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Categorical Abstract Algebraic Logic: Closure Operators on Classes of PoFunctors

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Abstract

Following work of Pałasińska and Pigozzi on partially ordered varieties and quasi-varieties of universal algebras, the author recently introduced partially ordered systems (posystems) and partially ordered functors (pofunctors) to cover the case of the algebraic systems arising in categorical abstract algebraic logic. Analogs of the ordered homomorphism theorems of universal algebra were shown to hold in the context of pofunctors. In the present work, operators on classes of pofunctors are introduced and it is shown that classes of pofunctors are closed under the **HSP** and the $\mathbf{SPP}_{\mathrm{U}}$ operators, forming analogs of the well-known variety and quasi-variety operators, respectively, of universal algebra.

Key Words: Varieties, Quasi-Varieties, Order Homomorphisms, Order Isomorphisms, Polarities, Polarity Translations, Order Translations, Algebraic Systems, Reduced Products, Closure Operators, Birkhoff's Theorem, Mal'cev's Theorem, π-Institutions, Protoalgebraic Logics, Algebraizable Logics, Protoalgebraic π-Institutions
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Introduction

The traditional theory of abstract algebraic logic (AAL) studies the process by which a class of *universal* algebras is associated to a given *sentential* logic to form its "algebraic counterpart". The methods developed in AAL allow the classification of sentential logics in an abstract algebraic hierarchy (also known as the Leibniz hierarchy), whose classes reflect how strong the ties are between the logics in each class and their associated classes of algebras. Stronger ties imply that one may draw more conclusions about metalogical properties of the logic by studying corresponding algebraic properties of its associated class of algebras. The theory of categorical abstract algebraic logic (CAAL) generalizes AAL by allowing the hierarchy to also cover logics formalized as π-institutions. The framework is broad enough to cover logics with multiple signatures and quantifiers as well as several logics that are not string-based and, therefore, could not be classified based on the original theory; see, e.g., [23, 24]. It turns out that the classes of algebras that are suitable for forming algebraic counterparts of π-institutional logics do not consist of universal algebras but rather of algebraic systems [27], which are algebraic entities based on set-valued functors rather than on single sets. In this paper it is these systems endowed with partial orderings that will be at the focus of our studies. In particular, we continue the work started in [30] on partially ordered algebraic systems, which form a generalization of the partially ordered algebras studied in [21]. The goal of this study is to make general methods and techniques of ordered universal algebra available in the context of the algebraic systems arising in CAAL. The present work may also be taken as a promotion of the potential usefulness of these partially ordered algebraic systems in a part of the theory of CAAL dealing with logical implication à la Pałasińska and Pigozzi (see also Raftery's recent work [22]), rather than logical equivalence, as is done in the Leibniz and Tarski operator approaches.

As background information, a quick summary of the work of Pałasińska and Pigozzi, presented in [21], is provided and, then, the precursor of the present work [30], which contains some of the definitions and early results in the categorical setting, is briefly reviewed.

The motivation for the development of the theory of partially ordered varieties and quasi-varieties of algebras in the context of AAL in [21] stems from Pałasińska and Pigozzi's belief that they form the right class of structures to consider on the algebraic side, when the focus on the logic side is on the abstract algebraization process of logical implication as opposed to the traditional treatment, using the operator approach (see, e.g., [2, 3]), that focuses on logical equivalence. A bulk of previous work has paved the way for developing the theory of [21]. Sample references include the work of Bloom [4] on varieties of ordered algebras, Mal'cev's work [19, 20] on quasi-varieties of first-order structures, Dellunde and Jansana's [9, 8] and Elgueta's [13, 14] work on first-order structures defined without equality, a special case of which are the structures defined using universal Horn logic without equality, and Dunn's work [11, 12] on gaggle theory. The book on partially ordered algebraic structures by Fuchs [17] should also be mentioned.

Given an algebraic signature $\mathcal{L}$ the notion of a polarity ρ, that allows treating operations that may be monotone in some arguments and antimonotone in others, is introduced in [21]. A ρ-poalgebra $\mathcal{A} = \langle \mathbf{A}, \leq^{\mathcal{A}} \rangle$ is defined to be a pair consisting of an $\mathcal{L}$-algebra $\mathbf{A} = \langle A, \mathcal{L}^{\mathbf{A}} \rangle$ together with a partial ordering $\leq^{\mathcal{A}}$ on A, such that every algebraic operation in $\mathcal{L}^{\mathbf{A}}$ is monotone in the arguments having positive polarity and antimonotone in those with negative polarity with respect to the partial ordering $\leq^{\mathcal{A}}$. A quasi-ordering $\lesssim$ on a ρ-poalgebra $\mathcal{A}$ is a quasi-ordering on A that includes the partial ordering $\leq^{\mathcal{A}}$. Congruences on a ρ-poalgebra that are compatible with a given ρ-quasi-ordering are then introduced, together with order homomorphisms between ρ-poalgebras. Based on these notions, analogs of the well-known Homomorphism, Isomorphism and Correspondence Theorems of universal algebra are established in the partially ordered setting.

The focus, then, shifts to operations on classes of ρ-poalgebras. The ordinary algebraic operations of taking homomorphic images, subalgebras, direct products, filtered products and direct limits are all shown to have valid counterparts for ρ-poalgebras. They are used to provide an Order Subdirect Representation Theorem, stating that every ρ-poalgebra is isomorphic to a subdirect product of order subdirectly irreducible ρ-poalgebras. They are also used to establish that, as in the case of ordinary algebras, taking homomorphic images of subalgebras of direct products of classes of ρ-poalgebras in the ordered setting, as well as taking subalgebras of filtered products, form closure operators on classes of ρ-poalgebras. After developing a syntactic apparatus, including inidentities and quasi-inidentities, key analogs of the notions of an identity and of a quasi-identity, respectively, analogs of Birkhoff's characterization theorem for varieties and Mal'cev's characterization theorem for quasi-varieties are proven for the case of ordered varieties and quasi-varieties of ρ-poalgebras.

The exposition in [21] concludes with the definition of algebraizable ρ-povarieties, taking after the theory of algebraizable logics [3]. Several results paralleling those proven earlier in the deductive system framework are now shown to be true for algebraizable ρ-povarieties.

In [25, 26, 27, 28, 29], an extension of the operator approach to AAL was developed to cover the case of logical systems formalized as π-institutions. As is evident in [27], in this abstract framework the role played by algebras in the universal algebraic setting is subsumed by more general algebraic structures which are termed algebraic systems. These are functors of the form $\mathrm{SEN} : \mathbf{Sign} \rightarrow \mathbf{Set}$, where **Sign** is an arbitrary category and **Set** denotes the category of small sets. It is, therefore, reasonable to expect that these systems will have a strong role to play if one attempts to lift a theory of algebraizability focusing on logical implication, rather than on logical equivalence, to the categorical level. With this motivation in mind, the author introduced in [30] the notion of a partially ordered algebraic system that generalizes the notion of a partially ordered algebra. Order translations and congruence systems compatible with given quasi-ordered systems, forming analogs of order homomorphisms and congruences compatible with given quasi-orderings, were also introduced and a Homomorphism, Isomorphism and Correspondence Theorem for partially ordered algebraic systems were formulated and proven.

This work is continued in the present paper along the lines of [21]. First, several operations on classes of partially ordered algebraic systems are introduced, some of them closely related to operations on π-institutions studied in [31]. It is then shown that the closure operators of taking homomorphic images of subalgebras of direct products and of taking subalgebras of filtered products give rise to analogous closure operators on classes of partially ordered algebraic systems.

It should be mentioned that additional work has been carried out in this direction [32], which deals with analogs of the Birkhoff variety theorem and the Mal'cev quasi-variety theorem in this context and explores the usefulness of all these results in the context of algebraizability of partially ordered algebraic systems.

For general concepts and notation from category theory the reader is referred to any of [1, 5, 18]. For an overview of the current state of affairs in abstract algebraic logic the review article [16], the monograph [15] and the book [7] are all excellent references. To follow recent developments on the categorical side of the subject the reader may refer to [25, 26, 27, 28, 29].

1 Operations on PoFunctors

We denote by **Set** the category of all small sets. Let **Sign** be a (arbitrary) category and SEN : **Sign** $\to$ **Set** a **Set**-valued functor. Any such functor will be referred to as a **sentence functor** in the sequel. It is thought of as giving, for each "signature" object Σ in the category **Sign**, perceived as a category of **signatures** with signature-changing morphisms between them, the corresponding set SEN(Σ) of "formulas" over the given signature. These formulas may vary significantly depending on the context, i.e., on the syntactic structure of the logical system that the functor SEN is intended to capture. It may consist of propositional formulas, of many-sorted formulas, of quantified formulas, of equations, or even of other entities (see, e.g., [23, 24] and, also, [10] for several examples). Recall from [28] that the **clone of all natural transformations on** SEN is defined to be the locally small category with collection of objects $\{\mathrm{SEN}^\alpha : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau_i : \mathrm{SEN}^\alpha \to \mathrm{SEN}$. Composition is defined by

$$\mathrm{SEN}^\alpha \xrightarrow{\langle \tau_i : i < \beta\rangle} \mathrm{SEN}^\beta \xrightarrow{\langle \sigma_j : j < \gamma\rangle} \mathrm{SEN}^\gamma$$

$$\langle \sigma_j : j < \gamma\rangle \circ \langle \tau_i : i < \beta\rangle = \langle \sigma_j(\langle \tau_i : i < \beta\rangle) : j < \gamma\rangle.$$

A subcategory N of this category containing *all* objects of the form SEN^k for $k < \omega$, and all projection morphisms $p^{k,i} : \mathrm{SEN}^k \to \mathrm{SEN}, i < k, k < \omega$, with $p^{k,i}_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}(\Sigma)$ given by

$$p^{k,i}_\Sigma(\vec{\phi}) = \phi_i, \quad \text{for all} \quad \vec{\phi} \in \mathrm{SEN}(\Sigma)^k,$$

and such that, for every family $\{\tau_i : \mathrm{SEN}^k \to \mathrm{SEN} : i < l\}$ of natural transformations in N, the sequence $\langle \tau_i : i < l\rangle : \mathrm{SEN}^k \to \mathrm{SEN}^l$ is also in N, is referred to as a **category of natural transformations on** SEN.

Recall, also, from [31] that, given a functor SEN : **Sign** $\to$ **Set**, a functor SEN$'$: **Sign**$'$ $\to$ **Set** is said to be a **subfunctor** of SEN if **Sign**$'$ is a subcategory of **Sign** and, for all $\Sigma, \Sigma' \in |\mathbf{Sign}'|, f \in \mathbf{Sign}'(\Sigma, \Sigma')$, we have

- $\mathrm{SEN}'(\Sigma) \subseteq \mathrm{SEN}(\Sigma)$, and

- $\mathrm{SEN}'(f)(\phi) = \mathrm{SEN}(f)(\phi)$, for all $\phi \in \mathrm{SEN}'(\Sigma)$.

Moreover, a subfunctor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ is called a **simple subfunctor** of $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ if $\mathbf{Sign}' = \mathbf{Sign}$. If N is a category of natural transformations on SEN, a subfunctor SEN' of SEN is said to be an **N-subfunctor**, if, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$, all $\Sigma \in |\mathbf{Sign}'|$ and all $\vec{\phi} \in \mathrm{SEN}'(\Sigma)^n$, we have $\sigma_\Sigma(\vec{\phi}) \in \mathrm{SEN}'(\Sigma)$.

