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Hyperidentities and Related Concepts, II

Yu. M. Movsisyan*

Abstract. This survey article illustrates many important current trends and perspectives for the field including classification of hyperidentities, characterizations of algebras with hyperidentities, functional representations of free algebras, structure results for bilattices, categorical questions and applications. However, the paper contains new results and open problems, too.

Key Words: Hyperidentity, Hypervariety, Variety, Termal Hyperidentity, Essential hyperidentity, Bilattice, De Morgan algebra, Boole-De Morgan algebra, De Morgan function, quasi-De Morgan function, Super-Boolean function, Super-Boolean algebra, Super-De Morgan algebra, Super-De Morgan function, Free algebra.
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1 Introduction and Preliminaries

Model theory and algebra study the connections between formal languages and their interpretations in models and algebras. The simplest and most widespread formal language is the first order language (A. Church [46], A. I. Mal'tsev [139, 141, 142], G. Grätzer [91], C. Chang with H. Keisler[43], S. Burris with H. P. Sankappanavar [42], B. I. Plotkin [231]). The founders of the first order language (logic) are Löwenheim, Skolem, Gödel, Tarski, Mal'tsev and Birkhoff.

However, there exist very commonly encountered, classical algebraic structures that are not axiomatizable by the first order formulae (logic). For example, rings, associative rings, commutative rings, associative-commutative rings, fields, or fields of fixed characteristics are axiomatized by the first order formulae, but their multiplicative groupoids, semigroups and groups are not, because these classes of groupoids, semigroups and groups are not closed under elementary equivalency (A. I. Mal'tsev, S. Kogalovskii [115], G. Sabbagh [248]). The situation is analogous for near-fields (M. Hall

[95]), Grätzer algebras (G-fields)[89, 72, 19, 186], topological rings and topological fields (L. S. Pontryagin [234]). Characterizations of such semigroups and groups are the most important problems in modern algebra, logic and topology. L. Fuchs [71] called the characterization of multiplicative groups of fields a big problem.

This is why it is necessary to widen the formal language to allow to express phenomena that the first order logic can not capture.

An important extension of the first order logic (language) is the second order logic (language), described in detail in [43], [46], [141], [142] (also see [110]). The second order formulae consist of the same logical symbols of $\&, \vee, \neg, \rightarrow, \exists, \forall$ of individual and functional (predicate) variables, which are used in the first order formulae. The difference is that in the second order formulae, the quantifiers $\forall, \exists$ can be applied not only to individual variables, but also to functional (or to predicate) variables. Investigations of the second order formulae (logic) go back to L. Henkin, A. I. Mal'tsev, A. Church, S. Kleene, A. Tarski. Many important mathematical concepts can be written in the second order language. Consequently, investigation of the theories of the second order languages (logic) is one of the central problems of algebra and mathematical logic.

Starting with the 1960's the following second order formulae were studied in various domains of algebra and its applications (see [140], [141], [254], [255], [249],[250], [18], [64], [165], [168], [199], [21], [45], [110], [285], [24],[25], [218], [266], [170], [173], [178], [180],[272], [219], [267], [175], [195], [196], [197], [203], [211], [212], [120], [256], [144], [145], [222], [50], [51], [52], [53], [54], [63], [85], [86], [214], [215], [288], [293], [294], [295], [296], [303], [307]).

$$\forall X_1, \ldots, X_m \forall x_1, \ldots, x_n (w_1 = w_2), \tag{1}$$

$$\forall X_1, \ldots, X_k \exists X_{k+1}, \ldots, X_m \forall x_1, \ldots, x_n (w_1 = w_2), \tag{2}$$

$$\exists x_1, \ldots, x_n \forall X_1, \ldots, X_m (w_1 = w_2), \tag{3}$$

$$\exists X_1, \ldots, X_k \forall X_{k+1}, \ldots, X_m \forall x_1, \ldots, x_n (w_1 = w_2), \tag{4}$$

$$\forall X_1, \ldots, X_k \exists X_{k+1}, \ldots, X_t \forall X_{t+1}, \ldots, X_m \forall x_1, \ldots, x_n (w_1 = w_2), \tag{5}$$

where w_1, w_2 are words (terms) in the functional variables $X_1, \ldots, X_m$ and the individual (object) variables $x_1, \ldots, x_n$. The first formula is called *hyperidentity* or $\forall(\forall)$-identity (see [170, 173, 256, 63, 116], and also [11]); the second (third, fourth, fifth) formula is called an $\forall\exists(\forall)$-identity ($(\exists)\forall$-identity, $\exists\forall(\forall)$-identity, $\forall\exists\forall(\forall)$- identity). Sometimes the $\forall\exists(\forall)$-identity is called a generalized identity [18], the $(\exists)\forall$-identity is called a *coidentity* [168, 170] (also see [11]) and $\exists\forall(\forall)$-identity is called a hybrid identity [21, 256, 200]. The satisfiability of these second order formulae in an algebra $\mathfrak{A} = (Q; \Sigma)$ is understood by functional quantifiers $(\forall X_i)$ and $(\exists X_j)$, meaning: "for every value $X_i = A \in \Sigma$ of the corresponding arity" and "there exists a value

$X_j = A \in \Sigma$ of the corresponding arity". It is assumed that such a replacement is possible, that is

$$\{|X_1|, \ldots, |X_m|\} \subseteq \{|A| \mid A \in \Sigma\} = T_{\mathfrak{A}},$$

where $|S|$ is the arity of S, and $T_{\mathfrak{A}}$ is called the arithmetic type of $\mathfrak{A}$.

For the categorical definition of hyperidentities and $\forall\exists(\forall)$-identities see [165].

Second order formulae with analogous predicative quantifiers in models and algebraic systems are also often used in mathematical logic. For example, finiteness, the axiom of well-ordering, the continuum hypothesis, the property of being countable and others can be formulated within the second order logic.

A variety (or equational class) is a class of algebras (all of the same similarity type or signature) closed under the formation of products, subalgebras and homomorphic images. Equivalently, a variety is a class of algebras defined by a set of equations (identities). A hypervariety is a class of algebras (all of the same arithmetic type) defined by a set of hyperidentities. Since 1954 the following-type second order formulae were studied in algebras of term functions of various classes of varieties

$$\exists X_1, \ldots, X_m \forall x_1, \ldots, x_n (w_1 = w_2), \tag{6}$$

which are called Mal'tsev (Mal'cev) conditions (see [139], [90], [283], [269], [216],[284], [268], [106]), reducing to the hyperidentities of the class of term functions' algebras (termal or term algebras). (Note that the formula (6) is called functional equation in the Set theory [2, 3, 5].)

The formulae (1)–(6) are usually written without quantifiers [1], if the structures of the quantifiers are understanding from the content. The formulae (2)–(6) are more general than hyperidentities. The numbers m and n in hyperidentity (1) are called the functional and object *rank*, respectively. A hyperidentity is said to be *non-trivial* if its functional *rank* is > 1, and it is called trivial otherwise (m=1). A hyperidentity is called n-ary, if its functional variables are n-ary. For $n = 1, 2, 3$ the n-ary hyperidentity is called unary, binary, ternary. A formula (hyperidentity, coidentity,...) is called a formula (hyperidentity, coidentity,...) of algebra $\mathfrak{A}$, if it is satisfied in algebra $\mathfrak{A}$. Hyperidentities (coidentities,...) are usually written without quantifiers: $w_1 = w_2$. Let V be a variety or a class of algebras. A hyperidentity (coidentity,...) $w_1 = w_2$ is called a hyperidentity (coidentity,...) of V if it is a hyperidentity (coidentity,...) for any algebra $\mathfrak{A} \in V$.

Examples 1. In any lattice the following hyperidentities are satisfied

$$X(x, x) = x,$$

[1] Sometimes the domains of functional variables are different (see section 4).

$$X(x,\, y) = X(y,\, x),$$
$$X(x,\, X(y,\, z)) = X(X(x,\, y), z),$$
$$X(Y(X(x,y),z),Y(y,z)) = Y(X(x,y),z).$$

Hence, these hyperidentities are hyperidentities of the variety of lattices. The last non-trivial hyperidentity is called hyperidentity of interlacity (see [184]).

2. In any commutative and associative ring the following hyperidentities are satisfied

$$X(x,\, y) = X(y,\, x),$$
$$X(x,\, X(y,\, z)) = X(X(x,\, y), z),$$
$$X\left(X\left(Y(x,\, x),\, Y(x,\, x)\right),\, Y\left(X(x,\, x),\, X(x,\, x)\right)\right) =$$
$$= X\left(Y\left(X(x,\, x),\, X(x,\, x)\right),\, X\left(Y(x,\, x),\, Y(x,\, x)\right)\right).$$

3. In the termal algebra (i.e., the algebra of term functions) of any group (semigroup, Moufang loop) the following non-trivial hyperidentity is satisfied (see [24])

$$X(Y(x,\, x),\, Y(x,\, x)) = Y(X(x,\, x),\, X(x,\, x)).$$

4. Let $B = \{0, 1\}$ and P be the set of all binary Boolean functions. In algebra $(B; P)$ the following hyperidentities are satisfied

$$X(X(X(x,y),y),y) = X(x,y),$$
$$X(x,X(x,X(x,y))) = X(x,y).$$

In particular, these hyperidentities are satisfied in two-element Boolean algebra $(\{0, 1\}; \&, \vee, ', 0, 1)$. Hence, these hyperidentities are satisfied in any Boolean algebra too, by Birkhoff's subdirect representation theorem. See [174] for corresponding hyperidentities of n-ary Boolean functions. On the application of the results of [174] in modal logic see [99].

5. In any De Morgan algebra $Q(+, \cdot, ^-, 0, 1)$ the following non-trivial hyperidentity is satisfied

$$F(X(F(Y(x,y)),z)) = Y(F(X(F(x),z)),F(X(F(y),z))).$$

The concept of hyperidentity is present in many well known notions. For example, an algebra $\mathfrak{A} = (Q; \Sigma)$ is said to be *Abelian* (A.G.Kurosh [125]) or *entropic* (medial) if the following non-trivial hyperidentity

$$X\Big(Y(x_{11}, \ldots, x_{1n}), \ldots, Y(x_{m1}, \ldots, x_{mn})\Big) =$$
$$= Y\Big(X(x_{11}, \ldots, x_{m1}), \ldots, X(x_{1n}, \ldots, x_{mn})\Big)$$

is valid for all $m, n \in T_{\mathfrak{A}}$. An algebra $\mathfrak{A} = (Q; \Sigma)$ is said to be *idempotent* if the following hyperidentity of idempotency

$$X(\underbrace{x, \ldots, x}_{n}) = x$$

is valid for all $n \in T_{\mathfrak{A}}$.

A *mode* is an idempotent and entropic algebra (studied in monographs [244, 245]). A *distributive bisemilattice* (multisemilattice) [114] is a binary algebra with semilattice operations satisfying the following non-trivial hyperidentity of distributivity

$$X(x, Y(y, z)) = Y(X(x, y), X(x, z)).$$

A *doppelsemigroup* (see [6, 129, 225, 243, 308, 310]) is an algebra with two binary operations satisfying the following hyperidentity of associativity

$$X(x, Y(y, z)) = Y(X(x, y), z).$$

The investigation of hyperidentities is a relatively new, actively developing field of pure and applied algebra. The concept of hyperidentity offers a high-level approach to algebraic questions, leading to new results, applications and problems. In particular, the investigation of hyperidentities is useful from the point of view of new technologies too, via optimization problems of block diagrams [180]. For applications of hyperidentities in discrete mathematics and topology see [61, 62, 74, 159, 190, 191, 192, 193, 194, 214, 307]. For characterization of Sheffer functions and primal algebras via hyperidentities see (K. Denecke, R. Pöschel [61, 62]).

