# SOLITARY WAVE-SERIES SOLUTIONS TO NON-LINEAR SCHRODINGER EQUATIONS

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***ABSTRACT:*** In this paper, higher-order dispersive non-linear Schrodinger equations are studied. Their solitary wave-series solutions with continuity of the derivatives and specific discontinuity of the derivatives at the crest are presented. Furthermore, convergence of the series' solutions is also validated and discussed with the help of graphs.

***ABSTRAK:*** Kertas ini mengkaji persamaan Schrodinger serakan taklinear turutan tinggi. Penyelesaian siri-gelombang tunggalnya dengan kamiran berterusan dan kamiran tak berterusan pada maksimum telah dibentangkan. Penumpuan penyelesaian siri juga telah diperiksa dan dibincangkan dengan bantuan graf-graf.

***KEYWORDS:*** *Schrodinger equation; solitary wave-series solution; continuity and discontinuity of derivatives at crest*

## 1. INTRODUCTION

It is well-known that the Schrodinger equation plays an important role in plasma physics, quantum mechanics and wave propagation in non-linear media [1-4] . In optical fibers propagation of short pulses is governed by the nonlinear Schrodinger equation [5]. Azzouzi et al. [6] recently presented Solitary wave solutions for high dispersive cubic-quintic nonlinear Schrodinger equation. Li Yao et al. [7] presented solution to nonlinear Schrodinger equation by variational principle method. The authors [8-11] discussed different ways of solutions for nonlinear Schrodinger equations.

Currently in this paper, first solitary wave technique is applied and then the homotopy analysis method (HAM) is employed for the series solution, which was introduced first by Liao [12-13] . The HAM is independent of a small or large parameter and has been applied successfully to solve nonlinear problems such as viscous flow, heat transfer, nonlinear oscillations and Thomas Fermi atom model [14-38]. Further the HAM has certain other advantages over the perturbation expansion method, the delta expansion method and the Lypanov's expansion method, that HAM allows us great freedom and flexibility: (i) to control the region of convergence; (ii) to choose the initial guess; (iii) to choose the auxiliary linear operator.

## 2. MODEL SCHRODINGER EQUATIONS AND THEIR SOLITARY WAVE-SERIES SOLUTIONS

First consider the non homogeneous linear Schrodinger equation [8] :

$$i\frac{\partial u(t,x)}{\partial x}+\alpha\frac{\partial^2 u(t,x)}{\partial t^2}+i\gamma u(t,x)=F(t,x) \tag{1}$$

Where $u(t,x)$ is a complex valued function. For solitary wave solution, under the transformation $\eta=x-ct$, $u(t,x)=a\ f(\eta)$ and $F(t,x)=g(\eta)$ , Eq. (1) reads

$$ia\ f'(\eta)+\alpha ac^2\ f''(\eta)+i\gamma af(\eta)=g(\eta), \tag{2}$$

Where $c$ is the wave speed, $a$ the wave amplitude, and the prime denotes the derivative with respect to $\eta$. For simplicity in case of $F(t,x)$ amplitude is choosen to be 1. Consider the case that $f(\eta)$ arrives its maximum at the origin. Obviously, $f(\eta)$ and its derivatives tend to zero as $\eta\to+\infty$. The corresponding boundary conditions of the solitary wave having discontinuity of derivatives at crest are

$$f(0)=1,\quad f(+\infty)=0. \tag{3}$$

We are considering only two cases of forcing function $g(\eta)$ i.e., $g(\eta)=0$ and $g(\eta)=e^{-\eta}$ . In order to obtain the series solution for $g(\eta)=0$ , we choose

$$f_0(\eta)=e^{-\eta}, \tag{4}$$

$$\mathbf{L}(f)=f''-f, \tag{5}$$

as initial approximation of $f$ and auxiliary linear operator $\mathbf{L}$ satisfying

$$\mathbf{L}\left[C_1e^{-\eta}+C_2e^{\eta}\right]=0. \tag{6}$$

where $C_1$ and $C_2$ are arbitrary constants.

If $p\in[0,1]$ is an embedding parameter and $\eta_1$ is auxiliary non zero parameter then

$$(1-p)\ \mathbf{L}[\phi(\eta,p)-f_0(\eta)]=p\,\eta_1\ \mathbf{N}[\phi(\eta,p)], \tag{7}$$

subject to boundary conditions

$$\phi(0,p)=1,\ \phi'(\infty,p)=0, \tag{8}$$

where

$$\mathbf{N}[\phi(\eta,p)]=ia\frac{\partial\phi(\eta,p)}{\partial\eta}+\alpha ac^2\frac{\partial^2\phi(\eta,p)}{\partial\eta^2}+i\gamma a\phi(\eta,p) \tag{9}$$

and when $p=0$ and $p=1$ , then

$$\phi(\eta,0)=f_0(\eta),\ \ \phi(\eta,1)=f(\eta), \tag{10}$$

As the embedding parameter $p$ increases from $0$ to $1$ , $\phi(\eta,p)$ varies (or deforms) from the initial approximation $f_0(\eta)$ to the solution $f(\eta)$ . Using Taylor's theorem and equation (10) , one obtains

$$\phi(\eta,p)=f_0(\eta)+\sum_{m=1}^{\infty}f_m(\eta)\ p^m, \tag{11}$$

in which

$$f_m(\eta) = \frac{1}{m!}\frac{\partial^m \phi(\eta, p)}{\partial p^m}\bigg|_{p=0}, \quad (m \geq 1). \tag{12}$$

Clearly, the convergence of the series (11) depends upon $\eta_1$. Assume that $\eta_1$ is selected such that the series (11) is convergent at $p = 1$, then due to equation (10) we have

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta). \tag{13}$$

For the *mth* order deformation problem, we differentiate equations (7) and (8) $m-times$ w.r.t $p$ and then setting $p = 0$ and finally dividing it by $m!$ the $mth-order$ deformation equation for $m \geq 1$ is given by

$$\mathbf{L}[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \eta_1 \mathbf{R}_m(\eta), \tag{14}$$

$$f_m(0) = 0, \quad f_m(+\infty) = 0, \tag{15}$$

where

$$\mathbf{R}_m(\eta) = iaf'_{m-1} + \alpha ac^2 f'''_{m-1} + i\gamma a f_{m-1}. \tag{16}$$