Suppose that $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is a functor, with N a category of natural transformations on SEN. A **polarity ρ for N** is an assignment of a polarity, either positive or negative, to each argument position i of each natural transformation σ in N that respects composition of natural transformations [30]. More formally, if we denote (by slightly abusing notation) by N the collection of morphisms of the category N, and by $r(\sigma) = \{0, 1, \ldots, r(\sigma) - 1\}$ the arity of the natural transformation $\sigma : \mathrm{SEN}^{r(\sigma)} \to \mathrm{SEN}$, then a **polarity ρ for N** is a function $\rho : (\bigcup_{\sigma \in N} \sigma \times r(\sigma)) \to \{+, -\}$ with the following **composition compatibility property:**

If $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}, \tau : \mathrm{SEN}^m \to \mathrm{SEN}$ are in N and k is fixed, $0 \leq k \leq m - 1$, the natural transformation $\omega : \mathrm{SEN}^{n+m-1} \to \mathrm{SEN}$ in N, defined by

$$\omega_\Sigma(\phi_0, \ldots, \phi_{m+n-2}) = \tau_\Sigma(\phi_0, \ldots, \phi_{k-1}, \sigma_\Sigma(\phi_k, \ldots, \phi_{k+n-1}), \phi_{k+n}, \ldots, \phi_{m+n-2}),$$

for all $\Sigma \in |\mathbf{Sign}|, \phi_0, \ldots, \phi_{n+m-2} \in \mathrm{SEN}(\Sigma)$, must satisfy, for all $0 \leq j \leq m + n - 2$,

$$\rho(\omega, j) = \begin{cases} \rho(\tau, j), & \text{if } j < k \text{ or } j \geq k + n, \\ \rho(\sigma, j - k), & \text{if } \rho(\tau, k) = + \text{ and } k \leq j < k + n, \\ -\rho(\sigma, j - k), & \text{if } \rho(\tau, k) = - \text{ and } k \leq j < k + n. \end{cases}$$

A natural transformation σ in N of arity $r(\sigma)$ is said to be of **positive** or of **negative polarity at the i-th argument** (**with respect to** ρ) if $\rho(\sigma, i)$ is $+$ or $-$, respectively. We let $\rho^+(\sigma) = \{i < r(\sigma) : \rho(\sigma, i) = +\}$ and $\rho^-(\sigma) = \{i < r(\sigma) : \rho(\sigma, i) = -\}$.

Given a a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, a **quasi-order system** (**qosystem** for short) $\lesssim = \{\lesssim_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ **on** SEN is a $|\mathbf{Sign}|$-indexed set of quasi-orderings $\lesssim_\Sigma$ on $\mathrm{SEN}(\Sigma)$, such that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)^2(\lesssim_\Sigma) \subseteq \lesssim_{\Sigma'}.$$

A quasi-order system $\lesssim = \{\lesssim_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ is a **partial order system** (**posystem** for short) if each $\lesssim_\Sigma$ is a partial ordering on $\mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$.

With these definitions in mind we may now formally define the notions of a ρ-quasi-ordered system (ρ-qosystem) and of a ρ-partially ordered system (ρ-posystem) as follows:

Definition 1 (Definition 1 of [30])**.** *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a functor,* N *a category of natural transformations on* SEN *and* ρ *a polarity for* N*. A qosystem* $\lesssim = \{\lesssim_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ *on* SEN *is said to be a* ρ-**qosystem** *if, for all* $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ *in* N*,* $\Sigma \in |\mathbf{Sign}|, \vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)^n$,

$$[(\forall i \in \rho^+(\sigma))(\phi_i \lesssim_\Sigma \psi_i) \wedge (\forall j \in \rho^-(\sigma))(\phi_j \gtrsim_\Sigma \psi_j)] \Rightarrow \sigma_\Sigma(\vec{\phi}) \lesssim_\Sigma \sigma_\Sigma(\vec{\psi}).$$

*A posystem $\lesssim$ that satisfies this ρ-**tonicity condition** is called a ρ-**posystem** and, in that case, $\langle \mathrm{SEN}, \lesssim\rangle$ is said to be a ρ-**pofunctor**.*

Suppose that $\lesssim$ is a ρ-posystem on SEN. A ρ-pofunctor $\langle \mathrm{SEN}', \lesssim'\rangle$, with $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ will be said to be a ρ-**subpofunctor** of the ρ-pofunctor $\langle \mathrm{SEN}, \lesssim\rangle$ if

- SEN' is an N-subfunctor of SEN and
- $\lesssim'_\Sigma = \lesssim_\Sigma \cap \mathrm{SEN}'(\Sigma)^2$, for all $\Sigma \in |\mathbf{Sign}'|$.

Next, we recall the definition of an order translation (Definition 7 of [30]). Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}, \mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ be two functors. A (**singleton**) **translation** $\langle F, \alpha\rangle : \mathrm{SEN} \to^s \mathrm{SEN}'$ consists of a functor $F : \mathbf{Sign} \to \mathbf{Sign}'$ and a natural transformation $\alpha : \mathrm{SEN} \to \mathrm{SEN}' \circ F$. Given categories N, N' of natural transformations on $\mathrm{SEN}, \mathrm{SEN}'$, respectively, a translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^s \mathrm{SEN}'$ is called an (N, N')-**epimorphic translation** if there exists a correspondence $\sigma \leftrightarrow \sigma'$ between the natural transformations in N and those in N', preserving the projection natural transformations (and, therefore, also, the arities of all natural transformations involved), such that, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^n$, $\alpha_\Sigma(\sigma_\Sigma(\vec{\phi})) = \sigma'_{F(\Sigma)}(\alpha^n_\Sigma(\vec{\phi}))$.

$$\begin{array}{ccc}
\mathrm{SEN}(\Sigma)^n & \xrightarrow{\ \sigma_\Sigma\ } & \mathrm{SEN}(\Sigma) \\
{\scriptstyle \alpha^n_\Sigma}\downarrow & & \downarrow{\scriptstyle \alpha_\Sigma} \\
\mathrm{SEN}'(F(\Sigma))^n & \xrightarrow[\ \sigma'_{F(\Sigma)}\ ]{} & \mathrm{SEN}'(F(\Sigma))
\end{array}$$

In this case, we sometimes write $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$. Suppose, next, that ρ, ρ' are polarities for N, N', respectively. An (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ is said to be a **polarity translation from** SEN **to** SEN', denoted $\langle F, \alpha\rangle : \mathrm{SEN} \to^p \mathrm{SEN}'$, if, for all corresponding $\tau : \mathrm{SEN}^n \to \mathrm{SEN}$ in N and $\tau' : \mathrm{SEN}'^n \to \mathrm{SEN}'$ via the (N, N')-epimorphic property,

$$\rho'(\tau', i) = \rho(\tau, i), \quad \text{for all} \quad i < n.$$

Given a ρ-posystem $\lesssim$ on SEN and a ρ'-posystem $\lesssim'$ on SEN', a polarity translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^p \mathrm{SEN}'$ is said to be an **order translation from** the ρ-pofunctor $\langle \mathrm{SEN}, \lesssim\rangle$ **to** the ρ'-pofunctor $\langle \mathrm{SEN}', \lesssim'\rangle$, denoted $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \lesssim\rangle \to^p \langle \mathrm{SEN}', \lesssim'\rangle$, if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \lesssim_\Sigma \psi \quad \text{implies} \quad \alpha_\Sigma(\phi) \lesssim'_{F(\Sigma)} \alpha_\Sigma(\psi).$$

The following proposition is easy to prove. The proof is left to the reader.

Proposition 1. *Suppose that $\langle \mathrm{SEN}, \lesssim\rangle$, with N a category of natural transformations on N, is a ρ-pofunctor and $\langle \mathrm{SEN}', \lesssim'\rangle$ a ρ-subpofunctor of $\langle \mathrm{SEN}, \lesssim\rangle$. Then, the inclusion translation $\langle J, j\rangle : \mathrm{SEN}' \to^{se} \mathrm{SEN}$ is an order translation $\langle J, j\rangle : \langle \mathrm{SEN}', \lesssim'\rangle \to^p \langle \mathrm{SEN}, \lesssim\rangle$.*

Given a collection of functors $\mathrm{SEN}^i : \mathbf{Sign}^i \to \mathbf{Set}$, with N^i a category of natural transformations on $\mathrm{SEN}^i, i \in I$, the $N^i, i \in I$, will be said to be **compatible categories** if there exists a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and a category of natural transformations N on SEN, such that, for all $i \in I$, N^i is a homomorphic image of N via a surjective functor $F^i : N \to N^i$ that preserves all projections. This implies that F^i also preserves the arities of all natural transformations involved. In this case, we will tacitly assume that, given $\sigma : \mathrm{SEN}^i \to \mathrm{SEN}$ in N, by $\sigma^i : (\mathrm{SEN}^i)^n \to \mathrm{SEN}^i$ in N^i is denoted the image of σ under F^i. It may be shown that, if the $N^i, i \in I$, are compatible categories of natural transformations, then there exists a category of natural transformations $\prod_{i\in I} N^i$ on the product functor $\prod_{i\in I} \mathrm{SEN}^i : \prod_{i\in I} \mathbf{Sign}^i \to \mathbf{Set}$ that is also compatible with the $N^i, i \in I$. Corresponding to σ will be the natural transformation denoted by $\prod_{i\in I} \sigma^i$.

Suppose, now, that, on top of each N^i a polarity ρ^i is provided for $N^i, i \in I$. The $\rho^i, i \in I$, will be said to be **compatible polarities** if

$$\rho^i(\sigma^i, k) = \rho^j(\sigma^j, k), \text{ for all } i, j \in I, k < r(\sigma),$$

where by $r(\sigma)$ is denoted the arity of σ in N. In this case, a polarity $\prod_{i\in I} \rho^i$ may be defined on $\prod_{i\in I} \mathrm{SEN}^i$ that is compatible with the $\rho^i, i \in I$, by

$$\prod_{i\in I} \rho^i(\prod_{i\in I} \sigma^i, k) = \rho(\sigma, k), \text{ for all } \sigma \text{ in } N, k < r(\sigma).$$

Suppose, next, that $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ are functors with compatible categories of natural transformations N and N' on SEN and SEN$'$, respectively, and ρ and ρ' compatible polarities for N and N', respectively. If $\precsim$ is a ρ-posystem on SEN and $\precsim'$ a ρ'-posystem on SEN$'$, then the ρ'-pofunctor $\langle \mathrm{SEN}', \precsim'\rangle$ is said to be an (**ordered**) **homomorphic image of** the ρ-pofunctor $\langle \mathrm{SEN}, \precsim\rangle$ if there exists a surjective order translation $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \precsim\rangle \to^p \langle \mathrm{SEN}', \precsim'\rangle$. Here, surjective means that $F : \mathbf{Sign} \to \mathbf{Sign}'$ is surjective both at the object and at the morphism level and, moreover, $\alpha_\Sigma : \mathrm{SEN}(\Sigma) \to \mathrm{SEN}'(F(\Sigma))$ is surjective, for all $\Sigma \in |\mathbf{Sign}|$. Given a class K of compatible pofunctors (i.e., endowed with compatible categories of natural transformations and compatible polarities on them), by $\mathbf{H}(\mathrm{K})$ is denoted the class of all ordered homomorphic images of pofunctors in the class K.

Next, suppose that $\mathrm{SEN}^i : \mathbf{Sign}^i \to \mathbf{Set}$ is a collection of functors with compatible categories of natural transformations N^i on $\mathrm{SEN}^i, i \in I$, and ρ^i a compatible family of polarities for the $N^i, i \in I$. If $\precsim^i$ is a ρ^i-posystem on SEN^i, $i \in I$, then by the **product** $\prod_{i\in I} \rho^i$**-pofunctor** $\prod_{i\in I}\langle \mathrm{SEN}^i, \precsim^i\rangle$ of the ρ^i-pofunctors $\langle \mathrm{SEN}^i, \precsim^i\rangle$ we mean the $\prod_{i\in I} \rho^i$-pofunctor $\langle \prod_{i\in I} \mathrm{SEN}^i, \prod_{i\in I} \precsim^i\rangle$, where

- $\prod_{i\in I} \mathrm{SEN}^i$ is the product of the SEN^i with $\prod_{i\in I} N^i$ as its category of natural transformations, with polarity $\prod_{i\in I} \rho^i$, and
- $\vec{\phi} \prod_{i\in I} \precsim^i_{\prod_{i\in I}\Sigma_i} \vec{\psi}$ iff $\phi_i \precsim^i_{\Sigma_i} \psi_i$, for all $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i, \psi_i \in \mathrm{SEN}^i(\Sigma_i)$, and $i \in I$.

Suppose that $\mathrm{SEN}^i : \mathbf{Sign}^i \to \mathbf{Set}$ is a collection of functors, with compatible categories of natural transformations N^i on $\mathrm{SEN}^i, i \in I$, and $\rho^i, i \in I$, a compatible family of polarities for the $N^i, i \in I$. Define the translation $\langle P^i, \pi^i \rangle : \prod_{i \in I} \mathrm{SEN}^i \to \mathrm{SEN}^i$, for all $i \in I$, by setting, for all $\Sigma_i \in |\mathbf{Sign}^i|, i \in I$,

$$P^i(\prod_{i \in I} \Sigma_i) = \Sigma_i,$$

and, similarly, for morphisms, and, for all $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$,

$$\pi^i_{\prod_{i \in I} \Sigma_i}(\vec{\phi}) = \phi_i.$$

In the following proposition, it is shown that $\langle P^i, \pi^i \rangle$ is an order translation, for all $i \in I$.

