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CHARACTERISTICS OF THE TSUNAMI WAVE REFLECTION FROM THE BEACH

R.Kh. Mazova[1], I.V. Remizov[1], N.A. Baranova[1], Yu.F. Orlov[1], S.M. Nikulin[1], A.V. Andriyanov[1], V.D. Kuzin[1]

[1]Nizhny Novgorod State Technical University n.a. R.E. Alekseev, Nizhny Novgorod, RUSSIA

ABSTRACT

It is well known that reflection from the complex relief of the continental slope can significantly affect the runup of a long wave on the beach and significantly increase the destructive impact on the coast. In this paper, we consider the reflection of long waves from a plane slope and from a slope conjugated with even bottom. The problem is considered both for a monochromatic wave and for an arbitrary initial perturbation. Using the "vertical wall" and "plane slope" approximations, the reflection coefficients and the solution for the reflected wave are obtained. Numerical simulation is carried out for the case of reflection of the "Lorentz pulse" from the slope, which is conjugated with even bottom. The results obtained are in good agreement with the available results, both an analytical study of the process and numerical modeling of runup of an arbitrary initial perturbation with taking into account of reflection from the shore.

Keywords: *long gravitational wave, reflection wave from the beach, analytical solution, reflection coefficient, reflection of the pulse of arbitrary form.*

1. INTRODUCTION

The study of the characteristics of the long gravitational wave (tsunami) reflected from the beach is interesting not only for a more complete understanding of the processes taking place in the coastal zone during the wave runup, but also for solution specific tasks of coastal engineering. The main point in coastal engineering is to understand the coastal behavior of wave movements: in particular, to assess both the maximum runup in abnormally large waves on the beach, and the problems of water withdrawal from the shore.

It is well known [Belokon, Semkin, 1998] that reflection from the continental slope and other irregularities of the bottom relief in many cases can significantly affect the runup of the long wave and their destructive impact on the coast. In general, the coastal relief is quite complex, so the problem of reflection is solved approximately. However, often large inhomogeneities such as an underwater ridge or a continental slope are elongated along one of the directions, what makes it possible to simplify the formulation of the problem. One of the important issues that makes it possible to solve the problem of wave reflection from the shore is the task of interpretation of a record of a tide gauge in the coastal zone that records a complex superposition of incident, reflected, and diffracted waves [Volzinger et al., 1989]. The coefficients of wave reflection from the underwater slope as well as from a plane slope were considered in a number of works (see, for example, [Kozlov, 1981, Pelinovsky, 1982, Mazova, 1984]). The problem of reflection of long sea waves from flat slopes and slopes conjugated with even bottom has also been many times considered in the literature [Sugimoto, Kakutani, 1984, 1988; Synolakis, 1987; Jeffrey, Day, 1988, 1989; Day, Jeffrey, 1989].

2. STATEMENT OF THE PROBLEM

After the wave climbed the bench, the rundown process begins and a reflected wave is generated that propagates to the sea. As shown in [Sugimoto, Kakutani, 1988], a reflected wave can be observed only at distances from the shore $x > 2.5d$, where d is the depth of water above a flat bottom. Close to the shoreline, at $x < 2.5d$, the incident and reflected waves can not be differentiated - they merge into one wave. The reflected wave has a dipole character, and although both the linear and nonlinear theories predict perfect reflection for non-breaking waves, the dipole nature of the reflected wave is clearly expressed. Because linear and nonlinear theories are in good agreement with laboratory data at the base of the coast, apparently, the dipole wave is transformed into a wave of one sign only after passing over a region of constant depth. The results of numerical modeling of wave reflection from a vertical wall lead to similar conclusions [Zheleznyak, 1985]: when reflected from a wall, the wave at small distances is essentially asymmetrical and a pronounced trough follows the crest. Over time, the waveform is approaching the original, but complete recovery does not occur - behind the single wave it is a train and an oscillating "tail". Analogous phenomena are observed at high-amplitude waves in laboratory experiments (see Zheleznyak, 1985). Very long waves "see" the shore as a vertical wall: the wavelength changes rapidly and the reflection appears immediately. Shorter, steeper waves, at first, at least, go ashore, as if there would be no shore. The wavelength remains constant and the reflection begins only when the total wave has come ashore [Synolakis, 1987].

When unbroken waves run to the shore, the reflected wave is generated continuously, this reflection is manifested as a "trail" between incoming and outgoing waves. However, the magnitude of the reflected wave, generated at a wave height in the open ocean of $H = 0.3d$, is negligible, until the wave reaches the maximum runup and the reflection process begins [Synolakis, 1987]. The process of reflection from a sloping beach is usually characterized by a reflection coefficient, which is the ratio of the incident and reflected waves. Because the height of the reflected wave can not be determined exactly due to the dipole nature of the wave, it is possible to determine the reflection coefficient from the ratio of the height of the positive dipole wave to the height of the incident wave [Synolakis, 1987].

2. REFLECTED WAVE FROM A PLANE SLOPE

It is obvious that the problem of reflection of a long wave from the shore is solved in direct connection with the problem of wave runup onto a slope. In the works [Mazova, Pelinovsky, 1982; Mazova 1984], the problems of runup in the framework of a linear and nonlinear formulation of the problem were considered in neglecting dispersion and dissipation. In this case, the nonlinear equations of shallow water by means of the Carrier-Greenspan transformations are reduced to a linear equation for the elevation of the water surface [Pelinovsry, 1982; Sugimoto and Kakutani, 1984; Sugimoto and Kakutani, 1988; Synolakis 1987; Jeffrey and.Dai 1988; Dai and A.Jeffrey 1989; Zheleznyak 1982; Mazova and Pelinovsky, 1982; Mazova et al, 1982].
The linear wave equation is solved

$$\frac{\partial^2 \eta}{\partial t^2} - g\frac{\partial}{\partial x}(h(x)\frac{\partial \eta}{\partial x}) = 0, \tag{1}$$