To obtain the solution of above equation up to first few order of approximations, the symbolic computation software MATHEMATICA is used. The series solution up to *first few order of approximations* is

$$\begin{aligned}
F[m] = {} & e^{-\eta} + 5\,i\,a\,e^{-\eta}\,hl\,\eta + \frac{45}{8}a^2 e^{-\eta} hl^2 \eta + \frac{105}{64}a^4 e^{-\eta} hl^4 \eta + \frac{105}{512}a^6 e^{-\eta} hl^6 \eta + \\
& \frac{225\,a^8 e^{-\eta} hl^8 \eta}{32768} + \frac{7\,a^{10} e^{-\eta} hl^{10} \eta}{262144} - 5\,a\,c^2 e^{-\eta}\,hl\,\alpha\,\eta + \frac{45}{2}\,i\,a^2 c^2 e^{-\eta} hl^2 \alpha \eta + \\
& \frac{45}{2}a^3 c^2 e^{-\eta} hl^3 \alpha \eta + \frac{315}{64}a^5 c^2 e^{-\eta} hl^5 \alpha \eta + \frac{105}{256}a^7 c^2 e^{-\eta} hl^7 \alpha \eta + \\
& \frac{225\,a^9 c^2 e^{-\eta} hl^9 \alpha \eta}{32768} - \frac{135}{8}a^2 c^4 e^{-\eta} hl^2 \alpha^2 \eta + 60\,i\,a^3 c^4 e^{-\eta} hl^3 \alpha^2 \eta + \\
& \frac{1575}{32}a^4 c^4 e^{-\eta} hl^4 \alpha^2 \eta + \frac{3675}{512}a^6 c^4 e^{-\eta} hl^6 \alpha^2 \eta + \frac{2835\,a^8 c^4 e^{-\eta} hl^8 \alpha^2 \eta}{8192} + \\
& \frac{495\,a^{10} c^4 e^{-\eta} hl^{10} \alpha^2 \eta}{262144} - \frac{75}{2}a^3 c^6 e^{-\eta} hl^3 \alpha^3 \eta + 105\,i\,a^4 c^6 e^{-\eta} hl^4 \alpha^3 \eta + \\
& \frac{2205}{32}a^5 c^6 e^{-\eta} hl^5 \alpha^3 \eta + \frac{1575}{256}a^7 c^6 e^{-\eta} hl^7 \alpha^3 \eta + \frac{1155\,a^9 c^6 e^{-\eta} hl^9 \alpha^3 \eta}{8192} - \\
& \frac{3675}{64}a^4 c^8 e^{-\eta} hl^4 \alpha^4 \eta + 126\,i\,a^5 c^8 e^{-\eta} hl^5 \alpha^4 \eta + \frac{33075}{512}a^6 c^8 e^{-\eta} hl^6 \alpha^4 \eta + \ldots..
\end{aligned} \tag{17}$$

when $g(\eta) = e^{-\eta}$ then solitary wave series solution having discontinuity of derivatives at crest up to *first few order of approximations* is

$$
\begin{aligned}
F[m] = {} & e^{-\eta} + \frac{21}{2} e^{-\eta} hl\,\eta + 3\,i\,a\,e^{-\eta} hl\,\eta - \frac{35}{8} i\,a\,e^{-\eta} hl^2\,\eta + \frac{15}{8} a^2 e^{-\eta} hl^2\,\eta - \\
& \frac{21}{128} i\,a^3 e^{-\eta} hl^4\,\eta + \frac{15}{128} a^4 e^{-\eta} hl^4\,\eta - \frac{i\,a^5 e^{-\eta} hl^6\,\eta}{1024} + \frac{a^6 e^{-\eta} hl^6\,\eta}{1024} - 3\,a\,c^2 e^{-\eta} hl\,\alpha\,\eta + \\
& \frac{105}{8} a\,c^2 e^{-\eta} hl^2\,\alpha\,\eta + \frac{15}{2} i\,a^2\,c^2 e^{-\eta} hl^2\,\alpha\,\eta - \frac{105}{16} i\,a^2\,c^2 e^{-\eta} hl^3\,\alpha\,\eta + \frac{15}{4} a^3 c^2 e^{-\eta} hl^3\,\alpha\,\eta - \\
& \frac{21}{128} a^3 c^2 e^{-\eta} hl^4\,\alpha\,\eta - \frac{35}{256} i\,a^4 c^2 e^{-\eta} hl^5\,\alpha\,\eta + \frac{15}{128} a^5 c^2 e^{-\eta} hl^5\,\alpha\,\eta - \frac{a^5 c^2 e^{-\eta} hl^6\,\alpha\,\eta}{1024} - \\
& \frac{45}{8} a^2 c^4 e^{-\eta} hl^2\,\alpha^2\,\eta + \frac{175}{16} a^2 c^4 e^{-\eta} hl^3\,\alpha^2\,\eta + 10\,i\,a^3 c^4 e^{-\eta} hl^3\,\alpha^2\,\eta - \frac{609}{128} i\,a^3 c^4 e^{-\eta} hl^4\,\alpha^2\,\eta + \\
& \frac{225}{64} a^4 c^4 e^{-\eta} hl^4\,\alpha^2\,\eta - \frac{35}{256} a^4 c^4 e^{-\eta} hl^5\,\alpha^2\,\eta - \frac{17}{512} i\,a^5 c^4 e^{-\eta} hl^6\,\alpha^2\,\eta + \\
& \frac{35\,a^6 c^4 e^{-\eta} hl^6\,\alpha^2\,\eta}{1024} - \frac{25}{4} a^3 c^6 e^{-\eta} hl^3\,\alpha^3\,\eta + \frac{735}{128} a^3 c^6 e^{-\eta} hl^4\,\alpha^3\,\eta + \ldots\ldots\ldots\ldots
\end{aligned}
$$

(18)

For continuity of derivatives at crest the boundary conditions are

$$f(0)=1,\ f'(0)=0,\ f(+\infty)=0. \tag{19}$$

Taking the initial guess

$$f_0(\eta)=2e^{-\eta}-e^{-2\eta}, \tag{20}$$

the solitary wave series solution, having continuity of derivatives at crest, using Eqs. (5), (19) and (20) , up to *first few order of approximations* are, when $g(\eta)=0$ ,

$$
\begin{aligned}
F[m] = {} & -e^{-2\eta} + 2e^{-\eta} + 4\,i\,a\,e^{-2\eta} hl - 4\,i\,a\,e^{-\eta} hl + \frac{20}{3} a^2 e^{-2\eta} hl^2 - \\
& \frac{20}{3} a^2 e^{-\eta} hl^2 - \frac{160}{27} i\,a^3 e^{-2\eta} hl^3 + \frac{160}{27} i\,a^3 e^{-\eta} hl^3 - \frac{80}{27} a^4 e^{-2\eta} hl^4 + \frac{80}{27} a^4 e^{-\eta} hl^4 + \\
& \frac{64}{81} i\,a^5 e^{-2\eta} hl^5 - \frac{64}{81} i\,a^5 e^{-\eta} hl^5 + \frac{64}{729} a^6 e^{-2\eta} hl^6 - \frac{64}{729} a^6 e^{-\eta} hl^6 - 8\,a\,c^2 e^{-2\eta} hl\,\alpha + \\
& 8\,a\,c^2 e^{-\eta} hl\,\alpha + \frac{80}{3} i\,a^2\,c^2 e^{-2\eta} hl^2\,\alpha - \frac{80}{3} i\,a^2\,c^2 e^{-\eta} hl^2\,\alpha + \frac{320}{9} a^3 c^2 e^{-2\eta} hl^3\,\alpha - \\
& \frac{320}{9} a^3 c^2 e^{-\eta} hl^3\,\alpha - \frac{640}{27} i\,a^4 c^2 e^{-2\eta} hl^4\,\alpha + \frac{640}{27} i\,a^4 c^2 e^{-\eta} hl^4\,\alpha - \frac{640}{81} a^5 c^2 e^{-2\eta} hl^5\,\alpha + \\
& \frac{640}{81} a^5 c^2 e^{-\eta} hl^5\,\alpha + \frac{256}{243} i\,a^6 c^2 e^{-2\eta} hl^6\,\alpha - \frac{256}{243} i\,a^6 c^2 e^{-\eta} hl^6\,\alpha - \frac{80}{3} a^2 c^4 e^{-2\eta} hl^2\,\alpha^2 + \\
& \frac{80}{3} a^2 c^4 e^{-\eta} hl^2\,\alpha^2 + \frac{640}{9} i\,a^3 c^4 e^{-2\eta} hl^3\,\alpha^2 - \frac{640}{9} i\,a^3 c^4 e^{-\eta} hl^3\,\alpha^2 + \ldots\ldots\ldots\ldots
\end{aligned}
$$

