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Lower bounds on the signed (total) k-domination number depending on the clique number

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Abstract: Let G be a graph with vertex set $V(G)$. For any integer $k \geq 1$, a signed (total) k-dominating function is a function $f : V(G) \to \{-1, 1\}$ satisfying $\sum_{x \in N[v]} f(x) \geq k$ ($\sum_{x \in N(v)} f(x) \geq k$) for every $v \in V(G)$, where $N(v)$ is the neighborhood of v and $N[v] = N(v) \cup \{v\}$. The minimum of the values $\sum_{v \in V(G)} f(v)$, taken over all signed (total) k-dominating functions f, is called the signed (total) k-domination number. The clique number of a graph G is the maximum cardinality of a complete subgraph of G. In this note we present some new sharp lower bounds on the signed (total) k-domination number depending on the clique number of the graph. Our results improve some known bounds.

Keywords: signed k-dominating function, signed k-domination number, clique number

AMS Subject classification: 05C69

1. Terminology and introduction

Let G be a finite graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We use [4] for terminology and notations which are not defined here. The *order* of G is given by $n = n(G) = |V|$ and its *size* by $m = m(G) = |E|$. If $v \in V(G)$, then $N_G(v) = N(v)$ is the *open neighborhood* of v, and $N_G[v] = N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v. The *degree* $d_G(v) = d(v)$ of a vertex $v \in V(G)$ is defined by $d(v) = |N(v)|$. The *minimum degree* of a graph G is denoted by $\delta = \delta(G)$. The *clique number* $\omega(G)$ of a graph G is the maximum cardinality of a complete subgraph of G. If $S \subseteq V(G)$, then $G[S]$ is the subgraph of G induced by S. For disjoint subsets S and T of vertices of a graph G, we let $[S, T]$ denote the set of edges between S and T. For a real-valued

function $f : V(G) \to \mathbf{R}$ we define $f(S) = \sum_{v \in S} f(v)$. The weight of f is $f(V(G))$.
Let $k \geq 1$ be an integer, and let G be a graph with minmum degree $\delta \geq k-1$ ($\delta \geq k$). A *signed (total) k-dominating function*, abbreviated SkDF (STkDF), of G is defined by Changping Wang in [15] as a function $f : V(G) \to \{-1, 1\}$ such that $f(N[v]) \geq k$ ($f(N(v) \geq k$) for every $v \in V(G)$. The minimum of the values of $f(V(G))$, taken over all signed (total) k-domination functions f, is called the *signed (total) k-domination number*, abbreviated SkDN (STkDN), of G and is denoted by $\gamma_{sk}(G)$ ($\gamma^t_{sk}(G)$). As the condition $\delta \geq k-1$ ($\delta \geq k$) is clearly necessary, we will always assume that when we discuss $\gamma_{sk}(G)$ ($\gamma^t_{sk}(G)$) all graphs involved satisfy $\delta \geq k-1$ ($\delta \geq k$).
If $k = 1$, then $\gamma_{s1}(G) = \gamma_s(G)$ is the classical signed domination number, introduced by Dunbar, Hedetniemi, Henning and Slater [3]. Investigation and bounds on the signed k-domination number in graphs and digraphs can be found, for example, in [1, 2, 6, 8, 9, 14, 15, 17]. The signed total domination number $\gamma^t_{s1}(G) = \gamma^t_s(G)$ was introduced by Zelinka [16]. Results on the signed total k-domination number can be found in [5, 7–11, 13, 15].
In this note, we derive some new lower bounds on $\gamma_{sk}(G)$ and $\gamma^t_{sk}(G)$ in terms of order and clique number. We improve some results of Henning [5] and Wang [15]. In addition, examples will demonstrate that all our bounds are sharp.

2. Lower bounds

The main tool of our results is the famous theorem of Turán [12].

Theorem 1. *Let $r \geq 1$ be an integer, and let G be a graph of order n. If the clique number $\omega(G) \leq r$, then*

$$2|E(G)| \leq \frac{(r-1)n^2}{r}.$$

Theorem 2. *Let $r \geq 2$ and $k \geq 1$ be integers. If G is a graph of order n with clique number $\omega(G) \leq r$, then*

$$\gamma_{sk}(G) \geq \frac{2}{r-1}\sqrt{(r-1)r(k+1)n + r^2} - n - \frac{2r}{r-1}.$$

