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INFERENCE BASED ON *K*-RECORD VALUES FROM GENERALIZED EXPONENTIAL DISTRIBUTION

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1. Introduction

Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables having an absolutely continuous cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. An observation X_j is called a lower record if $X_j < X_i$ for every $i < j$. An analogous definition deals with upper record values. In a number of situations, only observations that exceed or only those that fall below the current extreme value are recorded. Examples include meteorology, hydrology, athletic events and mining. Interest in records has increased steadily over the years since Chandler's (1952) formulation. Useful surveys are given in Ahsanullah (1995) and Arnold *et al.* (1998). Estimation of parameters using record values and prediction of future record values have been studied by several authors, for details see Balakrishnan and Chan (1998), Raqab (2002), and Sultan *et al.* (2002). Bayesian estimation and prediction for some life distributions based on record values have been considered by Ahamadi and Doostparast (2006).

Serious difficulties for the statistical inference based on records arise due to the fact that the occurrences of record data are very rare in practical situations and the expected waiting time is infinite for every record after the first. These problems are avoided if we consider the model of *k*-record statistics introduced by Dziubdziela and Kopocinski (1976).

For a positive integer *k*, the lower *k*-record times $T_{n(k)}$ and the lower *k*-record values $R_{n(k)}$ are defined as follows

$$T_{1(k)} = k$$

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and for $n > 1$

$$T_{n(k)} = \min\left\{ j : j > T_{n-1(k)}\ ,\ X_j < X_{k:T_{n-1(k)}} \right\}\ ,$$

where $X_{i:m}$ denotes the i-th order statistic in a sample of size m. The sequence of lower k-records are then defined by for $n \geq 1$

$$R_{n(k)} = X_{k:T_{n(k)}}\ ,\ k \geq 1.$$

In an analogous way, one can define the upper k-record values. Since the ordinary record values are contained in the k-records, by putting $k = 1$, the results for usual records can be obtained as special case. Recently, a lot of works have been done on the statistical inference, based on k-records. See, for instance, Ahmadi *et al.* (2005), Ahmadi and Doostparast (2008), Mary and Chacko (2010),Chacko and Mary (2013a), Chacko and Mary (2013b), Paul and Thomas (2015) and Malinowska and Szynal (2004). The pdf of nth lower k-record value $R_{n(k)}$ for $n \geq 1$ is given by

$$f_{n(k)}(x) = \frac{k^n}{(n-1)!}[-\log F(x)]^{n-1}[F(x)]^{k-1} f(x),\ x > 0, \tag{1}$$

and the joint pdf of m th and n th lower k-record values, $R_{m(k)}$ and $R_{n(k)}$ for $m < n$ is given by

$$\begin{aligned} f_{m,n(k)}(x,y) =& \frac{k^n}{(m-1)!(n-m-1)!}[-\log F(x)]^{m-1} \\ &\times [-\log F(y) + \log F(x)]^{n-m-1} \\ &\times \frac{[F(y)]^{k-1}}{F(x)} f(x) f(y),\ y < x. \end{aligned} \tag{2}$$

Then the joint pdf of $R_{1(k)}, R_{2(k)}, ..., R_{n(k)}$ is given by

$$f(r_1, r_2, ..., r_n) = k^n [F(r_n)]^k \prod_{i=1}^{n} \frac{f(r_i)}{F(r_i)}, -\infty < r_n < r_{n-1} < \ldots < r_1 < \infty. \tag{3}$$

The generalized exponential (GE) distribution introduced by Gupta and Kundu (1999) can be used quite effectively in analyzing many lifetime data. The two parameter generalized exponential distribution (denoted by GE(σ, β)) has the pdf given by

$$f(y) = \frac{\beta}{\sigma}\left(1 - \exp\left\{-\left(\frac{y}{\sigma}\right)\right\}\right)^{\beta-1} \exp\left\{-\left(\frac{y}{\sigma}\right)\right\}, 0 < y < \infty,\ \sigma, \beta > 0. \tag{4}$$

The cdf corresponding to the above pdf is given by

$$F(y) = \left(1 - \exp\left\{-\left(\frac{y}{\sigma}\right)\right\}\right)^{\beta},\ 0 < y < \infty,\ \sigma, \beta > 0. \tag{5}$$

Gupta and Kundu (1999) have shown that the GE model fits better than the gamma or the Weibull model in certain cases. The GE distribution has increasing or decreasing hazard rate depending on the shape parameter β. Raqab (2002) derived exact expressions for means, variances and covariances of lower record values arising from GE distribution and also obtained the BLUE's of the location and scale parameters based on lower record values. The Bayesian estimation and prediction based on lower record values for the two-parameter generalized exponential distribution were discussed by Madi and Raqab (2007). The maximum likelihood and Bayesian estimation based on record values and inter-record times for the two-parameter generalized exponential distribution have been considered by Kizilaslan and Nadar (2015).

In this paper, we consider the lower *k*-record values arising from generalized exponential distribution. The paper is organized as follows. In Section 2, we obtain the maximum likelihood estimator (MLE) for scale parameter σ and shape parameter β of GE distribution. In Section 3, we consider Bayes estimation of scale parameter σ and shape parameter β of GE distribution. In Section 4, Bayesian Prediction of future record values are considered. Section 5 is devoted to some simulation studies and finally in Section 6 some concluding remarks.

