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An infeasible interior-point method for the P_*-matrix linear complementarity problem based on a trigonometric kernel function with full-Newton step

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Abstract: An infeasible interior-point algorithm for solving the P_*-matrix linear complementarity problem based on a kernel function with trigonometric barrier term is analyzed. Each (main) iteration of the algorithm consists of a feasibility step and several centrality steps, whose feasibility step is induced by a trigonometric kernel function. The complexity result coincides with the best result for infeasible interior-point methods for P_*-matrix linear complementarity problem.

Keywords: Linear complementarity problem, Full-Newton step, Infeasible interior-point method, Kernel function, Polynomial complexity

AMS Subject classification: 90C51, 90C22

1. Introduction

Since Karmarkar's landmark paper [4], interior point methods (IPMs) have became one of the most active research areas. They have been widely extended for solving linear optimization (LO), linear complementarity problems (LCPs) and many other problems. Due to the fact that LCP is closely related to LO, several IPMs designed for LO have been extended to P_*-LCP. Kojima et al. [11] first proved the existence of the central path for P_*-LCP and generalized the primal-dual interior-point algorithm for LO to P_*-LCP. The authors obtained polynomial iteration complexity for the algorithm, which is yet the best iteration bound for solving P_*-LCP. Miao [14] gave a generalization of Mizuno-Todd-Ye predictor-corrector algorithm [16] with

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the P_*-LCP assuming the existence of a strictly positive solution. Miao's algorithm uses the l_2-neighborhood of the central path and has both polynomial complexity and quadratic convergence. Later Potra and Sheng [19] presented a new predictor-corrector algorithm for P_*-LCPs from arbitrary positive starting points. Their algorithm is quadratically convergent for nondegenerate problems. Potra and Sheng [20] proposed a superlinearly convergent predictor-corrector method for P_*-LCPs and improved the results in Miao [14]. Illés and Nagy [2] proposed a version of the Mizuno-Todd-Ye predictor-corrector interior-point algorithm for the P_*-LCP and showed the polynomial convergence.

In the above mentioned algorithms it is assumed that the starting point satisfies exactly the equality constraints and lies in the interior of the region defined by the inequality constraints. Such a staring point is called strictly feasible. All the points generated by the algorithms are also strictly feasible. However, in practice it may be very difficult to obtain feasible starting points. Numerical experiments have shown that it is possible to obtain good practical performance by using starting points that lie in the interior of the region defined by the inequality constraints but do not satisfy the equality constraints [17]. The points generated by the method will remain in the interior of the region defined by the inequality constraints but in general will not satisfy the equality constraints. These methods are referred as infeasible interior-point methods (IIPMs), and feasibility is reached as optimality is approached. The first IIPM was proposed by Lustig [13]. Global convergence shown by Kojima et al. [10], whereas Zhang [25] and Mizuno [15] presented polynomial iteration complexity results for variants of this algorithm. In 2006, Roos [22] proposed a new IIPM for LO. It differs from the classical IIPMs (e.g Kojima et al. [11], Potra and Sheng [20], Lustig [13], Mizuno [15], Potra [18] and ect) in that the new method uses only full steps, which has the advantage that no line searches are needed. Furthermore, the iteration bound of the algorithm matches the best known iteration bound for this type of algorithms. Some kernel function-based version of the algorithm, were carried out by Liu and Sun [12] to LO and Kheirfam [7, 8] to LCP and SCO. Recently, Kheirfam [5, 9] presented full-Newton step IIPMs for P_* horizontal linear complementarity problem (HLCP) and P_*-LCP, and developed various analysis from existing methods.

Motivated by the above-mentioned works, we propose another extension of Roos' algorithm to P_*-LCP. The main iteration of the algorithm consists of a feasibility step and a few centering steps, whose feasibility step is induced by a kernel function with trigonometric barrier term. We used a norm-based proximity to measure distance of the iterates from the central path, and derive the currently best known iteration bound for P_*-LCPs. To our knowledge, this is the first infeasible interior-point algorithm for P_*-LCP based on the kernel function with full-Newton step.

The paper is organized as follows: In Sect. 2 we recall basic concepts and the notion of the central path. We review some results that provide the local quadratic convergence of the full-Newton step. Sect. 3 contains an extension of Roos' infeasible interior-point algorithm for P_*-LCP, whose feasibility step is induced by a kernel function. In Sect. 4 we analyze the feasibility step, and then derive the iteration bound for the algorithm. In Sect. 5 are reported some numerical results. Finally, the concluding remarks are drawn in Sect. 6.

2. Full-Newton Step Feasible IPM

The $P_*(\kappa)$-LCP requires the computation of a vector pair $(x,s)\in R^n\times R^n$ satisfying

$$-Mx+s=q,\ xs=0,\quad x,s\geq 0, \tag{1}$$

where $q\in R^n$ and $M\in R^{n\times n}$ is a $P_*(\kappa)$-matrix. The class of P_*-matrices was introduced by Kojima et al. [11] and it contains many types of matrices encountered in practical applications. Let κ be a nonnegative number. A matrix M is called a $P_*(\kappa)$-matrix iff it satisfies the following condition:

$$(1+4\kappa)\sum_{i\in I_+}x_i(Mx)_i+\sum_{i\in I_-}x_i(Mx)_i\geq 0,\quad \forall x\in R^n,$$

where $I_+=\{i: x_i(Mx)_i\geq 0\}$ and $I_-=\{i: x_i(Mx)_i<0\}$ are two index sets. The class of all $P_*(\kappa)$-matrices is denoted by $P_*(\kappa)$, and the class P_* is defined by $P_*=\bigcup_{\kappa\geq 0}P_*(\kappa)$, i.e., M is a P_*-matrix iff $M\in P_*(\kappa)$ for some $\kappa\geq 0$. Obviously, $P_*(0)$ is the class of positive semidefinite matrices.

The concept of the central path plays a critical role in the development of IPMs. Kojima et al. [11] first proved the existence and uniqueness of the central path for $P_*(\kappa)$-LCP. Throughout the paper, we assume that $P_*(\kappa)$-LCP satisfies the interior-point condition (IPC), i.e., there exists a pair $(x^0,s^0)>0$ such that $s^0=Mx^0+q$, which implies the existence of a solution for $P_*(\kappa)$-LCP [11]. The basic idea of the path-following IPMs is to replace the second equation in (1), the so-called complementarity condition for $P_*(\kappa)$-LCP, by the relaxed equation $xs=\mu e$ with $\mu>0$. Thus, we consider the system

$$-Mx+s=q,\ xs=\mu e,\ x,s\geq 0. \tag{2}$$

Since M is a $P_*(\kappa)$-matrix and the IPC holds, the system (2) has a unique solution for each $\mu>0$ (cf. Lemma 4.3 in [11]). This solution is denoted as $(x(\mu),s(\mu))$ and is called the μ-center of $P_*(\kappa)$-LCP. The set of μ-centers gives a homotopy path, which is called the central path of $P_*(\kappa)$-LCP. If $\mu\to 0$, then the limit of the central path exists and yields a solution for $P_*(\kappa)$-LCP (Theorem 4.4 in [11]).

