and $\mathbf{H}_{2,m-1}$ (resp. $\mathbf{H}_{1,m-1}$ and $\mathbf{H}_{2,m-1}$) associated with f_n are positive definite matrices.*

The following remark proved in [10] allows the calculation of the Markov parameters s_k from the coefficients a_j of the polynomial f_n.

Remark 1 *[10, Lemma 3.1] Let f_n be a real polynomial of degree n, and let h_n, g_n be as in (6) and (7). The Markov parameter sequence $(s_j)_{j=0}^{2m}$ (resp. $(s_j)_{j=0}^{2m-1}$) from the relations (8) and (9) is determined by the following equalities:*

$$(s_0, s_1, \ldots, s_{2m-1})^{\mathsf{T}} = \mathcal{A}_{2m}^{-1}(a_1, a_3, \ldots, a_{2m-1}, 0, \ldots, 0)^{\mathsf{T}}, \quad n = 2m, \tag{10}$$

$$(s_0, s_1, \ldots, s_{2m})^{\mathsf{T}} = \mathcal{A}_{2m+1}^{-1}(a_1, a_3, \ldots, a_{2m+1}, 0, \ldots, 0)^{\mathsf{T}}, \quad n = 2m+1, \tag{11}$$

where

$$\mathcal{A}_n := \begin{pmatrix} a_0 & 0 & \dots & 0 & 0 \\ a_2 & -a_0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ a_{2(n-1)} & -a_{2(n-2)} & \dots & (-1)^n a_2 & (-1)^{n+1} a_0 \end{pmatrix},$$

for $n \geq 2$ is the $n \times n$ matrix with $a_k = 0$ for $k > n$.

In [9, Theorem 6.1], it was proven that every Hurwitz polynomial can be written in terms of orthogonal polynomials $p_{k,j}$, $k = 1, 2$, on $[0, \infty)$ and their second kind polynomials $q_{k,j}$; see [7, Equality E.2]. We reformulate the latter as a proposition.

Proposition 1 *Every real Hurwitz polynomial f_n with $a_0 = 1$ admits the following representation*

$$f_n(z) = \begin{cases} (-1)^m (p_{1,m}(-z^2) - z\, q_{1,m}(-z^2)), & n = 2m, \\ (-1)^m (q_{2,m}(-z^2) + z\, p_{2,m}(-z^2)), & n = 2m+1. \end{cases} \tag{12}$$

Here $p_{k,j}$, $k = 1, 2$ are orthogonal polynomials on $[0, \infty)$, and $q_{k,j}$ are their second kind polynomials defined as in Definition 2.

To prove Proposition 1, the subsequent, explicit relation between polynomials h_n, g_n as in (6), (7) and orthogonal polynomials (3) was introduced in [9, Pages 78 and 79]:

$$h_{2m}(z) = (-1)^m p_{1,m}(-z), \quad g_{2m}(z) = (-1)^{m+1} q_{1,m}(-z), \tag{13}$$

$$g_{2m+1}(z) = (-1)^m p_{2,m}(-z), \quad h_{2m+1}(z) = (-1)^m q_{2,m}(-z). \tag{14}$$

2 Kharitonov's theorem via orthogonal polynomials

In this section, we propose a new form of the Kharitonov theorem which first appeared in [26] in 1978. This representation consists of writing the $h_n^{(r)}$ (resp. $g_n^{(r)}$) part of each of the four Kharitonov polynomials via a member of a family of orthogonal polynomials on $[0, \infty)$ and their second kind polynomials. Such a procedure is based on the Markov parameters generated by the Kharitonov polynomials $K_n^{(r)}$.

Let $\delta \in \mathcal{R}^{n+1}$, and let $\mathcal{P}_n$ be a family of monic interval polynomials:

$$p_n(z, \delta) := \sum_{j=0}^{n} \delta_{n-j} z^j, \tag{15}$$

with

$$x_j \leq \delta_{n-j} \leq y_j, \quad j = \{0, 1, \dots, n\}. \tag{16}$$

Denote

$$h_n^{(1)}(z) :=x_0 + y_2 z + x_4 z^2 + \dots, \tag{17}$$
$$g_n^{(1)}(z) :=x_1 + y_3 z + x_5 z^2 + \dots, \tag{18}$$
$$h_n^{(2)}(z) :=y_0 + x_2 z + y_4 z^2 + \dots, \tag{19}$$

and

$$g_n^{(2)}(z) := y_1 + x_3 z + y_5 z^2 + \dots. \tag{20}$$

Definition 5 *Let p_n be an interval polynomial as in (15), and let $h_n^{(k)}$, $g_n^{(k)}$ be polynomials as in (17)-(20). The following four polynomials*

$$K_n^{(1)}(z) =h_n^{(1)}(z^2) + z g_n^{(1)}(z^2), \tag{21}$$
$$K_n^{(2)}(z) =h_n^{(1)}(z^2) + z g_n^{(2)}(z^2), \tag{22}$$
$$K_n^{(3)}(z) =h_n^{(2)}(z^2) + z g_n^{(1)}(z^2), \tag{23}$$

and

$$K_n^{(4)}(z) = h_n^{(2)}(z^2) + z g_n^{(2)}(z^2) \tag{24}$$

are called Kharitonov polynomials of the interval polynomial p_n.