2. Maximum likelihood estimation

In this section we obtain maximum likelihood estimators of the scale parameter σ and shape parameter β involved in two parameter generalized exponential distribution $GE(\sigma,\beta)$ using lower k record values. Let $R_{i(k)}$ $i=1,2,\ldots,n$ be the first n lower k-record values arising from $GE(\sigma,\beta)$ with density function given in (4).

Let $\mathbf{D_{n(k)}} = (R_{1(k)}, R_{2(k)}, \cdots, R_{n(k)})$ be the vector of first n lower k record values arising from $GE(\sigma,\beta)$. Then from (3) the likelihood function of (σ,β) is given by

$$L(\sigma,\beta|\mathbf{d_{n(k)}}) = \left(\frac{k\beta}{\sigma}\right)^n \left(1-e^{-\frac{r_n}{\sigma}}\right)^{k\beta} \prod_{i=1}^{n} \frac{e^{-\frac{r_i}{\sigma}}}{1-e^{-\frac{r_i}{\sigma}}}, \tag{6}$$

where $\mathbf{d_{n(k)}} = (r_1, r_2, \cdots, r_n)$ is the realization of $\mathbf{D_{n(k)}}$. The logarithm of the likelihood function is given by

$$\log L(\sigma,\beta|\mathbf{d_{n(k)}}) = n\log(k\beta) - n\log\sigma + k\beta\log(1-e^{-\frac{r_n}{\sigma}}) - \sum_{i=1}^{n}\frac{r_i}{\sigma} - \sum_{i=1}^{n}\log(1-e^{-\frac{r_i}{\sigma}}).$$

Then

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} + k\log(1-e^{-\frac{r_n}{\sigma}})$$

and

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} - \frac{k\beta r_n e^{-\frac{r_n}{\sigma}}}{\sigma^2(1-e^{-\frac{r_n}{\sigma}})} + \sum_{i=1}^{n} \frac{r_i}{\sigma^2} + \sum_{i=1}^{n} \frac{r_i e^{-\frac{r_i}{\sigma}}}{\sigma^2(1-e^{-\frac{r_i}{\sigma}})}.$$

Then the corresponding normal equations are

$$\frac{n}{\beta} + k\log(1-e^{-\frac{r_n}{\sigma}}) = 0 \tag{7}$$

and

$$-\frac{n}{\sigma} - \frac{k\beta r_n e^{-\frac{r_n}{\sigma}}}{\sigma^2(1-e^{-\frac{r_n}{\sigma}})} + \sum_{i=1}^{n} \frac{r_i}{\sigma^2} + \sum_{i=1}^{n} \frac{r_i e^{-\frac{r_i}{\sigma}}}{\sigma^2(1-e^{-\frac{r_i}{\sigma}})} = 0. \tag{8}$$

From (7), we get

$$\beta = -\frac{n}{k\log(1-e^{-\frac{r_n}{\sigma}})}. \tag{9}$$

Putting the value of β in (8) we get

$$-\frac{n}{\sigma} + \frac{n}{\log(1-e^{-\frac{r_n}{\sigma}})} \frac{r_n e^{-\frac{r_n}{\sigma}}}{\sigma^2(1-e^{-\frac{r_n}{\sigma}})} + \sum_{i=1}^{n} \frac{r_i}{\sigma^2} + \sum_{i=1}^{n} \frac{r_i e^{-\frac{r_i}{\sigma}}}{\sigma^2(1-e^{-\frac{r_i}{\sigma}})} = 0.$$

Therefore, the MLE of β is given by

$$\hat{\beta}_{ML} = -\frac{n}{k\log(1-e^{-\frac{r_n}{\hat{\sigma}_{ML}}})} \tag{10}$$

and the MLE of σ, $\hat{\sigma}_{ML}$ can be obtained as a solution of the non-linear equation of the form

$$g(\sigma) = \sigma, \tag{11}$$

where

$$g(\sigma) = \frac{1}{n}\left[\frac{n r_n}{\log(1-e^{-\frac{r_n}{\sigma}})} \frac{e^{-\frac{r_n}{\sigma}}}{(1-e^{-\frac{r_n}{\sigma}})} + \sum_{i=1}^{n} r_i + \sum_{i=1}^{n} \frac{r_i e^{-\frac{r_i}{\sigma}}}{(1-e^{-\frac{r_i}{\sigma}})}\right].$$

Since $\hat{\sigma}_{ML}$ is a fixed point solution of non-linear equation(11),therefore ,it can be obtained by using a simple iterative scheme as follows.

$$g(\sigma_j) = \sigma_{j+1}, \tag{12}$$

were σ_j is the jth iterate of $\hat{\sigma}_{ML}$.The iteration procedure should be stopped when $|\sigma_{j+1}-\sigma_j|<\epsilon$ where ϵ is sufficiently small positive number.Once we obtain $\hat{\sigma}_{ML}$ from (11),and the MLE of β becomes

$$\hat{\beta}_{ML}=-\frac{n}{k\log(1-e^{-\frac{r_n}{\hat{\sigma}_{ML}}})} \tag{13}$$

3. Bayesian estimation

In this section, we consider the Bayesian estimation of parameters involved in the two parameter generalized exponential distribution with scale parameter σ and shape parameter β under symmetric as well as asymmetric loss functions.