A promising way to define a search direction is to follow Newton's approach and linearize the second equation in (2), which leads to the system

$$M\Delta x-\Delta s=0,\quad x\Delta s+s\Delta x=\mu e-xs. \tag{3}$$

Since M is a $P_*(\kappa)$-matrix, the system (3) uniquely defines $(\Delta x,\Delta s)$ for any $x>0$ and $s>0$. For ease of analysis, we define

$$v:=\sqrt{\frac{xs}{\mu}},\quad d_x:=\frac{v\Delta x}{x},\quad d_s:=\frac{v\Delta s}{s}. \tag{4}$$

This enables us to rewrite the system (3) as follows:

$$\overline{M}d_x - d_s = 0, \quad d_x + d_s = v^{-1} - v, \tag{5}$$

where $\overline{M} := DMD$ and $D := \operatorname{diag}\big(\sqrt{\frac{x}{s}}\big)$. Since M is $P_*(\kappa)$-matrix, it follows that $\overline{M}$ is also $P_*(\kappa)$-matrix. Thus, the system (5) has a unique solution. The new search directions d_x and d_s are obtained by solving (5) so that Δx and Δs are computed via (4). The new iterate is obtained by taking a full-Newton step according to

$$x^+ := x + \Delta x, \quad s^+ := s + \Delta s.$$

For the analysis of the algorithm, we define a norm-based proximity measure as follows:

$$\delta(v) := \delta(x, s; \mu) := \frac{1}{2}\|v^{-1} - v\|. \tag{6}$$

Here, we recall some results which are needed for the analysis of the algorithm.

Lemma 1. (Lemma II.62 in [23]) *Let $\delta := \delta(x, s; \mu)$. Then*

$$\frac{1}{\rho(\delta)} \leq v_i \leq \rho(\delta), \quad i = 1, 2, \ldots, n, \text{ where } \rho(\delta) := \delta + \sqrt{1 + \delta^2}.$$

Lemma 2. (Lemma 3 in [5]) *Let $\delta := \delta(x, s; \mu) < \frac{1}{\sqrt{2(1+2\kappa)}}$. Then, the full-Newton step is strictly feasible and*

$$\delta(x^+, s^+; \mu) \leq \frac{(1 + 2\kappa)\delta^2}{\sqrt{1 - 2(1 + 2\kappa)\delta^2}}.$$

Corollary 1. *If $\delta := \delta(x, s; \mu) \leq \frac{1}{2(1+2\kappa)}$, then*

$$\delta(x^+, s^+; \mu) \leq \sqrt{2}(\sqrt{1 + 2\kappa}\delta)^2,$$

which shows the local quadratic convergence of the full-Newton step.

3. Full-Newton Step IIPM

In the case of an IIPM, we call the pair (x, s) an ϵ-solution of $P_*(\kappa)$-LCP iff

$$\max\{\|s - Mx - q\|, x^T s\} \leq \epsilon.$$

As usual for IIPMs, we assume that the initial iterate (x^0, s^0) is as follows

$$(x^0, s^0) = (\rho_p e, \rho_d e) \text{ and } \mu^0 = \rho_d \rho_p, \tag{7}$$

where ρ_p and ρ_d are (positive) numbers such that

$$\|x^*\|_\infty \leq \rho_p, \quad \{\|s^*\|_\infty, \rho_p\|Me\|_\infty, \|q\|_\infty\} \leq \rho_d, \tag{8}$$

for some solution (x^*, s^*). The initial value of the residual vector is denoted as $r^0 := s^0 - q - Mx^0$. In general, $r^0 \neq 0$, i.e., the initial iterate is not feasible. However, a sequence of perturbed problems is generated below in such a way that the initial iterate is strictly feasible for the first perturbed problem in the sequence. For this purpose, for any ν with $0 < \nu \leq 1$, the perturbed problem pertaining to $P_*(\kappa)$-LCP is given by

$$s - q - Mx = \nu r^0, \quad (x, s) \geq 0. \tag{P_ν}$$

It is obvious that $(x, s) = (x^0, s^0)$ is a strictly feasible solution of (P_ν) when $\nu = 1$. This means that if $\nu = 1$, then (P_ν) satisfies the IPC.

Lemma 3. (Lemma 3.1 in [9]) *Let the problem* (1) *is feasible and* $0 < \nu \leq 1$. *Then, the perturbed problem* (P_ν) *satisfies the IPC.*

Let the problem (1) be feasible and $0 < \nu \leq 1$. Then Lemma 3 implies that the problem (P_ν) satisfies the IPC, and therefore its central path exists. This means that the system

$$s - q - Mx = \nu r^0, \ xs = \mu e, \quad (x, s) \geq 0, \tag{9}$$

has a unique solution $(x(\mu, \nu), s(\mu, \nu))$, for every $\mu > 0$ that is called a μ-center of the problem (P_ν). In the sequel, the parameters μ and ν always satisfy the relation $\mu = \nu\mu^0 = \nu\rho_p\rho_d$. It is also worth noting that, according to (7), $x^0 s^0 = \rho_p\rho_d e = \mu^0 e$. Hence (x^0, s^0) is the μ^0-center of the perturbed problem (P_ν) for $\nu = 1$. In other words, $(x(\mu^0, 1), s(\mu^0, 1)) = (x^0, s^0)$ and the algorithm can easily be started since we have the initial starting point that is exactly on the central path of (P_ν) for $\nu = 1$.
The outline of one iteration of the algorithm is as follows. Suppose that for some $\nu \in (0, 1]$ we have an iterate (x, s) which satisfies the feasibility condition, i.e., the first equation of the system (9) for $\mu = \nu\mu^0$, and such that $\delta(x, s; \mu) \leq \tau$. This is certainly true at the start of the first iteration, because initially we have $\delta(x, s; \mu) = 0$. Each main iteration of the algorithm consists of a feasibility step, a μ-update and a few centering steps. The feasibility step serves to get an iterate (x^f, s^f) that is strictly feasible for (P_{ν^+}) where $\nu^+ = (1 - \theta)\nu$ with $0 < \theta < 1$, and belongs to the quadratic convergence region with respect to the μ^+-center of (P_{ν^+}) with $\mu^+ = (1 - \theta)\mu$, i.e., $\delta(x^f, s^f; \mu^+) \leq \frac{1}{2(1+2\kappa)}$. After the feasibility step, we perform a few centering steps in order to get iterates (x^+, s^+) which satisfy $\delta(x^+, s^+; \mu^+) \leq \tau$.