Note that the Kharitonov polynomials are usually defined in the following form:

$$K_n^{(1)}(z) =x_0 + x_1 z + y_2 z^2 + y_3 z^3 + x_4 z^4 + x_5 z^5 + \dots, \tag{25}$$
$$K_n^{(2)}(z) =x_0 + y_1 z + y_2 z^2 + x_3 z^3 + x_4 z^4 + y_5 z^5 + \dots, \tag{26}$$
$$K_n^{(3)}(z) =y_0 + x_1 z + x_2 z^2 + y_3 z^3 + y_4 z^4 + x_5 z^5 + \dots, \tag{27}$$
$$K_n^{(4)}(z) =y_0 + y_1 z + x_2 z^2 + x_3 z^3 + y_4 z^4 + y_5 z^5 + \dots, \tag{28}$$

The equivalence between (21)-(23) and (25)-(28) is obvious.

Definition 6 *Let $\alpha := (\alpha_0, \alpha_1, \dots, \alpha_n)$ where α_j are real numbers. An interval polynomial $p_n(z, \delta)$ as in (15) is said to be a stable interval polynomial if for each $\alpha_j \in [x_j, y_j]$ all the zeros of $p_n(z, \alpha)$ are strictly in the left-hand complex plane.*

Let us recall the celebrated Kharitonov theorem [26].

Theorem 1 *Let p_n be an interval polynomial as in (15). Furthermore, let $K_n^{(r)}$ for $r = 1, 2, 3, 4$ be Kharitonov polynomials as in Definition 5. The interval polynomial p_n (15) is stable if and only if the four Kharitonov polynomials $K_n^{(r)}$ for $r = 1, 2, 3, 4$ are stable.*

In the present work, we restrict ourselves to the case where the leading interval coefficient δ_0 is equal to $[1,1]$.

Definition 7 *Let the polynomials $h_n^{(k)}$, $g_n^{(k)}$ for $k=1,2$ be defined as in (17)-(20). For $n=2m$, define*

$$s^{(1)}(z) := \frac{g_n^{(1)}(-z)}{h_n^{(1)}(-z)}, \quad s^{(2)}(z) := \frac{g_n^{(2)}(-z)}{h_n^{(1)}(-z)}, \tag{29}$$

$$s^{(3)}(z) := \frac{g_n^{(1)}(-z)}{h_n^{(2)}(-z)}, \quad s^{(4)}(z) := \frac{g_n^{(2)}(-z)}{h_n^{(2)}(-z)}. \tag{30}$$

Similarly for $n=2m+1$, define

$$s^{(1)}(z) := \frac{h_n^{(1)}(-z)}{(-z)g_n^{(1)}(-z)}, \quad s^{(2)}(z) := \frac{h_n^{(2)}(-z)}{(-z)g_n^{(1)}(-z)}, \tag{31}$$

$$s^{(3)}(z) := \frac{h_n^{(1)}(-z)}{(-z)g_n^{(2)}(-z)}, \quad s^{(4)}(z) := \frac{h_n^{(2)}(-z)}{(-z)g_n^{(2)}(-z)}. \tag{32}$$

Each of these rational functions $s^{(r)}$ can be expanded as in (8) and (9), respectively. Every sequence $(s_j^{(r)})_{j=0}^{n-1}$ corresponding to such expansions is called the Markov parameter sequence, which is associated with the polynomial $K_n^{(r)}$.

Under the assumption that $K_n^{(r)}(z)$ are monic Hurwitz polynomials, we will prove that the functions $s^{(r)}(z)$ are in fact extremal solutions of truncated Stieltjes moment problems.

Lemma 2 *Let the polynomials $K_n^{(r)}(z)$ for $r=1,2,3,4$ be monic Hurwitz polynomial, then the following is valid.*
a) The Markov parameter sequence $(s_j^{(r)})_{j=0}^{n-1}$ associated with the polynomial $K_n^{(r)}$ is a truncated Stieltjes positive definite sequence for $r=1,2,3,4$.
b) The functions $s^{(r)}(z)$ defined by (29)-(32) are extremal solutions of the truncated Stieltjes moment problem with $(s_j^{(r)})_{j=0}^{n-1}$ for $r=1,2,3,4$.