A symmetric loss function is the squared error loss (SEL) function which is defined as

$$L_1\left(d(\mu),\hat{d}(\mu)\right)=\left(\hat{d}(\mu)-d(\mu)\right)^2, \tag{14}$$

where $\hat{d}(\mu)$ is an estimate of $d(\mu)$. The Bayes estimate of μ under L_1 is the posterior mean of μ. An asymmetric loss function is the LINEX loss (LL) function which is defined as

$$L_2\left(d(\mu),\hat{d}(\mu)\right)=e^{h\left(\hat{d}(\mu)-d(\mu)\right)}-h\left(\hat{d}(\mu)-d(\mu)\right)-1,\ h\neq 0. \tag{15}$$

The Bayes estimate of $d(\mu)$ for the loss function L_2 can be obtained as

$$\hat{d}_{LB}(\mu)=-\frac{1}{h}\ \log\left\{E_{\mu}\left(e^{-h\mu}\middle|\underline{x}\right)\right\}, \tag{16}$$

provided $E_{\mu}(\cdot)$ exists. Another asymmetric loss function is the general entropy loss (EL) function given by

$$L_3\left(d(\mu),\hat{d}(\mu)\right)=\left(\frac{\hat{d}(\mu)}{d(\mu)}\right)^q-q\ \log\left(\frac{\hat{d}(\mu)}{d(\mu)}\right)-1,\ q\neq 0. \tag{17}$$

In this case Bayes estimate of $d(\mu)$ is obtained as

$$\hat{d}_{EB}(\mu)=\left(E_{\mu}\left(\mu^{-q}|\underline{x}\right)\right)^{-\frac{1}{q}}. \tag{18}$$

Let $R_{i(k)}$, $i=1,2,\ldots,n$ be the first n lower k-record values arising from $GE(\sigma,\beta)$ with density function given in (4). Then the likelihood function is given by

$$L(\sigma,\beta|\mathbf{d}_{\mathbf{n(k)}})=\left(\frac{k\beta}{\sigma}\right)^n\left(1-e^{-\frac{r_n}{\sigma}}\right)^{k\beta}\prod_{i=1}^{n}\frac{e^{-\frac{r_i}{\sigma}}}{1-e^{-\frac{r_i}{\sigma}}}, \tag{19}$$

where $\mathbf{d}_{\mathbf{n(k)}} = (r_1, r_2, \cdots, r_n)$. Assume that the prior distributions of σ and β follow independent inverse gamma distribution and gamma distribution respectively with density functions given by

$$\pi_1(\sigma|a,b) = \frac{b^a}{\Gamma a}\sigma^{-(a+1)}e^{-\frac{b}{\sigma}}, \sigma > 0 \tag{20}$$

and

$$\pi_2(\beta|c,d) = \frac{d^c}{\Gamma c}\beta^{c-1}e^{-d\beta}, \beta > 0. \tag{21}$$

Then the joint posterior density of σ and β given $\mathbf{D}_{\mathbf{n(k)}} = \mathbf{d}_{\mathbf{n(k)}}$ can be written as

$$\pi^*(\sigma,\beta|\mathbf{d}_{\mathbf{n(k)}}) = \frac{L(\sigma,\beta|\mathbf{d}_{\mathbf{n(k)}})\pi_1(\sigma|a,b)\pi_2(\beta|c,d)}{\int\int L(\sigma,\beta|\mathbf{d}_{\mathbf{n(k)}})\pi_1(\sigma|a,b)\pi_2(\beta|c,d)d\sigma d\beta}. \tag{22}$$

From (19) we have

$$\begin{aligned} L(\sigma,\beta|\mathbf{d}_{\mathbf{n(k)}})\pi_1(\sigma|a,b)\pi_2(\beta|c,d) =& \frac{k^n b^a d^c}{\Gamma a \Gamma c}\beta^{n+c-1}\sigma^{-(a+n+1)}e^{-\beta(d-kU_\sigma)} \\ & \exp\left(-\frac{b}{\sigma} - \sum_{i=1}^{n}\frac{r_i}{\sigma} - \sum_{i=1}^{n}\log(1-e^{\frac{r_i}{\sigma}})\right), \end{aligned}$$

where $U_\sigma = \log(1-e^{\frac{r_n}{\sigma}})$. Since it is not possible to compute the posterior density (19) explicitly, we propose MCMC method to generate samples from the posterior distributions of β and σ then find the Bayes estimates of β and σ.

