UNIFYING VARIOUS FUZZY STRUCTURES USING BRANDT GROUPOIDS

Mădălina Roxana BUNECI, University Constantin Brâncuşi of Târgu-Jiu, ROMÂNIA

Abstract. *The purpose of this note is to introduce a notion of T-fuzzy groupoid (in the sense of Brandt) that allows to study within a unified framework various fuzzy structures such as fuzzy sets, fuzzy groups, fuzzy equivalence relations and fuzzy transformation groups.*

Keywords: fuzzy set; fuzzy group; fuzzy equivalence relation; t-norm; fuzzy groupoid;

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1. Notation, terminology and preliminary remarks

There are at least two definitions of the term groupoid currently in use. One refers to a basic sort of algebraic structure: more precisely, a set closed under a binary operation without any other further properties. The other generalizes the notion of group replacing the binary operation with a partially defined binary operation that is associative and has inverses and identities. This is the notion of groupoid that is used in this paper. We include below the precise definition: a groupoid is a set G, together with a distinguished subset $G^{(2)}\subset G\times G$, and two maps:

a product map (partially defined multiplication)

$(x, y) \to xy \ [:G^{(2)} \to G]$,

and an inverse map

$x \to x^{-1} \ [:G \to G]$,

such that the following conditions are satisfied:

1. Inverses: $(x^{-1})^{-1} = x$
2. Associativity: If $(x,y)\in G^{(2)}$ and $(y,z)\in G^{(2)}$, then $(xy, z), (x,yz) \in G^{(2)}$ and $(xy)z= x(yz)$.
3. Identities: for all $x\in G$, $(x, x^{-1}), (x^{-1}, x)\in G^{(2)}$, and if $(x, y)\in G^{(2)}$ (respectively, $(z,x)\in G^{(2)}$), then $y = x^{-1}(xy)$ (respectively, $z = (zx)x^{-1}$).

The maps r and d on G, defined by $r(x) =xx^{-1}$ and respectively, $d(x) =x^{-1}x$, are called the range and the domain maps. It follows easily from the definition that they have a common image $r(G)=d(G)$ called the unit space of G, which is denoted $G^{(0)}$. For every $u\in G^{(0)}$, we denote the fibre of the range (respectively, domain) map over u by $G^u =r^{-1}(\{u\})$ (respectively, $G_u = d^{-1}(\{u\})$). For u and v in $G^{(0)}$, (r,d)-fibre is denoted $G_v^u = G^u\cap G_v$. It is easy to see that G_u^u is a group, called the isotropy group at u. The group bundle $\{x\in G: r(x)=d(x)\}$ is denoted G', and is called the isotropy group bundle of G.

According to [3], this notion of groupoid was introduced by Brandt in [2], although he always used the extra condition that for $u, v\in G^{(0)}$ there is $x\in G$ such that $r(x)=u$ and $d(x)=v$ (currently, such groupoids are called transitive).

The notion of fuzzy set was first proposed by Zadeh [12] and since then it have been used to model uncertain information in various areas: control, reasoning, pattern recognition, computer vision, medical diagnosis, etc. Subsequently the theory of fuzzy set was extended to other algebraic structure: groups ([10], [5]), equivalence relation ([4-6], [8-11]), group actions

[1], etc. In this paper, we propose a unifying approach to these fuzzy structures through the (Brandt) groupoids.

2. Fuzzy groupoids

Let I=[0,1] (or more generally, a bounded lattice) and T: $I \times I \to I$ be a function. In some context T will be a t-norm, i.e. a function T: $[0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following properties:

1. $T(a, b) = T(b, a)$ for all $a,b\in[0,1]$;
2. $T(a, b) \leq T(c, d)$ if $a \leq c$ and $b \leq d$;
3. $T(a, T(b, c)) = T(T(a, b), c)$ for all $a,b,c\in[0,1]$;
4. $T(a, 1) = a$ for all $a\in[01,]$

Let us note that if T is a t-norm, then $T \leq T_{min}$, where $T_{min}(a,b) = \min\{a,b\}$ for all $a,b\in[0,1]$ (see [7] for more examples of t-norms)

Definition 2.1. Let G be groupoid and $T:I\times I\to I$ a function. A function $\gamma : G \to I$ (fuzzy set on G) is said to be T-fuzzy subgroupoid of G if the following conditions are satisfied

1. $\gamma(xy)\geq T(\gamma(x),\gamma(y))$ for all $(x,y)\in G^{(2)}$.
2. $\gamma(x^{-1})\geq\gamma(x)$ for all $x\in G$.
3. $\gamma(r(x)) \geq \gamma(x)$ for all $x\in G$.