Then the solution for a monochromatic wave runup in the "classical" geometry: the plane slope $h = -\alpha x$, the wave moves along the normal to the shore, gives the wave field as the sum of the fields of the incoming and reflected waves (Fig. 1) (see, for example, [Mazova , 1984])

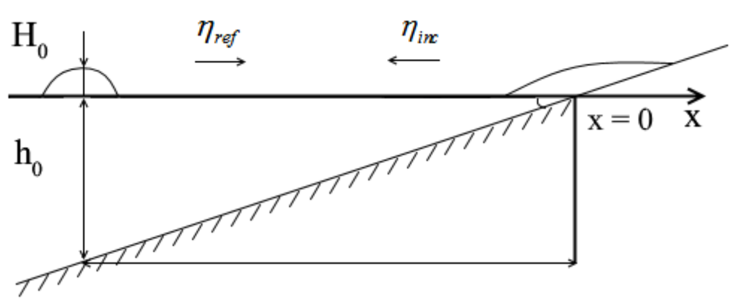


Fig. 1. The geometry of the problem for a plane slope.

$$\eta(x,t) = a(x)\left\{\sin\left[\omega\left(t - \int\frac{dx}{\sqrt{gh(x)}}\right) - \frac{\pi}{4}\right] + \sin\left[\omega\left(t + \int\frac{dx}{\sqrt{gh(x)}}\right)\right]\right\}, \quad (2)$$

where $a(x) = \frac{A}{2}\left[\frac{g\alpha^2}{\pi\omega^2 h(x)}\right]$; $t = \mp\int\frac{dx}{\sqrt{gh(x)}}$ is the time of wave movement from isobath h_0 to the shoreline (-) and back (+).

Those solution in the most general form is a superposition of two waves with a frequency ω and with a variable amplitude a (x). The coefficient A can be found by taking the amplitude of the incident wave at the depth h_0 to be equal H_0. Then we will have

$$\eta(x) = 2\pi H_0\sqrt{\frac{2L_0}{2\pi\sqrt{gh_0}}}\sqrt{\omega}J_0\left(4\pi\sqrt{\frac{L_0\,|\,x\,|}{\lambda_0}}\right), \quad (3)$$

where L_0 is the shelf length, λ_0 is the wavelength at isobath h_0, J_0 is the Bessel function.

The solution for runup of pulse wave can be obtained applying the Fourier superposition of solutions like (2), considering $H_0 = H(\omega)$

$$\eta(x,t)_{\pm} = 2\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}\left(\frac{h_0}{g}\right)^{\frac{1}{4}}\int_{-\infty}^{\infty}\sqrt{|\,\omega\,|}\,\mathrm{sgn}\,\omega H(\omega)J_0\left(\omega\sqrt{\frac{4\,|\,x\,|}{g\alpha}}\right)e^{i\omega t}d\omega, \quad (4)$$

or, in the form more convenient for further transformations [Mazova, 1984; Voltsinger et al., 1989]

$$\eta(x,t)_{\pm} = \frac{-i\,\mathrm{sign}\omega}{\sqrt{\pi g\alpha\sqrt{gh}}}\int_{-\infty}^{\infty}\sqrt{|\,\omega\,|}\,\mathrm{sgn}\,\omega H(\omega)e^{i\omega\left(t \mp \int\frac{dx}{\sqrt{gh(x)}}\right) - \frac{\pi}{4}\mathrm{sign}\omega}d\omega, \quad (5)$$

where $H(\omega)$ is the spectrum of a coming wave. This solution also consists of the sum for the fields of the incident and reflected waves. Knowing the shape of the incident wave at a depth h_0, we can calculate its spectrum

$$H(\omega) = \frac{-ig\sqrt{g\alpha\sqrt{gh_0}}}{2\sqrt{\pi}}\frac{1}{\sqrt{|\,\omega\,|}}\mathrm{sgn}\,\omega\cdot e^{-i\frac{\pi}{4}\mathrm{sgn}\,\omega}\int_{-\infty}^{\infty}\eta_+(t')e^{-i\omega t'}dt'. \quad (6)$$

Submitting (6) to (5) we have

$$\eta_{\pm}(x,t) = \frac{-i}{\pi}\mathrm{sgn}\,\omega\int_{-\infty}^{\infty}e^{i\omega\left(t \mp \int\frac{dx}{\sqrt{gh(x)}}\right)}d\omega\int_{-\infty}^{\infty}\eta_+(t)e^{-i\omega t'}dt'. \quad (7)$$

From formulas (2) and (5) it can be seen that the amplitude of the reflected signal does not change, and the phase changes by $\pi/2$. It can be shown that in this case

$$\eta_{-}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta_{+}(t')}{t' - t + \sqrt{\frac{4L}{g\alpha}}} dt' \quad . \tag{8}$$

The last term in the denominator can be eliminated by redefining the zero of the time. This solution was first obtained by using Hilbert transformations in works [Mazova, 1984; Volzinger et al., 1989]. Thus, for a plane slope, for an arbitrary initial perturbation, the solution for the reflected wave will have the form

$$\eta_{ref}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta_{inc}(t')}{t' - \tau} dt' \quad . \tag{9}$$

3. CALCULATION OF THE ANALYTICAL MODEL OF THE REFLECTED WAVE.

For example, consider the reflection of a single wave from a plane slope (the Lorentz momentum) [Mazova, 1984], whose spectrum has the following form

$$H(\omega) = \frac{1}{4} e^{-\frac{|\omega|}{2} + i\theta} \quad . \tag{10}$$

The reflected wave is obtained by means of the Hilbert transform, where the initial perturbation is calculated by formula

$$\eta_{inc} = \int_{-\infty}^{\infty} H(\omega) e^{i\left(\omega t - \frac{\pi}{4}\right)} d\omega \quad , \tag{11}$$