3.1. MCMC method

In this section, we consider the MCMC method to generate samples from the posterior distribution. The posterior distribution given in (22) can be written as

$$\begin{aligned} \pi^*(\beta,\sigma|\mathbf{d}_{\mathbf{n(k)}}) \quad \propto \quad & \frac{\beta^{n+c-1}e^{-\beta(d-kU_\sigma)}}{\sigma^{(a+n+1)}} \\ & \exp\left(-\frac{b}{\sigma} - \sum_{i=1}^{n}\frac{r_i}{\sigma} - \sum_{i=1}^{n}\log(1-e^{\frac{r_i}{\sigma}})\right), \end{aligned} \tag{23}$$

From (23) the conditional posterior distribution of β given σ and *data* is given by

$$\pi^*(\beta|\sigma,\mathbf{d}_{\mathbf{n(k)}}) \propto \beta^{n+c-1}e^{-\beta(d-kU_\sigma)}. \tag{24}$$

Again from (23) the conditional posterior distribution of σ given β and *data* is given by

$$\pi^*(\sigma|\beta,\mathbf{d}_{\mathbf{n(k)}}) \propto \frac{1}{\sigma^{(a+n+1)}}e^{-\frac{1}{\sigma}(b+\sum_{i=1}^{n}r_i)-\sum_{i=1}^{n}\log(1-e^{\frac{r_i}{\sigma}})+k\beta U_\sigma}. \tag{25}$$

Thus we can see that conditional posterior distribution of β follows a gamma distribution with parameters $(n+c)$ and $(d-kU_\sigma)$. That is $\beta \sim \text{gamma}(n+c, d-kU_\sigma)$. Therefore one can easily generate sample from the posterior distribution of β. But it is not possible to generate random variables from the posterior distribution of σ given in (25) using standard random number generation methods. Hence we use Metropolis-Hasting (M-H) algorithm to generate sample from (25) (Chib and Greenberg , 1995). Since plot of (25) is similar to a normal plot we take normal proposal density for σ for the M-H algorithm. For updating σ we have used adaptive MCMC method given in Roberts and Rosenthal (2009) to get an optimum acceptance rate (44%) for the Metropolis sampler.

By setting initial values $\beta^{(0)}$ and $\sigma^{(0)}$, let $\beta^{(t)}$ and $\sigma^{(t)}$, $t = 1, 2, \ldots, N$ be the observations generated from (24) and (25) respectively. Then the Bayes estimators under SEL function of β and σ, by taking first m iterations as burn-in period, are given by

$$\hat{\beta}_{SB} = \frac{1}{N-m} \sum_{j=m+1}^{N} \beta^{(j)} \tag{26}$$

and

$$\hat{\sigma}_{SB} = \frac{1}{N-m} \sum_{j=m+1}^{N} \sigma^{(j)}. \tag{27}$$

The Bayes estimate of β and σ under the LL function L_2 is obtained as

$$\hat{\beta}_{LB} = -\frac{1}{h} \log\left(\frac{1}{N-m} \sum_{j=m+1}^{N} e^{-h\beta^{(j)}} \right) \tag{28}$$

and

$$\hat{\sigma}_{LB} = -\frac{1}{h} \log\left(\frac{1}{N-m} \sum_{j=m+1}^{N} e^{-h\sigma^{(j)}} \right). \tag{29}$$

Finally, the Bayes estimate of β and σ under the EL function L_3 is obtained as

$$\hat{\beta}_{EB} = \left\{ \frac{1}{N-m} \sum_{j=m+1}^{N} \left(\beta^{(j)}\right)^{-q} \right\}^{-\frac{1}{q}} \tag{30}$$

and

$$\hat{\alpha}_{EB} = \left\{ \frac{1}{N-m} \sum_{j=m+1}^{N} \left(\alpha^{(j)}\right)^{-q} \right\}^{-\frac{1}{q}}. \tag{31}$$

4. Bayesian prediction

In this section, we consider the prediction problem under Bayesian context,namely one sampling prediction. Suppose we have n lower k-records $\mathbf{D_{n(k)}} = (R_{1(k)}, R_{2(k)}, \cdots, R_{n(k)})$ from GE(σ, β)). Based on this sample we are interested to find the s-th lower k record values, $1 < n < s$. Let $R_{s(k)}$ be the sth future lower k record value. Then the conditional distribution of $R_{(s)} = r_{(s)}$ given the observed lower k record value $\mathbf{d_{n(k)}} = (r_1, r_2, \cdots, r_n)$ is given by

$$f_1(r_s|\mathbf{d_{n(k)}}, \beta, \sigma) = \frac{k^{s-n}}{\Gamma(s-n)}[\log F(r_n) - \log F(r_s)]^{s-n-1}\frac{f(r_s)}{F(r_n)}\left(\frac{F(r_s)}{F(r_n)}\right)^{k-1}, \quad (32)$$

$$0 < r_s < r_n < \infty,$$

where $f(.)$ and $F(.)$ are the pdf and cdf defined in (4) and (5) respectively . Using binomial expansion, (32) can be expressed as

$$f_1(r_s|\mathbf{d_{n(k)}}, \beta, \sigma) = \frac{k^{s-n}}{\Gamma(s-n)}\sum_{i=0}^{s-n-1}\binom{s-n-1}{i}[\log F(r_n)]^i[-\log F(r_s)]^{s-n-1-i}\frac{f(r_s)}{F(r_n)} \left(\frac{F(r_s)}{F(r_n)}\right)^{k-1}. \quad (33)$$