Proposition 2. *Suppose that* $\mathrm{SEN}^i : \mathbf{Sign}^i \to \mathbf{Set}, i \in I$, *is a collection of functors with compatible categories of natural transformations* N^i *on* $\mathrm{SEN}^i, i \in I$, *and* ρ^i *a compatible family of polarities for the* $N^i, i \in I$. *Then* $\langle P^i, \pi^i \rangle : \langle \prod_{i \in I} \mathrm{SEN}^i, \prod_{i \in I} \precsim^i \rangle \to \langle \mathrm{SEN}^i, \precsim^i \rangle$ *is an order translation, for every collection of* ρ^i*-pofunctors* $\langle \mathrm{SEN}^i, \precsim^i \rangle$, $i \in I$, *and every* $i \in I$.

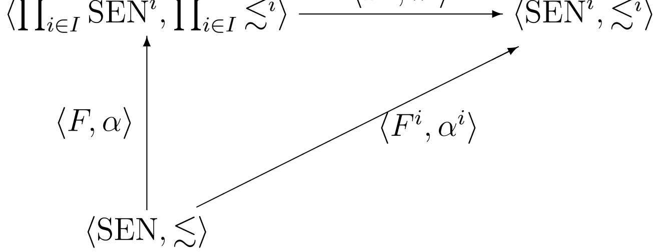
Proof. It has been shown in [31] (see remarks after Lemma 6) that $\langle P^i, \pi^i \rangle : \prod_{i \in I} \mathrm{SEN}^i \to^{se} \mathrm{SEN}^i$ is a surjective $(\prod_{i \in I} N^i, N^i)$-epimorphic translation and, by the definition of $\prod_{i \in I} \rho^i$, it is clearly a polarity translation, for all $i \in I$. So it suffices to show that, for all $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i, \psi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$,

$$\vec{\phi} \prod_{i \in I} \precsim^i_{\prod_{i \in I} \Sigma_i} \vec{\psi} \quad \text{implies} \quad \pi^i_{\prod_{i \in I} \Sigma_i}(\vec{\phi}) \precsim^i_{\Sigma_i} \pi^i_{\prod_{i \in I} \Sigma_i}(\vec{\psi}).$$

This, however, is fairly obvious, by the definition of $\prod_{i \in I} \precsim^i$. □

Finally, the universal mapping property of a product of pofunctors is shown to hold.

Proposition 3. *Let* $\mathrm{SEN}^i : \mathbf{Sign}^i \to \mathbf{Set}, i \in I$, *be a collection of functors with compatible categories of natural transformations* N^i *on* $\mathrm{SEN}^i, i \in I$, *and* ρ^i *a compatible family of polarities for the* $N^i, i \in I$. *If* $\langle \mathrm{SEN}, \precsim \rangle$ *is a* ρ*-pofunctor and* $\langle F^i, \alpha^i \rangle : \langle \mathrm{SEN}, \precsim \rangle \to^p \langle \mathrm{SEN}^i, \precsim^i \rangle, i \in I$, *are order translations, then, there exists a unique order translation* $\langle F, \alpha \rangle : \langle \mathrm{SEN}, \precsim \rangle \to^p \langle \prod_{i \in I} \mathrm{SEN}^i, \prod_{i \in I} \precsim^i \rangle$, *such that the following triangle commutes, for all* $i \in I$*:*



Proof. It has been shown in Lemma 6 of [31] that there exists an $(N, \prod_{i\in I} N^i)$-epimorphic translation

$$\langle F,\alpha\rangle = \prod_{i\in I}\langle F^i,\alpha^i\rangle : \mathrm{SEN} \to^{se} \prod_{i\in I}\mathrm{SEN}^i,$$

such that $\langle P^i,\pi^i\rangle \circ \langle F,\alpha\rangle = \langle F^i,\alpha^i\rangle$, for all $i \in I$. It is defined by $F(\Sigma) = \prod_{i\in I} F^i(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$, and, similarly for morphisms, and

$$\alpha_\Sigma(\phi) = \prod_{i\in I}\alpha^i_\Sigma(\phi), \quad \text{for all } \Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma).$$

So, it suffices to show that $\langle F,\alpha\rangle : \langle \mathrm{SEN}, \lesssim\rangle \to^p \langle \prod_{i\in I}\mathrm{SEN}^i, \prod_{i\in I}\lesssim^i\rangle$ is in fact an order translation. We do have, for all $\Sigma \in |\mathbf{Sign}|, \phi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll}
\phi \lesssim_\Sigma \psi & \text{implies} & (\forall i \in I)(\alpha^i_\Sigma(\phi) \lesssim^i_{F^i(\Sigma)} \alpha^i_\Sigma(\psi)) \\
& & \text{(since } \langle F^i,\alpha^i\rangle \text{ is an order translation)} \\
& \text{iff} & \prod_{i\in I}\alpha^i_\Sigma(\phi)\prod_{i\in I}\lesssim^i_{\prod_{i\in I}F^i(\Sigma)}\prod_{i\in I}\alpha^i_\Sigma(\psi) \\
& & \text{(by the definition of } \prod_{i\in I}\lesssim^i) \\
& \text{iff} & \alpha_\Sigma(\phi)\prod_{i\in I}\lesssim^i{}_{F(\Sigma)}\alpha_\Sigma(\psi) \\
& & \text{(by the definition of } \langle F,\alpha\rangle).
\end{array}$$

□

The order translation $\langle F,\alpha\rangle$ given in Proposition 3 will be denoted by $\prod_{i\in I}\langle F^i, \alpha^i\rangle$.

We switch now to the study of filtered products of pofunctors. Recall that, given a set I, a **filter** on I is a family F of subsets of I, such that

- $I \in F$,
- $K, L \in F$ imply $K \cap L \in F$ and
- $K \subseteq L$ and $K \in F$ imply $L \in F$.

Suppose that $\mathrm{SEN}^i : \mathbf{Sign}^i \to \mathbf{Set}, i \in I$, are functors, with compatible categories of natural transformations N^i on SEN^i and compatible polarities ρ^i for $N^i, i \in I$. Also suppose that $\lesssim^i$ is a ρ^i-posystem on SEN^i, $i \in I$. Define on the direct product $\prod_{i\in I}\mathrm{SEN}^i$ the relation system $\lesssim^F = \{\lesssim^F_{\prod_{i\in I}\Sigma_i}\}_{\prod_{i\in I}\Sigma_i\in|\prod_{i\in I}\mathbf{Sign}^i|}$ by setting, for all $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i, \psi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$,

$$\vec{\phi} \lesssim^F_{\prod_{i\in I}\Sigma_i} \vec{\psi} \quad \text{iff} \quad \{i \in I : \phi_i \lesssim^i_{\Sigma_i} \psi_i\} \in F.$$

It is shown next that $\lesssim^F$ is a $\prod_{i\in I}\rho^i$-qosystem on $\prod_{i\in I}\langle\mathrm{SEN}^i, \lesssim^i\rangle$.

Lemma 1. *Suppose that* $\mathrm{SEN}^i : \mathbf{Sign}^i \to \mathbf{Set}, i \in I$, *are functors, with compatible categories of natural transformations* N^i *on* SEN^i *and compatible polarities* ρ^i *for* $N^i, i \in I$. *Let* F *be a filter on* I *and* $\lesssim^i$ *a* ρ^i*-posystem on* SEN^i, $i \in I$. *The relation system* $\lesssim^F$ *is a* $\prod_{i\in I}\rho^i$*-qosystem on* $\prod_{i\in I}\langle\mathrm{SEN}^i, \lesssim^i\rangle$.

Proof. It is shown, first, that, for all $\prod_{i\in I}\Sigma_i \in |\prod_{i\in I}\mathbf{Sign}^i|$, $\precsim^F_{\prod_{i\in I}\Sigma_i}$ is a quasi-ordering on $\prod_{i\in I}\mathrm{SEN}^i(\Sigma_i)$, second, that $\precsim^F$ is a relation system on $\prod_{i\in I}\mathrm{SEN}^i$, and, finally, that it is a $\prod_{i\in I}\rho^i$-qosystem on $\prod_{i\in I}\mathrm{SEN}^i$.

For the first part, reflexivity is easy. So only transitivity will be shown. Suppose that $\Sigma_i \in |\mathbf{Sign}^i|$ and $\phi_i,\psi_i,\chi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$, such that $\vec{\phi} \precsim^F_{\prod_{i\in I}\Sigma_i} \vec{\psi}$ and $\vec{\psi} \precsim^F_{\prod_{i\in I}\Sigma_i} \vec{\chi}$. Then $\{i \in I : \phi_i \precsim^i_{\Sigma_i} \psi_i\} \in F$ and $\{i \in I : \psi_i \precsim^i_{\Sigma_i} \chi_i\} \in F$. Thus

$$\{i \in I : \phi_i \precsim^i_{\Sigma_i} \chi_i\} \supseteq \{i \in I : \phi_i \precsim^i_{\Sigma_i} \psi_i\} \cap \{i \in I : \psi_i \precsim^i_{\Sigma_i} \chi_i\} \in F$$

and, therefore, $\vec{\phi} \precsim^F_{\prod_{i\in I}\Sigma_i} \vec{\chi}$.

For the second part, suppose that $\Sigma_i,\Sigma_i' \in |\mathbf{Sign}^i|$, $f_i \in \mathbf{Sign}^i(\Sigma_i,\Sigma_i')$ and $\phi_i,\psi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$, such that $\vec{\phi} \precsim^F_{\prod_{i\in I}\Sigma_i} \vec{\psi}$. Then $\{i \in I : \phi_i \precsim^i_{\Sigma_i} \psi_i\} \in F$. But $\precsim^i$ is a qosystem on SEN^i, whence, for all $i \in I$, $\phi_i \precsim^i_{\Sigma_i} \psi_i$ implies $\mathrm{SEN}^i(f_i)(\phi_i) \precsim^i_{\Sigma_i'} \mathrm{SEN}^i(f_i)(\psi_i)$. This shows that

$$\{i \in I : \mathrm{SEN}^i(f_i)(\phi_i) \precsim^i_{\Sigma_i'} \mathrm{SEN}^i(f_i)(\psi_i)\} \supseteq \{i \in I : \phi_i \precsim^i_{\Sigma_i} \psi_i\} \in F$$

and, therefore, $\prod_{i\in I}\mathrm{SEN}^i(f_i)(\phi_i) \precsim^F_{\prod_{i\in I}\Sigma_i'} \prod_{i\in I}\mathrm{SEN}^i(f_i)(\psi_i)$, and $\precsim^F$ is in fact a qosystem on $\prod_{i\in I}\mathrm{SEN}^i$.