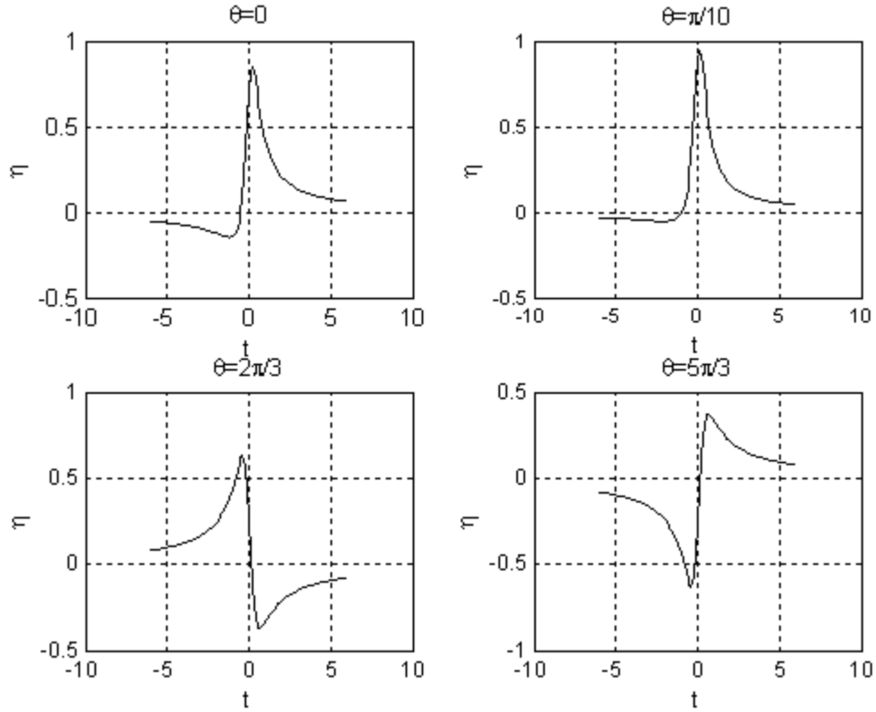


Fig. 2. Tide-gauge records of incident wave for θ: 0; π/10; 2π/3; 5π/3.

Figures 2 and 3 show tide-gauge records for the incident and reflected waves for various values of the parameter θ (see formulas (11) and (9)). As can be seen from the figures, the shape of the reflected wave changes significantly in comparison with the shape of the incident wave when θ changes [Mazova, 1984].

4. REFLECTED WAVES FROM THE SLOPE - CONJUGATED WITH PLANE BOTTOM

Reflection for a monochromatic wave. In the region of a constant depth $x > x_1$ (Fig. 4), the solution of equation (2) for a monochromatic wave is represented as a superposition of the incident and reflected waves [Mazova, 1984]

$$\eta_1(x,t) = Ae^{i(\omega t - kx_1)} + Be^{i(\omega t + kx_1)}, \tag{12}$$

where $k = \frac{2\pi}{\lambda}$, $\omega = k\sqrt{gh}$ (h = const), A and B are amplitudes of incident and reflected waves, respectively. Here, second term corresponds to the solution for reflected wave.

$$\eta_{ref} = Be^{i(\omega t + kx_1)}. \tag{13}$$

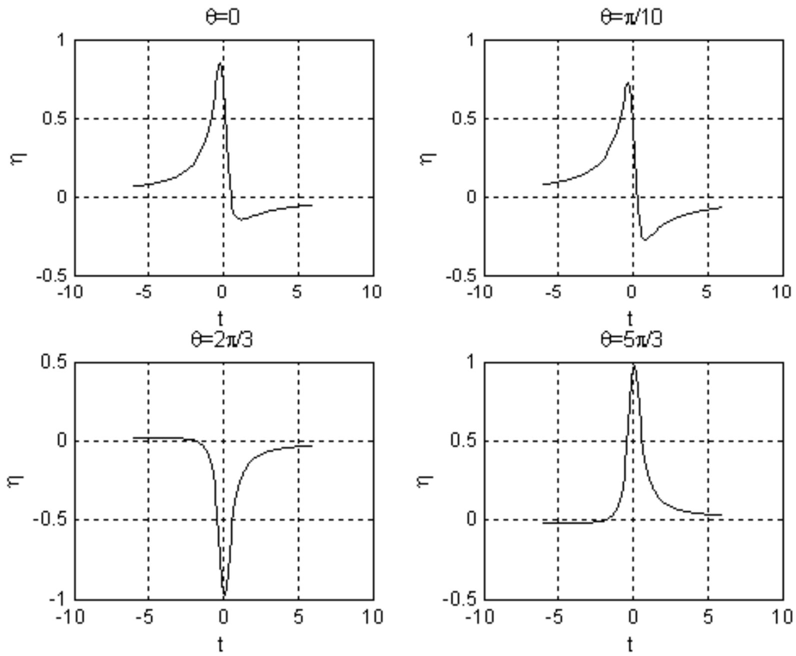


Fig. 3. Tide-gauge records of reflected wave for θ: 0; π/10; 2π/3; 5π/3

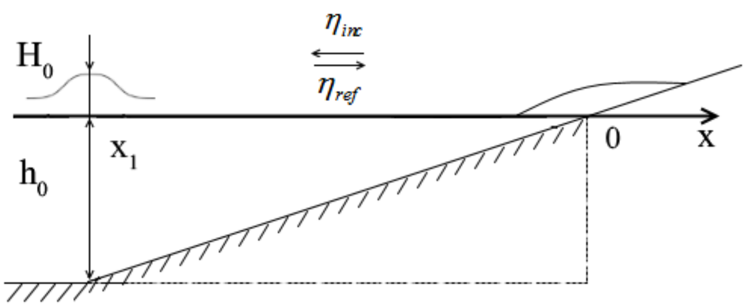


Fig.4. The geometry of the problem for a sloping beach - conjugated with even bottom.