P r o o f 1 Part a) is a direct consequence of lemma 1. Part b) is verified by employing (4), (5) and equalities in lines 12, 22 *on [9, Page 80].*

The following Proposition can be readily verified by applying Proposition 1 for every $r=1,2,3,4$.

Proposition 2 *The interval polynomial (15) with $\delta_0=[1,1]$ is stable if and only if the four Kharitonov polynomials $K_n^{(r)}$ for $r=1,2,3,4$ as in (21)-(24) admit the following representation*

$$K_n^{(r)}(z) = \begin{cases} (-1)^m\left(p_{1,m}^{(r)}(-z^2) - zq_{1,m}^{(r)}(-z^2)\right), & n=2m, \\ (-1)^m\left(q_{2,m}^{(r)}(-z^2) + zp_{2,m}^{(r)}(-z^2)\right), & n=2m+1 \end{cases} \quad r=1,2,3,4, \tag{33}$$

where $p^{(r)}_{1,m}$ and $q^{(r)}_{1,m}$ (resp. $p^{(r)}_{2,m}$ and $q^{(r)}_{2,m}$) are orthogonal polynomials on $[0,+\infty)$ and second kind polynomials.

To write Kharitonov's theorem of two sequences of Markov moments, we introduce the following notion.

Definition 8 *Let $n=2m$ (resp. $n=2m+1$). Let $((s^{(\min)}_j)^{n-1}_{j=0},\ (s^{(\max)}_j)^{n-1}_{j=0})$ be Stieltjes positive definite sequences such that $s^{(\min)}_j \le s^{(\max)}_j$, $0\le j\le n-1$ with at least one strict inequality. Furthermore, let $(p^{(\max)}_{k,m},\ q^{(\min)}_{k,m})$, for $k=1,2$, the polynomials as in Definition 2. The quadruple*

$$\mathcal{P}_{2m} := (p^{(\min)}_{1,m}, q^{(\min)}_{1,m}, p^{(\max)}_{1,m}, q^{(\max)}_{1,m}) \tag{34}$$

and

$$\mathcal{P}_{2m+1} := (p^{(\min)}_{2,m}, q^{(\min)}_{2,m}, p^{(\max)}_{2,m}, q^{(\max)}_{2,m}) \tag{35}$$

are called Kharitonov quadruple if the Markov parameter sequences

$$((s^{(i_1)}_j)^{2m-1}_{j=0},\quad (s^{(i_2)}_j)^{2m-1}_{j=0})\quad (resp.\quad ((s^{(i_1)}_j)^{2m}_{j=0}),\quad (s^{(i_2)}_j)^{2m}_{j=0})) \tag{36}$$

generated by

$$\left(-\frac{p^{(\min)}_{1,m}(z)}{q^{(\max)}_{1,m}(z)}, -\frac{p^{(\max)}_{1,m}(z)}{q^{(1\min)}_{1,m}(z)}\right)\quad (resp.\quad \left(-\frac{p^{(\min)}_{2,m}(z)}{zq^{(\max)}_{2,m}(z)}, -\frac{p^{(\max)}_{2,m}(z)}{zq^{(\min)}_{2,m}(z)}\right)) \tag{37}$$

are Stieltjes positive definite sequences.

Remark 2 *The Markov parameters (36) can be calculated by Laurent series expansion of the rational functions appearing in (37), respectively.*

Alternatively, to determine the Markov parameters (36) one can use remark 1 with

$$(h_n(z), g_n(z)) = ((-1)^m p^{(\min)}_{1,m}(-z), (-1)^{m+1} q^{(\min)}_{1,m}(-z)),\quad n=2m \tag{38}$$

and

$$(h_n(z), g_n(z)) = ((-1)^m q^{(\min)}_{2,m}(-z), (-1)^m p^{(\min)}_{2,m}(-z)),\quad n=2m+1. \tag{39}$$

Definition 9 *Let $n=2m$ (resp. $n=2m+1$), and let $K^{(r)}_n$ for $r=1,2,3,4$ be the monic Kharitonov polynomials as in Definition 5, which correspond to the interval polynomial (15) with the leading coefficient $\delta_0=[1,1]$. Furthermore, let $h^{(k)}_n$, $g^{(k)}_n$ for $k=1,2$ be as in (17)-(20). We say that the Kharitonov polynomials $K^{(r)}_n$ form a Kharitonov quadruple if between the polynomials $h^{(k)}_n$, $g^{(k)}_n$, $k=1,2$ there are quadruples*

$$\left((-1)^m h^{(i_1)}_{2m}(-z), (-1)^{m+1} g^{(i_2)}_{2m}(-z), (-1)^m h^{(i_3)}_{2m}(-z), (-1)^{m+1} g^{(i_4)}_{2m}(-z)\right)$$

and

$$\left((-1)^m g_{2m+1}^{(i_2)}(-z), (-1)^m h_{2m+1}^{(i_1)}(-z), (-1)^m g_{2m+1}^{(i_4)}(-z), (-1)^m h_{2m+1}^{(i_3)}(-z)\right),$$

that are Kharitonov quadruples. Here (i_j) *is one of the superscripts* (1) *or* (2).