Proof. Let $f : V(G) \to \{-1, 1\}$ be a SkDF of G such that $\gamma_{ks}(G) = f(V(G))$. Define the sets $P = \{v \in V(G) | f(v) = 1\}$ and $M = \{v \in V(G) | f(v) = -1\}$. This definition yields to $\gamma_{sk}(G) = |P| - |M| = 2|P| - n$. Since $f(N[v]) \geq k$ for $v \in V(G)$, we observe that each vertex $v \in M$ has at least $k+1$ neighbors in P. Furthermore, $|N(v) \cap M| \leq |N(v) \cap P| - k + 1$ for each vertex $v \in P$. Therefore Theorem 1 leads to

$$\begin{aligned}(k+1)(n-|P|) &= (k+1)|M| \leq |[M,P]| = \sum_{v \in P} |N(v) \cap M| \\ &\leq \sum_{v \in P} (|N(v) \cap P| - k + 1) = 2|E(G[P])| - (k-1)|P| \\ &\leq \frac{r-1}{r}|P|^2 - (k-1)|P|.\end{aligned}$$

We deduce that

$$|P|^2 + \frac{2r}{r-1}|P| \geq \frac{r(k+1)n}{r-1}$$

and thus

$$\left(|P| + \frac{r}{r-1}\right)^2 \geq \frac{r(k+1)n}{r-1} + \frac{r^2}{(r-1)^2}$$

and so

$$|P| + \frac{r}{r-1} \geq \frac{1}{r-1}\sqrt{(r-1)r(k+1)n + r^2}.$$

It follows that

$$\gamma_{ks}(G) = 2|P| - n \geq \frac{2}{r-1}\sqrt{(r-1)r(k+1)n + r^2} - n - \frac{2r}{r-1},$$

and the proof is complete. □

Theorem 2 is an extension of a result by Wang [15], who has proved the bound of Theorem 2 for bipartite graphs.

Example 1. Let $r \geq 2$ and $k \geq 1$ be integers, and let $T_{r,k+1}$ be the complete r-partite graph with the partite sets $X_1, X_2, \ldots, X_r$ such that $|X_1| = |X_2| = \cdots = |X_r| = k+1$. In addition, let $Y_1, Y_2, \ldots, Y_r$ be further vertex sets such that $|Y_1| = |Y_2| = \cdots = |Y_r| = k(r-2) + r$. Now define H as the union of $T_{r,k+1}$ with the vertex sets $Y_1, Y_2, \ldots, Y_r$ such that $H[X_i \cup Y_i]$ is the complete bipartite graph with the partite sets X_i and Y_i for $1 \leq i \leq r$. We define $f : V(H) \to \{-1, 1\}$ by $f(v) = 1$ for $v \in V(T_{r,k+1})$ and $f(v) = -1$ for $v \in \bigcup_{i=1}^{r} Y_i$. Since $f(N_H[v]) = k$ for each $v \in V(H)$, the function f is a signed k-domination function of H with weight

$$r(k+1) - r(k(r-2)+r) = r(3k+1-kr-r),$$

and therefore $\gamma_{sk}(H) \leq 3kr + r - kr^2 - r^2$. Since

$$n(H) = r(k+1) + r(k(r-2)+r) = r(kr+r+1-k),$$

Theorem 2 implies that

$$\begin{aligned}
\gamma_{sk}(H) &\geq \frac{2}{r-1}\sqrt{(r-1)r(k+1)n(H) + r^2} - n(H) - \frac{2r}{r-1} \\
&= \frac{2}{r-1}\sqrt{(r-1)r(k+1)r(kr+r+1-k) + r^2} - r(kr+r+1-k) - \frac{2r}{r-1} \\
&= \frac{2r}{r-1}\sqrt{(r-1)(k+1)(kr+r+1-k)+1} - r(kr+r+1-k) - \frac{2r}{r-1} \\
&= \frac{2r}{r-1}\sqrt{(kr+r-k)^2} - r(kr+r+1-k) - \frac{2r}{r-1} \\
&= \frac{2r}{r-1}(k(r-1)+r) - r(kr+r+1-k) - \frac{2r}{r-1} \\
&= 2kr + \frac{2r}{r-1}(r-1) - kr^2 - r^2 - r + kr \\
&= 3kr + r - kr^2 - r^2
\end{aligned}$$

and thus $\gamma_{sk}(H) = 3kr + r - kr^2 - r^2$.

Recently, Volkmann [14] has proved the following theorem.

Theorem 3. *Let $r \geq 2$ and $k \geq 1$ be integers, and let G be a graph of order n with clique number $\omega(G) \leq r$. If $c(G) = \lceil(\delta(G)+k+1)/2\rceil$, then*

$$\gamma_{sk}(G) \geq \frac{r}{r-1}\left(-(c(G)-k+1)+\sqrt{(c(G)-k+1)^2+4\frac{r-1}{r}c(G)n}\right)-n.$$

Using Example 1, we observe that $\delta(H) = k+1$ and hence $c(H) = k+1$. Applying Theorem 3, we conclude that

$$\begin{aligned}\gamma_{sk}(H) &\geq \frac{r}{r-1}\left(-(c(H)-k+1)+\sqrt{(c(H)-k+1)^2+4\frac{r-1}{r}c(H)n(H)}\right)-n(H)\\ &= \frac{r}{r-1}\left(-2+\sqrt{4+4\frac{r-1}{r}(k+1)r(kr+r+1-k)}\right)-n(H)\\ &= \frac{2r}{r-1}\sqrt{(r-1)(k+1)(kr+r+1-k)+1}-r(kr+r+1-k)-\frac{2r}{r-1}\\ &= 3kr+r-kr^2-r^2.\end{aligned}$$

Thus Example 1 shows that Theorem 3 is sharp too.