Any algebra $\mathfrak{A} = (Q; \Sigma)$ may be interpreted as a many-sorted algebra $(Q; \Sigma_i, \ldots, \Sigma_n, \ldots)$ (where Σ_n is a set of all n-ary operations of the given algebra) with the following operations $(f, x_1, \ldots, x_n) \to f(x_1, \ldots, x_n)$ where $f \in \Sigma_n$, $x_1, \ldots, x_n \in Q$, $n \in T_{\mathfrak{A}}$ ([31], [97]). Moreover, the hyperidentities of the given algebra become the identities of the corresponding many-sorted algebra and vice versa. In this way the theory of hyperidentities as a second order theory of algebras is converted into a first order theory of many-sorted algebras. Simultaneously there is a bijection between hyperidentities of termal (term) algebra $\mathcal{F}(\mathfrak{A})$ and identities of the clone $Cl(\mathfrak{A})$ of an algebra $\mathfrak{A}$ (a *clone* is also a many-sorted algebra, see [47], [126], [143], [157], [173], [231], [247], [279], [274]). One of the specifics of a hyperidentity (coidentity) is that if a hyperidentity (coidentity) is valid in algebra $\mathfrak{A}$ then it is also valid in every reduct $\mathfrak{B}$ of $\mathfrak{A}$ with the condition $T_{\mathfrak{B}} = T_{\mathfrak{A}}$.

Hyperidentities are also "identities" of algebras in the category of bihomomorphisms $(\varphi, \tilde{\psi})$, where

$$\varphi A(x_1, \ldots, x_n) = (\tilde{\psi} A)(\varphi x_1, \ldots, \varphi x_n),$$

which were studied in the monograph [170]. More about the application of such morphisms in the cryptography can be found in [9].

Hyperidentities in binary algebras with quasigroup operations were first considered by V. D. Belousov [18] (as a special case of $\forall\exists(\forall)$-identities which earlier is considered by R. Schauffler ([254], [255]) in coding theory) and then J. Aczel [4], about the classification of associative and distributive hyperidentities in binary algebras with quasigroup operations. Currently, more general results about these and other classifications of hyperidentities may be found in [170], [173], [178] and [180]. Observe that in algebras with quasigroup operations many $\forall\exists\forall(\forall)$- identities are equivalent to hyperidentities (see[173]).

The multiplicative groups of fields have been characterized in [172] and [178] by hyperidentities. On the base of these results the concept of binary G-spaces is developed in topology [74]. The hyperidentities of varieties of lattices, modular lattices, distributive lattices, Boolean algebras, De Morgan algebras and weakly idempotent lattices have been characterized in the works [174], [177], [178],[176],[195], [187],[188],[189], [202], [203], [207].

A hyperidentity $\omega_1 = \omega_2$ is called *termal* or *polynomial* hyperidentity of the algebra $\mathfrak{A}$ if it is valid in the term algebra $\mathcal{F}(\mathfrak{A})$. Let V be a variety. A hyperidentity $\omega_1 = \omega_2$ is called a termal hyperidentity of V if it is a termal hyperidentity for any algebra $\mathfrak{A} \in V$. Termal hyperidentities for varieties were first considered by W. Taylor ([285]) (as a special case of Mal'tsev conditions for varieties) for characterization of classes of varieties which are closed under formation of equivalent varieties, products of varieties, reducts of varieties and subvarieties. Since the operations of an algebra are included in the set of term operations (clone) of the algebra, the concept of termal hyperidentity of a variety is stronger than the concept of hyperidentity. In particular, the variety of rings (even commutative rings) does not have termal hyperidentities except $w = w$, but has hyperidentities (see section 8).

Termal hyperidentities of varieties of groups and semigroups have been characterized by G. Bergman [24] (also see [25]). Termal hyperidentities of the variety of lattices and of the variety of semilattices were studied by R. Padmanabhan and P. Penner ([218], [224], [219]).

Hyperidentities in algebras as an individual direction of investigations, were first presented in the monographs [170], [173]. The problem of characterization of termal hyperidentities of important classes of groups, semigroups, loops, quasigroups has been posed in the book [170] (p.129, problem 26). The hyperidentities of algebras and varieties, termal and essential hyperidentities, pre-hyperidentities of various varieties of groups, semigroups, quasigroups, loops and related algebras were also studied by many authors (see references of the current paper).

We briefly describe the structure of the paper. This paper is a survey of the results and problems on hyperidentities and related formulae (equations) and on related concepts. It may be divided into two parts. One part discusses primarily categorical questions, and the other part contains structure results, questions of classification of hyperidentities and characterization of algebras with hyperidentities. In section 2 we formulate the Mal'tsev-Gödel compactness theorem for hyperidentities. Sections 3,4,5,6,7,8 are devoted to the questions of classifications and characterizations of hyperidentities, termal hyperidentities, pre-hyperidentities and essential hyperidentities in individual algebras and varieties of algebras. In section 8 the concept of super-Boolean algebras and super-De Morgan algebras are introduced, and in the next two sections 9, 10 the concepts of super-Boolean function and super-De Morgan function are introduced with characterizations of finitely generated free super-Boolean algebras and finitely generated free super-De Morgan algebras. In the section 11 we characterize termal hyperidentities of the varieties of Boolean algebras, distributive lattices, lattices, semilattices, semigroups and groups. In the section 12 the concepts of bilattices, interlaced bilattices, distributive bilattices, Boolean bilattices are introduced (see [70, 73, 75, 76, 77, 78, 161, 184, 185, 201, 203, 211, 212, 237, 239, 246]) and finitely generated free distributive bilattices are characterized in the section 13. Section 14 is devoted to the extension and strengthening of Schauffler's theorem,which is applicable in coding theory. In section 15 distributive systems and their connection with functional equations and hyperidentities are discussed. Section 16 discusses the binary representations of semigroups and groups, and their applications; the topological version of which was started in [74]. And finally, section 17 discusses other open problems along with number of open problems presented in the current paper.

To limit the size of the paper the proofs of results are mostly omitted.

2 The Mal'tsev-Gödel compactness theorem for hyperidentities

If $\mathfrak{A} = (Q; \Sigma)$ is an algebra, then the set

$$T_{\mathfrak{A}} = \{|A| \mid A \in \Sigma\} \subseteq \mathcal{N}$$

is called an arithmetic type of the algebra $\mathfrak{A}$. A T-algebra is an algebra with the arithmetic type $T \subseteq \mathcal{N}$. A class of algebras is called a class of T-algebras if every algebra in it is a T-algebra. A T-reduct is a reduct with arithmetic type $T \subseteq \mathcal{N}$. The concept of arithmetic type of a relational structure and an algebraic system is defined analogously.

Let $T \subseteq N$ and $T \neq \emptyset$. The hyperidentity (1) (coidentity (3)) is called a T-hyperidentity (T-coidentity), if $\{|X_1|, \ldots, |X_m|\} \subseteq T$. We say that the T-hyperidentity (1) holds (is satisfied, valid, true) in the T-algebra $\mathfrak{A} =$

$(Q;\,\Sigma)$ if the equality $\omega_1 = \omega_2$ is valid when every object variable and every functional variable in it is replaced by an arbitrary element of Q and any operation of the corresponding arity from Σ respectively. Similarly, the T-coidentity (3) holds in the T-algebra $\mathfrak{A} = (Q;\,\Sigma)$ if there exist values for object variables $x_1, \ldots, x_n$ from Q, such that the equality $\omega_1 = \omega_2$ holds when every functional variable in it is replaced by any operation of the corresponding arity from Σ. In addition the object variables in the notation of the coidentity $\omega_1 = \omega_2$ are replaced by corresponding fixed values from Q.

A T-hyperidentity $\omega_1 = \omega_2$ is called a consequence of the system $\mathcal{L}$ of T-hyperidentities and is denoted by $\mathcal{L} \Rightarrow (\omega_1 = \omega_2)$ if the system $\mathcal{L}$ is valid in the T-algebra, then the hyperidentity $\omega_1 = \omega_2$ is also valid in it, that is, for any T-algebra $\mathfrak{A}$:

$$\mathfrak{A} \models \mathcal{L} \Rightarrow \mathfrak{A} \models (\omega_1 = \omega_2)$$

(the notation $\mathfrak{A} \models \mathcal{L}$ means that any hyperidentity from $\mathcal{L}$ is valid in the algebra $\mathfrak{A}$).

The hyperidentity $\omega_1 = \omega_2$ is called a termal consequence (briefly t-consequence) of a system of the hyperidentities $\mathcal{L}$ and is denoted by $\mathcal{L} \Rightarrow_t (\omega_1 = \omega_2)$ if for any algebra $\mathfrak{A}$:

$$\mathcal{F}(\mathfrak{A}) \models \mathcal{L} \Rightarrow \mathcal{F}(\mathfrak{A}) \models (\omega_1 = \omega_2);$$

In the category of T-algebras and bihomomorphisms $(\varphi, \tilde{\psi})$, we consider the concepts of subalgebra, quotient algebra, direct and filtered products.

An algebra $\mathfrak{A}' = (Q';\,\Sigma')$ is called a *subalgebra of the algebra* $\mathfrak{A} = (Q;\,\Sigma)$ if $Q' \subseteq Q$ and every operation from Σ' is the restriction of some operation from Σ (to the subset Q'). For example, every abelian group is a subalgebra of some ring, and, a fortiori, every groupoid is a subalgebra of some ring. Every semi-lattice is a subalgebra of some distributive lattice and every distributive lattice is a subalgebra of some Boolean algebra, etc. When we want to specify the arithmetic type of subalgebras, we call them *T-subalgebras.* A class of T-algebras is said to be *hereditary* if it contains all the T-subalgebras of any T-algebra of the given class.

Let $\mathfrak{A} = (Q;\,\Sigma)$ be an arbitrary T-algebra, and let $Q' \subseteq Q$, $\Sigma' \subseteq \Sigma$, where $Q' \neq \emptyset$ and $\Sigma' \neq \emptyset$. A pair $\mathfrak{A}' = (Q';\,\Sigma')$ is called a *subsystem of the T-algebra* $\mathfrak{A}$ if Q' is closed with respect to all the operations of Σ'. Subsystems of the form $(Q';\,\Sigma)$ are termed *principal.*

For every subsystem $(Q_1;\,\Sigma_1)$ of an algebra $(Q;\,\Sigma)$ there exists a corresponding subalgebra $(Q_1;\,\Sigma_1^*)$, if instead of the operations from Σ_1 (which are defined on the set Q) we consider their restrictions to the subset $Q_1 \subseteq Q$. It is clear that if $(Q_1;\,\Sigma_1)$ is a T-subsystem, then $(Q_1;\,\Sigma_1^*)$ is a T-algebra.

If $(\varphi,\,\tilde{\psi}) : \mathfrak{A} \Rightarrow \mathfrak{A}'$ is a bihomomorphism of a T-algebra $\mathfrak{A}$ into a T-algebra $\mathfrak{A}'$, then the pair $(\varphi Q;\,\tilde{\psi}\Sigma)$ is a T-subsystem of $\mathfrak{A}'$ and is called the *bihomomorphic image* of $\mathfrak{A}$ under the bihomomorphism $(\varphi,\,\tilde{\psi})$. The subalgebra corresponding to the subsystem $(\varphi Q;\, \tilde{\psi}\Sigma)$ is also called a *bihomomorphic image* of the algebra $\mathfrak{A}$ under the bihomomorphism $(\varphi,\, \tilde{\psi})$.

A class of T-algebras is said to be *homomorphically closed (abstract)* if along with each T-algebra it contains every bihomomorphic (respectively biisomorphic) image of it under any bihomomorphism (respectively biisomorphism) $(\varphi,\, \tilde{\psi})$.

Let r and $\tilde{t}$ be equivalence relations defined respectively on the sets Q and Σ of a T-algebra $\mathfrak{A} = (Q;\, \Sigma)$. A pair $q = (r,\, \tilde{t})$ is called a *congruence* of the T-algebra $\mathfrak{A}$ if, firstly, $\tilde{t}$ preserves the arity of the operations and, secondly, the relations r and $\tilde{t}$ are compatible in the following sense:

$$x_1 r x_1',\, \ldots,\, x_n r x_n',\; A\tilde{t}B \to A(x_1,\, \ldots,\, x_n) r B(x_1',\, \ldots,\, x_n'),$$

where $x_i,\ x_i' \in Q,\ A,\ B \in \Sigma,\ |A| = |B| = n$.