For the last part, suppose, for the sake of simplicity, that $\sigma^i : (\mathrm{SEN}^i)^2 \to \mathrm{SEN}^i$ is in N^i such that $\rho^i(\sigma^i,0) = +$, for all $i \in I$. The case with arbitrarily many arguments or with negative polarities may be treated similarly. Let $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i,\psi_i,\chi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$. Lemma 2 of [30] will be used. Suppose $\vec{\phi} \precsim^F_{\prod_{i\in I}\Sigma_i} \vec{\psi}$. Then $\{i \in I : \phi_i \precsim^i_{\Sigma_i} \psi_i\} \in F$. Now, since $\rho^i(\sigma^i,0) = +$, for all $i \in I$, we have that, if $\phi_i \precsim^i_{\Sigma_i} \psi_i$, then $\sigma^i_{\Sigma_i}(\phi_i,\chi_i) \precsim^i_{\Sigma_i} \sigma^i_{\Sigma_i}(\psi_i,\chi_i)$, for all $i \in I$. Therefore

$$\{i \in I : \sigma^i_{\Sigma_i}(\phi_i,\chi_i) \precsim^i_{\Sigma_i} \sigma^i_{\Sigma_i}(\psi_i,\chi_i)\} \supseteq \{i \in I : \phi_i \precsim^i_{\Sigma_i} \psi_i\} \in F$$

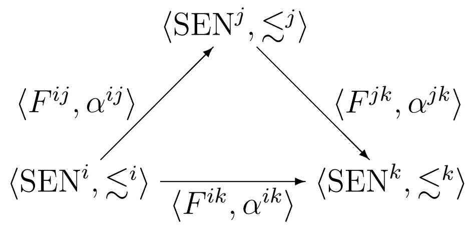
and, therefore, $\prod_{i\in I}\sigma^i_{\prod_{i\in I}\Sigma_i}(\vec{\phi},\vec{\chi}) \precsim^F_{\prod_{i\in I}\Sigma_i} \prod_{i\in I}\sigma^i_{\prod_{i\in I}\Sigma_i}(\vec{\psi},\vec{\chi})$. Hence, $\precsim^F$ is a $\prod_{i\in I}\rho^i$-qosystem on $\prod_{i\in I}\mathrm{SEN}^i$. □

As a consequence of Lemma 1 and of Proposition 4 and Definition 14 of [30], the quotient $\prod_{i\in I}\langle\mathrm{SEN}^i,\precsim^i\rangle/\sim^F$ is a $(\prod_{i\in I}\rho^i)^{\sim^F}$-pofunctor. It is called the **ordered reduced product of** the collection $\langle\mathrm{SEN}^i,\precsim^i\rangle, i \in I$, **by the filter** F and is denoted by $\prod^F_{i\in I}\langle\mathrm{SEN}^i,\precsim^i\rangle$. In case F is an ultrafilter on I, the ordered reduced product will be termed an **order ultraproduct**. $\prod_{i\in I}\mathrm{SEN}^i/\sim^F$ will sometimes be denoted by $\prod^F_{i\in I}\mathrm{SEN}^i$ and $\precsim^F/\sim^F$ by $\prod^F_{i\in I}\precsim^i$. With this notation, we then have $\prod^F_{i\in I}\langle\mathrm{SEN}^i,\precsim^i\rangle := \langle\prod^F_{i\in I}\mathrm{SEN}^i,\prod^F_{i\in I}\precsim^i\rangle$.

Finally, the case of order direct limits of pofunctors is treated. Recall that a partially ordered set $\langle I,\leq\rangle$ is **upward directed** if, for all $i,j \in I$, there exists $k \in I$, such that $i \leq k$ and $j \leq k$.

Let $\langle I,\leq\rangle$ be an upward directed partially ordered index set. Suppose that $\mathrm{SEN}^i : \mathbf{Sign}^i \to \mathbf{Set}$, $i \in I$, are functors, with compatible categories of natural transformations N^i on SEN^i and compatible polarities ρ^i for $N^i, i \in I$. Suppose, also, that $\precsim^i$ is a ρ^i-posystem on SEN^i, $i \in I$, and that $\langle F^{ij}, \alpha^{ij}\rangle : \langle \mathrm{SEN}^i, \precsim^i\rangle \to^p \langle \mathrm{SEN}^j, \precsim^j\rangle, i \leq j$, is a surjective order translation, satisfying

- $\langle F^{ii}, \alpha^{ii}\rangle = \langle \mathrm{I}_{\mathbf{Sign}^i}, \iota^i\rangle : \langle \mathrm{SEN}^i, \precsim^i\rangle \to^p \langle \mathrm{SEN}^i, \precsim^i\rangle$ is the identity order translation,
- $\langle F^{jk}, \alpha^{jk}\rangle \circ \langle F^{ij}, \alpha^{ij}\rangle = \langle F^{ik}, \alpha^{ik}\rangle$, for all $i \leq j \leq k$.



Set, for all $i \in I$, $[i) = \{j \in I : j \geq i\}$. The directedness of I implies that, for all $i, j \in I$, there exists $k \in I$, such that $[k) \subseteq [i) \cap [j)$. Let F be the collection of all subsets of I, that include $[i)$, for some $i \in I$. F is a filter on I and it may be shown that, for all $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i, \psi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$,

$$\vec{\phi} \precsim^F_{\prod_{i\in I}\Sigma_i} \vec{\psi} \quad \text{iff} \quad (\exists j \in I)(\forall i \geq j)(\phi_i \precsim^i_{\Sigma_i} \psi_i).$$

As a consequence, we have that for all $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i, \psi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$,

$$\vec{\phi} \sim^F_{\prod_{i\in I}\Sigma_i} \vec{\psi} \quad \text{iff} \quad (\exists j \in I)(\forall i \geq j)(\phi_i = \psi_i).$$

Consider the $\prod_{i\in I}\rho^i$-subpofunctor $\langle \mathrm{SEN}', \precsim'\rangle$ of $\prod_{i\in I}\langle \mathrm{SEN}^i, \precsim^i\rangle$ formed by taking $\mathrm{SEN}' : \prod_{i\in I}\mathbf{Sign}^i \to \mathbf{Set}$ to be the subfunctor of $\prod_{i\in I}\mathrm{SEN}^i$, defined by

$$\mathrm{SEN}'(\prod_{i\in I}\Sigma_i) = \{\vec{\phi} \in \prod_{i\in I}\mathrm{SEN}^i(\Sigma_i) : (\exists i \in I)(\forall j \geq i)(\alpha^{ij}_{\Sigma_i}(\phi_i) = \phi_j)\}.$$

SEN' is compatible with $\sim^F$ in the sense that, for all $\Sigma_i \in |\mathbf{Sign}^i|, i \in I$,

$$\vec{\phi} \in \mathrm{SEN}'(\prod_{i\in I}\Sigma_i) \quad \text{and} \quad \vec{\phi} \sim^F_{\prod_{i\in I}\Sigma_i} \vec{\psi} \quad \text{imply} \quad \vec{\psi} \in \mathrm{SEN}'(\prod_{i\in I}\Sigma_i),$$

for all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\prod_{i\in I}\Sigma_i)$. Therefore, the quotient $(\prod_{i\in I}\rho^i)^{\sim^F}$-pofunctor $\langle \mathrm{SEN}', \precsim'\rangle/\sim^F$ of $\langle \mathrm{SEN}', \precsim'\rangle$ by the restriction of $\sim^F$ on SEN' is a $(\prod_{i\in I}\rho^i)^{\sim^F}$-subpofunctor of the order reduced product $\prod^F_{i\in I}\langle \mathrm{SEN}^i, \precsim^i\rangle$. It is called the **order direct limit** of the $\langle \mathrm{SEN}^i, \precsim^i\rangle, i \in I$, by $\mathcal{F} = \{\langle F^{ij}, \alpha^{ij}\rangle : i, j \in I, i \leq j\}$ and is denoted by $\lim^{\mathcal{F}}_{i\in I}\langle \mathrm{SEN}^i, \precsim^i\rangle$.

2 Closure Operators

All sentence functors considered in this section will be assumed endowed with compatible categories of natural transformations and with compatible polarities for these categories. As already mentioned in the preceding section, such functors will be termed **compatible**.

Suppose that K is a class of compatible pofunctors. By $\mathbf{S}(\mathrm{K})$ is denoted the class of all pofunctors isomorphic to order subpofunctors of members of K, by $\mathbf{P}(\mathrm{K})$ the class of all pofunctors order isomorphic to a product pofunctor of pofunctors in K. Finally, by $\mathbf{P}_{\mathrm{R}}(\mathrm{K}), \mathbf{P}_{\mathrm{U}}(\mathrm{K})$ and $\mathbf{L}(\mathrm{K})$ will be denoted, respectively, the class of all order isomorphic copies of an order reduced product, an order ultraproduct or an order direct limit, respectively, of a collection of members of K.

2.1 The HSP operator

Work will now start that will culminate to an analog of Theorem 2.14 of [21], expressing closure of a class K with respect to the **HSP** combination of the operators introduced above. Because the framework of partially ordered algebraic systems is more complicated and abstract than that of universal algebras, we proceed step by step establishing carefully some of the results that are obvious and/or well-known in the basic theory of universal algebra and that were adapted to the context of partially ordered algebras in [21].

Consider, first, an order translation $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \precsim\rangle \rightarrow^p \langle \mathrm{SEN}', \precsim'\rangle$ and a ρ'-subpofunctor $\langle \mathrm{SEN}'', \precsim''\rangle$ of the ρ'-pofunctor $\langle \mathrm{SEN}', \precsim'\rangle$. Define the pair

$$\langle \mathrm{SEN}''', \precsim'''\rangle = \langle F, \alpha\rangle^{-1}(\langle \mathrm{SEN}'', \precsim''\rangle)$$

by setting:

- $\mathbf{Sign}''' = F^{-1}(\mathbf{Sign}'')$,
- $\mathrm{SEN}''' : \mathbf{Sign}''' \rightarrow \mathbf{Set}$, where $\mathrm{SEN}''' : \mathbf{Sign}''' \rightarrow \mathbf{Set}$, is defined by

  $$\mathrm{SEN}'''(\Sigma) = \alpha_\Sigma^{-1}(\mathrm{SEN}''(F(\Sigma))), \quad \text{for all } \Sigma \in |\mathbf{Sign}'''|,$$

  and, given $\Sigma, \Sigma' \in |\mathbf{Sign}'''|, f \in \mathbf{Sign}'''(\Sigma, \Sigma')$,

  $$\mathrm{SEN}'''(f)(\phi) = \mathrm{SEN}(f)(\phi), \quad \text{for all } \phi \in \alpha_\Sigma^{-1}(\mathrm{SEN}''(F(\Sigma))).$$

- $\precsim'''_\Sigma = \precsim_\Sigma \restriction_{\alpha_\Sigma^{-1}(\mathrm{SEN}''(F(\Sigma)))}$, for all $\Sigma \in |\mathbf{Sign}'''|$.

It is shown, next, that $\langle F, \alpha\rangle^{-1}(\langle \mathrm{SEN}'', \precsim''\rangle)$ is a ρ-subpofunctor of the ρ-pofunctor $\langle \mathrm{SEN}, \precsim\rangle$. Lemma 2 is the essential ingredient in the proof of Proposition 4, that follows, which shows that $\mathbf{SH} \leq \mathbf{HS}$.

Lemma 2. *Let $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \precsim\rangle \rightarrow^p \langle \mathrm{SEN}', \precsim'\rangle$ be a surjective order translation and $\langle \mathrm{SEN}'', \precsim''\rangle$ a ρ'-subpofunctor of $\langle \mathrm{SEN}', \precsim'\rangle$. Then, the pair $\langle \mathrm{SEN}''', \precsim'''\rangle := \langle F, \alpha\rangle^{-1}(\langle \mathrm{SEN}'', \precsim''\rangle)$ is a ρ-subpofunctor of the ρ-pofunctor $\langle \mathrm{SEN}, \precsim\rangle$.*

$$\begin{array}{ccc}
\langle \mathrm{SEN}, \precsim\rangle & \xrightarrow{\langle F, \alpha\rangle} & \langle \mathrm{SEN}', \precsim'\rangle \\
 & & \geq \\
\langle F, \alpha\rangle^{-1}(\langle \mathrm{SEN}'', \precsim''\rangle) & \xrightarrow[\langle F, \alpha\rangle]{} & \langle \mathrm{SEN}'', \precsim''\rangle
\end{array}$$

Proof. First, it is obvious that $\mathbf{Sign}'''$ is a subcategory of $\mathbf{Sign}$. It suffices then to show that SEN''' is well-defined on morphisms, it is a functor and a subfunctor of SEN and, finally, that $\lesssim'''$ is a ρ-posystem on SEN''', such that $\lesssim'''_\Sigma = \lesssim_\Sigma \cap \mathrm{SEN}'''(\Sigma)^2$, for all $\Sigma \in |\mathbf{Sign}'''|$.