The solution for plane slope $0 < x < x_1$ can be wrote via the Bessel function

$$\eta_2(x,t) = \left[C \cdot J_0(z) + D \cdot N_0(z) \right] e^{i\omega t}, \tag{14}$$

where $z = \sqrt{\frac{4\omega^2 |x|}{g\alpha}}$, α is the inclination angle of the plane slope, J_0 is the first kind Bessel function, N_0 is the Neumann function, and C and D are arbitrary constants. Because the solution must be bounded everywhere, including at the shoreline ($x = 0$), then it is necessary to require $D = 0$, since the function N_0 has a logarithmic singularity for $x = 0$. Then

$$\eta_2(x,t) = C \cdot J_0(z) \cdot e^{i\omega t} . \tag{15}$$

The boundary conditions in the point $x = x_1$ can be obtained using system of linear shallow water equations

$$\begin{cases} \dfrac{\partial \eta}{\partial t} + \dfrac{\partial}{\partial x}(hu) = 0, \\ \dfrac{\partial u}{\partial t} + g\dfrac{\partial \eta}{\partial x} = 0 \end{cases} \tag{16}$$

by integrating (16) we will obtain

$$\begin{gathered} \frac{\partial}{\partial t}\int \eta\, dx + hu\Big|_{-}^{+} \Rightarrow h_1 u_1^{left} = h_1 u_1^{right} \\ \frac{\partial}{\partial t}\int u dx + g\eta\Big|_{-}^{+} \Rightarrow \eta_1^{left} = \eta_1^{right}, \end{gathered} \tag{17}$$

where $\dfrac{\partial}{\partial t}\int \eta\, dx \to 0$ or $\dfrac{\partial}{\partial t}\int u dx \to 0$.

Using the continuity conditions at the point x_1 for the displacement of the water points and the equality of the water mass flows, we "sew" these solutions at the point $x = x_1$ (or $x = L$, where L is the shelf length)

$$\begin{cases} \eta^{left} = \eta^{right}, \\ h_1 u_1^{left} = h_1 u_1^{right}. \end{cases} \tag{18}$$

In the point $x = x_1$, using condition of equality of water point displacement $\eta^{left} = \eta^{right}$

$$Ae^{-i(\omega t - kx)} + Be^{-i(\omega t + kx)} = J_0(z_1)Ce^{i\omega t} , \tag{19}$$

where $k = \dfrac{\omega}{\sqrt{gh}}$

$$Ae^{-ikx_1} + Be^{ikx_1} = CJ_0(z_1) ,$$

rewrote in the form

$$Be^{ikx_1} - CJ_0(z_1) = -Ae^{-ikx_1} . \tag{20}$$

In the point $x = x_1$, using condition of equality of water mass $h_1 u_1^{left} = h_1 u_1^{right}$ and from (16) at $u(x,t) = u(x)e^{i\omega t}$, we will obtain

$$i \cdot \omega \cdot u = -g\frac{\partial \eta}{\partial x} \Rightarrow u = -\frac{g}{i\omega}\frac{\partial \eta}{\partial x},$$

then

$$u_1^{left} = -\frac{g}{i\omega}\frac{\partial \eta_1^{left}}{\partial x} = -\frac{g}{i\omega}\left[-ikAe^{i(\omega t - kx_1)} + ikBe^{i(\omega t + kx_1)}\right],$$

$$u_1^{right} = -\frac{g}{i\omega}\frac{\partial \eta_1^{right}}{\partial x} = -\frac{g}{i\omega}\cdot C\frac{\partial J_0}{\partial z}\frac{dz}{dx}.$$

Since

$$\frac{\partial J_0}{\partial z} = -J_1(z_1),$$

we will obtain

$$\frac{dz}{dx} = -\frac{1}{2}x^{-1/2}\sqrt{\frac{4\omega^2}{g\alpha}} = \frac{\omega}{\sqrt{g\alpha\, x_1}},$$

and from (16)

$$u_1^{right} = -\frac{g}{i\omega}C \cdot J_1(z_1)\frac{\omega}{\sqrt{g\alpha x_1}} = \frac{g}{i\omega}C \cdot J_1(z_1)\cdot k.$$

Since

$$h_1^{left} = h_1^{right},$$

then

$$u_1^{left} = u_2^{right}.$$

By equating,

$$-\frac{g}{i\omega}\left[-ikAe^{i(\omega t - kx_1)} + ikBe^{i(\omega t + kx_1)}\right] = -\frac{g}{i\omega}\cdot C \cdot J_1(z_1)\cdot k$$

we have

$$-iAe^{-ikx_1} + iBe^{ikx_1} = C \cdot J_1(z_1), \quad iBe^{ikx_1} - C \cdot J_1(z_1) = iAe^{-ikx_1}$$

or

$$Be^{ikx_1} + i \cdot C \cdot J_1(z_1) = Ae^{-ikx_1}. \qquad (21)$$

By writing solutions (20) or (21)

$$\begin{cases} Be^{ikx_1} - C \cdot J_0(z_1) = -Ae^{-ikx_1}, \\ Be^{ikx_1} + i \cdot C \cdot J_1(z_1) = Ae^{-ikx_1}, \end{cases}$$

we can find coefficient C

$$C = \frac{\begin{vmatrix} e^{ikx_1} & -Ae^{-ikx_1} \\ e^{ikx_1} & Ae^{-ikx_1} \end{vmatrix}}{\begin{vmatrix} e^{ikx_1} & -J_0(z_1) \\ e^{ikx_1} & iJ_1(z_1) \end{vmatrix}} = \frac{2A}{iJ_1(z_1)e^{ikx_1} + J_0(z_1)e^{ikx_1}} = \frac{2A}{J_0(z_1) + iJ_1(z_1)} \cdot \frac{1}{e^{ikx_1}} =$$

$$= \frac{2A}{\sqrt{J_0^2(z_1) + iJ_1^2(z_1)}} \cdot \frac{1}{e^{iarctg(-J_1/J_0)}} \cdot \frac{1}{e^{ikx_1}} = \frac{2A}{\sqrt{J_0^2(z_1) + iJ_1^2(z_1)}} e^{-i\left[kx_1 - arctg\left(\frac{J_1(z_1)}{J_0(z_1)}\right)\right]} =$$

$$= \frac{2A}{\sqrt{J_0^2 + iJ_1^2}} e^{-i\left[kx_1 - arctg\left(\frac{J_1}{J_0}\right)\right]}.$$