(21)

when $g(\eta)=e^{-\eta}$ ,

$$F[\mathrm{m}] = -e^{-2\eta} + 2e^{-\eta} + 4\,i\,a\,e^{-2\eta}\,hl - 4\,i\,a\,e^{-\eta}\,hl + \frac{20}{3}\,a^2 e^{-2\eta}\,hl^2 - \frac{20}{3}\,a^2 e^{-\eta}\,hl^2 - \frac{160}{27}\,i\,a^3 e^{-2\eta}\,hl^3 + \frac{160}{27}\,i\,a^3 e^{-\eta}\,hl^3 - \frac{80}{27}\,a^4 e^{-2\eta}\,hl^4 + \frac{80}{27}\,a^4 e^{-\eta}\,hl^4 + \frac{64}{81}\,i\,a^5 e^{-2\eta}\,hl^5 - \frac{64}{81}\,i\,a^5 e^{-\eta}\,hl^5 + \frac{64}{729}\,a^6 e^{-2\eta}\,hl^6 - \frac{64}{729}\,a^6 e^{-\eta}\,hl^6 - 8\,a\,c^2 e^{-2\eta}\,hl\,\alpha + 8\,a\,c^2 e^{-\eta}\,hl\,\alpha + \frac{80}{3}\,i\,a^2 c^2 e^{-2\eta}\,hl^2\,\alpha - \frac{80}{3}\,i\,a^2 c^2 e^{-\eta}\,hl^2\,\alpha + \frac{320}{9}\,a^3 c^2 e^{-2\eta}\,hl^3\,\alpha - \frac{320}{9}\,a^3 c^2 e^{-\eta}\,hl^3\,\alpha - \frac{640}{27}\,i\,a^4 c^2 e^{-2\eta}\,hl^4\,\alpha + \frac{640}{27}\,i\,a^4 c^2 e^{-\eta}\,hl^4\,\alpha - \frac{640}{81}\,a^5 c^2 e^{-2\eta}\,hl^5\,\alpha + \frac{640}{81}\,a^5 c^2 e^{-\eta}\,hl^5\,\alpha + \frac{256}{243}\,i\,a^6 c^2 e^{-2\eta}\,hl^6\,\alpha - \cdots\cdots\cdots\cdots$$

(22)

Consider the nonlinear Schrodinger equation [8] :

$$i\frac{\partial u(t,x)}{\partial x} + \alpha\frac{\partial^2 u(t,x)}{\partial t^2} + \varepsilon\left|u(t,x)\right|^2 u(t,x) + i\gamma u(t,x) = 0, \tag{23}$$

where $u(t,x)$ is a complex valued function. For solitary wave solution Eq. (23) reduces to

$$i\,af'(\eta) + \alpha ac^2\ f''(\eta) + \varepsilon\left|af(\eta)\right|^2 af(\eta) + i\gamma af(\eta) = 0, \tag{24}$$

Under certain assumptions, as in the solution of Eq. (1), the solitary wave-series solution up to *first few order of approximations at* $\eta_2 = -1/4$ , is of the form, when discontinuity of derivatives at crest,

$$F[\mathrm{m}] = e^{-\eta} - \left(\frac{253}{2048} - \frac{4611\,i}{131072}\right) a^2 e^{-3\eta}\,\epsilon + \left(\frac{253}{2048} - \frac{4611\,i}{131072}\right) a^2 e^{-\eta}\,\epsilon - \left(\frac{3}{32768} + \frac{3\,i}{1024}\right) a^2 c^2 e^{-3\eta}\,\alpha\,\epsilon + \left(\frac{3}{32768} + \frac{3\,i}{1024}\right) a^2 c^2 e^{-\eta}\,\alpha\,\epsilon + \frac{9\,i\,a^2 c^4 e^{-3\eta}\,\alpha^2\,\epsilon}{65536} - \frac{9\,i\,a^2 c^4 e^{-\eta}\,\alpha^2\,\epsilon}{65536} - \left(\frac{9}{1024} + \frac{765\,i}{16384}\right) a^2 e^{-3\eta}\,\gamma\,\epsilon + \left(\frac{9}{1024} + \frac{765\,i}{16384}\right) a^2 e^{-\eta}\,\gamma\,\epsilon + \frac{9\,a^2 c^2 e^{-3\eta}\,\alpha\,\gamma\,\epsilon}{32768} - \frac{9\,a^2 c^2 e^{-\eta}\,\alpha\,\gamma\,\epsilon}{32768} + \left(\frac{1}{128} - \frac{45\,i}{65536}\right) a^2 e^{-3\eta}\,\gamma^2\,\epsilon - \left(\frac{1}{128} - \frac{45\,i}{65536}\right) a^2 e^{-\eta}\,\gamma^2\,\epsilon + \frac{i\,a^2 e^{-3\eta}\,\gamma^3\,\epsilon}{2048} - \frac{i\,a^2 e^{-\eta}\,\gamma^3\,\epsilon}{2048} + \left(\frac{13667}{2359296} - \frac{19\,i}{12288}\right) a^4 e^{-5\eta}\,\epsilon^2 - \left(\frac{4563}{262144} - \frac{9\,i}{2048}\right) a^4 e^{-3\eta}\,\epsilon^2 + \left(\frac{3425}{294912} - \frac{35\,i}{12288}\right) a^4 e^{-\eta}\,\epsilon^2 + \frac{19\,i\,a^4 c^2 e^{-5\eta}\,\alpha\,\epsilon^2}{393216} - \frac{9\,i\,a^4 c^2 e^{-3\eta}\,\alpha\,\epsilon^2}{65536} + \frac{35\,i\,a^4 c^2 e^{-\eta}\,\alpha\,\epsilon^2}{393216} + \left(\frac{95}{393216} + \frac{i}{512}\right) a^4 e^{-5\eta}\,\gamma\,\epsilon^2 - \cdots\cdots$$