3.1. The Feasibility Step

Suppose that (x, s) is a strictly feasible solution for (P_ν). This means that (x, s) satisfies the first equation of (9). We need displacements $\Delta^f x$ and $\Delta^f s$ such that

$$(x^f, s^f) := (x + \Delta^f x, s + \Delta^f s) \tag{10}$$

is feasible for (P_{ν^+}), this implies that the first equation in the following system is satisfied

$$M\Delta^f x - \Delta^f s = \theta\nu r^0, \;\; x\Delta^f s + s\Delta^f x = \mu e - xs. \tag{11}$$

The system (11) defines the feasibility iterates uniquely since the coefficients matrix of the resulting system is exactly the same as in the feasible case. We define the scaled search directions

$$d_x^f := \frac{v\Delta^f x}{x}, \quad d_s^f := \frac{v\Delta^f s}{s}, \tag{12}$$

where v is defined as in (4). The system (11) can be expressed as follows:

$$\overline{M} d_x^f - d_s^f = \theta\nu v s^{-1} r^0, \quad d_x^f + d_s^f = v^{-1} - v. \tag{13}$$

It is clear that the right-hand side of the second equation in (13) coincides with the negative gradient of the logarithmic barrier function

$$\Phi(v) = \sum_{i=1}^{n} \big(\frac{v_i^2 - 1}{2} - \log v_i\big).$$

This coincidence motivates a new feasibility step, which is defined by the following system:

$$\overline{M} d_x^f - d_s^f = \theta\nu v s^{-1} r^0, \quad d_x^f + d_s^f = -\nabla\Psi(v), \tag{14}$$

where $\Psi(v)$ is a kernel function-based barrier function [6] as follows

$$\psi(t) := \frac{t^2 - 1}{2} + \frac{4}{\pi}\cot(h(t)), \quad \text{where} \;\; h(t) = \frac{\pi t}{1+t}, \;\; t > 0.$$

Since

$$\psi^{'}(t) = t - \frac{4}{\pi} h^{'}(t)(1 + \cot^2(h(t))) = t - \frac{4}{(1+t)^2}\csc^2(h(t)),$$

the second equation in (14) can be written as follows

$$d_x^f + d_s^f = 4(e+v)^{-2}\csc^2(h(v)) - v.$$

A solution of the system (14) returns d_x^f and d_s^f, and then $\Delta^f x$ and $\Delta^f s$ compute via (12). The new iterates are obtained by taking a full step, as given by (10). We conclude that after the feasibility step, we have iterate (x^f, s^f) that satisfies the first equation of (P_ν) with ν replaced by ν^+. In the analysis, we should also guarantee that x^f and s^f are positive and

$$\delta(x^f, s^f; \mu^+) \leq \frac{1}{2(1+2\kappa)}. \tag{15}$$

If this is satisfied then, by using Corollary 1, the required number of centering steps can easily be obtained. Starting at (x^f, s^f), after k centering steps we will have the iterate $(x^+, s^+) := (x^k, s^k)$ that is still feasible for (P_{ν^+}) and satisfies

$$\begin{aligned}
\delta(x^+, s^+; \mu^+) &\leq \left(\sqrt[4]{2}\sqrt{1+2\kappa}\,\delta(v^{k-1})\right)^2 \leq \left(\sqrt[4]{2}\sqrt{1+2\kappa}\left[\left(\sqrt[4]{2}\sqrt{1+2\kappa}\,\delta(v^{k-2})\right)^2\right]\right)^2 \\
&\leq \left(\sqrt[4]{2}\sqrt{1+2\kappa}\right)^{2^{k+1}-2}\delta(v^f)^{2^k} = \frac{1}{\sqrt{2}(1+2\kappa)}\left(\sqrt{2}(1+2\kappa)\delta(v^f)\right)^{2^k} \\
&\leq \frac{1}{\sqrt{2}(1+2\kappa)}\left(\frac{1}{\sqrt{2}}\right)^{2^k}.
\end{aligned}$$

From this one easily deduces that $\delta(x^+, s^+; \mu^+) \leq \tau$ will hold after at most

$$\left\lceil 1 + \log_2\left(\log_2 \frac{1}{\sqrt{2}\tau(1+2\kappa)}\right)\right\rceil = 1 + \left\lceil \log_2\left(\log_2 \frac{1}{\sqrt{2}\tau(1+2\kappa)}\right)\right\rceil \tag{16}$$

centering steps.

3.2. Algorithm

The steps of the algorithm are summarized as Algorithm 1.

Algorithm 1 : The full − Newton step IIPM .

```
Input : accuracy parameter ε > 0;
    barrier update parameter θ, 0 < θ < 1;
    threshold parameter 0 < τ < 1.
begin
    x := ρ_p e; s := ρ_d e; μ := ρ_p ρ_d; ν = 1;
    while max (x^T s, ν‖r^0‖) > ε
            (x, s) := (x, s) + (Δ^f x, Δ^f s);
        μ and ν − update :
            μ := (1 − θ)μ, ν := (1 − θ)ν;
        while δ(x, s; μ) > τ
                (x, s) := (x, s) + (Δx, Δs);
        endwhile
    endwhile
end.
```

4. Analysis of the Algorithm

Let x and s denote the iterates at the start of an iteration such that $\delta(x, s; \mu) \leq \tau$. Recall that at the start of the first iteration this is certainly true, because $\delta(x^0, s^0; \mu^0) = 0$.

4.1. The Effect of the Feasibility Step

Recall that the feasibility step generates new iterates (x^f, s^f) that satisfy (P_ν) with $\nu = \nu^+$, except possibly the positive conditions. A crucial element in the analysis is to show that after the feasibility step $\delta(x^f, s^f; \mu^+) \leq \frac{1}{2(1+2\kappa)}$ and x^f and s^f are positive. Note that, by using the second equation in (14) we have

$$x^f s^f = \frac{xs}{v^2}(v + d_x^f)(v + d_s^f) = \mu\big(v^2 + v(d_x^f + d_s^f) + d_x^f d_s^f\big)$$
$$= \mu\big(4v(e+v)^{-2}\csc^2(h(v)) + d_x^f d_s^f\big). \qquad (17)$$

The lemma below provides a sufficient condition for the strict feasibility of the feasibility step (x^f, s^f).