It follows easily from the definition that

1. $\gamma(x^{-1})=\gamma(x)$ for all $x\in G$. Hence if we denote by i the inversion on G, then $i[\gamma]=\gamma$.
2. $\gamma(d(x))\geq\gamma(x)$ for all $x\in G$ (we replace x with x^{-1} in 3).
3. $\gamma(r(x))=\gamma(xx^{-1})\geq T(\gamma(x),\gamma(x^{-1}))=T(\gamma(x), \gamma(x))$ for all $x\in G$.
4. $\gamma(d(x))=\gamma(x^{-1}x)\geq T(\gamma(x^{-1}),\gamma(x))=T(\gamma(x), \gamma(x))$ for all $x\in G$.
5. $T(\gamma(x), \gamma(d(x)) \leq \gamma(xd(x)) = \gamma(x)$ for all $x\in G$.
6. $T(\gamma(r(x)), \gamma(x)) \leq \gamma(r(x)x) = \gamma(x)$ for all $x\in G$.

Let us denote by $\gamma^{(0)} : G \to I$ the fuzzy set defined by

$$\gamma^{(0)}(x)=\begin{cases}\gamma(x), x\in G^{(0)}\\ 0, \text{ otherwise}\end{cases}$$

For all z with $r(z) = u$, $r[\gamma](u) = \sup\{\gamma(x): r(x) = u\} \leq \gamma(r(z)) = \gamma(u)=\gamma^{(0)}(u)$. On the other hand, $\gamma(u) \leq \sup\{\gamma(x): r(x) = u\}$. Thus $r[\gamma]=\gamma^{(0)}$. Similarly, $\gamma^{(0)}=d[\gamma]$.

Let us denote by $\gamma^{(2)} : G^{(2)} \to I$ the fuzzy set defined by

$$\gamma^{(2)}(x,y)=\begin{cases}T(\gamma(x),\gamma(y)),(x,y)\in G^{(2)}\\ 0, \text{ otherwise}\end{cases}$$

If m is the product map of G, then

$$m[\gamma^{(2)}](z)=\sup\{\gamma^{(2)}(x,y): xy=z\}=\sup\{T(\gamma(x),\gamma(y)): xy=z\} \leq\sup\{\gamma(xy): xy=z\}=\gamma(z)$$

Thus $m[\gamma^{(2)}]\leq\gamma$.

Let us denote by γ' : $G \to I$ the fuzzy set defined by

$$\gamma'(x)=\begin{cases}\gamma(x), x\in G'\\ 0, \text{ otherwise}\end{cases}$$

It is easy to see that $r[\gamma']=\gamma^{(0)}$ and $d[\gamma']=\gamma^{(0)}$.

For every $u,v\in G^{(0)}$ let us define

$$\gamma^{u}(x)=\begin{cases}\gamma(x), x\in G^{u}\\ 0, \text{ otherwise}\end{cases}$$

$$\gamma_{u}(x)=\begin{cases}\gamma(x), x\in G_{u}\\ 0, \text{ otherwise}\end{cases}$$

$$\gamma_{v}^{u}=\min\left\{\gamma^{u},\gamma_{v}\right\}$$

Since $\min\{\gamma, r^{-1}[1_{\{u\}}]\}(x) = \min\{\gamma(x), 1_{\{u\}}(r(x))\}$, it follows that

$$\gamma\cap r^{-1}[1_{\{u\}}] =\min\{\gamma, r^{-1}[1_{\{u\}}]\}=\gamma^{u}.$$

Similarly, $\gamma\cap d^{-1}[1_{\{u\}}]\} = \gamma_{u}$ and $\gamma\cap r^{-1}[1_{\{u\}}]\cap d^{-1}[1_{\{v\}}] = \gamma_{v}^{u}$.

Definition 2.2. Let G be groupoid and $T:I\times I\rightarrow I$ be a function. A T-fuzzy subgroupoid $\gamma:G\rightarrow I$ of G is said to be

- T-fuzzy group if and only if $\gamma^{(0)}$ is singleton (its support contains only one point).
- T-fuzzy group bundle if and only if $\gamma=\gamma'$.
- T-fuzzy equivalence relation if and only if γ_{v}^{u} is singleton or null for all $u,v\in G^{(0)}$.