Then the Bayesian predictive density function of $R_{s(k)}$ given the past n lower k record values is given by

$$f_1^*(r_s|\mathbf{d_{n(k)}}) = \int_0^\infty\int_0^\infty f_1(r_s|\mathbf{d_{n(k)}}, \beta, \sigma)\pi^*(\sigma, \beta|\mathbf{d_{n(k)}})d\sigma d\beta, \quad (34)$$

where $\pi^*(\sigma, \beta/\mathbf{d_{n(k)}})$ is the posterior density defined in (23). Thus the predictive estimate for the s the lower k record under SEL function is given by

$$\begin{aligned}\hat{r}_s &= \int_0^{r_n} r_s f_1^*(r_s|\mathbf{d_{n(k)}})dr_s \\ &= \int_0^{r_n} r_s\left[\int_0^\infty\int_0^\infty f_1(r_s|\mathbf{d_{n(k)}}, \beta, \sigma)\pi^*(\sigma, \beta|\mathbf{d_{n(k)}})d\sigma d\beta\right]dr_s \\ &= \int_0^\infty\int_0^\infty\left[\int_0^{r_n} r_s f_1(r_s|\mathbf{d_{n(k)}}, \beta, \sigma)dr_s\right]\pi^*(\sigma, \beta|\mathbf{d_{n(k)}})d\sigma d\beta.\end{aligned}$$

By putting $u = -\log F(r_s)$ in $f_1(r_s|\mathbf{d_{n(k)}}, \beta, \sigma))$, we get

$$\hat{r}_s = \int_0^\infty \int_0^\infty I(\beta,\sigma)\pi^*(\sigma,\beta|\mathbf{d_{n(k)}})d\sigma d\beta, \tag{35}$$

where

$$I(\beta,\sigma) = \frac{k^{s-n}\sigma}{\Gamma(s-n)} \sum_{i=0}^{s-n-1} \binom{s-n-1}{i} \frac{\beta^i \left[\log(1-e^{\frac{-r_n}{\sigma}})\right]^i}{\left[(1-e^{\frac{-r_n}{\sigma}})^\beta\right]^k} \int_{-\beta\log(1-e^{-\frac{r_n}{\sigma}})}^{\infty} -\log(1-e^{-\frac{u}{\sigma}})u^{s-n-i-1}e^{-uk}du \tag{36}$$

One may observe that (35) can not be evaluated explicitly. So we use MCMC method to obtain $\hat{r}_s$. Let $\beta^{(t)}$ and $\sigma^{(t)}$,t=1,2,...,N be the observation generated from conditional posterior densities given in (24) and (25) respectively. Then $\hat{r}_s$ can be obtained by taking first m iterations as burn period as

$$\hat{r}_s = \frac{1}{N-m} \sum_{i=m+1}^{N} I(\beta^{(t)}, \sigma^{(t)}). \tag{37}$$

4.1. *Interval prediction*

In this section we consider the equal tail prediction interval for the sth lower k record value. From (32), the one sample distribution function is obtained as

$$F_1(t|\mathbf{d_{n(k)}}, \beta, \sigma) = \frac{P(R_{s(k)} \le t|\mathbf{d_{n(k)}}, \beta, \sigma)}{P(R_{s(k)} \le r_n|\mathbf{d_{n(k)}}, \beta, \sigma)}, \tag{38}$$

where

$$\begin{aligned} P(R_{s(k)} \le w|\mathbf{d_{n(k)}}, \beta, \sigma) &= \int_0^w f_1(r_s|\mathbf{d_{n(k)}}, \beta, \sigma)dr_s \\ &= \frac{k^{s-n}}{\Gamma(s-n)} \sum_{i=0}^{s-n-1} \binom{s-n-1}{i} \frac{\beta^i \left[\log(1-e^{\frac{-r_n}{\sigma}})\right]^i}{\left[(1-e^{\frac{-r_n}{\sigma}})^\beta\right]^k} \frac{\Gamma(s-n-i)}{k^{s-n-i}} \\ &\quad \left[1 - G(-\log(1-e^{\frac{-w}{\sigma}}), s-n-i, k)\right], \end{aligned} \tag{39}$$

where $G(x,a,b)$ is the cdf of a gamma distribution given by

$$\int_0^x \frac{b^a}{\Gamma a} t^{a-1} e^{-bt} dt, x > 0.$$

Thus the posterior predictive distribution function is given by

$$F_1^*(r_s|\mathbf{d_{n(k)}}) = \int_0^\infty \int_0^\infty F_1(r_s|\mathbf{d_{n(k)}}, \beta, \sigma)\pi^*(\sigma, \beta|\mathbf{d_{n(k)}})d\sigma d\beta. \tag{40}$$

(40) can be approximated by MCMC method. Then the prediction interval (L, U) with $1-\alpha$ confidence level is obtained by solving the non-linear equations

$$F_1^*(L|\mathbf{d_{n(k)}}) = \frac{\alpha}{2} \text{ and } F_1^*(U|\mathbf{d_{n(k)}}) = 1 - \frac{\alpha}{2}. \tag{41}$$

5. Simulation study

In this section we carry out a simulation study for illustrating the estimation procedures developed in previous sections. We have obtained the bais and MSE of MLEs of β and σ for different values of n using simulated sample for different combinations of σ and β and are given in Tables 1 and 2. For the simulation studies for Bayes estimators we take the hyper parameters for the prior distributions of β and σ as $a = 2$, $b = 2$, $c = 2$ and $d = 2$. We have obtained the Bayes estimators under squared error loss function and its MSE. For the simulation study we do the following