(25)

when continuity of derivatives at crest,

$$F[m] = \left(-\frac{11}{12} - \frac{107\,i}{216}\right) e^{-2\eta} + \left(\frac{23}{12} + \frac{107\,i}{216}\right) e^{-\eta} + \left(\frac{35}{36} + \frac{i}{3}\right) c^2 e^{-2\eta}\alpha - \left(\frac{35}{36} + \frac{i}{3}\right) c^2 e^{-\eta}\alpha - \left(\frac{1}{3} + \frac{i}{18}\right) c^4 e^{-2\eta}\alpha^2 + \left(\frac{1}{3} + \frac{i}{18}\right) c^4 e^{-\eta}\alpha^2 + \frac{1}{27} c^6 e^{-2\eta}\alpha^3 - \frac{1}{27} c^6 e^{-\eta}\alpha^3 - \left(\frac{1}{12} - \frac{35\,i}{144}\right) e^{-2\eta}\gamma + \left(\frac{1}{12} - \frac{35\,i}{144}\right) e^{-\eta}\gamma + \left(\frac{1}{36} - \frac{i}{6}\right) c^2 e^{-2\eta}\alpha\gamma - \left(\frac{1}{36} - \frac{i}{6}\right) c^2 e^{-\eta}\alpha\gamma + \frac{1}{36} i c^4 e^{-2\eta}\alpha^2\gamma - \frac{1}{36} i c^4 e^{-\eta}\alpha^2\gamma + \left(\frac{1}{48} + \frac{i}{288}\right) e^{-2\eta}\gamma^2 - \left(\frac{1}{48} + \frac{i}{288}\right) e^{-\eta}\gamma^2 - \frac{1}{144} c^2 e^{-2\eta}\alpha\gamma^2 + \frac{1}{144} c^2 e^{-\eta}\alpha\gamma^2 - \frac{i e^{-2\eta}\gamma^3}{1728} + \frac{i e^{-\eta}\gamma^3}{1728} + \left(\frac{10327}{514500} + \frac{57\,i}{4900}\right) a^2 e^{-6\eta}\epsilon - \left(\frac{1105}{6144} + \frac{5\,i}{64}\right) a^2 e^{-5\eta}\epsilon + \ldots\ldots$$

(26)

The nonlinear Schrodinger equation as considered by Xu L. and Zhang J. [10] :

$$i\frac{\partial u(t,x)}{\partial x} - \frac{1}{2}\alpha\frac{\partial^2 u(t,x)}{\partial t^2} + \beta|u(t,x)|^2 u(t,x) + i\varepsilon\frac{\partial^3 u(t,x)}{\partial t^3} + i\delta|u(t,x)|^2\frac{\partial u(t,x)}{\partial t} + i\gamma u^2(t,x)\frac{\partial u(t,x)}{\partial t} = 0, \quad (27)$$

where $u(t, x)$ is a complex valued function.

Under certain assumptions, as in the solution of Eq. (1), the solitary wave-series solution up to *first few order of approximations* at $\eta_3 = -1/4$ of Eq.(27) is of the form, when discontinuity of derivatives at crest,

$$F[m] = e^{-\eta} - \left(\frac{765}{8192} - \frac{9\,i}{512}\right) a^2 e^{-3\eta}\beta + \left(\frac{765}{8192} - \frac{9\,i}{512}\right) a^2 e^{-\eta}\beta + \frac{3\,i\,a^2 c^2 e^{-3\eta}\alpha\beta}{8192} - \frac{3\,i\,a^2 c^2 e^{-\eta}\alpha\beta}{8192} + \left(\frac{3}{1024} - \frac{19\,i}{49152}\right) a^4 e^{-5\eta}\beta^2 - \left(\frac{9}{1024} - \frac{9\,i}{8192}\right) a^4 e^{-3\eta}\beta^2 + \left(\frac{3}{512} - \frac{35\,i}{49152}\right) a^4 e^{-\eta}\beta^2 - \frac{a^6 e^{-7\eta}\beta^3}{32768} + \frac{5\,a^6 e^{-5\eta}\beta^3}{32768} - \frac{9\,a^6 e^{-3\eta}\beta^3}{32768} + \frac{5\,a^6 e^{-\eta}\beta^3}{32768} - \left(\frac{3}{512} + \frac{769\,i}{8192}\right) a^2 c e^{-3\eta}\gamma + \left(\frac{3}{512} + \frac{769\,i}{8192}\right) a^2 c e^{-\eta}\gamma - \left(\frac{1}{4096} + \frac{3\,i}{512}\right) a^2 c^3 e^{-3\eta}\alpha\gamma + \left(\frac{1}{4096} + \frac{3\,i}{512}\right) a^2 c^3 e^{-\eta}\alpha\gamma - \frac{3\,i\,a^2 c^5 e^{-3\eta}\alpha^2\gamma}{16384} + \frac{3\,i\,a^2 c^5 e^{-\eta}\alpha^2\gamma}{16384} + \left(\frac{13}{18432} + \frac{i}{128}\right) a^4 c e^{-5\eta}\beta\gamma - \left(\frac{3}{2048} + \frac{9\,i}{512}\right) a^4 c e^{-3\eta}\beta\gamma + \left(\frac{7}{9216} + \frac{5\,i}{512}\right) a^4 c e^{-\eta}\beta\gamma + \frac{i\,a^4 c^3 e^{-5\eta}\alpha\beta\gamma}{6144} - \frac{3\,i\,a^4 c^3 e^{-3\eta}\alpha\beta\gamma}{8192} + \frac{5\,i\,a^4 c^3 e^{-\eta}\alpha\beta\gamma}{24576} - \frac{7\,i\,a^6 c e^{-7\eta}\beta^2\gamma}{49152} \ldots\ldots\ldots\ldots$$

(28)

When continuity of derivatives at crest,

$$F[m] = \left(-\frac{35}{36} - \frac{i}{3}\right) e^{-2\eta} + \left(\frac{71}{36} + \frac{i}{3}\right) e^{-\eta} - \left(\frac{1}{3} + \frac{i}{18}\right) c^2 e^{-2\eta}\alpha + \left(\frac{1}{3} + \frac{i}{18}\right) c^2 e^{-\eta}\alpha - \frac{1}{36} c^4 e^{-2\eta}\alpha^2 + \frac{1}{36} c^4 e^{-\eta}\alpha^2 + \left(\frac{1}{70} + \frac{19\,i}{4900}\right) a^2 e^{-6\eta}\beta - \left(\frac{1}{8} + \frac{5\,i}{192}\right) a^2 e^{-5\eta}\beta + \left(\frac{2}{5} + \frac{4\,i}{75}\right) a^2 e^{-4\eta}\beta - \left(\frac{1}{2} + \frac{i}{64}\right) a^2 e^{-3\eta}\beta + \left(\frac{59}{280} - \frac{457\,i}{29400}\right) a^2 e^{-\eta}\beta + \frac{11\,a^2 c^2 e^{-6\eta}\alpha\beta}{2450} - \frac{25}{768} a^2 c^2 e^{-5\eta}\alpha\beta + \frac{2}{25} a^2 c^2 e^{-4\eta}\alpha\beta - \frac{1}{16} a^2 c^2 e^{-3\eta}\alpha\beta + \frac{9937\,a^2 c^2 e^{-\eta}\alpha\beta}{940800} - \frac{a^4 e^{-10\eta}\beta^2}{18480} + \frac{153\,a^4 e^{-9\eta}\beta^2}{179200} - \frac{67\,a^4 e^{-8\eta}\beta^2}{11760} + \frac{13}{640} a^4 e^{-7\eta}\beta^2 - \frac{27}{700} a^4 e^{-6\eta}\beta^2 + \ldots\ldots$$

(29)

Sun et al. [11]

$$i\frac{\partial \psi(t,x)}{\partial t}+\frac{\partial^2 \psi(t,x)}{\partial x^2}+\left|\psi(t,x)\right|^2\psi(t,x)=0, \tag{30}$$

where $\psi(t,x)$ is a complex valued function.