If $q = (r,\, \tilde{t})$ is a congruence of a T-algebra $\mathfrak{A} = (Q;\, \Sigma)$ and $\tilde{0}$ is the identity relation of Σ, then $(r,\, \tilde{0})$ is ordinary congruence of $\mathfrak{A}$.

If $q = (r,\, \tilde{t})$ is a congruence of a T-algebra $\mathfrak{A} = (Q;\, \Sigma)$ then every element $[A]\tilde{t}$ of the quotient set $\Sigma/\tilde{t}$ defines an operation on the quotient set Q/r in the following way:

$$[A]\tilde{t}([x_1]r,\, \ldots,\, [x_n]r) = [A(x_1,\, \ldots,\, x_n)]r,$$

where $A \in \Sigma$, $|A| = n$, $x_1,\, \ldots,\, x_n \in Q$ and $[\mathfrak{x}]s$ denotes the equivalence class of an element $\mathfrak{x}$ modulo the equivalence relation s.

The definition of congruence implies that the operation $[A]\tilde{t}$ is well defined. As a result we obtain a quotient algebra of the T-algebra $\mathfrak{A}$ modulo the congruence $q = (r,\, \tilde{t})$, denoted by $\mathfrak{A}/q$.

A congruence $q = (r,\, \tilde{t})$ of a T-algebra $(Q;\, \Sigma)$ is said to be *fully invariant* if it preserves any biendomorphism $(\varphi,\, \tilde{\psi})$ of this algebra, that is,

$$xry \to \varphi(x) r \varphi(y)$$

and

$$A\tilde{t}B \to \tilde{\psi}(A)\tilde{t}\tilde{\psi}(B),$$

where $x,\, y \in Q$ and $A,\, B \in \Sigma$.

We proceed to direct and filtered products in our category. Let $\mathfrak{A}_i = (Q_i;\, \Sigma_i)$, $i \in I$ be T-algebras of the same arithmetic type. We form the Cartesian product $\widehat{Q} = \prod_{i\in I} Q_i$ as the set of all functions of the form $f : I \to \bigcup_{i\in I} Q_i$ for which $f(i) \in Q_i$ for all $i \in I$. In addition we form the Cartesian product $\prod_{i\in I} \Sigma_i$ and define the subset $\widehat{\Sigma} \subseteq \prod_{i\in I} \Sigma_i$ as the set of all possible functions $F : I \to \bigcup_{i\in I} \Sigma_i$ satisfying the following two conditions:

a) $F(i) \in \Sigma_i$ for all $i \in I$;

b) $|F(i)| = |F(j)|$ for all $i,\, j \in I$, that is, for fixed F the arity of an operation $F(i) \in \Sigma_i$ does not depend on $i \in I$.

The set $\widehat{\Sigma}$ can be identified with the set of all possible systems $\hat{A} = (A_i)_{i\in I}$ of operations of the same arity, taking one operation from each set Σ_i. We write $\widehat{\Sigma} = \widetilde{\prod}_{i\in I}\Sigma_i$. If $F \in \tilde{\Sigma}$ and $|F(i)| = n$, then F defines an n-ary operation, defined component-wise on the set $\widehat{Q} = \prod_{i\in I} Q_i$:

$$F(f_1, \ldots, f_n)(i) = F(i)(f_1(i), \ldots, f_n(i)), \quad i \in I,$$

where $f_1, \ldots, f_n \in \widehat{Q}$.

As a result we obtain a T-algebra $\widehat{\mathfrak{A}} = (\widehat{Q}; \widehat{\Sigma})$, which is called the superproduct of the T-algebras $\mathfrak{A}_i$, $i \in I$, or their (direct) T-product and is written as: $\widehat{\mathfrak{A}} = \prod_{i\in I} \mathfrak{A}_i$. Subdirect products of the T-algebras $\mathfrak{A}_i$, $i \in I$ are defined in the natural way, as a subdirectly irreducibility in the category of T-algebras.

Example. The superproduct of two lattices $Q_1(+, \cdot)$ and $Q_2(+, \cdot)$ is an algebra $Q_1 \times Q_2((+,+),(\cdot,\cdot),(+,\cdot),(\cdot,+))$ with four binary operations, which is a bilattice [184], because $Q_1 \times Q_2((+,+),(\cdot,\cdot))$ and $Q_1 \times Q_2((+,\cdot),(\cdot,+))$ are lattices.

A class of T-algebras is said to be *multiplicatively closed* if it contains the superproduct of any family of T-algebras from this class.

Let $\mathfrak{A}_i = (Q_i; \Sigma_i)$, $i \in I$ be T-algebras of the same arithmetic type and let $\mathcal{D}$ be a filter[2] on I. We introduce the relation $\equiv_{\mathcal{D}}$ defined on $\prod_{i\in I} Q_i$ and the relation $\sim_{\mathcal{D}}$ defined on $\widetilde{\prod}_{i\in I}\Sigma_i$ by setting

$$f \equiv_{\mathcal{D}} g \Leftrightarrow \{\alpha \in I \mid f(\alpha) = g(\alpha)\} \in \mathcal{D},$$

where f, $g \in \prod_{i\in I} Q_i$, and

$$F \sim_{\mathcal{D}} G \Leftrightarrow \{\alpha \in I \mid F(\alpha) = G(\alpha)\} \in \mathcal{D},$$

where F, $G \in \widetilde{\prod}_{i\in I}\Sigma_i$, $|F| = |G|$.

According to the definition of a filter, the relations $\equiv_{\mathcal{D}}$ and $\sim_{\mathcal{D}}$ are equivalence relations. In addition, it is easy to prove that the pair of equivalence relations $(\equiv_{\mathcal{D}}, \sim_{\mathcal{D}})$ also are a congruence of the product of the T-algebras $\mathfrak{A}_i = (Q_i; \Sigma_i)$, $i \in I$. The corresponding quotient algebra is denoted by $\prod_{i\in I} \mathfrak{A}_i/\mathcal{D}$ and is called *filtered product* (with respect to the filter $\mathcal{D}$) of

[2] A *filter* on a non-empty set I is a non-empty set $\mathcal{D}$ of subsets of I satisfying the requirements:

a) the intersection of any two subsets from $\mathcal{D}$ belongs to $\mathcal{D}$;

b) all the supersets of any subset belonging to $\mathcal{D}$ also belong to $\mathcal{D}$;

c) the empty set $\emptyset$ does not belong to $\mathcal{D}$.

A maximal filter on I, that is, a filter on I that does not lie in any other filter on I, is usually called an *ultrafilter* on I.

the T-algebras $\mathfrak{A}_i = (Q_i; \Sigma_i)$, $i \in I$. A filtered product of T-algebras with respect to an ultrafilter is called an ultraproduct of T-algebras.

For the congruence ($\equiv_{\mathcal{D}}$, $\sim_{\mathcal{D}}$) we have:

$F \sim_{\mathcal{D}} G \Leftrightarrow \{\alpha \in I \mid F(\alpha) = G(\alpha)\} \in \mathcal{D}$

$\Leftrightarrow \{\alpha \in I \mid F(\alpha)(f_1(\alpha), \ldots, f_n(\alpha)) = G(\alpha)(f_1(\alpha), \ldots, f_n(\alpha))\} \in \mathcal{D}$

$\Leftrightarrow \{\alpha \in I \mid F(f_1, \ldots, f_n)(\alpha) = G(f_1, \ldots, f_n)(\alpha)\} \in \mathcal{D}$

$\Leftrightarrow F(f_1, \ldots, f_n) \equiv_{\mathcal{D}} G(f_1, \ldots, f_n)$.

A set of T-hyperidentities is said to be *satisfiable* if there exists a non-trivial T-algebra in which every T-hyperidentity from this set is true. The Mal'tsev-Gödel compactness theorem for first-order languages extends to the hyperidentities, using ultraproduct of T-algebras.

Theorem 1 ([178]) *If every finite part of an infinite set of T-hyperidentities is satisfiable, then all the set of T-hyperidentities also is satisfiable.*

3 On the Birkhoff type theorems

Let $\mathcal{L}$ be some non-empty set of T-hyperidentities, and let $\mathfrak{M}_{\mathcal{L}}^T$ be the class of all T-algebras in which every hyperidentity from $\mathcal{L}$ is valid. A class of T-algebras $\mathfrak{N}$ is called a hypervariety of T-algebras if there exists a system $\mathcal{L}$ of T-hyperidentities with the property

$$\mathfrak{N} = \mathfrak{M}_{\mathcal{L}}^T.$$

In this case $\mathcal{L}$ is called a defining system of hyperidentities for $\mathfrak{N}$.

Let Ω be a signature with an arithmetical type T_Ω, i.e.

$$T_\Omega = \{|\omega| \mid \omega \in \Omega\},$$

and let $\mathcal{L}$ be some non-empty set of T_Ω-hyperidentities, and let $\mathfrak{N}_{\mathcal{L}}^\Omega$ be the class of all Ω-algebras, in which every hyperidentity from $\mathcal{L}$ is valid. It's easy to prove that $\mathfrak{N}_{\mathcal{L}}^\Omega$ is the variety for every Ω and $\mathcal{L} \neq \emptyset$. The variety V of Ω-algebras is called hypervariety of Ω-algebras, if there exists a system $\mathcal{L}$ of T_Ω-hyperidentities, such that

$$V = \mathfrak{N}_{\mathcal{L}}^\Omega.$$

Let L be some non-empty set of T-hyperidentities, where $T = \mathcal{N}$, and S_L^Ω is the class of all Ω-algebras, in which every hyperidentity from L is termally valid. It's easy to note that S_L^Ω is a variety for any Ω and $L \neq \emptyset$. The variety V of Ω-algebras is said to be solid ([87]), if

$$V = S_L^\Omega$$

for some $L \neq \emptyset$. For characterization of all solid varieties of semigroups see [233].

The concept of solid hypervariety is defined analogously. Let $\mathcal{Z}$ be some non-empty set of hyperidentities ($\mathcal{N}$-hyperidentities), and $P_{\mathcal{Z}}^{T}$ be the class of all T-algebras, in which every hyperidentity from $\mathcal{Z}$ is termally valid. It's easy to note that the class $P_{\mathcal{Z}}^{T}$ is the hypervariety of T-algebras for every $\mathcal{Z} \neq \emptyset$ and $T \subseteq \mathcal{N}$. The hypervarieties $\mathcal{W}$ of T-algebras are called solid, if

$$\mathcal{W} = P_{\mathcal{Z}}^{T}$$

for some $\mathcal{Z} \neq \emptyset$ [180].

Hypervarieties, solid varieties and solid hypervarieties are characterized in a natural way by the categorical notions introduced above.

Theorem 2 *A class of T-algebras is a hypervariety of T-algebras if and only if it is hereditary, homomorphicaly and multiplicatively closed.*

Theorem 3 *For any variety V of Ω-algebras the following conditions are equivalent:*

1) The variety V is a hypervariety of Ω-algebras;

2) Every identity of V is a hyperidentity for V;

3) The variety V along with any algebra $\mathfrak{A} \in V$ contains any T_{Ω}-reduct of $\mathfrak{A}$.

Theorem 4 *For any variety V of Ω-algebras the following conditions are equivalent:*

4) The variety V is solid;

5) Every identity of V is a termal hyperidentity for V;

6) The variety V along with any algebra $\mathfrak{A} \in V$ contains any Ω-reduct of termal algebra $\mathcal{F}(\mathfrak{A})$ (see [87], [60]).

Theorem 5 *For every hypervariety $\mathcal{W}$ of T-algebras the following conditions are equivalent:*

7) The hypervariety $\mathcal{W}$ of T-algebras is solid;

8) Every hyperidentity of $\mathcal{W}$ is termal hyperidentity for $\mathcal{W}$;

9) The hypervariety $\mathcal{W}$ of T-algebras along with any algebra $\mathfrak{A} \in \mathcal{W}$ contains any T-reduct of termal algebra $\mathcal{F}(\mathfrak{A})$;

Naturally here also arises the following notion of a quasi-solid variety of Ω-algebras.