By designating for $H_0 = \dfrac{2A}{\sqrt{J_0^2 + iJ_1^2}}$, we will obtain solution for reflected wave

$$\eta_{ref} = H_0 e^{-2i\left[kx_1 - arctg\frac{J_1}{J_0}\right]} e^{i\omega t} = H_0 e^{-i\left[(2kx_1 - \omega t) - 2arctg\frac{J_1}{J_0}\right]}. \tag{22}$$

Reflection of the pulse of arbitrary form. Using the Fourier transformation

$$\int_{-\infty}^{\infty} \eta_{ref}(t')e^{-i\omega t'}dt' = 2\pi \cdot H(\omega) \quad , \qquad (23)$$

we will obtain

$$\eta_{ref}(t) = \int_{-\infty}^{\infty} H(\omega)\cdot e^{-i\left[\left(2kx_1 - \omega t\right) - 2arctg\frac{J_1}{J_0}\operatorname{sgn}\omega\right]}d\omega \quad . \qquad (24)$$

By substituting (23) to (24), we will have

$$\eta_{ref}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta_{inc}(t')e^{i\omega(t-t')}dt' \int_{-\infty}^{\infty} e^{2i\omega(t-t')}e^{2iarctg\frac{J_1}{J_0}\operatorname{sgn}\omega} d\omega \qquad (25)$$

or

$$\eta_{ref}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta_{inc}(t')dt' \int_{-\infty}^{\infty} e^{i\left[2arctg\frac{J_1}{J_0}\operatorname{sgn}\omega - \omega(t-t')\right]}d\omega \quad . \qquad (26)$$

By taking inner integral for Green function

$$G(t-t') = \int_{-\infty}^{\infty} e^{i\left[2arctg\frac{J_1}{J_0}\operatorname{sgn}\omega - \omega(t-t')\right]}d\omega \quad , \qquad (27)$$

We will solve in the form

$$\eta_{ref} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta_{inc}(t')dt' \cdot G(t-t') \quad . \qquad (28)$$

Rewriting Green function in the following form

$$G(t-t') = \int_{-\infty}^{\infty} K_{ref}(\omega)e^{i\omega\left(t-t'\right)}d\omega \quad , \qquad (29)$$

where $K_{ref}(\omega) = e^{-i2arctg\frac{J_1}{J_0}}$ is the reflection coefficient.

Using asymptotic expansions of Bessel functions at infinity and for small values of the argument

$$J_0(z) = \begin{cases} 1 - \dfrac{z^2}{4}, & z \to 0, \\ \sqrt{\dfrac{2}{\pi \cdot z}} \cos(z - \dfrac{\pi}{4}), & z \to \infty, \end{cases} \qquad J_1(z) = \begin{cases} \dfrac{z}{2}, & z \to 0, \\ \sqrt{\dfrac{2}{\pi \cdot z}} \sin(z - \dfrac{\pi}{4}), & z \to \infty, \end{cases} \tag{30}$$

after a simple transformation

$$\frac{J_1}{J_0} \approx \begin{cases} \dfrac{z}{2}, & z \to 0, \\ tg(z - \dfrac{\pi}{4}), & z \to \infty, \end{cases} \tag{31a}$$

$$arctg \frac{J_1}{J_0} \approx \begin{cases} arctg \dfrac{z}{2} \approx \dfrac{z}{2}, & z \to 0, \\ z - \dfrac{\pi}{4}, & z \to \infty, \end{cases} \tag{31b}$$

it is possible to evaluate the obtained solution (26) for two limiting cases: $z \to 0$ - reflection of a wave from a vertical wall and $z \to \infty$ - reflection from a plate slope. The first solution is well known for linear formulation of the problem, and the second is obtained above. So, for $z \to 0$ ($x \to \infty$) from (27) we have

$$G(t - t') = \int_{-\infty}^{\infty} e^{i2arctg \frac{J_1}{J_0}} e^{i\omega\left(t - t'\right)} d\omega \quad . \tag{32}$$

Further, with taking into account (30), first multiplier under in integral becomes to be equal unity, and remaining integral is well known

$$\int_{-\infty}^{\infty} e^{i\omega\left(t - t'\right)} d\omega = 2\pi \cdot \delta(t - t') \quad . \tag{33}$$

Hence, from (26) we obtain that at $z \to 0$ the Green function is transformed to the δ-function

$$G(t - t') = 2\pi \cdot \delta(t - t') \tag{34}$$

and, substituting this expression to (25), we finally have

$$\eta_{ref}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \eta_{inc}(t^{'})dt^{'} \cdot 2\pi \cdot \delta(t-t') = \int_{-\infty}^{\infty} \eta_{inc}(t^{'})\delta(t-t^{'})dt^{'} = \eta_{inc}(t^{'}) . \quad (35)$$

Thus, at $z \to 0$ we obtain a classical solution to the problem of a long wave runup to a vertical wall in a linear formulation [Zheleznyak, 1985]

$$\eta_{ref} = \eta_{inc} . \quad (36)$$

For the case of the plane slope ($z \to \infty$), using (30), we will consider

$$G(t-t^{'}) = 2 \int_{-\infty}^{\infty} e^{2iarctg\frac{J_1}{J_0}} e^{i\omega\left(t-t^{'}\right)} d\omega = -i \int_{-\infty}^{\infty} e^{i\omega(t-t^{'})} d\omega = 2(-i)\frac{1}{t-t^{'}} \frac{(-1)}{(-i)} = \frac{2}{(t-t^{'})} . \quad (37)$$

Substituting (37) to (26), we obtain [Pelinovsky, 1982; Mazova, 1984]

$$\eta_{ref}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta_{inc}(t^{'})}{t-t^{'}} dt^{'} . \quad (38)$$

Thus, from general solution (22) the solution for two limited cases was obtained: $z \to 0$ and $z \to \infty$. Let us now consider the case of small deviations from a vertical wall and compare it with particular solutions for such a case (see [Mazova, 1984, Voltsinger et al., 1989]). Rewrite the reflection coefficient (see (28)

$$K_{ref}(\omega) = e^{-i2arctg\frac{J_1}{J_0}} . \quad (39)$$

Taking into account the expansion of exponent in small parameter series, we will have

$$K_{ref}(\omega) = 1 - i \cdot 2arctg\frac{J_1}{J_0} - \frac{4}{2!} arctg^2 \frac{J_1}{J_0} + ... , \quad (40)$$

then, taking into account the expansion (31a) and (31b), and restricting by members of the second order, we will obtain:

$$\operatorname{Re} K_{ref}(\omega) = \begin{cases} 1 - z^2, & z \to 0, \\ 1 - 2(z - \frac{\pi}{4})^2 & z \to \infty. \end{cases} \quad (41)$$

Designating for $\gamma = 2\sqrt{\frac{|x|}{g\alpha}}$, we have $z = 2\omega\sqrt{\frac{|x|}{g\alpha}} = \gamma\omega$.