3. Examples of fuzzy structures which fit naturaly into the fuzzy groupoids

Let $T:I\times I\rightarrow I$ be an arbitrary function.

3.1. T-Fuzzy groups: If G is a group, then G can be viewed as groupoid with $G^{(2)} = G\times G$ and $G^{(0)} = \{e\}$ (the unit element). Then $\gamma : G\rightarrow I$ is T-fuzzy subgroupoid of G if and only if

1. $\gamma(xy)\geq T(\gamma(x),\gamma(y))$ for all $x,y\in G$
2. $\gamma(x^{-1})=\gamma(x)$ for all $x\in G$
3. $\gamma(e)\geq\gamma(x)$ for all $x\in G$

Since

$$\gamma^{(0)}(x)=\begin{cases}\gamma(e), x=e\\ 0, \text{ otherwise}\end{cases}$$

it follows that any T-fuzzy subgroupoid γ of a group G is a T-fuzzy group in the sense of Definition 2.2. Also if T is a t-norm, then notion of T-fuzzy subgroup use in [5] coincides with our notion for $\gamma(e)=1$. It $T=T_{min}$, then the T-fuzzy subgroup in this example is a fuzzy group in G in the sense of Definition 5.1 [10].

3.2. Fuzzy sets. A set X can be viewed as a groupoid letting

$$X^{(2)} = \mathrm{diag}(X) = \{(x,x), x \in X\}$$

and defining the operations by $xx = x$, and $x^{-1} = x$. If γ is a fuzzy set on X, then γ can be viewed as a T-fuzzy subgroupoid of X for every function $T:I\times I\rightarrow I$ that satisfies $T(a,a)\leq a$ for all a (in particular, for every t-norm T)

3.3. T-Fuzzy group bundle: A groupoid G is a group bundle if and only if r(x)=d(x) for all x G. Sets and groups are particular cases of group bundles. Obviously, any T-fuzzy subgroupoid of a group bundle is a T-fuzzy group bundle in the sense of Definition 2.2.

3.4. T-Fuzzy equivalence relations. Let $E \subset X \times X$ be the graph of an equivalence relation on the set X. Let

$$E^{(2)} = \{((u_1,v_1), (u_2,v_2)): v_1 = u_2\}$$

Under the operations (u, v)(v,w) = (u, w), $(u,v)^{-1} = (v,u)$, E is a groupoid whose unit space $E^{(0)}$ may be identified with X. Let γ: $\to$I be a fuzzy set on E. Then γ is a T-fuzzy subgroupoid of E if and only if

i) $\gamma(u,v) \geq T(\gamma(u,w),\gamma(w,v))$ for all (u,w), (w,v)E.

ii) $\gamma(u,v) = \gamma(v,u)$ for all (u,v)E.

iii) $\gamma(u,u) \geq \gamma(u,v)$ for all (u,v)E

For all u, v X we have

$$\gamma^u(w,v) = \begin{cases} \gamma(u,v), w = u \text{ and } (u,v) \in E \\ 0, \text{ otherwise} \end{cases}$$

$$\gamma_u(v,w) = \begin{cases} \gamma(v,u), w = u \text{ and } (v,u) \in E \\ 0, \text{ otherwise} \end{cases}$$

$$\gamma_v^u = \min\{\gamma^u, \gamma_v\} = \begin{cases} \gamma(u,v), (u,v) \in E \\ 0, \text{otherwise} \end{cases}$$

Therefore any a T-fuzzy subgroupoid of E is a T-fuzzy equivalence relation in the sense of Definition 2.2.

In particular, if $E = X \times X$, then E is a groupoid called the pair groupoid. Thus we can take into consideration T-fuzzy subgroupoids of $E = X \times X$. Trillas introduced in [11] the notion of indistinguishability operator with the purpose of fuzzifying the classical notion of equivalence relation. That notion of indistinguishability as well as the notion of fuzzy equivalence relation used in [5] is a particular case of a T-fuzzy subgroupoid γ: X×X$\to$I of E=X×X for T a t-norm and γ satisfying the more restrictive condition $\gamma(u,u)=1$ for all u X. Also the notion of fuzzy relation used in [4] fits in the T-fuzzy subgroupoid framework (taking T(a,b) = a $\otimes$b and imposing $\gamma(u,u)=1$ for all u). The notion of fuzzy equivalence relation [9] is the same as in [5] for $T=T_{min}$. The notion of G-fuzzy equivalence relation [8] corresponds to a T-fuzzy subgroupoid γ: X×X$\to$I of E=X×X satisfying for $\gamma(u,u)>0$ for all u and $T=T_{min}$. Also the notion of I-fuzzy relation used in [6] (here I is an implicator function) fits in the settings of T-fuzzy subgroupoid (taking T(a,b) = 1-I(a,b), I the implicator function and imposing $\gamma(u,u)=1$ for all u).