We work first with SEN'''. Suppose that $\Sigma, \Sigma' \in |\mathbf{Sign}'''|$, $f \in \mathbf{Sign}'''(\Sigma, \Sigma')$ and $\phi \in \alpha_\Sigma^{-1}(\mathrm{SEN}''(F(\Sigma)))$. Then we have

$$\alpha_{\Sigma'}(\mathrm{SEN}(f)(\phi)) = \mathrm{SEN}'(F(f))(\alpha_\Sigma(\phi)) \in \mathrm{SEN}''(F(\Sigma')),$$

whence

$$\mathrm{SEN}'''(f)(\phi) = \mathrm{SEN}(f)(\phi) \in \alpha_{\Sigma'}^{-1}(\mathrm{SEN}''(F(\Sigma'))) = \mathrm{SEN}'''(\Sigma')$$

and SEN''' is well-defined on morphisms. Moreover, we have, for all $\Sigma \in |\mathbf{Sign}'''|$,

$$\begin{array}{rcl} \mathrm{SEN}'''(i_\Sigma) & = & \mathrm{SEN}(i_\Sigma) \restriction_{\alpha_\Sigma^{-1}(\mathrm{SEN}''(F(\Sigma)))} \\ & = & i_{\mathrm{SEN}(\Sigma)} \restriction_{\alpha_\Sigma^{-1}(\mathrm{SEN}''(F(\Sigma)))} \\ & = & i_{\mathrm{SEN}'''(\Sigma)} \end{array}$$

and, for all $f \in \mathbf{Sign}'''(\Sigma, \Sigma'), g \in \mathbf{Sign}'''(\Sigma', \Sigma'')$,

$$\begin{array}{rcl} \mathrm{SEN}'''(gf) & = & \mathrm{SEN}(gf) \restriction_{\alpha_\Sigma^{-1}(\mathrm{SEN}''(F(\Sigma)))} \\ & = & (\mathrm{SEN}(g)\mathrm{SEN}(f)) \restriction_{\alpha_\Sigma^{-1}(\mathrm{SEN}''(F(\Sigma)))} \\ & = & \mathrm{SEN}(g) \restriction_{\alpha_{\Sigma'}^{-1}(\mathrm{SEN}''(F(\Sigma')))} \mathrm{SEN}(f) \restriction_{\alpha_\Sigma^{-1}(\mathrm{SEN}''(F(\Sigma)))} \\ & = & \mathrm{SEN}'''(g)\mathrm{SEN}'''(f). \end{array}$$

Therefore, SEN''' is indeed a functor $\mathrm{SEN}''' : \mathbf{Sign}''' \to \mathbf{Set}$. It is obvious from the definition that $\mathrm{SEN}'''(\Sigma) \subseteq \mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}'''|$, and that $\mathrm{SEN}'''(f) = \mathrm{SEN}(f) \restriction_{\mathrm{SEN}'''(\Sigma)}$, for all $f \in \mathbf{Sign}'''(\Sigma, \Sigma')$, i.e., SEN''' is a subfunctor of SEN.

That $\lesssim'''$ is a ρ-posystem on SEN''' follows easily from the fact that $\lesssim$ is a ρ-posystem on SEN. Finally, by the definition of $\lesssim'''$, we have that $\lesssim'''_\Sigma = \lesssim_\Sigma \restriction_{\alpha_\Sigma^{-1}(\mathrm{SEN}''(F(\Sigma)))} = \lesssim_\Sigma \cap \mathrm{SEN}'''(\Sigma)^2$, for all $\Sigma \in |\mathbf{Sign}'''|$. Hence $\langle \mathrm{SEN}''', \lesssim''' \rangle$ is a ρ-subpofunctor of $\langle \mathrm{SEN}, \lesssim \rangle$. □

Lemma 2 leads directly to Proposition 4, which shows that, given any class K of compatible pofunctors, $\mathbf{SH}(\mathrm{K}) \subseteq \mathbf{HS}(\mathrm{K})$.

Proposition 4. *Let* K *be a class of compatible pofunctors. Then* $\mathbf{SH}(\mathrm{K}) \subseteq \mathbf{HS}(\mathrm{K})$.

Proof. Suppose that $\langle \mathrm{SEN}, \lesssim \rangle \in \mathbf{SH}(\mathrm{K})$. That is, there exists $\langle \mathrm{SEN}''', \lesssim''' \rangle \in \mathrm{K}$, and a surjective order translation $\langle F, \alpha \rangle : \langle \mathrm{SEN}''', \lesssim''' \rangle \to^p \langle \mathrm{SEN}'', \lesssim'' \rangle$, such that $\langle \mathrm{SEN}, \lesssim \rangle$ is a ρ''-subpofunctor of the ρ''-pofunctor $\langle \mathrm{SEN}'', \lesssim'' \rangle$.

$$\begin{array}{ccc} \langle \mathrm{SEN}''', \lesssim''' \rangle & \xrightarrow{\langle F, \alpha \rangle} & \langle \mathrm{SEN}'', \lesssim'' \rangle \\ & & \geq \\ \langle F, \alpha \rangle^{-1}(\langle \mathrm{SEN}, \lesssim \rangle) & \xrightarrow[\langle F, \alpha \rangle]{} & \langle \mathrm{SEN}, \lesssim \rangle \end{array}$$

Consider the ρ'''-pofunctor $\langle F,\alpha\rangle^{-1}(\langle \mathrm{SEN},\precsim\rangle)$, which, by Lemma 2, is a ρ'''-subpofunctor of the ρ'''-pofunctor $\langle \mathrm{SEN}''',\precsim'''\rangle$. Then, the restriction $\langle F,\alpha\rangle\restriction_{\langle F,\alpha\rangle^{-1}(\langle \mathrm{SEN},\precsim\rangle)}$ of the translation $\langle F,\alpha\rangle$ is a surjective order translation from $\langle F,\alpha\rangle^{-1}(\langle \mathrm{SEN}, \precsim\rangle)$ onto $\langle \mathrm{SEN},\precsim\rangle$, whence $\langle \mathrm{SEN},\precsim\rangle \in \mathbf{HS}(\mathrm{K})$. □

We proceed now to show that, given a collection of compatible pofunctors $\langle \mathrm{SEN}'^i, \precsim'^i\rangle$ and a collection of subpofunctors $\langle \mathrm{SEN}^i,\precsim^i\rangle$ of $\langle \mathrm{SEN}'^i,\precsim'^i\rangle, i \in I$, the product pofunctor $\prod_{i\in I}\langle \mathrm{SEN}^i,\precsim^i\rangle$ is also a subpofunctor of the product $\prod_{i\in I}\langle \mathrm{SEN}'^i,\precsim'^i\rangle$. This result paves the way for the proof of Proposition 5, that follows, which shows that $\mathbf{PS} \leq \mathbf{SP}$.

Lemma 3. *Let $\langle \mathrm{SEN}'^i,\precsim'^i\rangle$ be a collection of ρ^i-pofunctors, with compatible categories of natural transformations N^i and compatible polarities ρ^i for $N^i, i \in I$. Suppose that $\langle \mathrm{SEN}^i,\precsim^i\rangle$ is a ρ^i-subpofunctor of the ρ^i-pofunctor $\langle \mathrm{SEN}'^i, \precsim'^i\rangle$, for all $i \in I$. Then $\prod_{i\in I}\langle \mathrm{SEN}^i,\precsim^i\rangle$ is a $\prod_{i\in I}\rho^i$-subpofunctor of the $\prod_{i\in I}\rho^i$-pofunctor $\prod_{i\in I}\langle \mathrm{SEN}'^i,\precsim'^i\rangle$.*

Proof. Suppose that $\langle \mathrm{SEN}^i,\precsim^i\rangle$ is a ρ^i-subpofunctor of the ρ^i-pofunctor $\langle \mathrm{SEN}'^i, \precsim'^i\rangle$, for all $i \in I$. Since $\mathbf{Sign}^i$ is a subcategory of $\mathbf{Sign}'^i$, we get that $\prod_{i\in I}\mathbf{Sign}^i$ is a subcategory of $\prod_{i\in I}\mathbf{Sign}'^i$. Moreover, since, for all $\Sigma_i \in |\mathbf{Sign}^i|, \mathrm{SEN}^i(\Sigma_i) \subseteq \mathrm{SEN}'^i(\Sigma_i), i \in I$, we have that

$$\prod_{i\in I}\mathrm{SEN}^i(\Sigma_i) \subseteq \prod_{i\in I}\mathrm{SEN}'^i(\Sigma_i),$$

for all $\Sigma_i \in |\mathbf{Sign}^i|, i \in I$.

Since, by hypothesis, the pofunctors $\langle \mathrm{SEN}'^i,\precsim'^i\rangle$, $i \in I$, are compatible, there exists a functor SEN, with a category of natural transformations N on SEN, together with a collection of surjective functors $F'^i : N \to N'^i$, $i \in I$, that preserve all projections in N. Since, for every $i \in I$, the pofunctor $\langle \mathrm{SEN}^i,\precsim^i\rangle$ is a ρ^i-subpofunctor of $\langle \mathrm{SEN}'^i,\precsim'^i\rangle$, there exist surjective functors $F^i : N \to N^i$, $i \in I$ (among the other conditions that need to be satisfied for a ρ^i-pofunctor). Thus, the collection of pofunctors $\langle \mathrm{SEN}^i,\precsim^i\rangle$, $i \in I$, consists of compatible pofunctors, which are all compatible, also, with the pofunctors in the collection $\langle \mathrm{SEN}'^i,\precsim'^i\rangle, i \in I$.

Finally, given $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i,\psi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$,

$$\begin{array}{lll}\vec{\phi}\prod_{i\in I}\precsim^i_{\prod_{i\in I}\Sigma_i}\vec{\psi} & \text{iff} & (\forall i)(\phi_i \precsim^i_{\Sigma_i} \psi_i)\\ & \text{iff} & (\forall i)(\phi_i \precsim'^i_{\Sigma_i} \psi_i \ \wedge\ \phi_i,\psi_i \in \mathrm{SEN}^i(\Sigma_i))\\ & \text{iff} & \vec{\phi}\prod_{i\in I}\precsim'^i{}_{\prod_{i\in I}\Sigma_i}\vec{\psi}\ \wedge\ \vec{\phi},\vec{\psi} \in \prod_{i\in I}\mathrm{SEN}^i(\Sigma_i)\\ & \text{iff} & \vec{\phi}(\prod_{i\in I}\precsim'^i{}_{\prod_{i\in I}\Sigma_i} \cap (\prod_{i\in I}\mathrm{SEN}^i(\Sigma^i))^2)\vec{\psi},\end{array}$$

and, thus, $\prod_{i\in I}\langle \mathrm{SEN}^i,\precsim^i\rangle$ is in fact a $\prod_{i\in I}\rho^i$-subpofunctor of $\prod_{i\in I}\langle \mathrm{SEN}'^i,\precsim'^i\rangle$. □

Lemma 3 is the critical component in showing that, given a class K of compatible pofunctors, $\mathbf{PS}(\mathrm{K}) \subseteq \mathbf{SP}(\mathrm{K})$.

Proposition 5. *Let* K *be a class of compatible pofunctors. Then* $\mathbf{PS}(\mathrm{K}) \subseteq \mathbf{SP}(\mathrm{K})$.

Proof. Suppose that $\langle \mathrm{SEN}, \lesssim\rangle \in \mathbf{PS}(\mathrm{K})$. Then $\langle \mathrm{SEN}, \lesssim\rangle = \prod_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle$, where $\langle \mathrm{SEN}^i, \lesssim^i\rangle$ is a ρ^i-subpofunctor of a ρ^i-pofunctor $\langle \mathrm{SEN}'^i, \lesssim'^i\rangle \in \mathrm{K}, i \in I$. But then, by Lemma 3, $\langle \mathrm{SEN}, \lesssim\rangle = \prod_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle$ is a $\prod_{i\in I}\rho^i$-subpofunctor of the $\prod_{i\in I}\rho^i$-pofunctor $\prod_{i\in I}\langle \mathrm{SEN}'^i, \lesssim'^i\rangle$. Therefore $\langle \mathrm{SEN}, \lesssim\rangle \in \mathbf{SP}(\mathrm{K})$. □

Before introducing the next lemma, a construction of a translation out of a collection of given translations is needed.

Suppose that $\langle F^i, \alpha^i\rangle : \langle \mathrm{SEN}^i, \lesssim^i\rangle \to^p \langle \mathrm{SEN}'^i, \lesssim'^i\rangle, i \in I$, are surjective order translations, where $\langle \mathrm{SEN}^i, \lesssim^i\rangle, \langle \mathrm{SEN}'^i, \lesssim'^i\rangle, i \in I$, are collections of compatible pofunctors. Define $\langle F, \alpha\rangle := \prod_{i\in I}\langle F^i, \alpha^i\rangle : \prod_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle \to \prod_{i\in I}\langle \mathrm{SEN}'^i, \lesssim'^i\rangle$ by letting, for all $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$,

$$F(\prod_{i\in I}\Sigma_i) = \prod_{i\in I}F^i(\Sigma_i),$$

and, similarly for morphisms, and

$$\alpha_{\prod_{i\in I}\Sigma_i}(\vec{\phi}) = \prod_{i\in I}\alpha^i_{\Sigma_i}(\phi_i).$$

It is shown, next, that $\langle F, \alpha\rangle : \prod_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle \to \prod_{i\in I}\langle \mathrm{SEN}'^i, \lesssim'^i\rangle$ is also a surjective order translation. This will constitute the main component in the proof of Proposition 6, showing that $\mathbf{PH} \leq \mathbf{HP}$.