Now we state the main result of the present work.

Theorem 2 *Let* $n = 2m$ *(resp.* $n = 2m + 1$*) and* $K_n^{(r)}$ *for* $r = 1, 2, 3, 4$ *be monic Kharitonov polynomials as in Definition 5. If the polynomials* $K_n^{(r)}$ *form a Kharitonov quadruple, then the corresponding interval polynomial* p_n *is a stable interval polynomial.*

P r o o f 2 The proof follows by using Propositon 2 and Equalities (13)-(14).

Note that the converse statement to Theorem 2 appears in Conjecture 3.

The following remark verifies, for $2 \le j \le 7$, some ordering of the pairs $(h_j^{(i_k)}, g_j^{(i_k)})$ appearing in (17)-(20). This ordering allows the identification of the pairs $(h_j^{(\max)}, g_j^{(\max)})$ and $(h_j^{(\min)}, g_j^{(\min)})$. For $j \ge 7$, the corresponding equalities are stated in Conjecture 2.

Remark 3 *Let* $h_n^{(1)}$, $g_n^{(1)}$, $h_n^{(2)}$, $g_n^{(2)}$ *be as in (17)-(20). Furthermore, let the pairs* $(h_n^{(i_1)}, g_n^{(i_2)})$ *for* $i_k = 1$ *or* $i_k = 2$ *with* $k = 1, 2$. *be Kharitonov quadruples as in definition 9. Thus, the following equalities hold.*

$$(h_2^{(\max)}, g_2^{(\max)}) = (h_2^{(2)}, g_2^{(2)}), \quad (h_2^{(\min)}, g_2^{(\min)}) = (h_2^{(1)}, g_2^{(1)}), \tag{40}$$
$$(h_3^{(\max)}, g_3^{(\max)}) = (h_3^{(1)}, g_3^{(2)}), \quad (h_3^{(\min)}, g_3^{(\min)}) = (h_3^{(2)}, g_3^{(1)}), \tag{41}$$
$$(h_4^{(\max)}, g_4^{(\max)}) = (h_4^{(1)}, g_4^{(1)}), \quad (h_4^{(\min)}, g_4^{(\min)}) = (h_4^{(2)}, g_4^{(2)}), \tag{42}$$
$$(h_5^{(\max)}, g_5^{(\max)}) = (h_5^{(2)}, g_5^{(1)}), \quad (h_5^{(\min)}, g_5^{(\min)}) = (h_5^{(1)}, g_5^{(2)}), \tag{43}$$
$$(h_6^{(\max)}, g_6^{(\max)}) = (h_6^{(2)}, g_6^{(2)}), \quad (h_6^{(\min)}, g_6^{(\min)}) = (h_6^{(1)}, g_6^{(1)}), \tag{44}$$
$$(h_7^{(\max)}, g_7^{(\max)}) = (h_7^{(1)}, g_7^{(2)}), \quad (h_7^{(\min)}, g_7^{(\min)}) = (h_7^{(2)}, g_7^{(1)}). \tag{45}$$

P r o o f 3 Equalities (40)-(45) can be verified by using lemma 1: see also [10, Remark 3.1].

Example 1 *Let the following Stieltjes positive definite sequences be given*

$$\{s_k^{(\max)}\}_{k=0}^{6} = \left\{\frac{19}{2}, \frac{913}{4}, \frac{49959}{8}, \frac{2753481}{16}, \frac{151846263}{32}, \frac{8374343913}{64}, \frac{461849056119}{128}\right\}$$
$$\{s_k^{(\min)}\}_{k=0}^{6} = \left\{9, \frac{415}{2}, 5538, \frac{596853}{4}, \frac{16095575}{4}, \frac{868194535}{8}, 2926929877\right\}.$$