Under certain assumptions, as in the solution of Eq. (1), the solitary wave-series solution up to *first few order of approximations* at $\eta_4=-1$ of Eq.(30) is of the form, when discontinuity of derivatives at crest,

$$\begin{aligned}F[m] = {} & -\frac{1}{512}a^9 e^{-7\eta}+\frac{3}{64}a^6 e^{-5\eta}+\frac{5}{512}a^9 e^{-5\eta}+\frac{19}{768}i a^6 c e^{-5\eta}-\frac{3}{8}a^3 e^{-3\eta}-\frac{9}{64}a^6 e^{-3\eta}- \\ & \frac{9}{512}a^9 e^{-3\eta}-\frac{15}{64}i a^3 c e^{-3\eta}-\frac{9}{128}i a^6 c e^{-3\eta}+\frac{3}{128}a^3 c^2 e^{-3\eta}+e^{-\eta}+\frac{3}{8}a^3 e^{-\eta}+\frac{3}{32}a^6 e^{-\eta}+ \\ & \frac{5}{512}a^9 e^{-\eta}+\frac{15}{64}i a^3 c e^{-\eta}+\frac{35}{768}i a^6 c e^{-\eta}-\frac{3}{128}a^3 c^2 e^{-\eta}+\frac{5}{128}a^6 e^{-5\eta}\eta+\frac{5}{128}i a^6 c e^{-5\eta}\eta- \\ & \frac{27}{64}a^3 e^{-3\eta}\eta-\frac{9}{128}a^6 e^{-3\eta}\eta-\frac{33}{64}i a^3 c e^{-3\eta}\eta-\frac{9}{128}i a^6 c e^{-3\eta}\eta+\frac{3}{32}a^3 c^2 e^{-3\eta}\eta+\frac{11 e^{-\eta}\eta}{16}+ \\ & \frac{9}{64}a^3 e^{-\eta}\eta+\frac{1}{64}a^6 e^{-\eta}\eta+\frac{1}{2}i c e^{-\eta}\eta+\frac{11}{64}i a^3 c e^{-\eta}\eta+\frac{1}{64}i a^6 c e^{-\eta}\eta+\frac{3}{16}c^2 e^{-\eta}\eta- \\ & \frac{1}{32}a^3 c^2 e^{-\eta}\eta-\frac{9}{64}a^3 e^{-3\eta}\eta^2-\frac{9}{32}i a^3 c e^{-3\eta}\eta^2+\frac{9}{64}a^3 c^2 e^{-3\eta}\eta^2+\frac{3}{16}e^{-\eta}\eta^2+\cdots\cdots\end{aligned} \tag{31}$$

when continuity of derivatives at crest,

$$\begin{aligned}F[m] = {} & -\frac{139\,a^{12} e^{-18\eta}}{180032125}+\frac{20813407\,a^{12} e^{-17\eta}}{904071168000}- \\ & \frac{3464497\,a^{12} e^{-16\eta}}{11006595600}+\frac{96544781\,a^{12} e^{-15\eta}}{37186688000}-\frac{5137424\,a^9 e^{-14\eta}}{23672273625}- \\ & \frac{504299\,a^{12} e^{-14\eta}}{35392500}-\frac{15453364\,i\,a^9 c e^{-14\eta}}{118361368125}+\frac{4417999709\,a^9 e^{-13\eta}}{1004040576000}+ \\ & \frac{595770163\,a^{12} e^{-13\eta}}{11063808000}+\frac{1325604169\,i\,a^9 c e^{-13\eta}}{502020288000}-\frac{4153255321\,a^9 e^{-12\eta}}{106828722000}- \\ & \frac{224309297\,a^{12} e^{-12\eta}}{1618617000}-\frac{2007976\,i\,a^9 c e^{-12\eta}}{86711625}+\frac{5288413\,a^9 e^{-11\eta}}{27440000}+ \\ & \frac{146962727\,a^{12} e^{-11\eta}}{658560000}+\frac{47787907\,i\,a^9 c e^{-11\eta}}{423360000}-\frac{174734806\,a^6 e^{-10\eta}}{13867189875}-\cdots\cdots\end{aligned} \tag{32}$$

The equation considered by Azzouzi et al. [6] is

$$\begin{aligned}&-\frac{\partial E(t,z)}{\partial z}-i\frac{\beta_2}{2}\frac{\partial^2 E(t,z)}{\partial t^2}+i\gamma_1\left|E(t,z)\right|^2E(t,z)+\frac{\beta_3}{6}\frac{\partial^3 E(t,z)}{\partial t^3} \\ &+i\frac{\beta_4}{24}\frac{\partial^4 E(t,z)}{\partial t^4}-i\gamma_2\left|E(t,z)\right|^4E(t,z)=0.\end{aligned} \tag{33}$$

The solutions in this case at $\eta_5=-3/4$ are, when discontinuity of derivatives at crest,