If E = diag(X), then E is a groupoid on X (and may be identified with the groupoid in example 2). Consequently, for E = diag(X), the T-fuzzy subgroupoids are the fuzzy subsets (for functions Tsatisfying T(a,a)≤a for all a).

If G is a groupoid, then

$$R = (r,d)(G) = \{(r(x), d(x)), x \in G\}$$

is the graph of an equivalence relation on $G^{(0)}$. The groupoid defined by this equivalence relation is called the principal groupoid associated with G. Let γ be a T-fuzzy subgroupoid of G. Then

$(r,d)([\gamma])(u,v)=\sup\{\gamma(x): (r,d)(x)=(u,v)\}$ for $(u,v)\in R$ and $(r,d)([\gamma])(u,v)=0$, otherwise. It is easy to see that if T is monotonic and for every $(u,v) \in R$, there is $x_v^u \in G_v^u$ such that

$$\sup\left\{\gamma(x) : x \in G_v^u\right\} = \gamma\left(x_v^u\right),$$

then $(r,d)([\gamma])$ is a T-fuzzy subgroupoid of R.

3.5. T-Fuzzy transformation groupoids. Let Γ be a group acting on a set X:

$$(x,g) \to xg \ [X \times \Gamma \to X]$$

(xg denotes the transform of $x \in X$ by $g \in \Gamma$). Let $G = X \times \Gamma$, $G^{(2)} = \{((x_1, g_1), (x_2, g_2)) : x_2 = x_1 g_1\}$. If we define

product : $(x, g)(xg, h) = (x, gh)$

inverse: $(x,g)^{-1} = (xg, g^{-1})$.

then G becomes a groupoid with the unit space $X \times \{e\}$. Usually $G^{(0)}$ is identified with X.

We say that a fuzzy set $\mu: X \to I$ on X is a fuzzy Γ-invariant (respectively, T-fuzzy Γ-invariant) subset of X if

$\mu(xg) \geq \mu(x)$ for all $(x, g) \in X \times \Gamma$

(respectively, $T(\mu(xg), a) \geq T(\mu(x), a)$ for all $(x, g) \in X \times \Gamma$ and $a \in I$).

Obviously, if T is a t-norm, then any fuzzy Γ-invariant subset is T-fuzzy Γ-invariant subset.

Let T be a t-norm, $\mu: X \to I$ a T-fuzzy Γ-invariant subset of X and $\nu: \Gamma \to I$ a T - fuzzy subgroup of Γ. Then $\gamma: X \times \Gamma \to I$

$$\gamma(x,g) = T(\mu(x), \nu(g)) \text{ for all } (x,g) \in G = X \times \Gamma,$$

is a T-fuzzy subgroupoid of the transformation groupoid $G = X \times \Gamma$. Indeed, we have

$\gamma((x,g)(xg, h)) = \gamma(x, gh) = T(\mu(x), \nu(gh)) \geq T(\mu(x), T(\nu(g), \nu(h))) = T(T(\mu(x), \nu(g)), \nu(h))$
$\geq T(T(\mu(x), \nu(g)), T(\mu(x), \nu(h)))$ for all $((x,g)(xg, h)) \in G^{(2)}$.

$\gamma((x,g)^{-1}) = \gamma(xg, g^{-1}) = T(\mu(xg), \nu(g^{-1})) \geq T(\mu(xg), \nu(g)) \geq T(\mu(x), \nu(g))$ for all $(x,g) \in G = X \times \Gamma$.

$\gamma(r(x,g)) = \gamma(x, e) = T(\mu(x), \nu(e)) \geq T(\mu(x), \nu(x))$ for all $(x,g) \in G = X \times \Gamma$.