Lemma 4. *The new iterate (x^f, s^f) is strictly feasible if and only if*

$$4v(e+v)^{-2}\csc^2(h(v)) + d_x^f d_s^f > 0.$$

Proof. We introduce a step length α with $0 \leq \alpha \leq 1$, and we define

$$x(\alpha) := x + \alpha\Delta^f x, \qquad s(\alpha) := s + \alpha\Delta^f s.$$

We hence have $x(0) = x, s(0) = s, x(1) = x^f, s(1) = s^f$ and $x(0)s(0) = xs > 0$. On the other hand, we have

$$\begin{aligned}
x(\alpha)s(\alpha) &= \mu(v + \alpha d_x^f)(v + \alpha d_s^f)\\
&= \mu\Big(v^2 + \alpha v(d_x^f + d_s^f) + \alpha^2 d_x^f d_s^f\Big)\\
&= \mu\Big(v^2 + \alpha v\left(4(e+v)^{-2}\csc^2(h(v)) - v\right) + \alpha^2 d_x^f d_s^f\Big)\\
&> \mu\Big((1-\alpha)v^2 + 4\alpha v(e+v)^{-2}\csc^2(h(v)) + \alpha^2\Big(-4v(e+v)^{-2}\csc^2(h(v))\Big)\Big)\\
&= \mu\Big((1-\alpha)v^2 + 4\alpha(1-\alpha)v(e+v)^{-2}\csc^2(h(v))\Big) \geq 0.
\end{aligned}$$

Hence, none of the entries of $x(\alpha)$ and $s(\alpha)$ vanishes, for $0 < \alpha \leq 1$. Since $x(0)$ and $s(0)$ are positive and $x(\alpha)$ and $s(\alpha)$ depend linearly on α, this implies that $x(\alpha) > 0$ and $s(\alpha) > 0$ for $0 < \alpha \leq 1$. Hence, $x(1)$ and $s(1)$ are positive and this completes the proof. $\square$

Lemma 5. *For $\frac{\pi}{2} \leq t \leq \pi$, one has*

$$\cos(t) \leq \frac{2(\pi-2)}{\pi^2}(\pi-t)^2 + \frac{4-\pi}{\pi}(\pi-t) - 1.$$

Proof. For $\frac{\pi}{2} \leq t \leq \pi$, we have $0 \leq \pi - t \leq \frac{\pi}{2}$. Using the inequality

$$\cos(z) \geq 1 - \frac{4-\pi}{\pi}z - \frac{2(\pi-2)}{\pi^2}z^2,$$

for $0 \leq z \leq \frac{\pi}{2}$ (see Remark 2.1 in [21]), we get

$$-\cos(t) = \cos(\pi - t) \geq 1 - \frac{4-\pi}{\pi}(\pi-t) - \frac{2(\pi-2)}{\pi^2}(\pi-t)^2.$$

This implies the desired result. $\square$

Lemma 6. *For $t > 0$, one has*

$$\frac{4t}{(1+t)^2}\csc^2(h(t)) - 1 \geq 0.$$

Proof. If $0 < t \leq 1$, then we have $0 < h(t) \leq \frac{\pi}{2}$. In this case, by using the inequality $\sin(z) \leq \frac{4}{\pi}z - \frac{4}{\pi^2}z^2$ for $0 < z \leq \frac{\pi}{2}$ (see Remark 2.2 in [21]), we obtain

$$\begin{aligned}\frac{4t}{(1+t)^2}\csc^2(h(t)) &= \frac{4t}{(1+t)^2\sin^2(h(t))}\\ &\geq \frac{4t}{(1+t)^2\big(\frac{4}{\pi}h(t) - \frac{4}{\pi^2}h^2(t)\big)^2}\\ &= \frac{(1+t)^2}{4t} \geq 1.\end{aligned}$$

If $t > 1$, to prove the statement, we need to show

$$f(t) := \frac{4t}{(1+t)^2\sin^2(h(t))} - 1 \geq 0.$$

As $\frac{\pi}{2} < h(t) < \pi$, for $t > 1$, then we have

$$f'(t) = \frac{4\big((1-t^2)\sin(h(t)) - 2\pi t\cos(h(t))\big)}{(1+t)^4\sin^3(h(t))}.$$

By using the following inequality

$$\sin(z) \leq \frac{4(\pi-2)}{\pi^3}z^3 - \frac{12(\pi-2)}{\pi^2}z^2 + \frac{11\pi-24}{\pi}z + 8 - 3\pi$$

for $\frac{\pi}{2} \le z \le \pi$ (see Remark 2.6 in [21]), we obtain

$$(1-t^2)\sin(h(t)) \ge (1-t^2)\Big(\frac{4(\pi-2)}{\pi^3}h^3(t) - \frac{12(\pi-2)}{\pi^2}h^2(t) + \frac{11\pi-24}{\pi}h(t) + 8 - 3\pi\Big) = (1-t)\Big(\frac{\pi t^2 + 2\pi t + 8 - 3\pi}{(1+t)^2}\Big), \quad (18)$$

and from Lemma 5 we have

$$-2\pi t\cos(h(t)) \ge -2\pi t\Big(\frac{2(\pi-2)}{\pi^2}(\pi - h(t))^2 + \frac{4-\pi}{\pi}(\pi - h(t)) - 1\Big) = -2\pi t\Big(\frac{-t^2 + (2-\pi)t + \pi - 1}{(1+t)^2}\Big). \quad (19)$$

Substituting (18) and (19) in $f^{'}(t)$, we get

$$f^{'}(t) \ge \frac{4\big(\pi t^3 + (2\pi^2 - 5\pi)t^2 + (-8 + 7\pi - 2\pi^2)t - 3\pi + 8\big)}{(1+t)^6\sin^3(h(t))} = \frac{4(t-1)(\pi t^2 + (2\pi^2 - 4\pi)t + 3\pi - 8)}{(1+t)^6\sin^3(h(t))} \ge 0,$$

which implies that $f(t)$ is increasing for $t > 1$, i.e., $f(t) \ge f(1) = 0$. This completes the proof. □

Lemma 7. *For $t > 0$, the following inequality holds.*

$$\Big|1 - \frac{4t}{(1+t)^2}\csc^2(h(t))\Big| \le \frac{1}{2}\Big|t - \frac{1}{t}\Big|.$$

Proof. From Lemma 6 it suffices to prove

$$\frac{4t}{(1+t)^2}\csc^2(h(t)) - 1 \le \frac{1}{2}\Big|t - \frac{1}{t}\Big|.$$

We consider two cases: If $0 < t \le 1$, then we have $0 < h(t) \le \frac{\pi}{2}$. Using $\sin(z) \ge \frac{3}{\pi}z - \frac{4}{\pi^3}z^3$ for $0 < z \le \frac{\pi}{2}$ (see Remark 2.4 in [21]), we get

$$\sin^2(h(t)) \ge \Big(\frac{3}{\pi}h(t) - \frac{4}{\pi^3}h^3(t)\Big)^2 = \frac{t^2(3+6t-t^2)^2}{(1+t)^6}.$$

The above inequality implies that

$$\frac{4t}{(1+t)^2}\csc^2(h(t)) = \frac{4t}{(1+t)^2\sin^2(h(t))} \le \frac{4(1+t)^4}{t(3+6t-t^2)^2}. \quad (20)$$

Now, for $0 < t \leq 1$, we have

$$\frac{4(1+t)^4}{t(3+6t-t^2)^2} - \frac{1}{2}\Big(\frac{1}{t} - t\Big) - 1 = \frac{(t-1)(t^5 - 13t^4 + 48t^3 + 68t^2 + 23t + 1)}{2t(3+6t-t^2)^2} \leq 0.$$