Clearly, $s_k^{(\min)} < s_k^{(\max)}$ *for* $0 \le k \le 6$*. The corresponding orthogonal polynomials and second kind polynomials (see Definition 2) are given by*

$$\begin{aligned} p_{2,3}^{(\max)}(z) =& z^3 - \frac{63z^2}{2} + 111z - \frac{153}{2}, \quad q_{2,3}^{(\max)}(z) = \frac{19z^3}{2} - 71z^2 + \frac{219z}{2} - 12, \\ p_{2,3}^{(\min)}(z) =& z^3 - 31z^2 + \frac{223z}{2} - 76, \quad q_{2,3}^{(\min)}(z) = 9z^3 - \frac{143z^2}{2} + 109z - \frac{25}{2}. \end{aligned}$$

Let

$$K_7^{(\min)}(z) := -\,(q_{2,3}^{(\min)}(-z^2) + z\, p_{2,3}^{(\min)}(-z^2)), \tag{46}$$

$$K_7^{(\max)}(z) := -\,(q_{2,3}^{(\max)}(-z^2) + z\, p_{2,3}^{(\max)}(-z^2)), \tag{47}$$

$$K_7^{(3)}(z) := -\,(q_{2,3}^{(\max)}(-z^2) + z\, p_{2,3}^{(\min)}(-z^2)), \tag{48}$$

and

$$K_7^{(4)}(z) := -\,(q_{2,3}^{(\min)}(-z^2) + z\, p_{2,3}^{(\max)}(-z^2)). \tag{49}$$

By applying remark 2, we calculate the Markov parameters

$$\{s_k^{(3)}\}_{k=0}^6 =\{\frac{19}{2}, \frac{447}{2}, \frac{23910}{4}, 161131, \frac{34763331}{8}, \frac{937569489}{8}, \frac{50573020801}{16}\} \tag{50}$$

$$\{s_k^{(4)}\}_{k=0}^6 =\{9, 212, 5788, 159466, 4396929, \frac{242490639}{2}, \frac{13373470377}{4}\}. \tag{51}$$

Next, we verify that (50) and (51) are Stieltjes positive definite sequences; see Definition 1. Furthermore, we construct the corresponding orthogonal polynomials $p_{2,3}^{(3)}$, $p_{2,3}^{(4)}$ *and their second kind polynomials* $q_{2,3}^{(3)}$, $q_{2,3}^{(4)}$*. These are the following:*

$$\begin{aligned} p_{2,3}^{(1)}(z) =& z^3 - 31z^2 + \frac{223z}{2} - 76, \quad q_{2,3}^{(1)}(z) = \frac{19z^3}{2} - 71z^2 + \frac{219z}{2} - 12, \\ p_{2,3}^{(4)}(z) =& z^3 - \frac{63z^2}{2} + 111z - \frac{153}{2}, \quad q_{2,3}^{(4)}(z) = 9z^3 - \frac{143z^2}{2} + 109z - \frac{25}{2}. \end{aligned}$$

By Proposition 1, the corresponding Kharitonov polynomials $K_7^{(\min)}$, $K_7^{(\max)}$, $K_7^{(3)}$ *and* $K_7^{(4)}$ *are Hurwitz polynomials. Finally, by Theorem 2 the interval polynomial*

$$f_7(z,\delta) :=\delta_0 z^7 + \delta_1 z^6 + \delta_2 z^5 + \delta_3 z^4 + \delta_4 z^3 + \delta_5 z^2 + \delta_6 z + \delta_7 \tag{52}$$

is a stable interval polynomial. Here $\delta_0 \in [1,1]$, $\delta_1 \in [9, 9.5]$, $\delta_2 \in [31, 31.5]$, $\delta_3 \in [71, 71.5]$, $\delta_4 \in [111, 111.5]$, $\delta_5 \in [109, 109.5]$, $\delta_6 \in [76, 76.5]$ *and* $\delta_7 \in [12, 12.5]$*.*

The interval coefficients δ_j are attained from the coefficients of $K_7^{(\min)}$, $K_7^{(\max)}$, $K_7^{(3)}$ and $K_7^{(4)}$, which in fact are the Kharitonov polynomials of f_7.

Note that the interval polynomial (52) was considered in [4, Example 5.4, Chapter 5].