Theorem 4. *Let $r \geq 2$ and $k \geq 1$ be integers. If G is a graph of order n with clique number $\omega(G) \leq r$, then*

$$\gamma_{sk}^t(G) \geq 2\sqrt{\frac{krn}{r-1}}-n.$$

Proof. Let $f : V(G) \to \{-1,1\}$ be a STkDF of G such that $\gamma_{ks}^t(G) = f(V(G))$. Define the sets $P = \{v \in V(G)|f(v)=1\}$ and $M = \{v \in V(G)|f(v)=-1\}$. This definition leads to $\gamma_{sk}^t(G) = |P|-|M| = 2|P|-n$. Since $f(N(v)) \geq k$ for $v \in V(G)$, we observe that each vertex $v \in M$ has at least k neighbors in P. Furthermore, $|N(v)\cap M| \leq |N(v)\cap P|-k$ for each vertex $v \in P$. Therefore Theorem 1 leads to

$$\begin{aligned}k(n-|P|) &= k|M| \leq |[M,P]| = \sum_{v\in P}|N(v)\cap M|\\ &\leq \sum_{v\in P}(|N(v)\cap P|-k) = 2|E(G[P])|-k|P|\\ &\leq \frac{r-1}{r}|P|^2-k|P|.\end{aligned}$$

We deduce that

$$|P|^2 \geq \frac{krn}{r-1}$$

and thus

$$\gamma_{ks}^t(G) = 2|P|-n \geq 2\sqrt{\frac{krn}{r-1}}-n,$$

as desired ☐

Theorem 4 is a generalization of a result by Wang [15], who has presented the lower bound of Theorem 4 for bipartite graphs. The special case $k = 1$ of this bound by Wang [15] can be found in the paper [5] of Henning. Note that the proof Theorem 4 is shorter and more transparent than these in [5] and [15]. Wang has given examples which show that Theorem 4 is sharp for $r = 2$. Next we present examples which show that Theorem 4 is also sharp for $r \geq 3$.

Example 2. Let $r \geq 3$ and $k \geq 1$ be integers, and let $T_{r,k}$ be the complete r-partite graph with the partite sets $X_1, X_2, \ldots, X_r$ such that $|X_1| = |X_2| = \cdots = |X_r| = k$. In addition, let $Y_1, Y_2, \ldots, Y_r$ be further vertex sets such that $|Y_1| = |Y_2| = \cdots = |Y_r| = k(r-2)$. Now define Q as the union of $T_{r,k}$ with the vertex sets $Y_1, Y_2, \ldots, Y_r$ such that $Q[X_i \cup Y_i]$ is the complete bipartite graph with the partite sets X_i and Y_i for $1 \leq i \leq r$. We define $f : V(Q) \to \{-1, 1\}$ by $f(v) = 1$ for $v \in V(T_{r,k})$ and $f(v) = -1$ for $v \in \bigcup_{i=1}^{r} Y_i$. Since $f(N_Q(v)) = k$ for each $v \in V(Q)$, the function f is a signed total k-domination function of Q with weight $3kr - kr^2$ and therefore $\gamma_{sk}^t(Q) \leq 3kr - kr^2$. Since $n(Q) = kr(r-1)$, Theorem 4 implies that

$$
\begin{aligned}
\gamma_{sk}^t(Q) &\geq 2\sqrt{\frac{krn(Q)}{r-1}} - n(Q) \\
&= 2\sqrt{\frac{krkr(r-1)}{r-1}} - kr(r-1) \\
&= 2kr - kr^2 + kr = 3kr - kr^2
\end{aligned}
$$

and thus $\gamma_{sk}^t(Q) = 3kr - kr^2$.

As a generalization of a result by Shan and Cheng [10], Samadi and Mojdeh [9] and indepently Volkmann [13] proved the following theorem.

Theorem 5. *Let $r \geq 2$ and $k \geq 1$ be integers, and let G be a graph of order n with clique number $\omega(G) \leq r$. If $c = \lceil (\delta(G) + k)/2 \rceil$, then*

$$
\gamma_{sk}(G) \geq \frac{r}{r-1} \left(-(c-k) + \sqrt{(c-k)^2 + 4\frac{r-1}{r}cn} \right) - n.
$$

Example 2 shows that Theorem 5 is also sharp.

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