Lemma 4. *Given surjective order translations* $\langle F^i, \alpha^i\rangle : \langle \mathrm{SEN}^i, \lesssim^i\rangle \to^p \langle \mathrm{SEN}'^i, \lesssim'^i\rangle, i \in I$, *the mapping* $\langle F, \alpha\rangle := \prod_{i\in I}\langle F^i, \alpha^i\rangle : \prod_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle \to \prod_{i\in I}\langle \mathrm{SEN}'^i, \lesssim'^i\rangle$ *is also a surjective order translation.*

Proof. The verification that $F : \prod_{i\in I}\mathbf{Sign}^i \to \prod_{i\in I}\mathbf{Sign}'^i$ is a functor is left to the reader. It is shown here that $\alpha : \prod_{i\in I}\mathrm{SEN}^i \to \prod_{i\in I}\mathrm{SEN}'^i$ is a natural transformation, that it is surjective and that $\langle F, \alpha\rangle$ is an order translation.

Let $\Sigma_i, \Sigma'_i \in |\mathbf{Sign}^i|, f \in \mathbf{Sign}^i(\Sigma_i, \Sigma'_i), i \in I$. Then, we have, for all $\phi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$,

$$\begin{array}{ccc}
\prod_{i\in I}\mathrm{SEN}^i(\Sigma_i) & \xrightarrow{\alpha_{\prod_{i\in I}\Sigma_i}} & \prod_{i\in I}\mathrm{SEN}'^i(F^i(\Sigma_i)) \\
{\scriptstyle \prod_{i\in I}\mathrm{SEN}^i(f_i)}\downarrow & & \downarrow{\scriptstyle \prod_{i\in I}\mathrm{SEN}'^i(F^i(f_i))} \\
\prod_{i\in I}\mathrm{SEN}^i(\Sigma'_i) & \xrightarrow[\alpha_{\prod_{i\in I}\Sigma'_i}]{} & \prod_{i\in I}\mathrm{SEN}'^i(F^i(\Sigma'_i))
\end{array}$$

$$\begin{array}{rcl}
\alpha_{\prod_{i\in I}\Sigma'_i}(\prod_{i\in I}\mathrm{SEN}^i(f_i)(\vec{\phi})) & = & \alpha_{\prod_{i\in I}\Sigma'_i}(\prod_{i\in I}\mathrm{SEN}^i(f_i)(\phi_i)) \\
& = & \prod_{i\in I}\alpha^i_{\Sigma'_i}(\mathrm{SEN}^i(f_i)(\phi_i)) \\
& = & \prod_{i\in I}\mathrm{SEN}'^i(F^i(f_i))(\alpha^i_{\Sigma_i}(\phi_i)) \\
& = & \prod_{i\in I}\mathrm{SEN}'^i(F^i(f_i))(\prod_{i\in I}\alpha^i_{\Sigma_i}(\phi_i)) \\
& = & \prod_{i\in I}\mathrm{SEN}'^i(F^i(f_i))(\alpha_{\prod_{i\in I}\Sigma_i}(\vec{\phi})).
\end{array}$$

Surjectivity of $F : \prod_{i\in I} \mathbf{Sign}^i \to \prod_{i\in I} \mathbf{Sign}'^i$ follows easily from the surjectivity of $F^i : \mathbf{Sign}^i \to \mathbf{Sign}'^i, i \in I$. Suppose, now, that $\Sigma'_i \in |\mathbf{Sign}'^i|, \psi_i \in \mathrm{SEN}'^i(\Sigma'_i), i \in I$. Then, by the surjectivity of F^i, there exists $\Sigma_i \in |\mathbf{Sign}^i|$, such that $\Sigma'_i = F^i(\Sigma_i)$ and, by the surjectivity of $\langle F^i, \alpha^i\rangle$, there exists $\phi_i \in \mathrm{SEN}^i(\Sigma_i)$, such that $\psi_i = \alpha^i_{\Sigma_i}(\phi_i), i \in I$. Therefore $\alpha_{\prod_{i\in I}\Sigma_i}(\vec{\phi}) = \prod_{i\in I} \alpha^i_{\Sigma_i}(\phi_i) = \vec{\psi}$ and, therefore, $\langle F, \alpha\rangle$ is also surjective.

Finally, we show that $\langle F, \alpha\rangle$ is an order translation. Suppose, to this end, that $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$. Then we have

$$\begin{aligned}
&\vec{\phi} \textstyle\prod_{i\in I} \lesssim^i_{\prod_{i\in I}\Sigma_i} \vec{\psi} \\
&\qquad \text{iff} \quad (\forall i)(\phi_i \lesssim^i_{\Sigma_i} \psi_i) \\
&\qquad \text{implies} \quad (\forall i)(\alpha^i_{\Sigma_i}(\phi_i) \lesssim'^i_{F^i(\Sigma_i)} \alpha^i_{\Sigma_i}(\psi_i)) \\
&\qquad \text{iff} \quad \textstyle\prod_{i\in I} \alpha^i_{\Sigma_i}(\phi_i) \ \prod_{i\in I} \lesssim'^i_{\prod_{i\in I} F^i(\Sigma_i)} \ \prod_{i\in I} \alpha^i_{\Sigma_i}(\psi_i) \\
&\qquad \text{iff} \quad \alpha_{\prod_{i\in I}\Sigma_i}(\vec{\phi}) \ \textstyle\prod_{i\in I} \lesssim'^i_{F(\prod_{i\in I}\Sigma_i)} \alpha_{\prod_{i\in I}\Sigma_i}(\vec{\psi}),
\end{aligned}$$

and, therefore, $\langle F, \alpha\rangle$ is an order translation. □

Proposition 6 undertakes the task of showing that, given a class K of compatible pofunctors, $\mathbf{PH}(\mathrm{K}) \subseteq \mathbf{HP}(\mathrm{K})$.

Proposition 6. *Let* K *be a class of compatible pofunctors. Then* $\mathbf{PH}(\mathrm{K}) \subseteq \mathbf{HP}(\mathrm{K})$.

Proof. Suppose that $\langle \mathrm{SEN}, \lesssim\rangle \in \mathbf{PH}(\mathrm{K})$. Then, there are ρ^i-pofunctors $\langle \mathrm{SEN}''^i, \lesssim''^i\rangle \in \mathrm{K}, i \in I$, and surjective order translations $\langle F^i, \alpha^i\rangle : \langle \mathrm{SEN}''^i, \lesssim''^i\rangle \to^p \langle \mathrm{SEN}'^i, \lesssim'^i\rangle$, $i \in I$, such that $\langle \mathrm{SEN}, \lesssim\rangle = \prod_{i\in I}\langle \mathrm{SEN}'^i, \lesssim'^i\rangle$. But then, by Lemma 4, there exists a surjective order translation $\langle F, \alpha\rangle : \prod_{i\in I}\langle \mathrm{SEN}''^i, \lesssim''^i\rangle \to^p \prod_{i\in I}\langle \mathrm{SEN}'^i, \lesssim'^i\rangle$ and, therefore, since $\langle \mathrm{SEN}''^i, \lesssim''^i\rangle \in \mathrm{K}, i \in I$, $\langle \mathrm{SEN}, \lesssim\rangle = \prod_{i\in I}\langle \mathrm{SEN}'^i, \lesssim'^i\rangle \in \mathbf{HP}(\mathrm{K})$. □

The last piece of the puzzle, before showing that **HSP** is a closure operator, is proving the idempotency of the operators $\mathbf{H}, \mathbf{S}$ and $\mathbf{P}$. We state the result without proof, since its proof is relatively easy to establish.

Proposition 7. *If* K *is a class of compatible pofunctors, then* $\mathbf{HH}(\mathrm{K}) \subseteq \mathbf{H}(\mathrm{K})$, $\mathbf{SS}(\mathrm{K}) \subseteq \mathbf{S}(\mathrm{K})$ *and* $\mathbf{PP}(\mathrm{K}) \subseteq \mathbf{P}(\mathrm{K})$.

Proof. The proof is relatively easy, following the corresponding universal algebraic leads, and is left to the reader. □

Finally, in the first main closure result of the paper, it is shown that, as in the case of classes of universal algebras and corresponding operators, the operator **HSP** is a closure operator on classes of compatible pofunctors.

Theorem 1. HSP *is a closure operator on classes of compatible pofunctors.*

Proof. Reflexivity and monotonicity are clear. For idempotency we have

$$\begin{array}{rcll}\mathbf{HSPHSP}(\mathrm{K}) & = & \mathbf{HS}(\mathbf{PH})\mathbf{SP}(\mathrm{K}) & \\ & \subseteq & \mathbf{HSHPSP}(\mathrm{K}) & \text{(by Proposition 6)} \\ & = & \mathbf{H}(\mathbf{SH})\mathbf{PSP}(\mathrm{K}) & \\ & \subseteq & \mathbf{HHSPSP}(\mathrm{K}) & \text{(by Proposition 4)} \\ & = & \mathbf{HS}(\mathbf{PS})\mathbf{P}(\mathrm{K}) & \text{(by Proposition 7)} \\ & \subseteq & \mathbf{HSSPP}(\mathrm{K}) & \text{(by Proposition 5)} \\ & = & \mathbf{HSP}(\mathrm{K}) & \text{(by Proposition 7).}\end{array}$$

□

2.2 The $\mathbf{SPP}_{\mathrm{U}}$ operator

Having shown that **HSP** is a closure operator on classes of compatible pofunctors, we turn now our focus on showing that the analog of the well-known quasi-variety operator $\mathbf{SPP}_{\mathrm{U}}$ of universal algebra is also a closure operator on classes of compatible pofunctors. Work starts by showing that $\mathbf{P} \leq \mathbf{P}_{\mathrm{R}}$.

Proposition 8. *If* K *is a class of compatible pofunctors, then* $\mathbf{P}(\mathrm{K}) \subseteq \mathbf{P}_{\mathrm{R}}(\mathrm{K})$.

Proof. We have that $\prod_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle \cong^p \prod_{i\in I}^{\{I\}}\langle \mathrm{SEN}^i, \lesssim^i\rangle$, whence the result follows. □

Proposition 9 shows that the operator $\mathbf{P}_{\mathrm{R}}$ is idempotent on classes of compatible pofunctors. Its proof is complicated but follows steps similar to the ones used for the corresponding proof of the universal algebraic analog (e.g., Lemma 2.22 of [6]).

Proposition 9. *If* K *is a class of compatible pofunctors,* $\mathbf{P}_{\mathrm{R}}\mathbf{P}_{\mathrm{R}}(\mathrm{K}) \subseteq \mathbf{P}_{\mathrm{R}}(\mathrm{K})$.

Proof. We take after the proof of the corresponding result from first-order logic structures that was presented in full detail as the proof of Lemma 2.22 of [6]. The reader is encouraged to consult that proof and compare the present setting with that of first-order logic.

Let J be a set, $I_j, j \in J$, a family of pairwise disjoint sets, $\langle \mathrm{SEN}^i, \lesssim^i\rangle$ be ρ^i-pofunctors, for all $i \in I_j, j \in J$, F a filter over J and, for all $j \in J$, F_j a filter over I_j. Define $I = \bigcup_{j\in J} I_j$ and

$$\hat{F} = \{S \subseteq I : \{j \in J : S \cap I_j \in F_j\} \in F\}.$$

Then $\hat{F}$ is a filter over I and it suffices to show that

$$\prod_{j\in J}^{F}(\prod_{i\in I_j}^{F_j}\langle \mathrm{SEN}^i, \lesssim^i\rangle) \cong^p \prod_{i\in I}^{\hat{F}}\langle \mathrm{SEN}^i, \lesssim^i\rangle.$$

It is clear that, as categories, $\prod_{j\in J}(\prod_{i\in I_j}\mathbf{Sign}^i) \cong \prod_{i\in I}\mathbf{Sign}^i$, where an isomorphism $H : \prod_{j\in J}(\prod_{i\in I_j}\mathbf{Sign}^i) \cong \prod_{i\in I}\mathbf{Sign}^i$ is given at the object level by

$$H(\prod_{j\in J}\prod_{i\in I_j}\Sigma_i) = \prod_{i\in I}\Sigma_i,$$

and, similarly for morphisms. Next, a natural transformation

$$\gamma : (\prod_{j\in J}(\prod_{i\in I_j}\mathrm{SEN}^i)^{\sim^{F_j}})^{\sim^F} \to (\prod_{i\in I}\mathrm{SEN}^i)^{\sim^{\hat{F}}}$$

is constructed.