$$F[m] = e^{-\eta} + \frac{117 e^{-\eta}\eta}{128} + \frac{189 e^{-\eta}\eta^2}{1024} + \frac{9 e^{-\eta}\eta^3}{1024} - \frac{369 i c^2 e^{-\eta}\eta\beta_2}{2048} - \frac{297 i c^2 e^{-\eta}\eta^2\beta_2}{2048} - \frac{27 i c^2 e^{-\eta}\eta^3\beta_2}{2048} + \frac{27}{256} c^4 e^{-\eta}\eta\beta_2^2 - \frac{27 c^4 e^{-\eta}\eta^2\beta_2^2}{4096} - \frac{27 c^4 e^{-\eta}\eta^3\beta_2^2}{4096} + \frac{135 i c^6 e^{-\eta}\eta\beta_2^3}{8192} - \frac{81 i c^6 e^{-\eta}\eta^2\beta_2^3}{8192} + \frac{9 i c^6 e^{-\eta}\eta^3\beta_2^3}{8192} + \frac{3}{256} c^3 e^{-\eta}\eta\beta_3 + \frac{81 c^3 e^{-\eta}\eta^2\beta_3}{2048} + \frac{9 c^3 e^{-\eta}\eta^3\beta_3}{2048} + \frac{153 i c^5 e^{-\eta}\eta\beta_2\beta_3}{2048} + \frac{9 i c^5 e^{-\eta}\eta^2\beta_2\beta_3}{2048} - \frac{9 i c^5 e^{-\eta}\eta^3\beta_2\beta_3}{2048} - \frac{27 c^7 e^{-\eta}\eta\beta_2^2\beta_3}{1024} + \frac{99 c^7 e^{-\eta}\eta^2\beta_2^2\beta_3}{8192} - \frac{9 c^7 e^{-\eta}\eta^3\beta_2^2\beta_3}{8192} - \frac{3}{256} c^6 e^{-\eta}\eta\beta_3^2 - \frac{9 c^6 e^{-\eta}\eta^2\beta_3^2}{4096} + \frac{3 c^6 e^{-\eta}\eta^3\beta_3^2}{4096} - \frac{105 i c^8 e^{-\eta}\eta\beta_2\beta_3^2}{8192} + \frac{39 i c^8 e^{-\eta}\eta^2\beta_2\beta_3^2}{8192} - \frac{3 i c^8 e^{-\eta}\eta^3\beta_2\beta_3^2}{8192} + \cdots\cdots$$

(34)

when continuity of derivatives at crest,

$$F[m] = -\frac{1}{4} e^{-2\eta} + \frac{5 e^{-\eta}}{4} + \frac{75 e^{-\eta}\eta}{64} + \frac{9}{64} e^{-\eta}\eta^2 - \frac{1}{2} i c^2 e^{-2\eta}\beta_2 + \frac{1}{2} i c^2 e^{-\eta}\beta_2 - \frac{3}{16} i c^2 e^{-\eta}\eta\beta_2 - \frac{9}{64} i c^2 e^{-\eta}\eta^2\beta_2 + \frac{1}{4} c^4 e^{-2\eta}\beta_2^2 - \frac{1}{4} c^4 e^{-\eta}\beta_2^2 + \frac{51}{256} c^4 e^{-\eta}\eta\beta_2^2 - \frac{9}{256} c^4 e^{-\eta}\eta^2\beta_2^2 + \frac{1}{3} c^3 e^{-2\eta}\beta_3 - \frac{1}{3} c^3 e^{-\eta}\beta_3 - \frac{3}{64} c^3 e^{-\eta}\eta\beta_3 + \frac{3}{64} c^3 e^{-\eta}\eta^2\beta_3 + \frac{1}{3} i c^5 e^{-2\eta}\beta_2\beta_3 - \frac{1}{3} i c^5 e^{-\eta}\beta_2\beta_3 + \frac{3}{16} i c^5 e^{-\eta}\eta\beta_2\beta_3 - \frac{3}{128} i c^5 e^{-\eta}\eta^2\beta_2\beta_3 - \frac{1}{9} c^6 e^{-2\eta}\beta_3^2 + \frac{1}{9} c^6 e^{-\eta}\beta_3^2 - \frac{31}{768} c^6 e^{-\eta}\eta\beta_3^2 + \frac{1}{256} c^6 e^{-\eta}\eta^2\beta_3^2 + \frac{1}{6} i c^4 e^{-2\eta}\beta_4 - \frac{1}{6} i c^4 e^{-\eta}\beta_4 - \frac{7}{128} i c^4 e^{-\eta}\eta\beta_4 + \frac{3}{256} i c^4 e^{-\eta}\eta^2\beta_4 - \frac{1}{6} c^6 e^{-2\eta}\beta_2\beta_4 + \frac{1}{6} c^6 e^{-\eta}\beta_2\beta_4 - \frac{35}{512} c^6 e^{-\eta}\eta\beta_2\beta_4 + \frac{3}{512} c^6 e^{-\eta}\eta^2\beta_2\beta_4 - \cdots\cdots$$

(35)

## 3. CONVERGENCE OF THE SERIES SOLUTIONS

Clearly Eqs. 17, 18, 21, 22, 25, 26, 28, 29, 31, 32, 34 and 35 contains the auxiliary parameters $\eta_1$, $\eta_2$, $\eta_3$, $\eta_4$ and $\eta_5$ which gives the convergence region and rate of approximation for the homotopy analysis method. For this purpose, the $\eta$ -curves are plotted for *f* for different *order of approximations*. Figures 1 to 12 are plotted for the solutions given in Eqs. 17, 18, 21, 22, 25, 26, 28, 29, 31, 32, 34 and 35, respectively. It is obvious from Fig. 1 that the range for the admissible value for $\eta_1$ is $-0.2 < \eta_1 < 0$. Figure 2 shows that the range for the admissible value for $\eta_1$ is $-0.52 < \eta_1 < 0$ . Figure 3 depicts that the range for the admissible value for $\eta_1$ is $-0.2 < \eta_1 < 0$.

Figure 4 indicates that the range for the admissible value for $\eta_1$ is $-0.8 < \eta_1 < -0.4$. Figure 5 shows that the range for the admissible value for $\eta_2$ is $-0.3 < \eta_2 < 0$. Figure 6

describes that the range for the admissible value for $\eta_2$ is $-0.3<\eta_2<0$. Figure 7 shows that the range for the admissible value for $\eta_3$ is $-0.3<\eta_3<0$. Figure 8 indicates that the range for the admissible value for $\eta_3$ is $-0.3<\eta_3<0$. Figure 9 shows that the range for the admissible value for $\eta_4$ is $-1.7<\eta_4<-0.5$. Figure 10 shows that the range for the admissible value for $\eta_4$ is $-1.2<\eta_4<-0.7$. Figure 11 describes that the range for the admissible value for $\eta_5$ is $-0.7<\eta_5<-0.6$. Figure 12 shows that the range for the admissible value for $\eta_5$ is $-1.2<\eta_5<-0.9$. These all prescribed values of $\eta_1$, $\eta_2$, $\eta_3$, $\eta_4$ and $\eta_5$ in their respective intervals shows region of convergence for their respective series solutions.

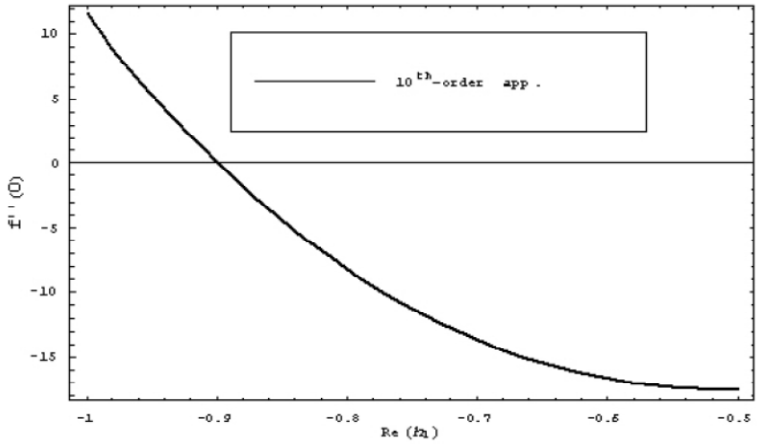


Fig. 2 $\hbar_1$-curve for discontinuity of derivative at the crest, for the $f(\eta)$ at $\alpha = 0.01$, $\gamma = 0.01$, $c = 0.01$, $a = 1$, $g(\eta) = e^{-\eta}$.