Using the above inequality, (20) and $|t - \frac{1}{t}| = \frac{1}{t} - t$ for $0 < t \leq 1$, we obtain

$$\frac{4t}{(1+t)^2}\csc^2(h(t)) - 1 \leq \frac{4(1+t)^4}{t(3+6t-t^2)^2} - 1 \leq \frac{1}{2}\Big(\frac{1}{t} - t\Big) = \frac{1}{2}\Big|t - \frac{1}{t}\Big|.$$

If $t > 1$, then we have $\frac{\pi}{2} \leq h(t) < \pi$. In this case, by using the inequality $\sin(z) \geq \frac{4}{\pi^3}z^3 - \frac{12}{\pi^2}z^2 + \frac{9}{\pi}z - 1$, for $\frac{\pi}{2} \leq z \leq \pi$ (see Remark 2.6 in [21]), we have

$$\sin^2(h(t)) \geq \Big(\frac{4t^3}{(1+t)^3} - \frac{12t^2}{(1+t)^2} + \frac{9t}{1+t} - 1\Big)^2 = \frac{(3t^2+6t-1)^2}{(1+t)^6},$$

which implies that

$$\frac{4t}{(1+t)^2}\csc^2(h(t)) = \frac{4t}{(1+t)^2\sin^2(h(t))} \leq \frac{4t(1+t)^4}{(3t^2+6t-1)^2}. \tag{21}$$

On the other hand, we have

$$\frac{4t(1+t)^4}{(3t^2+6t-1)^2} - 1 - \frac{1}{2}\Big(t - \frac{1}{t}\Big) = \frac{(1-t)(t^5 + 23t^4 + 68t^3 + 48t^2 - 13t + 1)}{2t(3t^2+6t-1)^2} \leq 0.$$

From the above inequality, (21) and $t - \frac{1}{t} = \big|t - \frac{1}{t}\big|$ for $t \geq 1$, it follows that

$$\frac{4t}{(1+t)^2}\csc^2(h(t)) - 1 \leq \frac{4t(1+t)^4}{(3t^2+6t-1)^2} - 1 \leq \frac{1}{2}\Big|t - \frac{1}{t}\Big|.$$

This completes the proof. □

Lemma 8. *For $t > 0$, one has*

$$\Big|t - \frac{4}{(1+t)^2}\csc^2(h(t))\Big| \leq \Big(1 + \frac{1}{2t}\Big)\Big|t - \frac{1}{t}\Big|.$$

Proof. If $0 < t \leq 1$, then by using the inequality $\sin(z) \geq \frac{3}{\pi}z - \frac{4}{\pi^3}z^3$ for $0 < z \leq \frac{\pi}{2}$ (see Remark 2.4 in [21]), we have

$$\begin{aligned}
&\frac{4}{(1+t)^2}\csc^2(h(t)) - t - \Big(1 + \frac{1}{2t}\Big)\Big(\frac{1}{t} - t\Big) \leq \\
&\qquad \frac{4}{(1+t)^2\Big(\frac{3}{\pi}h(t) - \frac{4}{\pi^3}h^3(t)\Big)^2} - t - \Big(1 + \frac{1}{2t}\Big)\Big(\frac{1}{t} - t\Big) \\
&\qquad = \frac{4(1+t)^4}{t^2(3+6t-t^2)^2} - t - \Big(1 + \frac{1}{2t}\Big)\Big(\frac{1}{t} - t\Big) \leq 0,
\end{aligned}$$

where the last inequality follows from fact that the left-hand side of the inequality is monotonically increasing for $t \le 1$. If $t > 1$, then $t - \frac{4}{(1+t)^2}\csc^2(h(t)) \ge 0$ and $t - \frac{1}{t} \ge 0$. In this case, we have

$$\begin{aligned} t - \frac{4}{(1+t)^2}\csc^2(h(t)) &= t - \frac{4}{(1+t)^2}\big(1+\cot^2(h(t))\big) \\ &\le t - \frac{4}{(1+t)^2} \le \Big(1+\frac{1}{2t}\Big)\Big(t-\frac{1}{t}\Big). \end{aligned}$$

The above two inequalities prove the desired result. $\square$

Lemma 9. *For $t > 0$, one has*

$$\frac{4t}{(1+t)^2}\csc^2(h(t)) \ge \frac{4}{\pi^2 t} + \frac{\pi^2 t}{36}.$$

Proof. As $0 < h(t) < \pi$, for $t > 0$, then by using the inequality $\csc(z) > \frac{1}{z} + \frac{\pi^2 z}{6(\pi^2 - z^2)}$ for $0 < z < \pi$ (see Theorem 1 in [1]), we have

$$\begin{aligned} \frac{4t}{(1+t)^2}\csc^2(h(t)) &\ge \frac{4t}{(1+t)^2}\left(\frac{1}{h(t)} + \frac{\pi^2 h(t)}{6(\pi^2 - h^2(t))}\right)^2 \\ &= \frac{4t}{(1+t)^2}\left(\frac{1+t}{\pi t} + \frac{\pi t(1+t)}{6(1+2t)}\right)^2 \\ &= \frac{4}{\pi^2 t} + \frac{t(\pi^2 t^2 + 24t + 12)}{9(1+2t)^2} \\ &\ge \frac{4}{\pi^2 t} + \frac{\pi^2 t}{36}, \end{aligned}$$

where the last inequality follows from the fact that $g(t) := \frac{\pi^2 t^2 + 24t + 12}{(1+2t)^2}$ is nonnegative and decreasing for $t > 0$, so $g(t) \ge \lim_{t\to\infty} g(t) = \dfrac{\pi^2}{4}$. Therefore, the proof is complete. $\square$

In the sequel, we denote

$$w := \frac{1}{2}\sqrt{\|d_x^f\|^2 + \|d_s^f\|^2},$$

which implies

$$\|d_x^f d_s^f\| \le \|d_x^f\|\|d_s^f\| \le \frac{1}{2}(\|d_x^f\|^2 + \|d_s^f\|^2) = 2w^2, \tag{22}$$

$$|d_{xi}^f d_{si}^f| \le \frac{1}{2}(|d_{xi}^f|^2 + |d_{si}^f|^2) \le 2w^2,\ i = 1, 2, \cdots, n. \tag{23}$$

In what follows, we denote $\delta(x^f, s^f; \mu^+)$ also briefly by $\delta(v^f)$, where $v^f := \sqrt{\frac{x^f s^f}{\mu^+}}$.

Lemma 10. *Let the iterate (x^f, s^f) be strictly feasible. Then, we have*

$$v^f_{\min} \geq \sqrt{\frac{3-10\rho(\delta)w^2}{5(1-\theta)\rho(\delta)}},$$

where $v^f_{\min} = \min_{1\leq i\leq n}\{v^f_i\}$.

