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ON METACYCLIC SUBGROUPS OF FINITE GROUPS

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Abstract. The aim of this survey article is to present some structural results about of groups whose Sylow p-subgroups are metacylic (p a prime). A complete characterisation of non-nilpotent groups whose 2-generator subgroups are metacyclic is also presented.

All groups considered here are finite.

The recovery of information about the structure of a finite group from information about its subgroups has a long history. This survey article is intended to present some contributions in this context, and it is organised around two questions concerning the influence of metacyclic Sylow subgroups and 2-generator subgroups on the group structure.

Over years there has been considerable literature studying global properties of groups which are determined by the structure or embedding of their Sylow p-subgroups, where p is a prime which is going to be fixed. Most of these results go back to Burnside's p-nilpotency criterion stating that a group is p-nilpotent, i.e. it has a normal Hall p'-subgroup provided that a Sylow p-subgroup is in the centre of its normaliser. As a consequence, a group with cyclic Sylow p-subgroups is p-nilpotent if its order is coprime to $p-1$. This result does not remain true for metacyclic Sylow p-subgroups as the alternating group of degree 5 shows. However, if the order of a group G is coprime to p^2-1 and its Sylow p-subgroups are metacyclic, then G is p-nilpotent (see for example [11, IV, 5.10]). Therefore a natural question arises.

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Question 1. *What can be said about the structure of a group with metacyclic Sylow p-subgroups?*

In general we cannot say too much: consider a semidirect product of any p'-subgroup of $\mathrm{GL}(2, p)$ with the natural module of dimension 2. However we cannot become alarmed over this remark, since we still have some interesting results.

We begin with *Sylow metacyclic groups*, or groups with all Sylow subgroups metacyclic. These groups were intensively studied by Chillag and Sonn in [7]. They proved the following structural result.

Theorem 2. *Let G be a Sylow metacyclic group. Then $G = [N]A$, where N is a normal Sylow tower subgroup of G of odd order, and either*

- *G is soluble and A is a Hall subgroup of G of order $2^a 3^b$, or*
- *N is the largest normal subgroup of G of odd order and A is one of the following groups: M_{11}, A_7, a metacyclic central extension A_7^+ of A_7, $\mathrm{PSL}(2, p^i)$, $\mathrm{SL}(2, p^i)$, $\mathrm{PGL}(2, p^i)$, two central extensions $\mathrm{PGL}(2, p^i)^+$, $\mathrm{PGL}(2, p^i)^-$ of $\mathrm{PGL}(2, p^i)$, ($i = 1, 2$ $p^i \geq 5$), and the unique metacyclic group $\mathrm{PGL}^*(2, p^2)$ lying between $\mathrm{PSL}(2, p^2)$ and $\mathrm{Aut}(\mathrm{PSL}(2, p^2))$.*

  In this case, if $q \in \pi(N) \cap \pi(A)$, then the Sylow q-subgroups of G are abelian and those of A and N are cyclic.

The class of Sylow metacyclic groups is closely related to the class of $\mathbb{Q}$-admissible groups: a group G is said to be $\mathbb{Q}$-*admissible* if there exists a $\mathbb{Q}$-central division algebra containing a maximal subfield K such that $\mathrm{G}(K/\mathbb{Q})$ is isomorphic to G. Schacher [15] proved that every $\mathbb{Q}$-admissible group is Sylow metacyclic. The validity of the converse is unknown at the time of writing. The following remarkable result was proved by Sonn [17].

Theorem 3. *A soluble group is Sylow metacyclic if and only if it is $\mathbb{Q}$-admissible*

Groups with metacyclic Sylow p-subgroups for a single p where also considered by some authors. They have a restricted structure in the p-soluble case as Monakhov and Gribovskaya proved in [14].

Theorem 4. *If G is a p-soluble group with metacyclic Sylow p-subgroups, then*

(1) *if $p = 2$, then $G/\mathrm{O}_{2',2}(G)$ is of odd order, or is isomorphic to Σ_3. In particular, the 2-length of G is at most 2.*

(2) *if $p > 2$, then the p-length of G is at most 1.*

The particular case when $p = 2$ was studied by Mazurov [13] and Camina and Gagen [6]. The latter authors showed the solubility of a group possessing a metacyclic Sylow 2-subgroup with a particular structure.

Theorem 5. *Let G be a group with a metacyclic Sylow 2-subgroup S possessing a cyclic normal subgroup N with $|S/N| > 2$. Then G is soluble.*

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The results I am going to present in the sequel spring from this sources and take the study of groups with metacyclic Sylow p-subgroups further. They were proved jointly with Su and Wang in [3].

Our next theorem gives a precise description of a group with metacyclic Sylow p-subgroups provided that $G/\operatorname{O}_{p'p}(G)$ is of odd order.

Theorem 6. *Let G be a group with metacyclic Sylow p-subgroups. Suppose that $G/\operatorname{O}_{p'p}(G)$ is of odd order, and let $G^* = G/\operatorname{O}_{p'}(G)$. Then G^* has a normal Sylow p-subgroup P^*, and $G^* = [P^*](H_1 \times H_2)$, where H_1 is an abelian group of exponent dividing $p-1$, H_2 is a cyclic group with exponent dividing $p+1$. In particular, G' is p-nilpotent.*

Let G be a group with metacyclic Sylow p-subgroups. In the following, we apply the above theorem to obtain some results on the structure of G when $|G/\operatorname{O}_{p'p}(G)|$ is coprime with $p+1$ or $p-1$. Recall that a group G is said to be *p-supersoluble* if every chief factor of G is either a cyclic group of order p or a p'-group.