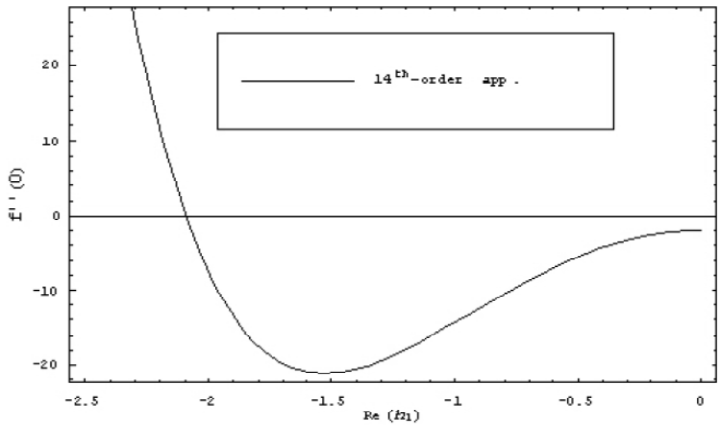


Fig. 3 $\hbar_1$-curve for continuity of derivative at the crest, for the $f(\eta)$ at $\alpha = 0.01$, $\gamma = 0.01$, $c = 0.01$, $a = 1$, $g(\eta) = 0$.

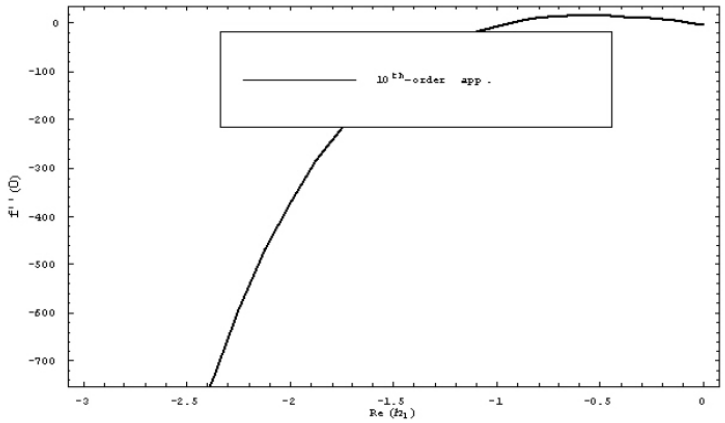


Fig. 4 $\hbar_1$-curve for continuity of derivative at the crest, for the $f(\eta)$ at $\alpha = 0.01$, $\gamma = 0.01$, $c = 0.01$, $a = 1$, $g(\eta) = e^{-\eta}$.

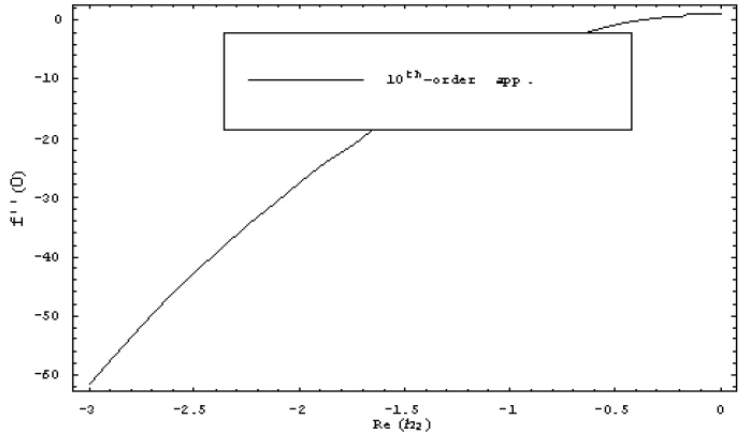


Fig. 5 $\hbar_2$-curve for discontinuity of derivative at the crest, for the $f(\eta)$ at $\alpha = 0.01$, $\gamma = 0.001$, $c = 0.01$, $\epsilon = 0.01$, $a = 1$.

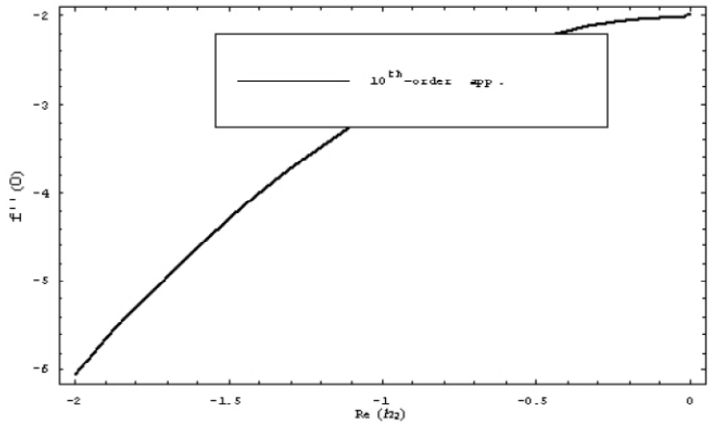


Fig. 6 $\hbar_2$-curve for continuity of derivative at the crest, for the $f(\eta)$ at $\alpha = 0.01$, $\gamma = 0.001$, $c = 0.01$, $\epsilon = 0.01$, $a = 1$.

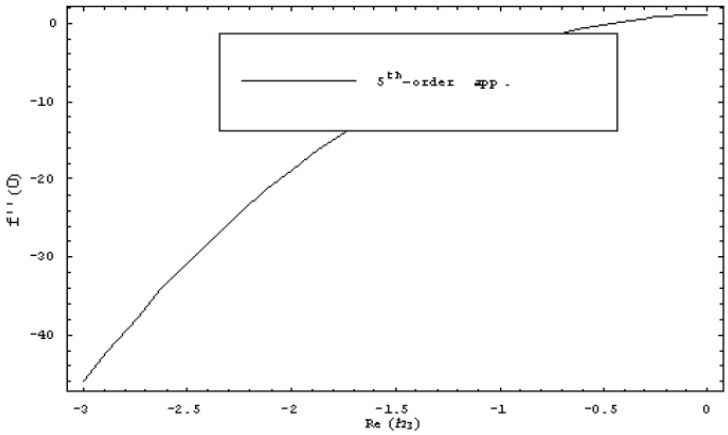


Fig. 7 $\hbar_3$-curve for discontinuity of derivative at the crest, for the $f(\eta)$ at $\alpha = 0.01$, $c = 1$, $\beta = 0.01$, $a = 1$, $\epsilon = 0.01$, $\delta = 0.01$, $\gamma = 0.01$.