Remark 4 *By using the moments of example 1 and Definition 2, we construct the polynomials* $((p^{(r)}_{1,m})^3_{m=0}, (q^{(r)}_{1,m})^3_{m=0}, (p^{(r)}_{2,m})^3_{m=0}, (q^{(r)}_{2,m})^3_{m=0})$. *With the help of these polynomials and (12), we establish four finite sequences of Hurwitz polynomials:*

$$f^{(r)}_k(z) := z^k + a^{(r)}_{k,1} z^{k-1} + \ldots + a^{(r)}_{k,k}, \quad r = 1, \ldots, 4,$$

and $k \in \mathbb{Z}^6_1$. *Here* $\mathbb{Z}^p_1 := \{1, 2, \ldots, p\}$. *For every* k, *each interval coefficient of the interval polynomial is defined by*

$$[\min_{r\in\mathbb{Z}^4_1} a^{(r)}_{k,j}, \max_{r\in\mathbb{Z}^4_1} a^{(r)}_{k,j}].$$

The family of stable interval polynomials in descending order with an initial interval polynomial (52) is then given by

$$\begin{aligned}
f_6(z) =& z^6 + [9, 9.5]z^5 + [30.57, 30.05]z^4 + [66.61, 67.74]z^3 + [97.24, 99.18]z^2 \\
&+ [83.30, 85.84]z + [47.27, 48.74], \\
f_5(z) =& z^5 + [9, 9.5]z^4 + [29.33, 29.94]z^3 + [54.87, 57.75]z^2 \\
&+ [61.43, 67.49]z + [26.00, 29.92], \\
f_4(z) =& z^4 + [9, 9.5]z^3 + [28.41, 29.11]z^2 \\
&+ [46.39, 50.06]z + [38.96, 43.07], \\
f_3(z) =& z^3 + [9, 9.5]z^2 + [26.69, 27.35]z + [30.63, 33.71], \\
f_2(z) =& z^2 + [9, 9.5]z + [23.05, 24.02], \quad f_1(z) = z + [9, 9.5].
\end{aligned}$$

3 Robust stabilization of the canonical system

Let $x := \text{column}(x_1, x_2, \ldots, x_n)$. Consider the linear system

$$\dot{x} = \mathbf{A}_n x, \tag{53}$$

where

$$\mathbf{A}_n := \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 1 \\ -\alpha_n & -\alpha_{n-1} & \ldots & -\alpha_2 & -\alpha_1 \end{pmatrix}$$

with $\alpha_j \in [\underline{\alpha_j}, \overline{\alpha_j}]$ for $1 \le j \le n$. System (53) represents a linear system subject to some uncertainties, which may be caused by unknown perturbations with entries within a given interval; see [25].

Definition 10 *Let* $\mathbf{A}_n$ *be a matrix as in (53).*
a) The interval polynomial

$$p_{\mathbf{A}_n}(t) := (-1)^n (t^n + \alpha_1 t^{n-1} + \alpha_1 t^{n-2} + \ldots + \alpha_{n-1} t + \alpha_n) \tag{54}$$

is called the characteristic interval polynomial of the matrix $\mathbf{A}_n$.
b) System (53) is called stable if $(-1)^n p_{\mathbf{A}_n}$ *is a stable interval polynomial.*

Now consider the linear control system

$$\dot{x} = \mathbf{A}_n x + b_n u_n, \tag{55}$$

with

$$b_n := \text{column}\,(0, \dots, 0, 1). \tag{56}$$

Definition 11 *The system (55) is robustly stabilizable if there exists a $1 \times n$ interval matrix $\gamma := -(\gamma_1, \gamma_2, \dots, \gamma_n)$ where $\gamma_j \in [\underline{\gamma_j}, \overline{\gamma_j}]$ for $1 \leq j \leq n$ such that the linear system $\dot{x} = (\mathbf{A}_n + b\gamma)x$ is stable. Here*

$$\mathbf{A}_n + b\gamma = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \\ -\delta_n & -\delta_{n-1} & \dots & -\delta_2 & -\delta_1 \end{pmatrix},$$

with $\delta_j = [x_j, y_j]$ and

$$[x_j, y_j] = [\underline{\alpha_j} + \underline{\gamma_j}, \overline{\alpha_j} + \overline{\gamma_j}]. \tag{57}$$

The linear interval function

$$u_n(x, \gamma) := -\gamma_n x_1 - \gamma_{n-1} x_2 - \dots - \gamma_1 x_n \tag{58}$$

is called the robust stabilizing control of the system (55).

In (57), we used interval arithmetic. For completeness, let us recall endpoint formulas for the arithmetic operations of intervals; see [31].

Remark 5 *Let $[a, b]$ and $[c, d]$ be closed intervals. The addition, subtraction, multiplication and division of intervals are defined respectively as follows:*

$$\begin{aligned} [a, b] + [c, d] :=& [a + c, b + d], \\ [a, b] - [c, d] :=& [a - d, b - c], \\ [a, b] \cdot [c, d] :=& [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}], \\ \frac{[a, b]}{[c, d]} :=& \left[\min\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\}, \max\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\} \right], \quad 0 \notin [c, d]. \end{aligned}$$

Remark 6 *System (55) with $\mathbf{A}_n = \mathbf{A}_n(\alpha)$ and $u_n = u_n(x, \gamma)$ a is parametric differential equation*

$$\dot{x} = \mathbf{A}_n(\alpha)x + b_n u_n(x, \gamma). \tag{59}$$

In turn, differential equation (59) is a special case of the differential equation

$$\dot{x} = f(x, \alpha, \gamma),$$

where α and γ are parameters taking certain given values within certain closed intervals. See for example [33, Equality (1)], [21] and [35].