Proof. Using (17), after dividing both sides by $\mu^+ = (1-\theta)\mu$, we have

$$(v^f)^2 = \frac{x^f s^f}{\mu^+} = \frac{4v(e+v)^{-2}\csc^2(h(v)) + d^f_x d^f_s}{1-\theta}. \tag{24}$$

Therefore, using (23), Lemma 9 and Lemma 1, we have

$$\begin{aligned}
(v^f_{\min})^2 &= \frac{1}{1-\theta}\min_i\left(4v_i(1+v_i)^{-2}\csc^2(h(v_i)) + d^f_{xi}d^f_{si}\right)\\
&\geq \frac{1}{1-\theta}\left(\min_i\left(4v_i(1+v_i)^{-2}\csc^2(h(v_i))\right) + \min_i\left(d^f_{xi}d^f_{si}\right)\right)\\
&\geq \frac{1}{1-\theta}\left(\min_i\left(4v_i(1+v_i)^{-2}\csc^2(h(v_i))\right) - 2w^2\right)\\
&\geq \frac{1}{1-\theta}\left(\min_i\left(\frac{4}{\pi^2}v_i^{-1} + \frac{\pi^2 v_i}{36}\right) - 2w^2\right)\\
&\geq \frac{1}{1-\theta}\left(\frac{4}{\pi^2 v_{\max}} + \frac{\pi^2}{36}v_{\min} - 2w^2\right)\\
&\geq \frac{1}{1-\theta}\left(\frac{4}{\pi^2\rho(\delta)} + \frac{\pi^2}{36\rho(\delta)} - 2w^2\right)\\
&\geq \frac{1}{1-\theta}\left(\frac{3}{5\rho(\delta)} - 2w^2\right) = \frac{1}{1-\theta}\left(\frac{3-10\rho(\delta)w^2}{5\rho(\delta)}\right).
\end{aligned}$$

This implies the desired result. □

Lemma 11. *The following inequality holds.*

$$\|e-(v^f)^2\| \leq \frac{1}{1-\theta}\left(\theta\sqrt{n} + \delta(v) + 2w^2\right).$$

Proof. Using (24), the triangular inequality, (22) and Lemma 7, we have

$$\begin{aligned}
\|e-(v^f)^2\| &= \left\|e - \frac{4v(e+v)^{-2}\csc^2(h(v)) + d^f_x d^f_s}{1-\theta}\right\|\\
&= \frac{1}{1-\theta}\left\|(1-\theta)e - 4v(e+v)^{-2}\csc^2(h(v)) - d^f_x d^f_s\right\|\\
&\leq \frac{1}{1-\theta}\left(\theta\sqrt{n} + \left\|e - 4v(e+v)^{-2}\csc^2(h(v))\right\| + 2w^2\right)\\
&\leq \frac{1}{1-\theta}\left(\theta\sqrt{n} + \frac{1}{2}\|v^{-1}-v\| + 2w^2\right)\\
&\leq \frac{1}{1-\theta}\left(\theta\sqrt{n} + \delta(v) + 2w^2\right).
\end{aligned}$$

This completes the proof. □

Lemma 12. *If* $4v(e+v)^{-2}\csc^2(h(v)) + d_x^f d_s^f > 0$*, then*

$$\delta(v^f) \leq \frac{\sqrt{5\rho(\delta)}\big(\theta\sqrt{n}+\delta+2w^2\big)}{2\sqrt{(1-\theta)\big(3-10\rho(\delta)w^2\big)}}.$$

Proof. We may write, using (6),

$$\begin{aligned} 2\delta(v^f) &= \left\|v^f-(v^f)^{-1}\right\| = \left\|(v^f)^{-1}\big(e-(v^f)^2\big)\right\| \\ &\leq \left\|(v^f)^{-1}\right\|_\infty \left\|e-(v^f)^2\right\| = \frac{1}{v^f_{\min}}\left\|e-(v^f)^2\right\|. \end{aligned}$$

Using the obtained bounds in Lemma 10 and Lemma 11 the lemma follows. □

In what follows, we want to choose $0<\theta<1$, as large as possible, and such that (x^f, s^f) lies in the quadratic convergence neighborhood with respect to the μ^+-center of (P_{ν^+}), i.e., $\delta(v^f) \leq \frac{1}{2(1+2\kappa)}$. According to Lemma 12, it suffices to have

$$\frac{\sqrt{5\rho(\delta)}\big(\theta\sqrt{n}+\delta+2w^2\big)}{\sqrt{(1-\theta)\big(3-10\rho(\delta)w^2\big)}} \leq \frac{1}{1+2\kappa}. \tag{25}$$

At this stage, we choose

$$\tau = \frac{1}{16(1+2\kappa)}, \qquad \theta \leq \frac{1}{5n(1+2\kappa)}. \tag{26}$$

Since the left-hand side of (25) is monotonically increasing with respect to w^2, then, for $\delta \leq \tau$, one can verify that for

$$w \leq \frac{1}{3\sqrt{1+2\kappa}} \tag{27}$$

the inequality (25) holds.

4.2. Upper Bound for w

We start by finding some bounds for the unique solution of the linear system (14).

Lemma 13. (Corollary 2.2 in [3]) *Let x, s, a, r be four n-dimensional vectors with $x>0$ and $s>0$, and let $M \in R^{n\times n}$ be a $P_*(\kappa)$-matrix. Then the solution (u,v) of the linear system*

$$Mu - v = b, \quad Su + Xv = a, \tag{28}$$

satisfies the following inequality:

$$\|Du\|^2 + \|D^{-1}v\|^2 \le \|\tilde{a}\|^2 + 2\kappa\|\tilde{c}\|^2 + 2\|\tilde{b}\|^2 + 2\|\tilde{b}\|\sqrt{\|\tilde{a}\|^2 + \|\tilde{b}\|^2 + 2\kappa\|\tilde{c}\|^2},$$

where $D = X^{-\frac{1}{2}}S^{\frac{1}{2}},\ \tilde{a} = (XS)^{-\frac{1}{2}}a,\ \tilde{b} = D^{-1}b,\ \tilde{c} = \tilde{a} + \tilde{b}.$