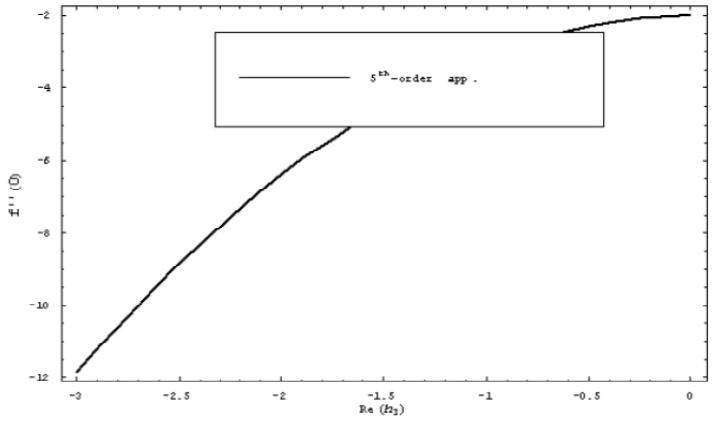


Fig. 8 $\hbar_3$-curve for continuity of derivative at the crest, for the $f(\eta)$ at $\alpha = 0.01$, $c = 1$, $\beta = 0.01$, $a = 1$, $\epsilon = 0.01$, $\delta = 0.01$, $\gamma = 0.01$.

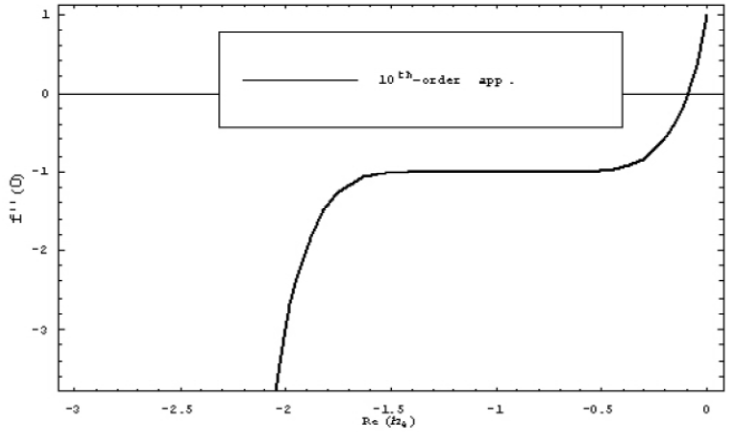


Fig. 9 $\hbar_4$-curve for discontinuity of derivative at the crest, for the $f(\eta)$ at $c = 0.01$, $a = 1$.

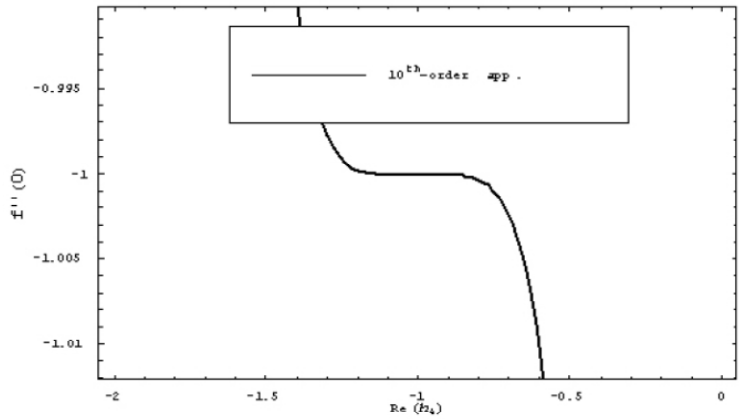


Fig. 10 $\hbar_4$-curve for continuity of derivative at the crest, for the $f(\eta)$ at $c = 0.01$, $a = 1$.

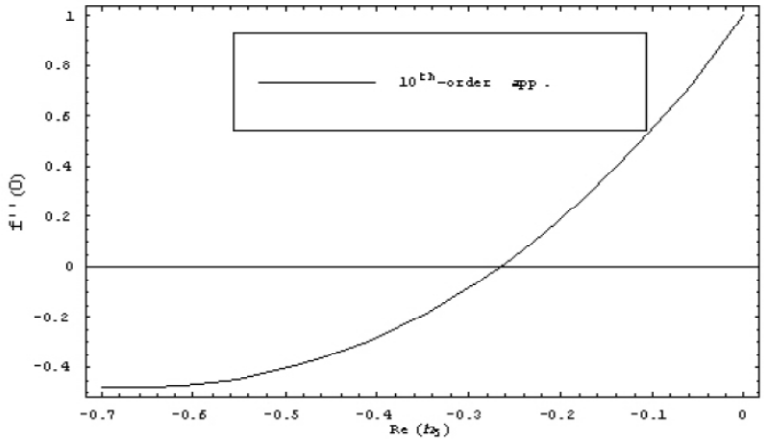


Fig. 11 $\hbar_5$-curve for discontinuity of derivative at the crest, for the $f(\eta)$ at $\beta_2 = 0.01$, $\beta_3 = 0.01$, $\beta_4 = 0.01$, $\gamma_1 = 0.01$, $\gamma_2 = 0.01$, $c = 1$, $a = 1$.

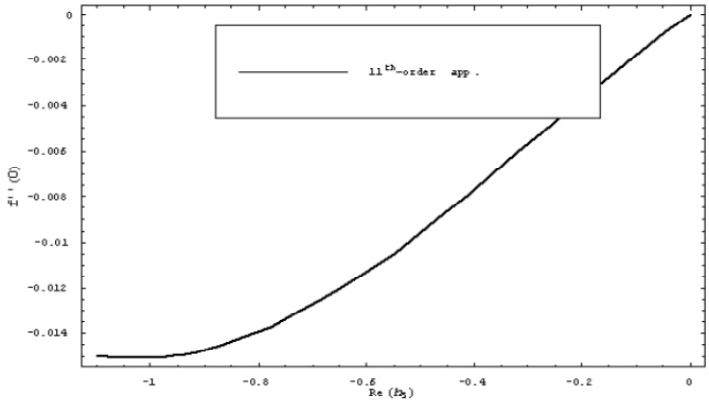


Fig. 12 $\hbar_5$-curve for continuity of derivative at the crest, for the $f(\eta)$ at $\beta_2 = 0.01$, $\beta_3 = 0.01$, $\beta_4 = 0.01$, $\gamma_1 = 0.01$, $\gamma_2 = 0.01$, $c = 1$, $a = 1$.

## 4. CONCLUSION

In this paper, solitary waves-series solutions are obtained with and without continuity of the first derivative at crest. The auxiliary linear operator in all the cases is same. The initial guess in all the cases of discontinuity of the derivative at the crest are same and for the cases having continuity of the derivative at the crest are similar. In all the cases series solutions are obtained by taking the values of homotopy parameters $\eta_1$, $\eta_2$, $\eta_3$, $\eta_4$ and $\eta_5$ from their interval of convergence. The advantage of this method over the other methods is that it itself provides us a convenient way to control the convergence of the approximation series, which shows the flexibility and potential of this method to apply it to nonlinear problems in engineering and science.

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