The variety of Ω-algebras is called quasi-solid, if its any hyperidentity is its termal hyperidentity.

The Birkhoff type theorems of completeness for $\forall\exists(\forall)$-identities, hyperidentities, termal hyperidentities and hybrid identities were considered in [165], [170], [173], [256], [63].

4 Varieties of clone-algebras. Second order algebras

The set Q is called an algebra over the clone Γ, if for every $\gamma \in \Gamma$, $|\gamma| = n$ and $a_1, \ldots, a_n \in Q$ the element $\gamma(a_1, \ldots, a_n) \in Q$ is defined, and the following two identities

$$\mu_n^m(\gamma, \gamma_1, \ldots, \gamma_n)(a_1, \ldots, a_m) =$$
$$= \gamma\left(\gamma_1(a_1, \ldots, a_m), \ldots, \gamma_n(a_1, \ldots, a_m)\right),$$
$$\delta_n^i(a_1, \ldots, a_n) = a_i$$

are valid.

Every clone homomorphism

$$\varphi : \Gamma \to O_p Q$$

converts the set Q into an algebra over clone Γ and vice versa: if Q is an algebra over clone Γ, then it determines the representation of clone Γ in the clone of all operations of the set Q.

If Q is an algebra over clone Γ, then Q will be called a clone Γ-algebra. For every fixed Γ we have a category of all clone Γ-algebras, in which homomorphisms are the mappings of sets, permutational with the actions of clone Γ:

$$\varphi\gamma(x_1, \ldots, x_n) = \gamma(\varphi x_1, \ldots, \varphi x_n).$$

All possible clone Γ-algebras (at a fixed clone Γ) form a variety of all clone Γ-algebra. The subvariety in the variety of all clone Γ-algebras is called the variety of clone Γ-algebras.

Theorem 6 *There exists a one-to-one correspondence between the varieties of clone Γ-algebras and the congruencies of clone Γ. The clone Γ is isomorphic to the clone of the variety of all clone Γ-algebras.*

The set Q is called a clone-algebra, if it is an algebra over some clone Γ. Now let's consider the category of clone-algebras over various clones.

Let Q be a clone Γ-algebra; we will denote this clone-algebra by $(Q; \Gamma)$. If $(Q_1; \Gamma_1)$ and $(Q_2; \Gamma_2)$ are two clone-algebras, then the homomorphism between them is defined as a pair $(\varphi, \tilde{\psi})$, where $\varphi : Q_1 \to Q_2$ and $\tilde{\psi} : \Gamma_1 \to \Gamma_2$ is a homomorphism of clones and the following condition

$$\varphi\omega(x_1, \ldots, x_n) = \left[\tilde{\psi}(\omega)\right](\varphi x_1, \ldots, \varphi x_n)$$

is true for every $\omega \in \Gamma_1$, $|\omega| = n$ and $x_1, \ldots, x_n \in Q_1$.

So we obtain the category of clone-algebras over various clones. According to this definition of homomorphism of clone-algebras, the notions of homomorphic images, congruences, fully-invariant congruences, subalgebras, direct products of clone-algebras etc. are understood.

The characterization of varieties of clone-algebras (in this category) is given in ([173]).

The class V of clone-algebras is called a variety of clone-algebras, if it is closed to homomorphic images, subalgebras and direct products of clone-algebras of V. The variety of clone-algebras is called a hypervariety of clone algebras, if it is defined by some system of hyperidentities.

The class of varieties is called hypervariety of varieties if it is defined by some system of termal hyperidentities ([285]).

We consider a free clone, freely-generated by set U which contains countable n-ary elements for every natural n. Such a free clone is called a standard free clone.

Theorem 7 ([173]). *There exists a one-to-one correspondence between the varieties of clone-algebras and the pare (p, q) of fully-invariant congruencies $q \subseteq p$ of a standard free clone.*

The homomorphism $(\varphi, \tilde{\psi})$ of clone-algebras is called a right-homomorphism if φ is the identical mapping. The variety V of clone-algebras is called saturated, if for every $(Q; \Gamma) \in V$ and for every right-epimorphism of clone-algebras $(\varphi, \tilde{\psi}) : (Q; \Gamma') \Rightarrow (Q; \Gamma)$ we have $(Q; \Gamma') \in V$.

Theorem 8 ([173]). *The saturated varieties of clone-algebras are hypervarieties of clone-algebras. There exists a one-to-one correspondence between the saturated varieties of clone-algebras and the fully-invariant congruencies of a standard free clone. There exists a one-to-one correspondence between the saturated varieties of clone-algebras and the varieties of clones. There exists a one-to-one correspondence between the saturated varieties of clone-algebras and the hypervarieties of varieties.*

In particular, the clone $\Gamma = Cl(\mathfrak{A})$ and its every isomorphic clone acts naturally on the set Q for every algebra $\mathfrak{A} = (Q; \Sigma)$, and Q converts to a clone-algebra which is called a natural clone-algebra. In that case the algebra $\mathfrak{A}$ is called the spoor for the corresponding natural clone-algebra. In this sense we also understand the spoor for varieties of natural clone-algebras.

The class of all algebras of hypervariety of varieties is a spoor for the variety of natural clone-algebras, which is defined by hyperidentities ([63], [242]).

For the proof of the corresponding Birkhoff type theorem of $\forall\exists(\forall)$-identities the concept of second order algebras is introduced in [165]. If $(Q; \Sigma)$ and $(\Sigma; \Omega)$ are algebras then $(Q; \Sigma; \Omega)$ is called second order algebra. Since Σ is naturally graduated by arities of operations, algebra $(\Sigma; \Omega)$

may also be considered as a many-sorted algebra. This concept of second order algebras also includes the concepts of natural clone-algebras, symmetric groups, symmetric semigroups, Boolean algebras of Boolean functions of n variables, De Morgan algebras of De Morgan functions of n variables [186], etc.

5 Classical hyperidentities in invertible and related algebras

A binary algebra $(Q;\Sigma)$ is called isotopic to the groupoid $Q(\cdot)$, if its every operation is isotopic to the groupoid $Q(\cdot)$, i.e. for any operation $A\in\Sigma$ there exist bijections $\alpha_A,\beta_A,\gamma_A : Q\to Q$, such that

$$A(x,y)=\alpha_A^{-1}(\beta_A x\cdot\gamma_A y)$$

for every $x,y\in Q$ [7, 8]. A binary algebra $(Q;\Sigma)$ is called isotopic to a groupoid (group, loop, semigroup) if $(Q;\Sigma)$ is isotopic to some groupoid (group, loop, semigroup) $Q(\cdot)$.

A binary algebra $(Q;\Sigma)$ is called left(right)-linear on the groupoid $Q(\cdot)$ if its every operation is left(right)-linear on the groupoid $Q(\cdot)$, i.e. for any operation $A\in\Sigma$ there exist automorphisms φ_A of the groupoid $Q(\cdot)$ and permutation α_A of Q with the equality

$$A(x,y)=\varphi_A x\cdot\alpha_A y$$

$$(A(x,y)=\alpha_A x\cdot\varphi_A y)$$

for every $x,y\in Q$. A binary algebra $(Q;\Sigma)$ is called left(right)-linear on a groupoid (group, loop, semigroup) if $(Q;\Sigma)$ is left(right)-linear on some groupoid (group, loop, semigroup) $Q(\cdot)$.

A binary algebra $(Q;\Sigma)$ is called linear on the groupoid $Q(\cdot)$ if its every operation is linear on the groupoid $Q(\cdot)$, i.e. for any operation $A\in\Sigma$ there exist automorphisms φ_A and ψ_A of the groupoid $Q(\cdot)$ and element $t_A\in Q$ with the equality

$$A(x,y)=(\varphi_A x\cdot t_A)\cdot\psi_A y$$

for every $x,y\in Q$.

A binary algebra $(Q;\Sigma)$ is called linear on a groupoid (group, loop, semigroup) if $(Q;\Sigma)$ is linear on some groupoid (group, loop, semigroup) $Q(\cdot)$. (A linear algebra on the commutative group is often called T-linear algebra [289].)

Proposition 1 *A binary algebra is linear on the group iff it is left-linear on a group and right-linear on a group.*

Lemma 1 *1) In the binary algebra $Q(A, B)$ with two binary quasigroup operations holds the identity*

$$A(B(x, y), z) = B(x, A(y, z))$$

iff there exist a group $Q(\circ)$ and permutations α, β of Q such that:

$$A(x, y) = x \circ \alpha y,$$

$$B(x, y) = \beta x \circ y;$$

2) In the binary algebra $Q(A, B)$ with two binary quasigroup operations holds the identity

$$A(B(x, y), z) = A(x, B(y, z))$$

iff there exist a group $Q(\circ)$ and a permutation α of Q such that:

$$A(x, y) = t \circ \alpha x \circ s \circ \alpha y,$$

$$B(x, y) = \alpha^{-1}(\alpha x \circ s \circ \alpha y),$$

where $t, s \in Q$;

3) In the binary algebra $Q(A, B)$ with two binary quasigroup operations holds the identity

$$A(A(x, y), z) = B(x, B(y, z))$$

iff there exist a group $Q(\circ)$ and its automorphisms α, β such that:

$$A(x, y) = \alpha x \circ t \circ \beta^2 y,$$

$$B(x, y) = \alpha^2 x \circ s \circ \beta y,$$

where $s, t \in Q$, and $\alpha t \circ t = s \circ \beta s$, $\beta\alpha^2 = I_{\beta s \circ t^{-1}} \alpha\beta^2$ ($I_u(x) = u \circ x \circ u^{-1}$ is an inner automorphism). Thus, quasigroups $Q(A)$ and $Q(B)$ are linear on the group $Q(\circ)$, i.e. the algebra $Q(A, B)$ is linear on the group $Q(\circ)$;

4) If the identity

$$A(A(x, y), z) = B(x, C(y, z))$$

is satisfied in the algebra $Q(A, B, C)$ with tree quasigroup operations, then the quasigroup $Q(A)$ is left-linear on a group;

5) If the identity

$$A(x, A(y, z)) = B(C(x, y), z)$$

is satisfied in the algebra $Q(A, B, C)$ with tree quasigroup operations, then the quasigroup $Q(A)$ is right-linear on a group;

6) In the binary algebra $Q(A, B)$ with two binary quasigroup operations the following identity

$$A(A(x, y), z) = A(x, B(y, z))$$

holds iff there exist a group $Q(\circ)$ and a permutations α of Q such that:

$$A(x, y) = x \circ \alpha y,$$

$$\alpha B(x, y) = \alpha x \circ \alpha y;$$

Hence, $Q(B)$ is a group (cf.[278]) isomorphic to $Q(\circ)$.

7) In the binary algebra $Q(A, B)$ with two binary quasigroup operations the following identity

$$A(x, A(y, z)) = A(B(x, y), z)$$

holds iff there exist a group $Q(\circ)$ and a permutations α of Q such that:

$$A(x, y) = \alpha x \circ y,$$

$$\alpha B(x, y) = \alpha x \circ \alpha y.$$

8) If the identity of mediality

$$A_1(A_1(x, y), A_1(u, v)) = A_2(A_2(x, u), A_2(y, v))$$

is satisfied in the algebra $Q(A_1, A_2)$ with two quasigroup operations, then the algebra $Q(A_1, A_2)$ is linear on an Abelian group;

9) If the identity of mediality

$$A_1(A_2(x, y), A_2(u, v)) = A_2(A_1(x, u), A_1(y, v))$$

is satisfied in the algebra $Q(A_1, A_2)$ with two quasigroup operations, then the algebra $Q(A_1, A_2)$ is linear on an Abelian group.

10) If the identity of mediality

$$A_1(A_2(x, y), A_3(u, v)) = A_4(A_1(x, u), A_1(y, v))$$

is satisfied in the algebra $Q(A_1, A_2, A_3, A_4)$ with four quasigroup operations, then the quasigroup $Q(A_4)$ is linear on an Abelian group [179].