Now we turn to the problem of the robust stabilization of the Brunovsky system. Let

$$\mathbf{A}_n^{(0)} := \begin{pmatrix} 0_{n-1\times 1} & \mathbf{I}_{n-1} \\ 0 & 0_{1\times n-1} \end{pmatrix},$$

where $\mathbf{I}_n$ and $0_{p\times q}$ denotes the identity matrix and $p\times q$ zero matrix. The system

$$\dot{x} = \mathbf{A}_n^{(0)}x + b_n u_n \tag{60}$$

is called the Brunovsky system or canonical system. System (60) is a widely used control system for the study of the controllability and feedback stabilizability of linear and nonlinear systems, with the latter after a certain transformation; see [37], [36]. The Brunovsky system as the basic control model is used for testing results or approximating more general systems for controllability, time optimal control and stability problems; see [5], [37], [40], [41], [42], [44], [15], [13], and [11]. In particular, we emphasize the relevance of the controllability function method created by B.I. Korobov in 1979 [27]. This method allows stabilization at a finite time of the Brunovksy system and more general control systems under bounded controls [28]. See also [14].

The following result allows the construction of a robust control that stabilizes system (60) by employing the Kharitonov quadruples as in Definition 8.

Theorem 3 *Let $n = 2m$ (resp. $n = 2m+1$). Let p_n be the interval polynomial of the form (15) with interval coefficients δ_j constructed via the Kharitonov quadruples $(p_{1,m}^{(\min)}, q_{1,m}^{(\min)}, p_{1,m}^{(\max)}, q_{1,m}^{(\max)})$, respectively $(p_{2,m}^{(\min)}, q_{2,m}^{(\min)}, p_{2,m}^{(\max)}, q_{2,m}^{\max})$. Thus, the linear interval function*

$$u_n(x) = -\delta_n x_1 - \delta_{n-1}x_2 - \ldots - \delta_1 x_n \tag{61}$$

is a robustly stabilizing control for system (60).

Proof 4 Let $\delta^{(n)} := -(\delta_1, \delta_2, \ldots, \delta_n)$. Write the positional control u_n (61) as $u_n(x) = \delta^{(n)}x$. Substitute u_n for $u_n(x) = \delta^{(n)}x$ in (60). The right-hand side of (60) can be written in the form $\dot{x} = \widetilde{\mathbf{A}}_n x$, where

$$\widetilde{\mathbf{A}}_n := \mathbf{A}_n^{(0)} + b_n\delta^{(n)}.$$

The characteristic polynomial of $\widetilde{\mathbf{A}}_n$ has the form

$$p_{\widetilde{\mathbf{A}}_n}(t) := \det(tI - \widetilde{\mathbf{A}}_n) = (-1)^n(t^n + \delta_1 t^{n-1} + \delta_1 t^{n-2} + \ldots + \delta_{n-1}t + \delta_n)$$

Clearly $(-1)^n p_{\widetilde{\mathbf{A}}_n}$ coincides with the stable interval polynomial p_n of the form (15) with coefficients $(1, \delta_1, \delta_2, \ldots, \delta_n)$. Consequently, the control (61) robustly stabilizes system (60).

3.1 An algorithm for constructing a robust control

Let $n = 2m$ (resp. $n = 2m+1$).

1) Find two Stieltjes positive sequences $(s_j^{(\min)})_{j=0}^{n-1}$, $(s_j^{(\max)})_{j=0}^{n-1}$ such that $s_j^{(\min)} \le s_j^{(\max)}$ with at least one strict inequality.

2) Construct polynomials $(p_{1,m}^{(\min)}, q_{1,m}^{(\min)}, p_{1,m}^{(\max)}, q_{1,m}^{(\max)})$, and $(p_{2,m}^{(\min)}$, $q_{2,m}^{(\min)}$, $p_{2,m}^{(\max)}$, $q_{2,m}^{(\max)})$ as in Definition 2.

3) In the case that the polynomials constructed in 2) form a Kharitonov quadruple, using lemma 1 and remark 2 calculate the interval coefficients. In the opposite case, return to Step 1).

4) With the help of (61), write the stabilizing robust control u_n.