Comparing system (28) with the system (14), which can be expressed equivalently as follows:

$$M\Delta^f x - \Delta^f s = \theta\nu r^0,\ \ s\Delta^f x + x\Delta^f s = -\mu v\nabla\Psi(v) \tag{29}$$

and considering $(u, v) = (\Delta^f x, \Delta^f s), b = \theta\nu r^0$ and $a = -\mu v\nabla\Psi(v)$ in the system (28), we obtain

$$\begin{aligned}
\big\|D\Delta^f x\big\|^2 + \big\|D^{-1}\Delta^f s\big\|^2 &\le \big\| - (XS)^{-\frac{1}{2}}\mu v\nabla\Psi(v)\big\|^2 + 2\big\|\theta\nu D^{-1}r^0\big\|^2 \\
&\quad + 2\kappa\big\| - (XS)^{-\frac{1}{2}}\mu v\nabla\Psi(v) + \theta\nu D^{-1}r^0\big\|^2 + 2\big\|\theta\nu D^{-1}r^0\big\| \\
&\sqrt{\big\| - (XS)^{-\frac{1}{2}}\mu v\nabla\Psi(v)\big\|^2 + \big\|\theta\nu D^{-1}r^0\big\|^2 + 2\kappa\big\| - (XS)^{-\frac{1}{2}}\mu v\nabla\Psi(v) + \theta\nu D^{-1}r^0\big\|^2} \\
&= \mu\big\|\nabla\Psi(v)\big\|^2 + 2\theta^2\nu^2\big\|D^{-1}r^0\big\|^2 + 2\kappa\big\|\theta\nu D^{-1}r^0 - \sqrt{\mu}\nabla\Psi(v)\big\|^2 \\
&\quad + 2\theta\nu\big\|D^{-1}r^0\big\|\sqrt{\mu\big\|\nabla\Psi(v)\big\|^2 + \theta^2\nu^2\big\|D^{-1}r^0\big\|^2 + 2\kappa\big\|\theta\nu D^{-1}r^0 - \sqrt{\mu}\nabla\Psi(v)\big\|^2}.
\end{aligned}$$

Let (x^*, s^*) be the optimal solution of (1) that satisfies (8) and suppose that the algorithm starts with $(x^0, s^0) = (\rho_p e, \rho_d e)$. Then, we have

$$x^* - x^0 \le \rho_p e, \quad s^* - s^0 \le \rho_d e. \tag{30}$$

Also, by using (8) and (30), we get

$$\begin{aligned}
\|D^{-1}r^0\| = \left\|\sqrt{\frac{x}{s}}r^0\right\| = \left\|\frac{xr^0}{\sqrt{xs}}\right\| &\le \frac{1}{\sqrt{\mu}v_{\min}}\|xr^0\|_1 \\
&\le \frac{1}{\sqrt{\mu}v_{\min}}\|(S^0)^{-1}r^0\|_\infty\|s^0\|_\infty\|x\|_1 \\
&\le \frac{1}{\sqrt{\mu}v_{\min}}\Big(1 + \frac{\rho_p}{\rho_d}\|Me\|_\infty + \frac{1}{\rho_d}\|q\|_\infty\Big)\rho_d\|x\|_1 \le \frac{3\rho_d}{\sqrt{\mu}v_{\min}}\|x\|_1.
\end{aligned} \tag{31}$$

Lemma 14. *Let* $\delta := \delta(v)$*. Then, one has*

$$\big\|\nabla\Psi(v)\big\| \le (2 + \rho(\delta))\delta.$$

Proof. Using Lemma 8 and Lemma 1, we have

$$\begin{aligned}\|\nabla\Psi(v)\| &= \|v - 4(e+v)^{-2}\csc^2(h(v))\| \le \left\|\left(e+\frac{1}{2}v^{-1}\right)\left(v^{-1}-v\right)\right\| \\ &\le \left\|e+\frac{1}{2}v^{-1}\right\|_\infty \|v^{-1}-v\| = 2\left(1+\frac{1}{2v_{\min}}\right)\delta \\ &\le 2\left(1+\frac{\rho(\delta)}{2}\right)\delta = (2+\rho(\delta))\delta.\end{aligned}$$

This completes the proof of lemma. □

Using (31) and Lemma 14, we get

$$\begin{aligned}\left\|\theta\nu D^{-1}r^0 - \sqrt{\mu}\nabla\Psi(v)\right\| &\le \theta\nu\left\|D^{-1}r^0\right\| + \sqrt{\mu}\left\|\nabla\Psi(v)\right\| \\ &\le \frac{3\theta\nu\rho_d\|x\|_1}{\sqrt{\mu}v_{\min}} + \sqrt{\mu}\delta\big(2+\rho(\delta)\big). \end{aligned} \tag{32}$$

Therefore, using bounds in (31), (32) and the equations $D^{-1}\Delta^f s = \sqrt{\mu}d_s^f$ and $D\Delta^f x = \sqrt{\mu}d_x^f$, we obtain

$$\begin{aligned}\|d_x^f\|^2 + \|d_s^f\|^2 \le{}& \delta^2(2+\rho(\delta))^2 + \frac{18\theta^2\nu^2\rho_d^2}{\mu^2 v_{\min}^2}\|x\|_1^2 + 2\kappa\left(\frac{3\theta\nu\rho_d}{\mu v_{\min}}\|x\|_1 + \delta(2+\rho(\delta))\right)^2 \\ &+ \frac{6\theta\nu\rho_d}{\mu v_{\min}}\|x\|_1\sqrt{\delta^2(2+\rho(\delta))^2 + \frac{9\theta^2\nu^2\rho_d^2}{\mu^2 v_{\min}^2}\|x\|_1^2 + 2\kappa\left(\frac{3\theta\nu\rho_d}{\mu v_{\min}}\|x\|_1 + \delta(2+\rho(\delta))\right)^2}.\end{aligned} \tag{33}$$

Lemma 15. (Lemma 12 in [5]) *Let (x,s) be feasible for the perturbed problem (P_ν) and let (x^0,s^0) and (x^*,s^*) be as defined in (7) and (8), respectively. Then,*

$$\|x\|_1 \le (1+4\kappa)\big(2+\rho(\delta)^2\big)n\rho_p.$$

Substituting the bounds of $\|x\|_1$ and $v_{\min}$ into (33) and using $\mu = \nu\rho_p\rho_p$, we obtain

$$\begin{aligned}\|d_x^f\|^2 + \|d_s^f\|^2 \le{}& \delta^2(2+\rho(\delta))^2 + 18(1+4\kappa)^2\theta^2 n^2\rho(\delta)^2(2+\rho(\delta)^2)^2 \\ &+2\kappa\Big(3(1+4\kappa)n\theta\rho(\delta)(2+\rho(\delta)^2) + \delta(2+\rho(\delta))\Big)^2 + \\ &6(1+4\kappa)n\theta\rho(\delta)(2+\rho(\delta)^2)\Bigg(\delta^2(2+\rho(\delta))^2 + 9(1+4\kappa)^2\theta^2n^2\rho(\delta)^2(2+\rho(\delta)^2)^2 + \\ &2\kappa\Big(3(1+4\kappa)n\theta\rho(\delta)(2+\rho(\delta)^2) + \delta(2+\rho(\delta))\Big)^2\Bigg)^{\frac{1}{2}}.\end{aligned} \tag{34}$$