The following order translations will be used in the construction: For all $j \in J$, the order translation $\langle G^j, \alpha^j\rangle : \prod_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle \to^p \prod_{i\in I_j}\langle \mathrm{SEN}^i, \lesssim^i\rangle$, given by

$$G^j(\prod_{i\in I}\Sigma_i) = \prod_{i\in I_j}\Sigma_i, \text{ for all } \Sigma_i \in |\mathbf{Sign}^i|, i \in I,$$

and, similarly for morphisms, and

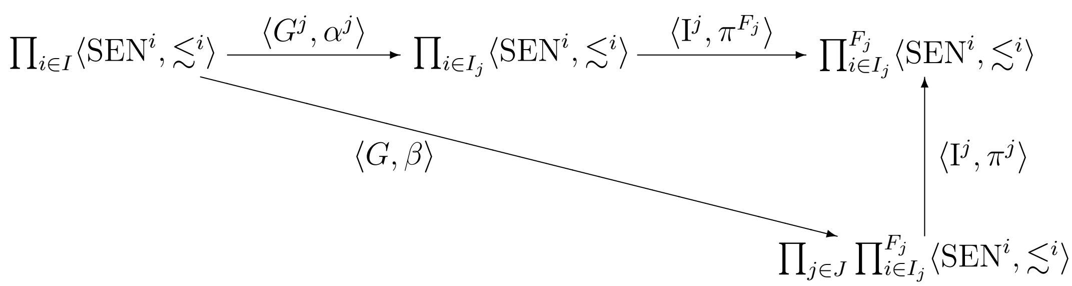
$$\alpha^j_{\prod_{i\in I}\Sigma_i}(\vec{\phi}) = \vec{\phi}\restriction_{I_j}, \text{ for all } \vec{\phi} \in \prod_{i\in I}\mathrm{SEN}^i(\Sigma_i).$$

The natural projection translation

$$\langle \mathrm{I}^j, \pi^{F_j}\rangle : \prod_{i\in I_j}\langle \mathrm{SEN}^i, \lesssim^i\rangle \to^p \prod_{i\in I_j}^{F_j}\langle \mathrm{SEN}^i, \lesssim^i\rangle.$$

The mapping

$$\langle G, \beta\rangle := \prod_{j\in J}\langle \mathrm{I}^j, \pi^{F_j}\rangle\langle G^j, \alpha^j\rangle : \prod_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle \to \prod_{j\in J}(\prod_{i\in I_j}^{F_j}\langle \mathrm{SEN}^i, \lesssim^i\rangle).$$



Consider also the natural projection translation

$$\langle \mathrm{I}, \pi^F\rangle : \prod_{j\in J}\prod_{i\in I_j}^{F_j}\langle \mathrm{SEN}^i, \lesssim^i\rangle \to^p \prod_{j\in J}^{F}\prod_{i\in I_j}^{F_j}\langle \mathrm{SEN}^i, \lesssim^i\rangle$$

and the natural projection translation

$$\langle \mathrm{I}, \pi^{\hat{F}}\rangle : \prod_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle \to^p \prod_{i\in I}^{\hat{F}}\langle \mathrm{SEN}^i, \lesssim^i\rangle.$$

Notice that we have, for all $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$,

$$\begin{array}{ll}
 & \vec{\phi}\ \mathrm{OrdKer}_{\prod_{i\in I}\Sigma_i}(\langle \mathrm{I}, \pi^F\rangle\langle G,\beta\rangle)\ \vec{\psi} \\
\text{iff} & \pi^F_{G(\prod_{i\in I}\Sigma_i)}(\beta_{\prod_{i\in I}\Sigma_i}(\vec{\phi}))\ \prod^F_{j\in J}\prod^{F_j}_{i\in I_j} \lesssim^i{}_{G(\prod_{i\in I}\Sigma_i)} \\
 & \qquad\qquad \pi^F_{G(\prod_{i\in I}\Sigma_i)}(\beta_{\prod_{i\in I}\Sigma_i}(\vec{\psi})) \\
\text{iff} & \{j \in J : \pi^j_{G(\prod_{i\in I}\Sigma_i)}(\beta_{\prod_{i\in I}\Sigma_i}(\vec{\phi}))\ \prod^{F_j}_{i\in I_j} \lesssim^i_{G(\prod_{i\in I}\Sigma_i)} \\
 & \qquad\qquad \pi^j_{G(\prod_{i\in I}\Sigma_i)}(\beta_{\prod_{i\in I}\Sigma_i}(\vec{\psi}))\} \in F \\
\text{iff} & \{j \in J : \{i \in I_j : \phi_i \lesssim^i_{\Sigma_i} \psi_i\} \in F_j\} \in F \\
\text{iff} & \{i \in I : \phi_i \lesssim^i_{\Sigma_i} \psi_i\} \in \hat{F} \\
\text{iff} & \vec{\phi}\ \prod^{\hat{F}}_{i\in I} \lesssim^i{}_{\prod_{i\in I}\Sigma_i}\ \vec{\psi}.
\end{array}$$

Therefore, matching the hypothesis of the Order Isomorphism Theorem (Corollary 16 of [30]), we obtain an order isomorphism

$$\langle H,\gamma\rangle : \prod^{\hat{F}}_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle \to^p \prod^{F}_{j\in J}\prod^{F_j}_{i\in I_j}\langle \mathrm{SEN}^i, \lesssim^i\rangle,$$

such that the following diagram commutes:

$$\begin{array}{ccc}
\prod_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle & \xrightarrow{\langle \mathrm{I}, \pi^{\hat{F}}\rangle} & \prod^{\hat{F}}_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle \\
\langle G,\beta\rangle \Big\downarrow & & \Big\downarrow \langle H,\gamma\rangle \\
\prod_{j\in J}\prod^{F_j}_{i\in I_j}\langle \mathrm{SEN}^i, \lesssim^i\rangle & \xrightarrow[\langle \mathrm{I}, \pi^F\rangle]{} & \prod^F_{j\in J}\prod^{F_j}_{i\in I_j}\langle \mathrm{SEN}^i, \lesssim^i\rangle
\end{array}$$

□

Once more, following the universal algebraic leads, it will now be shown that $\mathbf{P}_{\mathrm{R}} \leq \mathbf{SPP}_{\mathrm{U}}$.

Proposition 10. *If* K *is a class of compatible pofunctors, then* $\mathbf{P}_{\mathrm{R}}(\mathrm{K}) \subseteq \mathbf{SPP}_{\mathrm{U}}(\mathrm{K})$.

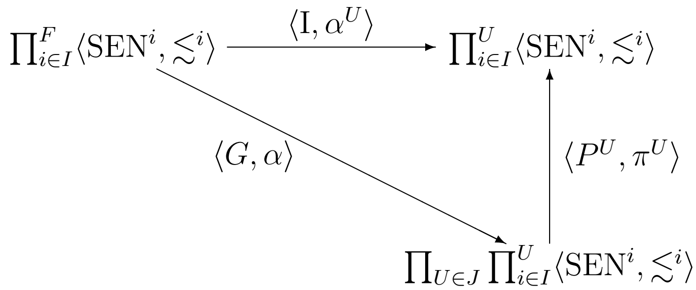
Proof. We follow again the proof of Part (b) of Lemma 2.22 of [6] that deals with the corresponding result for first-order logic.

Suppose F is a filter over I and consider the collection J of all ultrafilters containing F. Let, for all $U \in J$, $\langle \mathrm{I}, \alpha^U\rangle : \prod^F_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle \to \prod^U_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle$ be defined, for all $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$, by

$$\alpha^U_{\prod_{i\in I}\Sigma_i}(\vec{\phi}/\sim^F_{\prod_{i\in I}\Sigma_i}) = \vec{\phi}/\sim^U_{\prod_{i\in I}\Sigma_i}.$$

This is a well-defined order translation. Then let

$$\langle G,\alpha\rangle := \prod_{U\in J}\langle \mathrm{I}, \alpha^U\rangle : \prod^F_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle \to \prod_{U\in J}\prod^U_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle.$$



It will now be shown that $\langle G, \alpha\rangle$ is an embedding of the $(\prod_{i\in I}\rho^i)^{\sim^F}$-pofunctor $\prod^F_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle$ into the $\prod_{U\in J}(\prod_{i\in I}\rho^i)^{\sim^U}$-pofunctor $\prod_{U\in J}\prod^U_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle$. It will then follow, by the fact that $\langle \mathrm{SEN}^i, \lesssim^i\rangle \in \mathrm{K}$, that $\prod^F_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle \in \mathbf{SPP}_{\mathrm{U}}(\mathrm{K})$.

If $\Sigma_i, \Sigma'_i \in |\mathbf{Sign}^i|$, for all $i \in I$, such that $G(\prod_{i\in I}\Sigma_i) = G(\prod_{i\in I}\Sigma'_i)$, then

$$P^U(G(\prod_{i\in I}\Sigma_i)) = P^U(G(\prod_{i\in I}\Sigma'_i)),$$

for all $U \in J$, whence $I(\prod_{i\in I}\Sigma_i) = I(\prod_{i\in I}\Sigma'_i)$, i.e., $\prod_{i\in I}\Sigma_i = \prod_{i\in I}\Sigma'_i$. Similarly, it is shown that G is injective on morphisms. Now, let $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i, \psi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$, such that

$$\alpha_{\prod_{i\in I}\Sigma_i}(\vec{\phi}/{\sim^F_{\prod_{i\in I}\Sigma_i}}) = \alpha_{\prod_{i\in I}\Sigma_i}(\vec{\psi}/{\sim^F_{\prod_{i\in I}\Sigma_i}}).$$

Then, we get, for all $U \in J$,

$$\pi^U_{G(\prod_{i\in I}\Sigma_i)}(\alpha_{\prod_{i\in I}\Sigma_i}(\vec{\phi}/{\sim^F_{\prod_{i\in I}\Sigma_i}})) = \pi^U_{G(\prod_{i\in I}\Sigma_i)}(\alpha_{\prod_{i\in I}\Sigma_i}(\vec{\psi}/{\sim^F_{\prod_{i\in I}\Sigma_i}})),$$

whence, for all $U \in J$, $\alpha^U_{\prod_{i\in I}\Sigma_i}(\vec{\phi}/{\sim^F_{\prod_{i\in I}\Sigma_i}}) = \alpha^U_{\prod_{i\in I}\Sigma_i}(\vec{\psi}/{\sim^F_{\prod_{i\in I}\Sigma_i}})$, and, therefore, by the definition of α^U, for all $U \in J$, $\vec{\phi}/{\sim^U_{\prod_{i\in I}\Sigma_i}} = \vec{\psi}/{\sim^U_{\prod_{i\in I}\Sigma_i}}$. Following now the same argument as in the universal algebraic case, we obtain that $\vec{\phi}/{\sim^F_{\prod_{i\in I}\Sigma_i}} = \vec{\psi}/{\sim^F_{\prod_{i\in I}\Sigma_i}}$.