Theorem 7. *Let G be a group with metacyclic Sylow p-subgroups. Assume that $|G/\operatorname{O}_{p'p}(G)|$ and $p+1$ are coprime. Then G is p-supersoluble.*

The converse is not true: the symmetric group G of degree four has cyclic Sylow 3-subgroups, G is 3-supersoluble and $(|G/\operatorname{O}_{3'3}(G)|, 3+1) = 2$. However, we have:

Corollary 8. *Suppose G is a group of odd order with metacyclic Sylow p-subgroups. Then G is p-supersoluble if and only if $|G/\operatorname{O}_{p'p}(G)|$ and $p+1$ are coprime.*

The following result, due to Berkovich [4], follows easily from the above results. It was extended by Asaad and Monakhov in [1].

Corollary 9. *Let the group $G = AB$ be the product of the subgroups A and B. If G is of odd order and the Sylow p-subgroups of A and B are cyclic, then G is p-supersoluble.*

Next, we consider what happens if G is a group with metacyclic Sylow p-subgroups such that $(|G/\operatorname{O}_{p'p}(G)|, p-1) = 1$. In this case, G is not necessarily p-supersoluble as the following example shows:

Example 10. *Let p, q be two primes such that $2 < q$ divides $p+1$. Let C be a cyclic group of order q and let V be an irreducible and faithful C-module over the finite field of p-elements. Then V is an elementary abelian group of order p^2. Let $G = V \rtimes C$ be the corresponding semidirect product. Then G is a non p-supersoluble group with a metacyclic Sylow p-subgroup such that $(|G|, p-1) = 1$.*

Theorem 11. *Assume p is odd and let G be a group with metacyclic Sylow p-subgroups. Set $G^* = G/\operatorname{O}_{p'}(G)$. Then $(|G/\operatorname{O}_{p'p}(G)|, p-1) = 1$ if and only if G^* satisfies the following properties:*
(1) *A Sylow p-subgroup of G^* is normal in G^*.*
(2) *The Hall p'-subgroups of G^* are cyclic groups of odd order dividing $p+1$.*

The alternating group of degree 5 is an example of how the case $p = 2$ can differ from the case when p is an odd prime.

Let G be a group with metacyclic Sylow p-subgroups. Suppose that $(|G/\operatorname{O}_{p'p}(G)|, p^2 - 1) = 1$. What does G look like? The answer is contained in our next result which can be considered as an extension of [11, IV, 5.10].

Theorem 12. *Suppose that a Sylow p-subgroup of a group G is metacyclic. Then G is p-nilpotent if and only if* $(|G/\operatorname{O}_{p'p}(G)|, p^2 - 1) = 1$.

In the second part of the article, we are concerned with the influence of 2-generator subgroups on the structure of a group. The basic idea behind the results is the following: assume that a group G has 2-generator subgroups in a class $\mathfrak{X}$ of groups which is subgroup-closed and the minimal non-$\mathfrak{X}$-groups are 2-generator. Then G belongs to $\mathfrak{X}$.

This is true for the classes of soluble [10], supersoluble [9] and nilpotent [11, Satz III.5.2] groups.

Minimal non-metacyclic p-groups have been classified by Blackburn ([5, Theorem 3.2]). These groups are all 3-generator and so the class of groups with all 2-generator subgroups metacyclic contains non-metacyclic groups. For convenience, we denote the class of finite groups with 2-generator subgroups metacyclic by $\mathfrak{M}$. Note that $\mathfrak{M}$ is a subgroup and quotient closed class.

For odd primes p, the classification of p-groups in $\mathfrak{M}$ is easy. If $G \in \mathfrak{M}$, then G can not contain a non-abelian section of order p^3 and exponent p, since such a section is not metacyclic. It then follows by [16, Lemma 2.3.3] that G is a modular group. Conversely it follows from [16, Theorem 2.3.1] that a modular p-group has all 2-generator subgroups metacyclic. Mann [12] showed that the class of monotone 2-groups coincides with the class of 2-groups in $\mathfrak{M}$. These groups have been classified by Crestani and Menegazzo [8] and we refer the reader to that paper for details.

In a joint paper with Cossey [2], we complete the classification of groups in $\mathfrak{M}$ by classifying all the non-nilpotent groups in this class.

Theorem 13. *A non-nilpotent group G has 2-generator subgroups metacyclic if and only if*

(1) *G is supersoluble and metabelian, with Sylow subgroups modular for odd primes and monotone groups for the prime 2,*
(2) *$N = G^{\mathfrak{N}}$ (the nilpotent residual of G) is abelian (and $\neq 1$) and so $G = NK$, $N \cap K = 1$,*
(3) *K acts on N as power automorphisms. If π is the set of primes dividing N then K_p is cyclic if $p \in \pi$ and $K_{\pi'}/C_{K_{\pi'}}(K_\pi)$ is cyclic.*
(4) *If $q \in \pi'$, $x \notin C_{K_q}(N)$ and $y \in C_{K_q}(N)$ then $H = \langle x, y\rangle = U\langle x\rangle$ with U cyclic, normal in H and contained in $C_H(N)$.*

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