1. Generate n lower k-record values from generalized exponential distribution with scale parameter β and and shape parameter σ.
2. Calculate estimators of β and σ using the generated lower k-record values using MCMC method as described below.
   (a) start with initial values $\beta^{(0)}$ and $\sigma^{(0)}$
   (b) set $t = 1$
   (c) generate $\beta^{(t)}$ from gamma$(n + c, d - kU_{\sigma(t-1)})$
   (d) Using M-H algorithm, generate $\sigma^{(t)}$ from $\pi_2^*(\sigma|\beta^{(t)}, \mathbf{d_{n(k)}})$.
   (e) set $t = t + 1$
   (f) Repeat steps (c) to (e) for $N = 50,000$ times
   (g) Calculate the Bayes estimators of β and σ under different loss functions by taking burn-in period $m = 5000$.
3. Repeat the steps 1 and 2 for 1000 times.
4. Calculate the bias and MSE of all estimators.

Repeat the simulation study for $n = 5(1)8$, $k = 2, 3$ and for different values of σ and β. The bias and MSE for Bayes estimators under squared error loss function, LL function and EL function are given in Tables 3-6. We have also obtained the prediction interval for next $(n+1)^{th}$ future record values and average interval length (AIL) for $n = 5(1)8$ under different combinations of σ and β are given in Table 7 and Table 8. From the tables one can see that the bias and MSE of Bayes estimators are smaller than bias and MSE of MLEs, therefore we conclude that Bayes estimators perform better than MLE in terms of bias and MSE. The bias and MSE of both MLE's and Bayes estimators are decrease when n increases. Also, amoung the Bayes estimators, estimators under entropy loss function posses minimum bias and MSE.

6. Conclusion

Serious difficulties for the statistical inference based on records arise due to the fact that the occurrences of record data are very rare in practical situations and the expected waiting time is infinite for every record after the first. In this work we have considered k record values rising from generalized exponential distribution. We have obtained the MLEs for shape and scale parameters of GE distribution using k-record values. The Bayes estimates of scale parameter and shape parameter have been obtained. The prediction interval for next future record value has also been obtained. Since posterior distribution has not been obtained explicitly MCMC method has been performed to obtain the Bayes estimates under squared error loss function, LINEX loss function and entropy loss function. It is found that Bayes estimators perform much better than the MLEs in terms of bias and MSEs. Amoung the Bayes estimators, estimators under entropy loss function perform better.

APPENDIX

A. TABLES

TABLE 1
The bias and MSE of MLEs of β and σ for $k = 2$.

n	(β,σ)	$\hat{\beta}_{ML}$		$\hat{\sigma}_{ML}$	
		Bias	MSE	Bias	MSE
4	(1,2)	0.350	0.430	-0.400	0.610
	(1.5,2)	0.197	0.546	-0.481	0.501
	(2,2)	-0.116	0.752	-0.415	0.438
	(2,2.5)	0.020	0.776	-0.383	0.387
	(2,3)	0.131	0.801	-0.284	0.333
6	(1,2)	0.284	0.314	-0.361	0.534
	(1.5,2)	0.125	0.527	-0.462	0.476
	(2,2)	0.106	0.663	-0.350	0.345
	(2,2.5)	0.149	0.685	-0.360	0.360
	(2,3)	0.120	0.661	-0.289	0.314
8	(1,2)	0.281	0.247	-0.250	0.392
	(1.5,2)	0.121	0.437	-0.377	0.383
	(2,2)	0.100	0.589	-0.293	0.274
	2,2.5)	0.025	0.606	-0.305	0.110
	(2,3)	0.113	0.587	-0.281	0.282

TABLE 2
The bias and MSE of MLEs of β and σ for $k = 3$.

n	(β, σ)	$\hat{\beta}_{ML}$		$\hat{\sigma}_{ML}$	
		Bias	MSE	Bias	MSE
4	(1,2)	0.343	0.449	-0.630	0.840
	(1.5,2)	0.109	0.582	-0.639	0.781
	(2,2)	-0.134	0.528	-0.649	0.681
	(2,2.5)	-0.040	0.589	-0.562	0.502
	(2,3)	-0.006	0.624	-0.561	0.509
6	(1,2)	0.298	0.358	-0.552	0.790
	(1.5,2)	0.189	0.545	-0.496	0.706
	(2,2)	-0.016	0.460	-0.460	0.617
	(2,2.5)	0.136	0.469	-0.483	0.483
	(2,3)	0.178	0.568	-0.489	0.478
8	(1,2)	0.272	0.304	-0.505	0.755
	(1.5,2)	0.188	0.458	-0.394	0.611
	(2,2)	0.111	0.427	-0.351	0.561
	(2,2.5)	0.197	0.461	-0.452	0.288
	(2,3)	0.245	0.460	-0.366	0.324

TABLE 3

The bias and MSE for Bayes estimator for β of generalized exponential distribution based lower k-record for $k = 2$.