Then, (41) will be rewritten as

$$\operatorname{Re} K_{ref}(\omega) = \begin{cases} 1 - \gamma^2\omega^2, & z \to 0, \\ 1 - 2(\gamma\omega - \frac{\pi}{4})^2, & z \to \infty. \end{cases} \tag{42}$$

Returning to the Green function (27) and taking into account we will have

$$G(t - t') = \int_{-\infty}^{\infty} e^{-i\omega(t - t')}(1 - \gamma^2\omega^2)d\omega \tag{43}$$

or

$$G(t - t') = \int_{-\infty}^{\infty} e^{-i\omega(t - t')}d\omega - \gamma^2 \int_{-\infty}^{\infty} \omega^2 e^{-i\omega(t - t')}d\omega. \tag{44}$$

Since first term in this expression is

$$I_1 = \int_{-\infty}^{\infty} e^{-i\omega(t - t')}d\omega = 2\pi \cdot \delta(t - t'), \tag{45}$$

and second one can be easy transformed by following manner

$$I_2 = \int_{-\infty}^{\infty} \omega^2 e^{-i\omega(t - t')}d\omega = -\int_{-\infty}^{\infty} \frac{\partial^2}{\partial t^2} e^{-i\omega(t - t')}d\omega =$$

$$= -\frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} e^{-i\omega(t - t')}d\omega = -\frac{\partial^2}{\partial t^2} 2\pi \cdot \delta(t - t'), \tag{46}$$

then (44) can be rewrote in the following form

$$G(t - t') = \left(1 - \gamma^2 \frac{\partial^2}{\partial t^2}\right) \cdot 2\pi \cdot \delta(t - t').$$

From here, we have:

$$\eta_{ref} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_{inc}(t')\left(1+\gamma^2\frac{\partial^2}{\partial t^2}\right)\cdot 2\pi\cdot\delta(t-t')dt' =$$

$$= \frac{1}{2\pi}\cdot 2\pi\left(1+\gamma^2\frac{\partial^2}{\partial t^2}\right)\int_{-\infty}^{\infty}\eta_{inc}(t')\delta(t-t')dt' = \left(1+\gamma^2\frac{\partial^2}{\partial t^2}\right)\cdot\eta_{inc}(t). \quad (47)$$

Finally, we have

$$\eta_{ref}(t) = \eta_{inc}(t) + \gamma^2\frac{d^2\eta_{inc}}{dt^2} = (1+\gamma^2\frac{d^2}{dt^2})\eta_{inc}, \quad \gamma = 2\sqrt{\frac{|x|}{g\alpha}}. \quad (48)$$

This solution agrees well with the analogous result from the work on wave reflection in the boundary layer theory [Sugimoto, Kakutani, 1984, 1988] for the case of steep slopes, where the coefficient in front of the second-order term in the differentiation operator is equal to $\mu^2/2$, and $\mu = h/\text{ctg}\alpha$ is a dimensionless quantity. The quantity γ in (48) has the dimension of time. The relation between μ and γ is easy to find: $\gamma = 2\mu\tau^*$, where the introduced quantity $\tau^* = l/\sqrt{gh}$ has the meaning of a characteristic time, and l of a characteristic length.

Thus, the solution (48) is a solution for the reflected wave from the slope, which is conjugated with even bottom for any slope angles. The solution is valid for any form of the initial perturbation in the source to obtain extreme rundown characteristics.

Below are the figures (Fig. 5, Fig. 6) of the incident and reflected waves, calculated from the formula (26) for different values of the parameter θ.