In the monograph [20](p. 37) instead of the criterion 6) it is proved a necessary condition for the following partial case: $B(x, y) = A(x, \beta y)$, where β is a permutation of Q (cf. [278]).

A binary algebra $\mathfrak{A} = (Q; \Sigma)$ is called 1) invertible, if $Q(A)$ is a quasigroup for any operation $A \in \Sigma$; 2) a q-algebra, if $Q(A)$ is a quasigroup for some operation $A \in \Sigma$; 3) an e-algebra, if $Q(A)$ is a groupoid with unit for some operation $A \in \Sigma$; 4) a functionally non-trivial, if the cardinality of Σ is: $|\Sigma| > 1$.

Problem 1 *Characterize invertible algebras isotopic to Moufang loops.*

Problem 2 *Characterize invertible algebras isotopic to Commutative Moufang loops.*

Problem 3 *Characterize invertible algebras isotopic to Bol loops.*

Problem 4 *Characterize invertible algebras which are left(right)-linear on the Moufang loops.*

Problem 5 *Characterize invertible algebras which are left(right)-linear on the Commutative Moufang loops.*

Problem 6 *Characterize invertible algebras which are left(right)-linear on the Bol loops.*

Let $[\omega]$ mean the set of all object variables of the word ω. The functional variable X is called singular in the hyperidentity $w_1 = w_2$, if the symbol X occurs just once in this equality (and so only on one side) and at least one of the following conditions is true:

a) in the subword $w = X(\omega_1, \omega_2)$ there exist object variables $x \in [\omega_1]$ and $y \in [\omega_2]$ such that each of them occurs just once in the subword w;

b) the subword $w = X(\omega_1, \omega_2)$ has the form $X(\omega_1, x)$ or $X(x, \omega_2)$ and there exists an object variable $y \in [w]$, different from x and occurring only once in the subword w.

The functional variable X is called singular in the $\exists\forall(\forall)$-identity (4), if the symbol X occurs just once in the equality $w_1 = w_2$ of (4), $X = X_i$, where $k+1 \leqslant i \leqslant m$ and at least one of the conditions a), b) is valid.

The functional variable X is called singular in the $\forall\exists\forall(\forall)$-identity (5), if the symbol X occurs just once in the equality $w_1 = w_2$ of (5), $X = X_i$, where $t+1 \leqslant i \leqslant m$ and at least one of the conditions a), b) is valid.

Lemma 2 ([170]) *The hyperidentity with a singular functional variable can not be valid in the functional non-trivial q-algebra.*

Lemma 3 *The $\exists\forall(\forall)$-identity (i.e hybrid identity) with a singular functional variable can not be valid in the functional non-trivial q-algebra.*

Lemma 4 *The $\forall\exists\forall(\forall)$-identity with a singular functional variable can not be valid in the functional non-trivial invertible algebra.*

Theorem 9 *In the class of invertible algebras every $\forall\exists\forall(\forall)$-identity of associativity is equivalent to a hyperidentity of associativity.*

Let's move to the characterization of hyperidentities, defined by the classical identities of associativity ($x \cdot yz = xy \cdot z$), left distributivity ($x \cdot yz = xy \cdot xz$), right distributivity ($xy \cdot z = xz \cdot yz$), transitivity ($xz \cdot yz = xy$), and the identity of Kolmogoroff ($xy \cdot yz = xz$). Such hyperidentities are called hyperidentities of associativity, left distributivity, right distributivity, transitivity and Kolmogoroff hyperidentities, respectively.

Theorem 10 *If a non-trivial hyperidentity of associativity holds in a functional non-trivial q-algebra, then it can only be of the functional rank* 2 *and of one of the following forms:*

$$X\left(x,\, Y(y,\, z)\right) = Y\left(X(x,\, y),\, z\right), \tag{7}$$

$$X\left(x,\, Y(y,\, z)\right) = X\left(Y(x,\, y),\, z\right), \tag{8}$$

$$X\left(x,\, X(y,\, z)\right) = Y\left(Y(x,\, y),\, z\right). \tag{9}$$

Moreover, operations of q-algebra with a non-trivial hyperidentity of associativity are isomorphic. In the class of all q-algebras, from the hyperidentity (9) implies the hyperidentity (8), and from the hyperidentity (8) implies the hyperidentity (7). More precisely, the hyperidentity (8) is valid in a q-algebra $\mathfrak{A}$ *with hyperidentity (7) iff the following hyperidentity holds in* $\mathfrak{A}$*:*

$$X(x, Y(y, x)) = X(Y(x, y), x);$$

The hyperidentity (9) holds in q-algebra $\mathfrak{A}$ *with hyperidentity (8) iff the following coidentity holds in* $\mathfrak{A}$*:*

$$X(a, X(a, a)) = Y(Y(a, a), a).$$

The similar results for e-algebras can be found in [178].

The semigroups $Q(\cdot)$ and $Q(\circ)$ are called interassociative ([311]) if the algebra $Q(\cdot, \circ)$ satisfies the hyperidentity of associativity (7). It is proved that two interassociative groups are isomorphic ([66]) (which also follows from the Albert's ([7, 8]) theorem: isotopic groups are isomorphic (see Lemma 1)). Moreover, if a group and a semigroup are interassociative then they are isomorphic too ([170]). Interassociative semigroups $Q(\cdot)$ and $Q(\circ)$ are called strongly interassociative if the algebra $Q(\cdot, \circ)$ satisfies the hyperidentity of associativity (8) too. For the structure of strongly interassociative semigroups see [309].

Binary algebras with the hyperidentity of associativity (7) under the name of Γ-semigroups (or gamma-semigroups), doppelsemigroups and doppelalgebras also were considered by various authors [14, 130, 217, 259, 261, 262, 225, 243, 308, 310, 6, 129, 35, 36, 66, 79, 80, 81, 82, 83, 311] (see earlier papers [250, 45] too).

Theorem 11 *If a non-trivial hyperidentity of transitivity holds in a functional non-trivial q-algebra, then it can only be of the functional rank* 2 *and of one of the following forms:*

$$Y\left(X(x,\, z),\, Y(y,\, z)\right) = Y(x,\, y), \tag{10}$$

$$Y\left(X(x,\, z),\, Y(y,\, z)\right) = X(x,\, y), \tag{11}$$

$$X\left(X(x,\, z),\, Y(y,\, z)\right) = Y(x,\, y). \tag{12}$$

Moreover, in the class of all q-algebras, from the hyperidentity (12) implies the hyperidentity (11), and from the hyperidentity (11) implies the hyperidentity (10). More precisely, the hyperidentity (11) is valid in a q-algebra $\mathfrak{A}$ *with hyperidentity (10) iff the following hyperidentity holds in* $\mathfrak{A}$*:*

$$Y(X(x,x), Y(y,x)) = X(x,y);$$

The hyperidentity (12) holds in q-algebra $\mathfrak{A}$ *with hyperidentity (11) iff the following coidentity holds in* $\mathfrak{A}$*:*

$$X(X(a,a), Y(a,a)) = Y(a,a).$$

The similar result is valid for the hyperidentities defined by the identity: $zx \cdot zy = xy$.

Theorem 12 *If Kolmogoroff's non-trivial hyperidentity holds in a functional non-trivial q-algebra, then it can only be of the functional rank* 2 *and of one of the following forms:*

$$Y\left(X(x,\, y),\, X(y,\, z)\right) = Y(x,\, z), \tag{13}$$

$$Y\left(X(x,\, y),\, Y(y,\, z)\right) = X(x,\, z), \tag{14}$$

$$X\left(X(x,\, y),\, Y(y,\, z)\right) = Y(x,\, z). \tag{15}$$

Moreover, the hyperidentities (13), (14), (15) are equivalent in the class of all q-algebras.

Theorem 13 *If in a functional non-trivial q-algebra with the trivial hyperidentity of right distributivity*

$$X\left(X(x,\, y),\, z\right) = X\left(X(x,\, z),\, X(y,\, z)\right) \tag{16}$$

holds a non-trivial hyperidentity of left distributivity, then it will have only the functional rank 2 *and the form:*

$$X\left(x,\, Y(y,\, z)\right) = Y\left(X(x,\, y),\, X(x,\, z)\right). \tag{17}$$

If in a functional non-trivial q-algebra with the trivial hyperidentity of left distributivity

$$X\,(x,\,X(y,\,z)) = X\,(X(x,\,y),\,X(x,\,z)) \tag{18}$$

holds a non-trivial hyperidentity of right distributivity, then it will only have the functional rank 2 *and the form:*

$$X\,(Y(x,\,y),\,z) = Y\,(X(x,\,z),\,X(y,\,z))\,. \tag{19}$$

Corollary 1 *If the non-trivial hyperidentities of right and left distributivity hold in the functional non-trivial q-algebra, then the non-trivial hyperidentity of left distributivity will have the functional rank* 2 *and the form (17), and the non-trivial hyperidentity of right distributivity will have the functional rank* 2 *and the form (19).*

Corollary 2 *(*[18], [4]*). If the non-trivial hyperidentities of left or right distributivity holds in the functional non-trivial invertible algebra, then the non-trivial hyperidentity of left distributivity will have the functional rank* 2 *and the form (17), and the nontrivial hyperidentity of right distributivity will have the functional rank* 2 *and the form (19).*

The problem of the characterization of invertible algebras with hyperidentities (17) and (19) of distributivity has been posed by V.D. Belousov (1965) and was solved in [170]. A more general result on the characterization of q-algebras with hyperidentities (17) and (19) of distributivity is proved in [173]. Let us remind that the binary algebra $(Q;\,\Sigma)$ is called idempotent, if the hyperidentity of idempotency $X(x,\,x) = x$ is valid in it.

Theorem 14 ([173]). *If the hyperidentities (17) and (19) are valid in the q-algebra $(Q;\,\Sigma)$, then it is idempotent and there exists a commutative Moufang loop $Q(\circ)$, such that every operation $A \in \Sigma$ is defined by the rule:*

$$A(x,\,y) = \varphi_A(x) \circ \psi_A(y),$$

where φ_A, ψ_A are commutative endomorphisms of the commutative Moufang loop $Q(\circ)$ and algebra $(Q;\,\Sigma)$.

Theorem 15 ([178]). *The hyperidentities (16) and (17) of distributivity are valid in an invertible algebra $(Q;\,\Sigma)$ iff there exists a commutative Moufang loop $Q(\circ)$, such that every operation $A \in \Sigma$ is defined by the rule:*

$$A(x,\,y) = \varphi_A(x) \circ \psi_A(y),$$

where $\varphi_A,\,\psi_A \in AutQ(\circ)$, $\varphi_A \in AutQ(A)$, $(\psi_A,\,\tilde{\varepsilon}) \in Aut(Q;\,\Sigma)$, $\varphi_A(x) \circ \psi_A(x) = x$, $x \circ \varphi_A(x) \in K_Q$ (the kernel of the loop $Q(\circ)$) for all $A \in \Sigma$, $x,\,y \in Q$.

Theorem 16 *The hyperidentities (18) and (19) of distributivity are valid in an invertible algebra* $(Q;\,\Sigma)$ *iff there exists a commutative Moufang loop* $Q(\circ)$ *such that every operation* $A \in \Sigma$ *is defined by the rule:*

$$A(x,\,y) = \varphi_A(x) \circ \psi_A(y),$$

where $\varphi_A,\,\psi_A \in AutQ(\circ)$, $\psi_A \in AutQ(A)$, $(\varphi_A,\,\tilde{\varepsilon}) \in Aut(Q;\,\Sigma)$, $\varphi_A(x) \circ \psi_A(x) = x$, $x \circ \varphi_A(x) \in K_Q$ *for any* $A \in \Sigma$, $x,\,y \in Q$.