n	σ	β	SEL		LL				EL			
			Bias	MSE	h=-1		h=1		q=-.05		q=0.5	
					Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
4	1.5	2	0.206	0.124	0.076	0.092	0.161	0.099	0.154	0.098	0.127	0.095
4	1.5	2.5	-0.113	0.074	0.113	0.126	-0.256	0.113	-0.173	0.086	-0.285	0.135
4	2	2	-0.475	0.286	-0.201	0.152	-0.679	0.496	-0.541	0.348	-0.702	0.535
4	2	2.5	-0.423	0.224	-0.129	0.182	-0.620	0.416	-0.491	0.283	-0.637	0.444
4	2	3	-0.426	0.234	-0.131	0.161	-0.551	0.332	-0.494	0.293	-0.560	0.349
6	1.5	2	0.150	0.098	0.024	0.086	0.120	0.083	0.112	0.089	0.094	0.084
6	1.5	2.5	-0.076	0.118	0.104	0.113	-0.114	0.090	-0.124	0.120	-0.148	0.124
6	2	2	-0.351	0.222	-0.104	0.129	-0.495	0.285	-0.407	0.257	-0.495	0.293
6	2	2.5	-0.248	0.156	0.029	0.169	-0.469	0.265	-0.308	0.183	-0.465	0.272
6	2	3	-0.219	0.135	0.097	0.158	-0.477	0.291	-0.280	0.159	-0.473	0.301
8	1.5	2	0.136	0.094	0.022	0.074	0.111	0.086	0.106	0.083	0.090	0.070
8	1.5	2.5	0.006	0.092	0.095	0.107	-0.070	0.075	-0.035	0.087	-0.072	0.113
8	2	2	-0.285	0.149	-0.082	0.114	-0.399	0.245	-0.333	0.174	-0.387	0.252
8	2	2.5	-0.238	0.129	-0.018	0.071	-0.419	0.247	-0.287	0.179	-0.409	0.253
8	2	3	-0.157	0.114	0.088	0.190	-0.356	0.195	-0.208	0.146	-0.340	0.199

TABLE 4

The bias and MSE for Bayes estimator for β of generalized exponential distribution based lower k-record for $k = 3$.

n	σ	β	SEL		LL				EL			
			Bias	MSE	h=-1		h=1		q=-.05		q=0.5	
					Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
4	1.5	2	0.259	0.158	0.054	0.331	0.105	0.057	0.191	0.148	0.064	0.076
4	1.5	2.5	-0.072	0.081	0.287	0.138	-0.226	0.089	0.137	0.149	-0.254	0.109
4	2	2	-0.480	0.278	-0.190	0.156	-0.714	0.551	-0.549	0.345	-0.742	0.598
4	2	2.5	-0.453	0.247	-0.154	0.137	-0.635	0.429	-0.523	0.312	-0.655	0.460
4	2	3	-0.436	0.229	-0.130	0.121	-0.693	0.507	-0.507	0.292	-0.719	0.549
6	1.5	2	0.232	0.152	0.032	0.301	0.091	0.076	0.189	0.126	0.041	0.068
6	1.5	2.5	0.051	0.068	0.258	0.219	-0.194	0.081	-0.041	0.125	-0.211	0.095
6	2	2	-0.355	0.198	-0.092	0.139	-0.543	0.348	-0.414	0.239	-0.546	0.362
6	2	2.5	-0.282	0.147	0.064	0.121	-0.567	0.387	-0.344	0.181	-0.571	0.405
6	2	3	-0.350	0.228	-0.084	0.110	-0.491	0.304	-0.409	0.266	-0.488	0.315
8	1.5	2	0.208	0.136	0.032	0.263	0.081	0.061	0.174	0.136	0.030	0.039
8	1.5	2.5	0.038	0.054	0.277	0.202	-0.116	0.058	0.037	0.111	-0.124	0.067
8	2	2	-0.275	0.164	-0.046	0.121	-0.470	0.298	-0.326	0.189	-0.465	0.308
8	2	2.5	-0.199	0.138	0.055	0.114	-0.391	0.219	-0.253	0.156	-0.378	0.224
8	2	3	-0.244	0.142	-0.006	0.102	-0.364	0.227	-0.296	0.165	-0.347	0.234

TABLE 5
The bias and MSE for Bayes estimator for σ of generalized exponential distribution based lower k-record for $k = 2$.

			SEL		LL				EL			
n	σ	β	Bias	MSE	h=-1		h=1		q=-.05		q=0.5	
					Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
4	1.5	2	0.206	0.124	0.076	0.092	0.161	0.090	0.154	0.098	0.127	0.095
4	1.5	2.5	-0.113	0.124	0.113	0.096	-0.256	0.113	-0.173	0.126	-0.285	0.135
4	2	2	-0.475	0.286	-0.201	0.152	-0.679	0.496	-0.541	0.348	-0.702	0.535
4	2	3	-0.426	0.234	-0.131	0.116	-0.551	0.332	-0.494	0.293	-0.560	0.349
6	1.5	2	0.145	0.098	0.069	0.084	0.120	0.083	0.122	0.089	0.094	0.084
6	1.5	2.5	-0.076	0.118	0.104	0.093	-0.137	0.103	-0.124	0.120	-0.148	0.082
6	2	2	-0.351	0.222	-0.104	0.129	-0.495	0.285	-0.407	0.257	-0.495	0.293
6	2	2.5	-0.248	0.156	0.029	0.093	-0.469	0.265	-0.308	0.183	-0.465	0.272
6	2	3	-0.219	0.145	0.067	0.106	-0.477	0.291	-0.280	0.159	-0.473	0.301
8	1.5	2	0.136	0.094	0.058	0.074	0.111	0.086	0.106	0.083	0.090	0.070
8	1.5	2.5	0.006	0.092	0.109	0.087	-0.070	0.098	-0.035	0.087	-0.117	0.113
8	2	2	-0.285	0.149	-0.082	0.114	-0.399	0.245	-0.333	0.174	-0.387	0.252
8	2	2.5	-0.238	0.153	-0.018	0.071	-0.419	0.247	-0.287	0.179	-0.409	0.253
8	2	3	-0.157	0.134	0.059	0.090	-0.356	0.195	-0.208	0.146	-0.340	0.199