Finally, we check that, for all $\Sigma_i \in |\mathbf{Sign}^i|, i \in I$,

$$\alpha_{\prod_{i\in I}\Sigma_i}(\prod^F_{i\in I}\lesssim^i_{\prod_{i\in I}\Sigma_i}) = \prod_{U\in J}\prod^U_{i\in I}\lesssim^i_{\prod_{U\in J}\prod_{i\in I}\Sigma_i}\bigcap\alpha_{\prod_{i\in I}\Sigma_i}(\prod^F_{i\in I}\mathrm{SEN}^i(\Sigma_i))^2.$$

Since $\langle G, \alpha\rangle$ is an order translation, it suffices to show that, for all $\Sigma_i \in |\mathbf{Sign}^i|$, ϕ_i, $\psi_i \in \mathrm{SEN}^i(\Sigma_i), i \in I$, if

$$\alpha_{\prod_{i\in I}\Sigma_i}(\vec{\phi}/{\sim^F_{\prod_{i\in I}\Sigma_i}})\prod_{U\in J}\prod^U_{i\in I}\lesssim^i_{\prod_{U\in J}\prod_{i\in I}\Sigma_i}\alpha_{\prod_{i\in I}\Sigma_i}(\vec{\psi}/{\sim^F_{\prod_{i\in I}\Sigma_i}}),$$

then $\vec{\phi}/{\sim^F_{\prod_{i\in I}\Sigma_i}}\prod^F_{i\in I}\lesssim^i_{\prod_{i\in I}\Sigma_i}\vec{\psi}/{\sim^F_{\prod_{i\in I}\Sigma_i}}$. We have, indeed,

$$\alpha_{\prod_{i\in I}\Sigma_i}(\vec{\phi}/{\sim^F_{\prod_{i\in I}\Sigma_i}})\prod_{U\in J}\prod^U_{i\in I}\lesssim^i_{\prod_{U\in J}\prod_{i\in I}\Sigma_i}\alpha_{\prod_{i\in I}\Sigma_i}(\vec{\psi}/{\sim^F_{\prod_{i\in I}\Sigma_i}})$$

whence, for all $U \in J$,

$$\alpha^U_{\prod_{i\in I}\Sigma_i}(\vec{\phi}/{\sim^F_{\prod_{i\in I}\Sigma_i}})\prod^U_{i\in I}\lesssim^i_{\prod_{i\in I}\Sigma_i}\alpha^U_{\prod_{i\in I}\Sigma_i}(\vec{\psi}/{\sim^F_{\prod_{i\in I}\Sigma_i}}),$$

which yields that, for all $U \in J$, $\vec{\phi}/{\sim^U_{\prod_{i\in I}\Sigma_i}} \prod^U_{i\in I} \lesssim^i_{\prod_{i\in I}\Sigma_i} \vec{\psi}/{\sim^U_{\prod_{i\in I}\Sigma_i}}$, and, therefore, that $\{i \in I : \phi_i \lesssim^i_{\Sigma_i} \psi_i\} \in U$. This holding for all $U \in J$, we get that $\{i \in I : \phi_i \lesssim^i_{\Sigma_i} \psi_i\} \in F$, whence $\vec{\phi}/{\sim^F_{\prod_{i\in I}\Sigma_i}} \prod^F_{i\in I} \lesssim^i_{\prod_{i\in I}\Sigma_i} \vec{\psi}/{\sim^F_{\prod_{i\in I}\Sigma_i}}$, as was to be shown. □

The next lemma shows that, given a collection of compatible pofunctors $\langle \mathrm{SEN}'^i, \lesssim'^i\rangle, i \in I$, a collection $\langle \mathrm{SEN}^i, \lesssim^i\rangle, i \in I$, of subpofunctors of $\langle \mathrm{SEN}'^i, \lesssim'^i\rangle, i \in I$, respectively, and a filter F over I, the order reduced product $\prod^F_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle$ is also a subpofunctor of the order reduced product $\prod^F_{i\in I}\langle \mathrm{SEN}'^i, \lesssim'^i\rangle$. This lemma provides the main component in the proof of the inclusion $\mathbf{P}_{\mathrm{R}}\mathbf{S} \leq \mathbf{S}\mathbf{P}_{\mathrm{R}}$.

Lemma 5. *Let $\langle \mathrm{SEN}^i, \lesssim^i\rangle$ be ρ^i-subpofunctors of the compatible ρ^i-pofunctors $\langle \mathrm{SEN}'^i, \lesssim'^i\rangle$, $i \in I$, respectively, and F a filter over I. Then the order reduced product $\prod^F_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle$ is a $(\prod_{i\in I}\rho^i)^{\sim^F}$-subpofunctor of the order reduced product $\prod^F_{i\in I}\langle \mathrm{SEN}'^i, \lesssim'^i\rangle$.*

Proof. Since $\mathbf{Sign}^i$ is a subcategory of $\mathbf{Sign}'^i$, for all $i \in I$, we have that $\prod_{i\in I}\mathbf{Sign}^i$ is a subcategory of $\prod_{i\in I}\mathbf{Sign}'^i$. Suppose, now, that $\Sigma_i \in |\mathbf{Sign}^i|$, $i \in I$. Then $\mathrm{SEN}^i(\Sigma_i) \subseteq \mathrm{SEN}'^i(\Sigma_i)$, for all $i \in I$, whence $\prod_{i\in I}\mathrm{SEN}^i(\Sigma_i) \subseteq \prod_{i\in I}\mathrm{SEN}'^i(\Sigma_i)$. Therefore, we obtain that $\prod_{i\in I}\mathrm{SEN}^i(\Sigma_i)/{\sim^F_{\prod_{i\in I}\Sigma_i}}$ is isomorphic to a subset of $\prod_{i\in I}\mathrm{SEN}'^i(\Sigma_i)/{\sim^F_{\prod_{i\in I}\Sigma_i}}$, i.e., that the set $\prod^F_{i\in I}\mathrm{SEN}^i(\prod_{i\in I}\Sigma_i)$ is isomorphic to a subset of $\prod^F_{i\in I}\mathrm{SEN}'^i(\prod_{i\in I}\Sigma_i)$. The proof for sentence morphisms is similar. Finally, for the posystem relation, we have, under the preceding identification of the subset isomorphism, for all $\Sigma_i \in |\mathbf{Sign}^i|, \phi_i, \psi_i \in \mathrm{SEN}'^i(\Sigma_i), i \in I$,

$$\begin{array}{ll}
 & \vec{\phi}/{\sim^F_{\prod_{i\in I}\Sigma_i}} \ \prod^F_{i\in I} \lesssim^i_{\prod_{i\in I}\Sigma_i} \ \vec{\psi}/{\sim^F_{\prod_{i\in I}\Sigma_i}} \\
\text{iff} & (\forall i \in I)(\exists \phi'_i, \psi'_i \in \mathrm{SEN}^i(\Sigma_i))(\vec{\phi}' \sim^F_{\prod_{i\in I}\Sigma_i} \vec{\phi},\ \vec{\psi}' \sim^F_{\prod_{i\in I}\Sigma_i} \vec{\psi} \\
 & \qquad\qquad \text{and } \{i \in I : \phi'_i \lesssim^i_{\Sigma_i} \psi'_i\} \in F) \\
\text{iff} & \vec{\phi}/{\sim^F_{\prod_{i\in I}\Sigma_i}}, \vec{\psi}/{\sim^F_{\prod_{i\in I}\Sigma_i}} \in \prod^F_{i\in I}\mathrm{SEN}^i(\prod_{i\in I}\Sigma_i) \text{ and} \\
 & \qquad\qquad \{i \in I : \phi_i \lesssim'^i_{\Sigma_i} \psi_i\} \in F \\
\text{iff} & \vec{\phi}/{\sim^F_{\prod_{i\in I}\Sigma_i}}, \vec{\psi}/{\sim^F_{\prod_{i\in I}\Sigma_i}} \in \prod^F_{i\in I}\mathrm{SEN}^i(\prod_{i\in I}\Sigma_i) \text{ and} \\
 & \qquad\qquad \vec{\phi}/{\sim^F_{\prod_{i\in I}\Sigma_i}} \ \prod^F_{i\in I} \lesssim'^i_{\prod_{i\in I}\Sigma_i} \ \vec{\psi}/{\sim^F_{\prod_{i\in I}\Sigma_i}}
\end{array}$$

and, therefore $\prod^F_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle$ is a $(\prod_{i\in I}\rho^i)^{\sim^F}$-subpofunctor of $\prod^F_{i\in I}\langle \mathrm{SEN}'^i, \lesssim'^i\rangle$. □

Lemma 5 leads directly to Proposition 11, showing that, given a class K of compatible pofunctors, $\mathbf{P}_{\mathrm{R}}\mathbf{S}(\mathrm{K}) \subseteq \mathbf{S}\mathbf{P}_{\mathrm{R}}(\mathrm{K})$.

Proposition 11. *If* K *is a class of compatible pofunctors, then* $\mathbf{P}_{\mathrm{R}}\mathbf{S}(\mathrm{K}) \subseteq \mathbf{S}\mathbf{P}_{\mathrm{R}}(\mathrm{K})$.

Proof. Suppose that $\langle \mathrm{SEN}, \lesssim\rangle \in \mathbf{P}_{\mathrm{R}}\mathbf{S}(\mathrm{K})$. Then, there exist $\langle \mathrm{SEN}'^i, \lesssim'^i\rangle \in \mathrm{K}, i \in I$, and ρ^i-subpofunctors $\langle \mathrm{SEN}^i, \lesssim^i\rangle$ of $\langle \mathrm{SEN}'^i, \lesssim'^i\rangle, i \in I$, such that $\langle \mathrm{SEN}, \lesssim\rangle = \prod^F_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle$, for some filter F on I. But, then, by Lemma 5, we obtain that $\langle \mathrm{SEN}, \lesssim\rangle = \prod^F_{i\in I}\langle \mathrm{SEN}^i, \lesssim^i\rangle$ is a $(\prod_{i\in I}\rho^i)^{\sim^F}$-subpofunctor of the order filtered product $\prod^F_{i\in I}\langle \mathrm{SEN}'^i, \lesssim'^i\rangle$, and, since $\langle \mathrm{SEN}'^i, \lesssim'^i\rangle \in \mathrm{K}, i \in I$, we obtain that $\langle \mathrm{SEN}, \lesssim\rangle \in \mathbf{S}\mathbf{P}_{\mathrm{R}}(\mathrm{K})$. □

Finally, in the second main closure result of the paper, it is shown that the operator $\mathbf{SP}_{\mathrm{R}} = \mathbf{SPP}_{\mathrm{U}}$ is a closure operator on classes of compatible pofunctors.

Theorem 2. $\mathbf{SP}_{\mathrm{R}} = \mathbf{SPP}_{\mathrm{U}}$ *is a closure operator on classes of compatible pofunctors.*

Proof. We first prove equality of the two operators. For right-to-left inclusion

$$\begin{aligned}\mathbf{SPP}_{\mathrm{U}}(\mathrm{K}) &\subseteq \mathbf{SP}_{\mathrm{R}}\mathbf{P}_{\mathrm{R}}(\mathrm{K}) \quad \text{(by Proposition 8)}\\ &= \mathbf{SP}_{\mathrm{R}}(\mathrm{K}) \quad \text{(by Proposition 9)}\end{aligned}$$

For the left-to-right inclusion

$$\begin{aligned}\mathbf{SP}_{\mathrm{R}}(\mathrm{K}) &\subseteq \mathbf{SSPP}_{\mathrm{U}}(\mathrm{K}) \quad \text{(by Proposition 10)}\\ &= \mathbf{SPP}_{\mathrm{U}}(\mathrm{K}) \quad \text{(by Proposition 7)}\end{aligned}$$

Finally, to show that $\mathbf{SP}_{\mathrm{R}}$ is a closure operator, it suffices to show that $\mathbf{SP}_{\mathrm{R}}\mathbf{SP}_{\mathrm{R}} \subseteq \mathbf{SP}_{\mathrm{R}}$. We indeed have

$$\begin{aligned}\mathbf{SP}_{\mathrm{R}}\mathbf{SP}_{\mathrm{R}}(\mathrm{K}) &= \mathbf{S}(\mathbf{P}_{\mathrm{R}}\mathbf{S})\mathbf{P}_{\mathrm{R}}(\mathrm{K})\\ &\subseteq \mathbf{SSP}_{\mathrm{R}}\mathbf{P}_{\mathrm{R}}(\mathrm{K}) \quad \text{(by Proposition 11)}\\ &\subseteq \mathbf{SP}_{\mathrm{R}}(\mathrm{K}) \quad \text{(by Propositions 7 and 9).}\end{aligned}$$

□

In [32] the results presented here on closure operators on partially ordered algebraic systems were used to extend the Birkhoff and Mal'cev style characterization theorems for ordered varieties and quasivarieties of partially ordered algebras (Theorems 3.14 and 3.17 of [21]) to the framework of partially ordered algebraic systems. The work in [32] introduces and uses syntactical tools (similar to equations and quasi-equations) in the theory of partially ordered algebraic systems.

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