4.3. Fixing θ and Complexity Analysis

Since $\delta \leq \tau$ and the right-hand side of (34) is monotonically increasing in δ, so it follows that

$$\begin{aligned}\|d_x^f\|^2 + \|d_s^f\|^2 \leq \tau^2(2+\rho(\tau))^2 &+ 18(1+4\kappa)^2\theta^2 n^2\rho(\tau)^2(2+\rho(\tau)^2)^2 \\ &+2\kappa\Big(3(1+4\kappa)n\theta\rho(\tau)(2+\rho(\tau)^2)+\tau(2+\rho(\tau))\Big)^2 + \\ 6(1+4\kappa)n\theta\rho(\tau)(2+\rho(\tau)^2)\Bigg(&\tau^2(2+\rho(\tau))^2 + 9(1+4\kappa)^2\theta^2n^2\rho(\tau)^2(2+\rho(\tau)^2)^2 + \\ &2\kappa\Big(3(1+4\kappa)n\theta\rho(\tau)(2+\rho(\tau)^2)+\tau(2+\rho(\tau))\Big)^2\Bigg)^{\frac{1}{2}}.\end{aligned}$$

Using (26), we have found that $\delta(v^f) \leq \frac{1}{2(1+2\kappa)}$ holds if the inequality (27) is satisfied. Then, by the above inequality, (27) holds if

$$\begin{aligned}\tau^2(2+\rho(\tau))^2 &+ 18(1+4\kappa)^2\theta^2 n^2\rho(\tau)^2(2+\rho(\tau)^2)^2 \\ &+2\kappa\Big(3(1+4\kappa)n\theta\rho(\tau)(2+\rho(\tau)^2)+\tau(2+\rho(\tau))\Big)^2 + \\ 6(1+4\kappa)n\theta\rho(\tau)(2+\rho(\tau)^2)\Bigg(&\tau^2(2+\rho(\tau))^2 + 9(1+4\kappa)^2\theta^2n^2\rho(\tau)^2(2+\rho(\tau)^2)^2 + \\ &2\kappa\Big(3(1+4\kappa)n\theta\rho(\tau)(2+\rho(\tau)^2)+\tau(2+\rho(\tau))\Big)^2\Bigg)^{\frac{1}{2}} \leq \frac{4}{9(1+2\kappa)}.\end{aligned}$$

One may easily verify that, by some elementary calculation, the above inequality is satisfied if

$$\tau = \frac{1}{16(1+2\kappa)}, \qquad \theta = \frac{1}{33n(1+2\kappa)^3}, \tag{35}$$

which is in agreement with (26). Note that we have found that if at the start of an iteration, $\delta(x, s; \mu) \leq \tau$, then after the feasibility step, $\delta(x^f, s^f; \mu^+) \leq \frac{1}{2(1+2\kappa)}$. According to (16), at most

$$1 + \left\lceil \log_2\left(\log_2 \frac{1}{\sqrt{2}\tau(1+2\kappa)}\right)\right\rceil = 2$$

centering steps are needed to get iterates (x^+, s^+) such that $\delta(x^+, s^+; \mu^+) \leq \tau$. In each main iteration both the duality gap and the norm of the residual vector are reduced by the factor $1-\theta$. Hence, the total number of main iterations is bounded above by

$$\frac{1}{\theta}\log\frac{\max\{(x^0)^T s^0, \|r^0\|\}}{\epsilon}.$$

So, due to (35) the total number of inner iterations is bounded above by

$$99n(1+2\kappa)^3 \log \frac{\max\{(x^0)^T s^0, \|r^0\|\}}{\epsilon}.$$

In the following we state our main result without further proof.

Theorem 1. *If the problem* (1) *has a solution* (x^*, s^*) *such that* $\|x^*\|_\infty \leq \rho_p$ *and* $\|s^*\|_\infty \leq \rho_d$, *then after at most*

$$99n(1+2\kappa)^3 \log \frac{\max\{(x^0)^T s^0, \|r^0\|\}}{\epsilon},$$

inner iterations, the algorithm finds an ϵ*-optimal solution of* (1).

5. Numerical results

We test our algorithm on some instances. We write simple MATLAB codes for our Algorithm and the algorithm of Zhang et al. [24]. In our experiments, we choose $x = \rho_p e, s = \rho_d e$ and $\mu = \rho_p \rho_d$ as the starting data. In order to guarantee the convergence property of these algorithms, we take the parameters τ and θ as $\tau = \frac{1}{16(1+2\kappa)}, \theta = \frac{1}{33n(1+2\kappa)^3}$ and $\tau = \frac{1}{16}, \theta = \frac{1}{33n}$ respectively. We terminate the algorithms if $x^T s \leq \epsilon = 10^{-4}$. Table 1 shows the required number of iterations for $P_*(0)$-LCP problems corresponding to positive semidefinite matrices, with various sizes, as follows.

$$M_{1,n} = \begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}, \quad M_{2,n} = \begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 5 & 6 & \cdots & 6 \\ \vdots & \vdots & \vdots & & \vdots \\ 2 & 6 & 10 & \cdots & 4n-3 \end{bmatrix}$$

Table 1

problem	$\|x^*\|_\infty \leq \rho_p$	$\|s^*\|_\infty \leq \rho_d$	$\rho_p\|Me\|_\infty \leq \rho_d$	Iter.	
				Algor. 1	Algor. [24]
$M_{2,5}$	$0.9998 \leq 2$	$1.0006 \leq 50$	$49 \leq 50$	1615	2425
$M_{2,10}$	$0.9998 \leq 2$	$1.0016 \leq 200$	$199 \leq 200$	3692	5769
$M_{2,15}$	$0.9998 \leq 1$	$1.0026 \leq 450$	$449 \leq 450$	5942	8916
$M_{2,20}$	$0.9998 \leq 1$	$1.0036 \leq 800$	$799 \leq 800$	8304	12460
$M_{1,5}$	$1 \leq 2$	$1.0002 \leq 10$	$9 \leq 10$	1514	2274
$M_{1,10}$	$1 \leq 2$	$1.0002 \leq 20$	$19 \leq 20$	3338	5010
$M_{1,20}$	$1 \leq 2$	$1.0002 \leq 40$	$39 \leq 40$	7292	10941

One can easily see that the assumptions in the theoretical results are satisfied, i.e., $\|x^*\|_\infty \leq \rho_p, \{\|s^*\|_\infty, \rho_p\|Me\|_\infty\} \leq \rho_d$ and $\mu = \rho_p\rho_d$. Based on the obtained numerical results, as is shown in Table 1, our proposed algorithm appears to have a competitive edge over the algorithm in [24].

6. Concluding Remarks

In this paper, we have proposed a full-Newton step infeasible interior-point method based on a kernel function for the P_*-matrix LCP. The iteration bound coincides with the currently best known one for IIPM. For further research, this algorithm may be possible extended to the Cartesian P_*-matrix LCP over symmetric cones.

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