3.6. T-Fuzzy reductions. Let G be a groupoid and A be a subset of $G^{(0)}$. Let

$$G|A = \{x \in G : r(x) \in A \text{ and } d(x) \in A\}.$$

If we define $(G|A)^{(2)} = G^{(2)} \cap (G|A \times G|A)$, then $G|A$ becomes a groupoid with the unit space A. Let T be a t-norm, ν be a T-fuzzy subgroupoid of G and μ be a fuzzy subset on $A \subset G^{(0)}$ ($\mu \leq \nu^{(0)}$) such that $T(\mu(u), \mu(u)) = \mu(u)$ for all $u \square G^{(0)}$, then $\gamma: G \to I$

$$\gamma(x) = T(\nu(x), T(\mu(r(x), \mu(d(x))) \text{ for all } x \square G$$

is a T-fuzzy subgroupoid of G. Indeed, we have

$\gamma(xy) = T(\nu(xy), T(\mu(r(x), \mu(d(y)))$
$\geq T(T(\nu(x), \nu(y)), T(\mu(r(x), \mu(d(y)))$
$= T(\nu(x), T(\nu(y)), T(\ \mu(d(y)), \mu(r(x))$
$= T(\nu(x), T(T(\nu(y)),\ \mu(d(y)), \mu(r(x))$
$\geq T(\nu(x), T(T(\nu(y)), T(\mu(r(y)), \mu(d(y)))), \mu(r(x))$
$= T(\nu(x), T(\mu(r(x), T(\nu(y)), T(\mu(r(y)), \mu(d(y)))))$
$= T(T(\nu(x), \mu(r(x))), T(\nu(y)), T(\mu(r(y)), \mu(d(y)))))$
$\geq T(T(\nu(x), T(\mu(r(x)), \mu(d(x)))), T(\nu(y)), T(\mu(r(y)), \mu(d(y)))))$
$= T(\gamma(x), \gamma(y))$ for all $(x,y) \in G^{(2)}$.

$\gamma(x^{-1}) = T(\nu(x^{-1}), T(\mu(d(x)), \mu(r(x))) \geq T(\nu(x), T(\mu(d(x)), \mu(r(x)))$ for all $x \in G$.

$$\begin{aligned}\gamma(r(x)) &= T(\nu(r(x)), T(\mu(r(x), \mu(r(x))) \geq T(\nu(x), \mu(r(x)))\\ &\geq T(\nu(xd(x)), \mu(r(x))) \geq T(T(\nu(x),\nu(d(x))), \mu(r(x)))\\ &=T(T(\nu(x),\nu^{(0)}(d(x))), \mu(r(x))) \geq T(T(\nu(x), \mu(d(x))), \mu(r(x)))\\ &= T(\nu(x), T(\mu(d(x)), \mu(r(x)))) \text{ for all } x\in G.\end{aligned}$$

3.7. T-Fuzzy disjoint unions. Let $\{G_j\}_{j\in J}$ be a family of groupoids. Let G = $\bigcup_{j\in J}\left(\{j\}\times G_j\right)$. If we define

$$G^{(2)} = \bigcup_{j\in J}\left\{\left((j,x),(j,y)\right):(x,y)\in G_j^{(2)}\right\}$$

$$(j,x)(j,y) = (j,xy) \text{ for all } ((j,x), (j,y)) \in G^{(2)}$$

$$(j,x)^{-1} = (j,x^{-1}) \text{ for all } (j,x)\in G,$$

then G becomes a groupoid.

Let $T:I\times I\to I$ be a function and for every $j\in J$ let $\gamma_j : G_j \to I$ be a T-fuzzy subgroupoid of G_j. If we define $\gamma: G \to I$ by

$$\gamma(j,x) = \gamma_j(x) \text{ for all } x \in G_j \text{ and all } j \in J,$$

then γ is a T-fuzzy subgroupoid of G. Indeed

$\gamma((j,x)(j,y)) = \gamma(j, xy) = \gamma_j(xy) \geq T(\gamma_j(x), \gamma_j(y)) =T(\gamma(j,x), \gamma(j,y))$ for all $((j,x), (j,y)) \in G^{(2)}$

$\gamma((j,x)^{-1}) = \gamma(j, x^{-1}) = \gamma_j(x^{-1}) \geq \gamma_j(x) = \gamma(j,x)$ for all $(j,x)\in G$.

$\gamma(r(j,x))=\gamma(j, r(x))= \gamma_j(r(x)) \geq \gamma_j(x) = \gamma(j,x)$ for all $(j, x)\in G$.

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