Corollary 3 *The non-trivial hyperidentities (17) and (19) of distributivity are valid in an invertible algebra* $(Q;\,\Sigma)$ *iff there exists a commutative Moufang loop* $Q(\circ)$ *such that every operation* $A \in \Sigma$ *is defined by the rule:*

$$A(x,\,y) = \varphi_A(x) \circ \psi_A(y),$$

where $\varphi_A,\,\psi_A \in AutQ(\circ)$, $(\varphi_A,\,\tilde{\varepsilon})$, $(\psi_A,\,\tilde{\varepsilon}) \in Aut(Q;\,\Sigma)$, $\varphi_A(x)\circ\psi_A(x) = x$, $x \circ \varphi_A(x) \in K_Q$ *for any* $A \in \Sigma$, $x,\,y \in Q$.

However, the problem of the characterization of q-algebras with hyperidentities (16) and (17) (or with hyperidentities (18) and (19)) still remains open. The characterization of invertible algebras (q-algebras) with the hyperidentity (17) of distributivity (or with the hyperidentity (19)) still remains open too.

See also [54],[145],[288].

The problem, whether the invertible algebra (q-algebra) with the non-trivial hyperidentities (17) and (19) of distributivity is isotopic to the algebra with the non-trivial Moufang hyperidentity:

$$X(x, Y(x, X(y, z))) = X(Y(x, y), Y(x, z)),$$

also remains open (see Theorem 69). Moreover, binary algebras $(Q;\,\Sigma)$ and $(Q';\,\Sigma')$ are called isotopic if there exist bijections $\alpha,\,\beta,\,\gamma : Q \to Q'$, $\tilde{\psi} : \Sigma \to \Sigma'$ such that the following condition

$$\alpha A(x,\,y) = \left[\tilde{\psi}(A)\right](\beta x,\,\gamma y)$$

is true for any $A \in \Sigma$, $x,\,y \in Q$. This definition goes back to the monographs [170, 173], [269] and paper [94], which are different from the definitions of A. Albert [7] and A. G. Kurosh [124] for the rings.

Theorem 17 *If a ring with an identity element is isotopic to the associative ring, then they are isomorphic.*

Theorem 18 *If a binary algebra with associative operations is isotopic to a binary algebra in which every operation has an identity element, then they are isomorphic.*

Let us remind that a binary algebra $\mathfrak{A} = (Q;\, \Sigma)$ is called an e-algebra, if Σ contains an operation with identity element.

Theorem 19 ([173]) *If an e-algebra is isotopic to a binary algebra satisfying a non-trivial associative hyperidentity, then they are isomorphic.*

This is a wide generalization of the classical results of A. Albert([7, 8]), N. J. S. Hughes([98]), and R. Bruck([38]).

The above classifications of hyperidentities are mostly valid for e-algebras too (also see [167, 168, 169, 171, 1, 93, 118]). On conditional hyperidentities see [158].

6 Classical termal hyperidentities and pre-hyperidentities of semigroups

Proposition 2 *In the non-trivial binary algebra* $(Q;\, \Sigma)$ *with identical operations* $\delta_2^1(x,\ y) = x$ *and* $\delta_2^2(x,\ y) = y$ *(and consequently in the termal algebra* $\mathcal{F}(\mathfrak{A})$ *of any non-trivial algebra* $\mathfrak{A}$*)*

1) the hyperidentity of commutativity $X(x,\, y) = X(y,\, x)$ *is not valid;*

2) the hyperidentity of transitivity $X\left(X(x,\, z),\, X(y,\, z)\right) = X(x,\, y)$ *is not valid;*

3) no non-trivial hyperidentity of associativity is valid;

4) no non-trivial Kolmogoroff's hyperidentity is valid.

Proposition 3 *If in the non-trivial binary algebra* $(Q;\, \Sigma)$ *with identical operations* $\delta_2^1(x,\ y) = x$ *and* $\delta_2^2(x,\, y) = y$ *(and consequently if in the termal algebra* $\mathcal{F}(\mathfrak{A})$ *of any non-trivial algebra* $\mathfrak{A}$*) holds:*

5) a non-trivial hyperidentity of mediality, then it will have the functional rank 2 *and the form*

$$X\left(Y(x,\, y),\, Y(u,\, v)\right) = Y\left(X(x,\, u),\, X(y,\, v)\right); \tag{20}$$

6) a non-trivial hyperidentity of left distributivity, then it will have the functional rank 2 *and the form (17);*

7) a non-trivial hyperidentity of right distributivity, then it will have the functional rank 2 *and the form (19).*

According to the proposition 2 a non-trivial semigroup does not termally satisfy the non-trivial hyperidentity of associativity. The semigroup $Q(\cdot)$ is called hyperassociative, if it termally satisfies the hyperidentity of associativity

$$X\left(X(x,\, y),\, z\right) = X\left(x,\, X(y,\, z)\right).$$

It is shown in the paper [59], that the variety of all hyperassociative semigroups is defined by the finite system of identities. Their finite basis of identities consists of about 1000 identities. It is found in the works [232] and [221] a basis of identities of this variety consisting of only four identities.

Theorem 20 *(*[232] *and* [221]*). The semigroup $Q(\cdot)$ is hyperassociative iff the following identities:*

$$x^4 = x^2,$$

$$xyxzxyx = xyzyx,$$

$$xy^2z^2 = xyz^2yz^2,$$

$$x^2y^2z = x^2yx^2yz$$

are valid in $Q(\cdot)$.

The following more general result is proved in [206]. The semigroup $Q(\cdot)$ is called left hyperalternative (cf. [180]) if it termally satisfies the hyperidentity of left alternativity:

$$X\left(X(x,\, x),\, z\right) = X\left(x,\, X(x,\, z)\right).$$

The semigroup $Q(\cdot)$ is called right hyperalternative if it termally satisfies the hyperidentity of right alternativity:

$$X\left(X(x,\, y),\, y\right) = X\left(x,\, X(y,\, y)\right).$$

The semigroup $Q(\cdot)$ is called hyperalternative if it both left and right hyperalternative. A semigroup is right hyperalternative iff it is left hyperalternative. Consequently the right or left hyperalternative semigroup is hyperalternative.

First we note that every idempotent semigroup is hyperalternative, since all the binary terms of an idempotent semigroup are the following: x, y, $x \cdot y$, $y \cdot x$, $x \cdot y \cdot x$ and $y \cdot x \cdot y$.

Theorem 21 ([206]). *The semigroup $Q(\cdot)$ is hyperalternative iff the following identities are valid in $Q(\cdot)$:*

$$x^4 = x^2, \tag{21}$$

$$x^3yx^3 = x^2yx^2, \tag{22}$$

$$x^2y^2x^2y^2 = x^2y^2, \tag{23}$$

$$yx^3y = yxyxyxy. \tag{24}$$

$\cdot$	**1**	**2**	**3**	**4**	**5**
1	1	1	1	1	1
2	1	2	1	4	5
3	3	3	3	3	3
4	1	2	5	4	5
5	5	5	5	5	5

There also exists a hyperalternative semigroup, which is not hyperassociative.

An example of an idempotent semigroup with 5 elements, which is hyperalternative but not hyperassociative, is given by the first Cayley table below.

As the semigroup having this Cayley table is idempotent, hence it is hyperalternative. On the other hand, the identity $xyxzxyx = xyzyx$ is not satisfied for $x = 2$, $y = 4$ and $z = 3$ (as $xyxzxyx = 1$ and $xyzyx = 5$). Thus, this semigroup is not hyperassociative according to the above mentioned theorem.

Note, that there exist non-idempotent hyperalternative semigroups too. For example, this semigroup is given by the second Cayley table.

$\cdot$	**1**	**2**	**3**	**4**	**5**
1	1	1	1	1	1
2	1	1	1	2	3
3	3	3	3	3	3
4	1	2	3	4	5
5	5	5	5	5	5

In non-associative ring theory, the classical Artin theorem states that in an alternative algebra the subalgebra generated by any two elements is associative (see [263]). In the hyperalternative semigroups the following result is valid.

Theorem 22 ([206]) *If the semigroup $Q(\cdot)$ is hyperalternative, then any two elements in $Q(\cdot)$ generate a hyperassociative subsemigroup, i.e. the following identity*

$$X(X(A(x,y),B(x,y)),C(x,y)) = X(A(x,y),X(B(x,y),C(x,y))), \quad (25)$$

holds for any binary polynomials X, A, B, C of $Q(\cdot)$, that is the semigroup $Q(\cdot)$ termally (polynomially) satisfies the following hyperidentity of the functional rank 4:

$$X(X(Y(x,y),Z(x,y)),U(x,y)) = X(Y(x,y),X(Z(x,y),U(x,y))). \quad (26)$$

The semigroup $Q(\cdot)$ is called left hyperdistributive (right hyperdistributive), if it termally satisfies the hyperidentity of left distributivity (18) (correspondingly the hyperidentity of right distributivity (16)). The semigroup $Q(\cdot)$ is called hyperdistributive, if it is both left and right hyperdistributive. A semigroup is right hyperdistributive iff it is left hyperdistributive. Consequently, the right or left hyperdistributive semigroup is hyperdistributive.

The next result shows that the variety of all left hyperdistributive semigroups has a finite basis of identities.

Theorem 23 *The semigroup $Q(\cdot)$ is left hyperdistributive iff the following identities*

$$x^2 = x^3, \tag{27}$$

$$xyz = xyxz, \tag{28}$$

$$xyz = xzyz \tag{29}$$

are valid in $Q(\cdot)$.

Corollary 4 *Every left hyperdistributive semigroup is a hyperassociative.*

Proof. It is required to check the conditions of theorem 20. The identity $x^2 = x^4$ is clear. Then:

$$\begin{aligned} &xyxzxyx = (xyxz)xyx = xyzxyx = \\ &\quad = xy(zxyx) = xyzyx, \\ &xyz^2yz^2 = x(yz^2yz^2) = xy^2z^2, \\ &x^2yx^2yz = (x^2yx^2y)z = x^2y^2z. \end{aligned}$$

Corollary 5 *An medial semigroup (i.e. with the identity of mediality: $xy \cdot uv = xu \cdot yv$) is left hyperdistributive iff it satisfies the following identities:*

$$x^2 = x^4,$$

$$xyz = xyz^2,$$

$$xyz = x^2yz;$$

Corollary 6 *A commutative semigroup is left hyperdistributive iff it satisfies the following identities:*

$$x^2 = x^4,$$

$$xyz = x^2yz.$$

Corollary 7 *An idempotent semigroup is left hyperdistributive iff it satisfies the following identities:*

$$xyz = xyxz,$$

$$xyz = xzyz.$$

Theorem 24 *For every semigroup $Q(\cdot)$ the following conditions are equivalent:*

i) $Q(\cdot)$ termally satisfies the non-trivial hyperidentity of left distributivity (17);

ii) $Q(\cdot)$ termally satisfies the non-trivial hyperidentity of right distributivity (19);

iii) $Q(\cdot)$ satisfies the identities:

$$x^2 = x,$$

$$xyz = xyxz,$$

$$xyz = xzyz.$$

About the termal hyperidentities of semigroups see also [298], [299], [48].

Let $Q(\cdot)$ be a semigroup, $\mathcal{F}(Q)$ be the set of its termal operations, $\mathcal{F}^*(Q) = \mathcal{F}(Q)\backslash P$, where P is the set of all identical operations (projections) of Q. A hyperidentity $\omega_1 = \omega_2$ is called a pre-hyperidentity of the semigroup Q, if the hyperidentity $\omega_1 = \omega_2$ is satisfied in algebra $(Q; \mathcal{F}^*(Q))$. In this case we say that the pre-hyperidentity $\omega_1 = \omega_2$ is valid in the semigroup Q.

Two pre-hyperidentities are called equivalent if in every semigroup either both of them or none of them is satisfied.

Theorem 25 *Every non-trivial pre-hyperidentity of associativity of a non-trivial semigroup is equivalent to one of the following pre-hyperidentities: (7), (8), (9).*

Theorem 26 *Every non-trivial pre-hyperidentity of left or right distributivity of a non-trivial semigroup is equivalent to one of the following pre-hyperidentities: (17) and*

$$X\left(x,\, Y(y,\, z)\right) = Z\left(U(x,\, y),\, V(x,\, z)\right). \tag{30}$$

Moreover, the pre-hyperidentities (30) and (9) are equivalent.