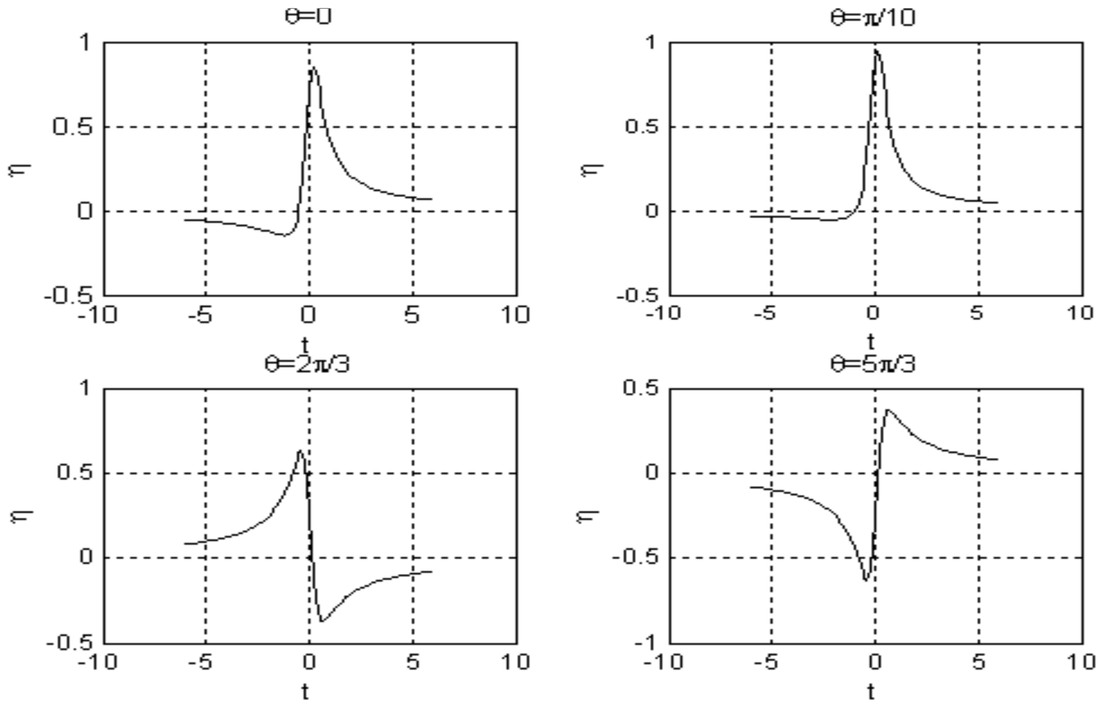


Fig. 5. Tide-gauge records of incident wave for *θ*: 0; π/10; 2π/3; 5π/3.

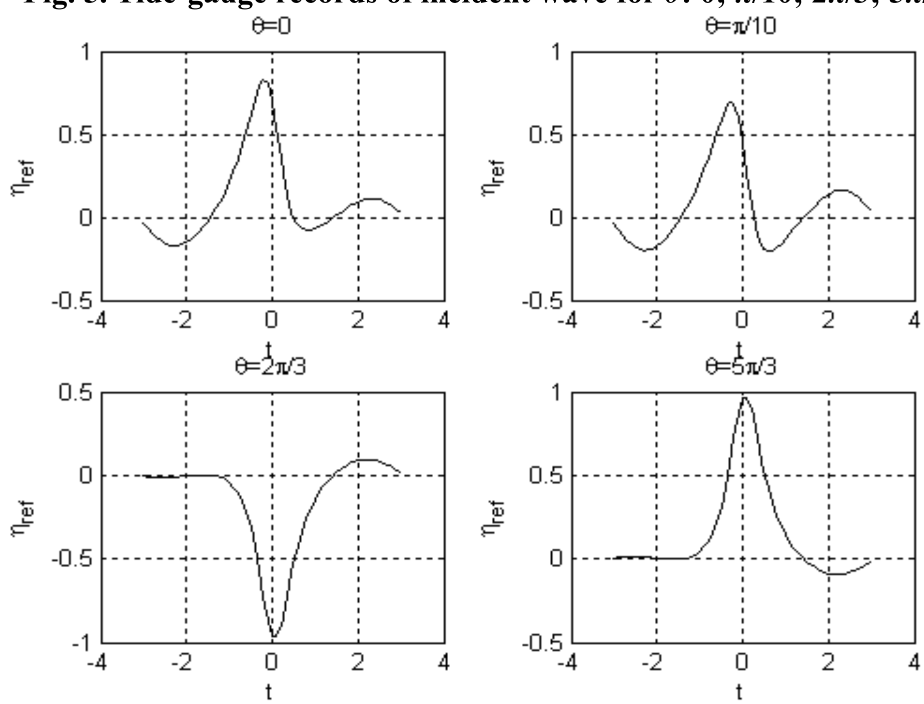


Fig. 6. Tide-gauge records of reflected wave for *θ*: 0; π/10; 2π/3; 5π/3.

In Fig.7 it is presented an incident wave given by expression (11) and computation on analytical formula (28).

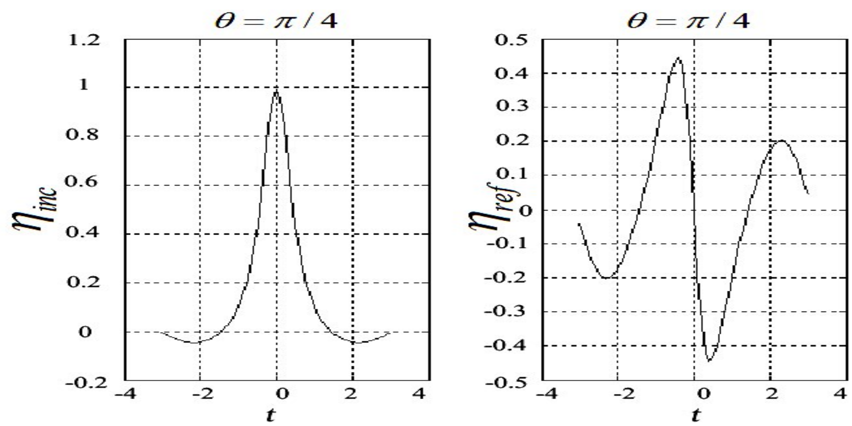


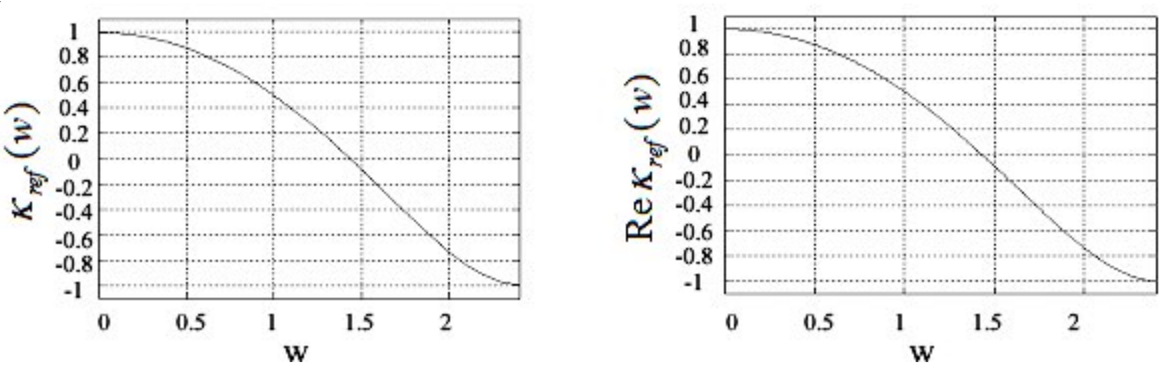
Fig. 7. The incident wave given by expression (11) and calculation by the analytical formula (47) for θ = π / 4.