TABLE 6

The bias and MSE for Bayes estimator for σ of generalized exponential distribution based lower k-record for $k = 3$.

n	σ	β	SEL		LL				EL			
			Bias	MSE	h=-1		h=1		q=-.05		q=0.5	
					Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
4	1.5	2	0.246	0.158	0.039	0.063	0.105	0.096	0.195	0.118	0.094	0.116
4	1.5	2.5	-0.072	0.096	0.187	0.138	-0.226	0.089	-0.137	0.117	-0.254	0.109
4	2	2	-0.480	0.278	-0.190	0.126	-0.714	0.551	-0.549	0.345	-0.742	0.598
4	2	2.5	-0.453	0.247	-0.154	0.137	-0.635	0.429	-0.523	0.312	-0.655	0.460
4	2	3	-0.436	0.229	-0.130	0.091	-0.693	0.507	-0.507	0.292	-0.719	0.549
6	1.5	2	0.232	0.152	0.038	0.053	0.071	0.076	0.189	0.103	0.081	0.108
6	1.5	2.5	0.039	0.088	0.126	0.119	-0.194	0.081	-0.084	0.091	-0.211	0.095
6	2	2	-0.355	0.198	-0.092	0.114	-0.543	0.348	-0.414	0.239	-0.546	0.362
6	2	2.5	-0.282	0.147	0.064	0.112	-0.567	0.387	-0.344	0.181	-0.571	0.405
6	2	3	-0.350	0.228	-0.084	0.080	-0.491	0.304	-0.409	0.266	-0.488	0.315
8	1.5	2	0.208	0.152	0.032	0.043	0.061	0.061	0.174	0.091	0.060	0.094
8	1.5	2.5	0.038	0.074	0.118	0.093	-0.116	0.058	0.073	0.081	-0.124	0.067
8	2	2	-0.275	0.164	-0.046	0.105	-0.470	0.298	-0.326	0.189	-0.465	0.308
8	2	2.5	-0.199	0.138	0.055	0.102	-0.391	0.219	-0.253	0.156	-0.378	0.224
8	2	3	-0.244	0.142	-0.006	0.074	-0.364	0.227	-0.296	0.165	-0.347	0.234

TABLE 7
The lower and upper bound for $s th$ record and AIL for $k = 2$.

n	s	β	σ	Lower	Upper	AIL
4	5	1.5	2	0.766	0.811	0.045
4	5	1.5	2.5	0.539	0.570	0.031
4	5	2	2	1.076	1.149	0.072
4	5	2	2.5	0.908	0.965	0.057
4	5	2	3	0.585	0.620	0.035
6	7	1.5	2	0.311	0.330	0.019
6	7	1.5	2.5	0.367	0.388	0.022
6	7	2	2	0.209	0.220	0.012
6	7	2	2.5	0.345	0.366	0.020
6	7	2	3	0.323	0.341	0.018
8	9	1.5	2	0.084	0.088	0.004
8	9	1.5	2.5	0.275	0.290	0.015
8	9	2	2	0.067	0.071	0.004
8	9	2	2.5	0.171	0.180	0.009
8	9	2	3	0.664	0.703	0.039

TABLE 8
The lower and upper bound for $s th$ record and AIL for $k = 3$.

n	s	β	σ	Lower	Upper	AIL
4	5	1.5	2	0.777	0.813	0.035
4	5	1.5	2.5	0.423	0.440	0.017
4	5	2	2	0.627	0.652	0.025
4	5	2	2.5	2.599	2.723	0.125
4	5	2	3	2.313	2.434	0.121
6	7	1.5	2	0.100	0.103	0.003
6	7	1.5	2.5	0.190	0.197	0.007
6	7	2	2	1.103	1.152	0.049
6	7	2	2.5	0.585	0.608	0.023
6	7	2	3	0.819	0.853	0.035
8	9	1.5	2	0.531	0.552	0.021
8	9	1.5	2.5	0.167	0.173	0.006
8	9	2	2	0.489	0.508	0.019
8	9	2	2.5	1.100	1.149	0.048
8	9	2	3	1.313	1.366	0.052

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SUMMARY

In this paper, the lower k-record values arising from a two parameter generalized exponential distribution is considered. The maximum likelihood estimators for the shape parameter and scale parameter are obtained. The Bayes estimates of the parameters are also developed by using Markov chain Monte Carlo method under symmetric and asymmetric loss functions. Finally, a simulation study is performed to find the performance of different estimators developed in this paper.

Keywords: k-record values; Generalized exponential distribution; Maximum likelihood estimation; Bayes estimation; MCMC method.