Theorem 27 *i) The pre-hyperidentity (7) is valid in the semigroup $Q(\cdot)$ iff the following identities*

$$xyz = xzy,$$

$$xyz = yxz,$$

$$x^2 = y^2,$$

$$x^2 = x^3$$

hold in $Q(\cdot)$;

ii) The pre-hyperidentity (8) is valid in the semigroup $Q(\cdot)$ iff the following identities

$$xyz = zxy,$$

$$xyz = yzx,$$

$$x^2 = y^2,$$

$$x^2 = x^3$$

hold in $Q(\cdot)$;

iii) The pre-hyperidentity (9) is valid in $Q(\cdot)$ iff the following identities

$$xyz = x^2,$$

$$x^2 = y^2$$

hold in $Q(\cdot)$;

Consequently, if the pre-hyperidentity (9) is valid in a semigroup, then the pre-hyperidentity (7) is also valid in it; And if the pre-hyperidentity (7) is valid in a semigroup, then the pre-hyperidentity (8) is also valid in it (compare with the corollary 10).

The characterization of semigroups with pre-hyperidentity of commutativity $X(x, y) = X(y, x)$ is contained in the paper [58].

Theorem 28 *Each of the pre-hyperidentities (10)-(12), (13)-(15) holds in the semigroup $Q(\cdot)$ iff the identities*

$$x^2 = y^4,$$

$$xy = x^2,$$

$$x^2 = x^3$$

hold in it. Hence the pre-hyperidentities (10)-(12) and (13)-(15) are equivalent.

7 Essential hyperidentities of semigroups

Let $Q(\cdot)$ be a semigroup. Every *binary polynomial* (term) of $Q(\cdot)$ has the following form:

$$F(x,y) = z_1^{\varepsilon_1} z_2^{\varepsilon_2} \dots z_n^{\varepsilon_n}, \tag{31}$$

where $n \in \mathbb{N}, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in \mathbb{N}$, $z_1, z_2, \dots, z_n \in \{x, y\}$ and $z_i \neq z_{i+1}$. The number n is called the length of this representation of the polynomial $F(x, y)$. However, due to the identities in the semigroup $Q(\cdot)$, the same polynomial $F(x, y)$ can have different representations of the form (31).

Definition 1 *The polynomial $F(x,y)$ essentially depends on the variable x in the semigroup $Q(\cdot)$ if there are elements $x_1, x_2, y \in Q$ such that $F(x_1,y) \neq F(x_2,y)$. In the same way the essentially dependence of the polynomial $F(x,y)$ on the variable y is defined.*

Definition 2 *The polynomial $F(x,y)$ is called essential if it essentially depends on both variables x and y.*

We let Q^2_{epol} be the collection of all binary essential polynomials of the semigroup $Q(\cdot)$.

We say that the hyperidentity $(*)$ is *essentially satisfied* (valid) or is *satisfied for essential polynomials* in the semigroup $Q(\cdot)$ if this hyperidentity is satisfied in the binary algebra $\left(Q; Q^2_{epol}\right)$. The hyperidentity $(*)$ is called *essential hyperidentity* of semigroup $Q(\cdot)$ if this hyperidentity is essentially satisfied in this semigroup.

Definition 3 *We say that two hyperidentities are* essentially equivalent *(written as $\Leftrightarrow_e$), if they simultaneously are either essentially satisfied or none of them is essentially satisfied in any semigroup $Q(\cdot)$. It is said that the hyperidentity (h_1) essentially implies the hyperidentity (h_2), written as $(h_1) \Rightarrow_e (h_2)$, if in all semigroups where the hyperidentity (h_1) is essentially satisfied, the hyperidentity (h_2) is also essentially satisfied.*

Theorem 29 *Any non-trivial associative essential hyperidentity of semigroup is essentially equivalent to one of following hyperidentities:*

$$X\left(X\left(x,y\right),z\right) = X\left(x,Y\left(y,z\right)\right), \tag{32}$$

$$X\left(X\left(x,y\right),z\right) = Y\left(x,X\left(y,z\right)\right), \tag{33}$$

$$X\left(X\left(x,y\right),z\right) = Y\left(x,Y\left(y,z\right)\right), \tag{34}$$

$$X\left(Y\left(x,y\right),z\right) = X\left(x,Y\left(y,z\right)\right), \tag{35}$$

$$X\left(Y\left(x,y\right),z\right) = Y\left(x,X\left(y,z\right)\right). \tag{36}$$

Moreover, we have the following implications: (32) $\Rightarrow_e$ (34), (32) $\Rightarrow_e$ (33) $\Rightarrow_e$ (36) $\Rightarrow_e$ (35) *(cf.* [204]*).*

Theorem 30 *Any left distributive essential hyperidentity of semigroup is essentially equivalent either to the hyperidentity:*

$$X(x, X(y,z)) = X(X(x,y), X(x,z)), \tag{37}$$

or to the hyperidentity:

$$X(x, X(y,z)) = X(X(x,y), Y(x,z)). \tag{38}$$

Moreover: (38) $\Rightarrow_e$ (37) *(cf.* [205, 213]*).*

The similar results are valid for hyperidentities of right distributivity, transitivity and Kolmogoroff hyperidentities too.

8 Hyperidentities of varieties

It's obvious that every variety of Ω-algebras is contained in the least hypervariety of Ω-algebras, which is defined by all hyperidentities of the given variety and is called the hypervariety of Ω-algebras generated by the given variety. The characterization of a hypervariety means the description of its hyperidentities and algebras. For example, the problem of characterization of hypervarieties generated by the variety of all rings is missing until now. The next example shows that non-trivial hyperidentities are valid in the variety of all rings.

Example. The following hyperidentities are satisfied in any ring:

$$X\left(X\left(Y(x,x),Y(x,x)\right),Y\left(X(x,x),X(x,x)\right)\right)=$$

$$=X\left(Y\left(X(x,x),X(x,x)\right),X\left(Y(x,x),Y(x,x)\right)\right),$$

$$X\left(Y\left(Y(x,x),X(x,x)\right),Y\left(X(x,x),Y(x,x)\right)\right)=$$

$$=X\left(Y\left(X(x,x),Y(x,x)\right),Y\left(Y(x,x),X(x,x)\right)\right).$$

It's obvious that every hyperidentity $w_1=w_2$ of a non-trivial lattice is regular, i.e. the same object variables occur in w_1 and w_2.

Theorem 31 ([177]) *Any hyperidentity of the variety of all lattices is a consequence of the following four hyperidentities:*

$$X(x,x)=x, \tag{39}$$

$$X(x,y)=X(y,x), \tag{40}$$

$$X\left(x,X(y,z)\right)=X\left(X(x,y),z\right), \tag{41}$$

$$X\left(Y\left(X(x,y),z\right),Y(y,z)\right)=Y\left(X(x,y),z\right). \tag{42}$$

Theorem 32 ([177]) *Any hyperidentity of the variety of all modular lattices is a consequence of the hyperidentities (39)-(42) and hyperidentity:*

$$X\left(Y\left(x,X(y,z)\right),Y(y,z)\right)=Y\left(X\left(x,Y(y,z)\right),X(y,z)\right). \tag{43}$$

Theorem 33 *(*[174], [177]*) . Any hyperidentity of the variety of all distributive lattices is a consequence of the hyperidentities (39)-(41), (17).*

Since the variety of all distributive lattices doesn't have its own subvarieties, then the theorem 33 induces the characterization of hyperidentities of an arbitrary individual distributive lattice.

Corollary 8 *Any hyperidentity of a non-trivial distributive lattice is a consequence of the hyperidentities (39)-(41), (17). Hence, every hyperidentity of a non-trivial bounded commutative BCK-algebra (as a distributive lattice) is a consequence of the hyperidentities (39)-(41), (17) ([101], [102]). Any hyperidentity of the class of all bounded commutative BCK-algebras is a consequence of the hyperidentities (39)-(41), (17).*

However, the problem of the characterization of hyperidentities of commutative BCK-algebras (with two operations $*$ and $\wedge$) is open.

Theorem 34 *(*[178], [174]*) Any hyperidentity of the variety of all Boolean algebras is a consequence of the hyperidentities (39)-(41), (17) as well as hyperidentities:*

$$F\left(F(x)\right) = x, \tag{44}$$

$$X\left(F(x),\, y\right) = X\left(F\left(X(x,\, y)\right),\, y\right), \tag{45}$$

$$F\left(X\left(F\left(X(x,\ y)\right),\, F\left(X\left(x,\, F(y)\right)\right)\right)\right) = x. \tag{46}$$

All hyperidentities of the variety of Boolean algebras are consequences of one of its hyperidentities, i.e. the hyperequational theory of the variety of Boolean algebras is one-based.

The characterization of hyperidentities of the variety of Boolean algebras, given in [174] reduces also to the description of free algebras of the corresponding hypervariety.

Since the variety of Boolean algebras does not have its own subvarieties, then it follows from here the characterization of hyperidentities of an arbitrary Boolean algebra.

Corollary 9 *Any hyperidentity of a Boolean algebra is a consequence of the hyperidentities (39)-(41), (17),(44)-(46). Hence, any hyperidentity of an arbitrary bounded implicative BCK-algebra is a consequence of the hyperidentities (39)-(41), (17),(44)-(46). Any hyperidentity of the class of all bounded implicative BCK-algebras is a consequence of the hyperidentities (39)-(41), (17),(44)-(46).*

However, the problem of the characterization of hyperidentities of bounded BCK-algebras remains open.

Theorem 35 ([187]) *The variety of De Morgan algebras satisfies the following hyperidentities (39)-(41), (17),(44) as well as hyperidentities:*

$$F(x) = G(x), \tag{47}$$

$$F(Y(F(X(x,y)),z)) = X(F(Y(F(x),z)), F(Y(F(y),z))), \tag{48}$$

$$X(F(X(x,y)), F(x)) = F(x), \tag{49}$$

$$X(x, X(y,z)) = X(Y(x, Y(y,z)), F(Y(F(x), Y(F(y), F(z))))), \tag{50}$$

$$\begin{gathered} X(F(X(Y(x,e), Y(x, Y(y,e)))), F(X(Y(z,e), Y(z, Y(t,e))))) = \\ X(F(X(Y(x, Y(z,e))), Y(x, Y(y, Y(z, Y(u,e)))))), \\ F(X(F(Y(F(Y(x, Y(y,e))), F(Y(z, Y(t,e))))), e)))). \end{gathered} \tag{51}$$

And conversely, every hyperidentity of the variety of De Morgan algebras is a consequence of the hyperidentities: (39)-(41), (17),(44), (47), (48), (49), (50), (51).

The hyperequational theory of the variety of De Morgan algebras is not one-based. □

As noted above, in the study of hyperidentities usually two questions are posed. The first one is the problem of characterization of hyperidentities of the given algebra (class of algebras or variety), and the second one is the problem of characterization of algebras with these hyperidentities. The proofs of Theorems 31-33, 34 have an advantage that allows to receive the characterization of algebras with hyperidentities of the corresponding varieties simultaneously. Another structural characterization of algebras with hyperidentities of the variety of Boolean algebras is given in the work [177]. For that reason the following concept of a Boolean sum is introduced there. First we give the following natural definition.