Example 2 *Consider the system (60) with $n = 7$. We use example 1, which in fact follows the suggested algorithm. Thus, we attain the positional control*

$$\begin{aligned} u_7(x) =& -[12.12.5]x_1 - [76, 76.5]x_2 - [109, 109.5]x_3 - [111, 111.5]x_4 - [71, 71.5]x_5 \\ & - [31, 31.5]x_6 - [9, 9.5]x_7, \end{aligned}$$

which robustly stabilizes system (60).

Example 3 *As in a similar manner for $2 \le n \le 6$, system (60) can be robustly stabilized by*

$$\begin{aligned} u_6(x) =& -[47.27, 48.47]x_1 - [83.3, 85.84]x_2 - [97.24, 99.18]x_3 \\ & - [66.61, 67.74]x_4 - [30.57, 30.05]x_5 - [9, 9.5]x_6, \\ u_5(x) =& -[26, 29.92]x_1 - [61.43, 67.49]x_2 - [54.87, 57.75]x_3 \\ & - [29.33, 29.94]x_4 - [9, 9.5]x_5, \\ u_4(x) =& -[38.96, 43.07]x_1 - [46.39, 50.06]x_2 - [28.41, 29.11]x_3 - [9, 9.5]x_4, \\ u_3(x) =& -[30.63, 33.71]x_1 - [26.69, 27.35]x_2 - [9, 9.5]x_3 \end{aligned}$$

and

$$u_2(x) = -[23.05, 24.02]x_1 - [9, 9.5]x_2.$$

4 Conclusion and conjectures

In the present work, a reformulation of the Kharitonov theorem via quadruple polynomials is given. A family of decreasing degrees stable interval polynomials is proposed. With the help of constructed stable interval polynomials, a family of robust controls is formulated.

Next we present three conjectures concerning the results of section 1.

Conjecture 1 *Let $n = 2m$ (resp. $n = 2m+1$) and let p_n be a stable interval polynomial of the form (15). Furthermore, for $r = 1,2,3,4$ let $\left(s_j^{(r)}\right)_{j=0}^{n-1}$ be Markov parameters*

corresponding to Kharitonov polynomials $K_n^{(r)}$ of p_n. Thus, the following order yields

$$s_j^{(\min)} \leq s_j^{(i_2)} \leq s_j^{(i_3)} \leq s_j^{(\max)}, \quad 0 \leq j \leq n-1 \tag{62}$$

where $(\min)$, (i_2), (i_3), *and* $(\max)$ *take one of the values* $1, 2, 3$ *or* 4. *Furthermore, at least one of the inequalities in (62) is a strict inequality.*

Conjecture 2 *Let $h_n^{(1)}$, $g_n^{(1)}$, $h_n^{(2)}$, $g_n^{(2)}$ be as in (17)-(20). The following equalities hold.*

$$(h_{4\ell-2}^{(\max)}, g_{4\ell-2}^{(\max)}) = (h_{4\ell-2}^{(2)}, g_{4\ell-2}^{(2)}), \quad (h_{4\ell-2}^{(\min)}, g_{4\ell-2}^{(\min)}) = (h_{4\ell-2}^{(1)}, g_{4\ell-2}^{(1)}), \tag{63}$$

$$(h_{4\ell-1}^{(\max)}, g_{4\ell-1}^{(\max)}) = (h_{4\ell-1}^{(1)}, g_{4\ell-1}^{(2)}), \quad (h_{4\ell-1}^{(\min)}, g_{4\ell-1}^{(\min)}) = (h_{4\ell-1}^{(2)}, g_{4\ell-1}^{(1)}), \tag{64}$$

$$(h_{4\ell}^{(\max)}, g_{4\ell}^{(\max)}) = (h_{4\ell}^{(1)}, g_{4\ell}^{(1)}), \quad (h_{4\ell}^{(\min)}, g_{4\ell}^{(\min)}) = (h_{4\ell}^{(2)}, g_{4\ell}^{(2)}), \tag{65}$$

$$(h_{4\ell-3}^{(\max)}, g_{4\ell-3}^{(\max)}) = (h_{4\ell-3}^{(2)}, g_{4\ell-3}^{(1)}), \quad (h_{4\ell-3}^{(\min)}, g_{4\ell-3}^{(\min)}) = (h_{4\ell-3}^{(1)}, g_{4\ell-3}^{(2)}). \tag{66}$$

This conjecture is a generalization of remark 3. It says that the superindex (min) and (max) can be related to the degree of the interval polynomial p_n (15).

Conjecture 3 *Let $n = 2m$ (resp. $n = 2m+1$). The interval polynomial p_n is a stable if and only if the Kharitonov polynomials $K_n^{(r)}$ for $r = 1,2,3,4$ form Kharitonov quadruples.*

Note that the sufficient condition of Conjecture 3 is proven in Theorem 2.

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