5. CALCULATION OF THE REFLECTION COEFFICIENT

For the Lorentz pulse, the reflection coefficient determined by the ratio of the incident and reflected waves was also calculated (see, for example, [Synolakis, 1987])

$$\eta_{ref} = \eta_{inc} \cdot K(\omega), \tag{49}$$

and also by formula (38) obtained in the analytic solution of our problem. As seen from Fig. 8, both dependences are in good agreement, including in the initial part of the curves, where the dependence is quadratic in character.



a) ***b)***

Fig. 8. The reflection coefficient for complex slope for *θ*: a) 0; π/10; 2π/3; 5π/3 according to the formulas (39); b) by formula (49).

6. NUMERICAL SIMULATION OF THE REFLECTED WAVE FROM THE SLOPE, CONJUGATED WITH EVEN BOTTOM

Figure 9 shows an example of the results of numerical modeling of the runup process and the formation of a reflected wave for the Lorentz pulse (see above) of a specific shape and typical parameters of the coastal slope. The calculation was carried out in an oblique coordinate system (scheme 35 [Marchuk et al., 1983]). As can be seen from the figure, in this case the observed value of the crest in the dipolar wave forming in the process of rundown ($t \approx 45, 46$) of the order of its amplitude in a coming wave in the runup phase ($t = 40$). At the same time, behind the crest, leaving in the direction of deep water, there is a well-observed depression. It is also clearly seen that at smaller times ($t \approx 43, 44$), as already mentioned above, the incident and reflected waves can not be differentiated from each other. At long calculation times, although the waveform approaches the initial wave, but with smaller amplitude, and oscillations at the trailing edge of the cavity formed during the reflection process. The observed picture is in good agreement with the available results of both the analytical study of the process and the numerical modeling of the runup of a single pulse with taking into account of reflection (see, for example, [Zheleznyak, 1985, Synolakis, 1987]).

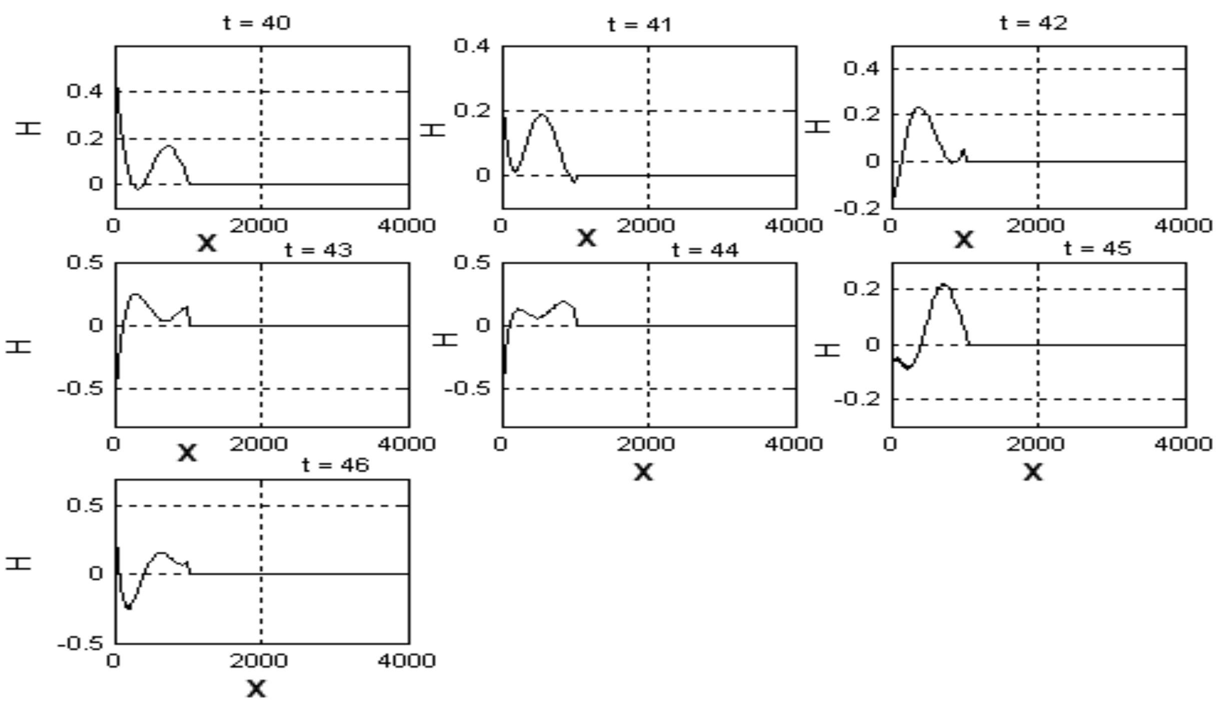


Fig. 9. Numerical simulation of the reflection process of the Lorentz pulse from the slope.

7. CONCLUSION

The analysis of the characteristics of the long gravitational wave (tsunami) reflected from the sloping beach in the framework of the theory of shallow water allows us to indicate the main features of the process of runup in the coastal region. The results obtained were tested for the case of a Lorentzian pulse (with different phases of the wave), and a good agreement was obtained with the data available in the literature. In the case of a slope, conjugated with a even bottom, limiting cases of low and high frequencies are analytically investigated, and numerical calculations of the behavior of the reflection coefficient on the whole frequency interval are also carried out. Analytical and numerical results are consistent; the reflection coefficient of low frequencies is a quadratic function of frequency.

Such a result agrees with the conclusions of similar studies of the reflection process in the framework of the "edge layer" theory. A comparison is made with numerical calculations of the reflection of a soliton from a vertical wall, and a qualitative agreement is obtained. Numerical modeling of the reflection process from a sloping beach conjugated with even bottom is carried out according to available programs. Good agreement with analytical results and available results of laboratory modeling is obtained.

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