Definition 4 *An algebra is called super-Boolean algebra if it satisfies the hyperidentities of the variety of Boolean algebras. An algebra is called super-De Morgan algebra if it satisfies the hyperidentities of the variety of De Morgan algebras.*

Let $\mathfrak{A} = (Q; \Omega \bigcup \{F\})$ be a T-algebra with one unary operation F. Let $(Q_i; \Omega)$, $i \in I$, be subsystems (subalgebras) for the algebra $\mathfrak{A}$, and let $\mathfrak{A}_i = (Q_i; \Omega \bigcup \{F_i\})$ be an algebra with one unary operation F_i for any $i \in I$. The algebra $\mathfrak{A} = (Q; \Omega \bigcup \{F\})$ is called the Boolean sum of the algebras $\mathfrak{A}_i = (Q_i; \Omega \bigcup \{F_i\})$, $i \in I$, if

a) $Q = \bigcup_{i \in I} Q_i$, $Q_i \bigcap Q_j = \emptyset$, where $i,\ j \in I$, $i \neq j$;

b) one can define a partial order "$\leqslant$" on the index set I such that $I(\leqslant)$ is a Boolean algebra;

c) if $i \leqslant j$, then there exists an isomorphism:

$$(\varphi_{i,j},\, \tilde{\varepsilon}) : \mathfrak{A}_i \Rightarrow \mathfrak{A}_j, \qquad i,\, j \in I,$$

where $\tilde{\varepsilon}(F_i) = F_j$, $\tilde{\varepsilon}(A) = A$ for any $A \in \Omega$, and $\varphi_{i,i} = \varepsilon$ and $\varphi_{i,j} \cdot \varphi_{j,k} = \varphi_{i,k}$, where $i \leqslant j \leqslant k$;

d) for any $i \in I$ there exists an isomorphism:

$$\left(h_{i,\bar{i}},\, \tilde{\varepsilon}\right) : \mathfrak{A}_i \Rightarrow \mathfrak{A}_{\bar{i}},$$

where $h^{-1}_{i,\bar{i}} = h_{\bar{i},i}$ and $\varphi_{i,1} = h_{i,\bar{i}} \cdot \varphi_{\bar{i},1}$, and 1 is the unit of the Boolean algebra $I(\leqslant)$ and $\bar{i}$ is the complement in this lattice;

e) for any $A \in \Omega$, $|A| = n \geqslant 2$, and for any $x_1, \ldots, x_n \in Q$ the following equality holds:

$$A(x_1, \ldots, x_n) = A\left(\varphi_{i_1, i_0}(x_1), \ldots, \varphi_{i_n, i_0}(x_n)\right),$$

where $x_1 \in Q_{i_1}, \ldots, x_n \in Q_{i_n}$, $i_1, \ldots, i_n \in I$, and $i_0 = i_1 + \ldots + i_n$ in the lattice $I(\leqslant)$;

f) for the unary operation F and any $x \in Q$:

$$F(x) = h_{i,\bar{i}}\left(F_i(x)\right),$$

where $x \in Q_i$.

Theorem 36 *For every two Boolean algebras $\mathfrak{B}$ and I there exists an algebra $\mathfrak{A}$ with the same signature (type), which is the Boolean sum of Boolean algebras $\mathfrak{A}_i$, $i \in I$, where any $\mathfrak{A}_i$ is isomorphic to $\mathfrak{B}$.*

Proof. Let $\mathfrak{A}_i = (A_i, +, \cdot, ', 0, 1)$, $i \in I$ be a Boolean algebra isomorphic to $\mathfrak{B}$, where $A_i \cap A_j = \emptyset$, $i, j \in I$, $i \neq j$. Let 0 and 1 be the identity elements of Boolean algebra I, and $\varphi_{0,i} : A_0 \to A_i$ be an isomorphism from $\mathfrak{A}_0$ to $\mathfrak{A}_i$, $i \in I$. For $i \leqslant k$, where $i, k \in I$, we define isomorphism $\varphi_{i,k} : A_i \to A_k$ by the equality:

$$\varphi_{i,k} = \varphi^{-1}_{0,i} \cdot \varphi_{0,k},$$

and isomorphism $h_{i,\bar{i}} : A_i \to A_{\bar{i}}$ by the equality:

$$h_{i,\bar{i}} = \varphi_{i,1} \cdot \varphi^{-1}_{\bar{i},1}$$

for every $i \in I$. So, for any $i \in I$ we have:

$$\varphi_{i,i} = \varphi^{-1}_{0,i} \cdot \varphi_{0,i} = \varepsilon \ (= id_{A_i}).$$

If $i \leqslant j \leqslant k$, where $i, j, k \in I$, we have:

$$\varphi_{i,j} \cdot \varphi_{j,k} = \left(\varphi^{-1}_{0,i} \cdot \varphi_{0,j}\right) \cdot \left(\varphi^{-1}_{0,j} \cdot \varphi_{0,k}\right) = \varphi^{-1}_{0,i} \cdot \varphi_{0,k} = \varphi_{i,k}.$$

Note the following equalities too:

$$\varphi_{i,1} = h_{i,\bar{i}} \cdot \varphi_{\bar{i},1},$$

$$h_{\bar{i},i} = \varphi_{\bar{i},1} \cdot \varphi_{i,1}^{-1} = \left(\left(\varphi_{i,1}^{-1}\right)^{-1} \cdot \varphi_{\bar{i},1}^{-1} \right)^{-1} = \left(\varphi_{i,1} \cdot \varphi_{\bar{i},1}^{-1} \right)^{-1} = h_{i,\bar{i}}^{-1}.$$

Now we can define the required algebra $\mathfrak{A}$ on the set $\bigcup\limits_{i \in I} A_i$ with the following unary operation $\bar{a}$ and the binary operations $a + b$ and $a \cdot b$:

$$\bar{a} = h_{i,\bar{i}}(a'),$$

where $a \in A_i$, $i \in I$,

$$\begin{aligned} a + b &= \varphi_{i,i+j}(a) + \varphi_{j,i+j}(b), \\ a \cdot b &= \varphi_{i,i+j}(a) \cdot \varphi_{j,i+j}(b), \end{aligned}$$

where $a \in A_i$ and $b \in A_j$, $i, j \in I$.

Theorem 37 ([177]) *An algebra* $\mathfrak{A} = Q(+, \cdot, ^{-}, 0, 1)$ *with one unary, two binary and two nullary operations is a super-Boolean algebra iff* $\mathfrak{A}$ *is a Boolean algebra or a Boolean sum of Boolean algebras.*

The similar result is valid for the super-De Morgan algebras too [188].

The next results is the widely generalization of A. Tarski's and Yu. Yershov's [301] classical result.

Theorem 38 *(Yu. Movsisyan and L. Budaghyan* ([196], [197])*) Elementary theory of any super-Boolean algebra with one unary and one binary operations is decidable.*

Theorem 39 *(Yu. Movsisyan and L. Budaghyan* ([196], [197])*) Elementary theory of the variety of super-Boolean algebras with one unary and one binary operations is decidable.*

Theorem 40 *(Yu. Movsisyan and L. Budaghyan* ([196], [197])*) Elementary theory of any super-Boolean algebra with one unary and two binary operations is decidable.*

Theorem 41 *(Yu. Movsisyan and L. Budaghyan* ([196], [197])*) Elementary theory of the variety of super-Boolean algebras with one unary and two binary operations is decidable.*

9 Free super-Boolean algebras with two binary one unary and two nullary operations, and super-Boolean functions

Note that from the hyperidentity $X(x) = Y(x)$ of Boolean algebras follows that any super-Boolean algebra has a unique unary operation, which we will denote by $\,'$. We will consider the super-Boolean algebras with two binary, one unary and two nullary operations (i.e. constants) that satisfy some natural identities (as given below). We will denote the variety of such super-Boolean algebras by $\mathfrak{Q}\mathfrak{B}(2,2,1,0,0)$. Namely, an algebra $\mathfrak{A} = (Q; \{+, \cdot, \,', 0, 1\})$ belongs to $\mathfrak{Q}\mathfrak{B}(2,2,1,0,0)$ if and only if it satisfies all the hyperidentities of the variety of Boolean algebras and also the following two identities:

$$x \cdot 1 = x,$$
$$1' = 0.$$

The free algebras of the variety $\mathfrak{Q}\mathfrak{B}(2,2,1,0,0)$ are called the free super-Boolean algebras with two binary, one unary and two nullary operations. Our main result in this section is the characterization of the finitely generated free super-Boolean algebras with two binary, one unary and two nullary operations.

Recall that $B = \{0, 1\}$, $D = \{0, 1, a, b\}$. Let us construct a one-to-one correspondence between the sets D and $B \times B$ as follows:

$$0 \leftrightarrow (0,0),\ a \leftrightarrow (1,0),\ b \leftrightarrow (0,1),\ 1 \leftrightarrow (1,1).$$

We define the operations $+, \cdot, \vee, \wedge, {}^-, \,'$ on the set $B \times B$ as follows:

$$(u, v)' = (u', v'),$$
$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2),\ (u_1, v_1) \cdot (u_2, v_2) = (u_1 \cdot u_2, v_1 \cdot v_2),$$
$$(u_1, v_1) \vee (u_2, v_2) = (u_1 + u_2, v_1 \cdot v_2),\ (u_1, v_1) \wedge (u_2, v_2) = (u_1 \cdot u_2, v_1 + v_2),$$

where the operations on the right hand side are the operations of the Boolean algebra **2**. These operations are isomorphic to the corresponding operations on the set D (the one-to-one correspondence described above is an isomorphism).

However, if the tuple $(y, z) \in B \times B$ corresponds to $x \in D$ then we will write $x = (y, z)$ (this causes no confusion).

Definition 5 A function $f : B^n \to D$ is called a *super-Boolean function* of n variables.

We will consider the set B as a subset of D. And so all Boolean functions are super-Boolean functions.

Lemma 5 *For any super-Boolean function $f : B^n \to D$ there exist two Boolean functions $f_1, f_2 : B^n \to B$ such that for all $x_1, \ldots, x_n \in B$ the following equality holds:*

$$f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), f_2(x_1, \ldots, x_n)). \tag{52}$$

These Boolean functions f_1, f_2 are uniquely determined by the super-Boolean function f.

Proof: As we said above we will not distinguish the sets $B \times B$ and D. So we can consider the projection functions f_1 and f_2 such that for any $x_1, \ldots, x_n \in B$ $f_i(x_1, \ldots, x_n)$ is the i-th coordinate of $f(x_1, \ldots, x_n)$ in the set $B \times B$, where $i = 1, 2$. Then f_1 and f_2 map B^n into B, i.e. they are Boolean functions and obviously the equality (52) holds. The uniqueness of such functions is obvious. □

Note that here for a Boolean function $f : B^n \to B$ we have $f_1 = f_2 = f$.

Theorem 42 *For any super-Boolean function $f : B^n \to D$ there exist two Boolean functions $f_1, f_2 : B^n \to B$ with identity:*

$$f(x_1, \ldots, x_n) = (a \cdot f_1(x_1, \ldots, x_n)) + (b \cdot f_2(x_1, \ldots, x_n)), \tag{53}$$

where the operations are the operations on the set D defined above. These Boolean functions f_1, f_2 are uniquely determined by the super-Boolean function f.

Proof: From equality (52) of Lemma 5 we have (we omit the variables):

$$f = (f_1, f_2) = ((1, 0) \cdot (f_1, f_1)) + ((0, 1) \cdot (f_2, f_2)) = (a \cdot f_1) + (b \cdot f_2).$$

The uniqueness follows from Lemma 5. □

Taking into account the equalities in D:

$$a \cdot x = 0 \vee x, \ b \cdot y = 0 \wedge y, \ (x \vee y)' = x' \wedge y',$$
$$(x \wedge y)' = x' \vee y', \ x + y = (1 \wedge (x \vee y)) \vee (0 \wedge x \wedge y), \ x, y \in D,$$

we get:

$$a \cdot x + b \cdot y = (0 \vee x) + (0 \wedge y) = (1 \wedge (0 \vee x \vee (0 \wedge y))) \vee (0 \wedge y \wedge (0 \vee x)) =$$
$$(1 \wedge x) \vee (0 \wedge y) = (0 \vee x')' \vee (1 \vee y')'.$$

Thus we conclude that any super-Boolean function f can be represented in the following form:

$$f = (0 \vee f_1')' \vee (1 \vee f_2')',$$

where f_1 and f_2 are the corresponding Boolean functions with the Boolean operations $\vee$ and $'$. Note that this form of super-Boolean functions is the analogue of the b_2-canonical form of terms with two binary functional variables introduced in [174].

Now, we conclude that any super-Boolean function can be represented in the following form:

$$f(x_1,\dots,x_n) = \left(0 \vee \prod_{s\in S_1}\left(\prod_{i\in s} x_i \cdot \prod_{i\in \overline{s}} x_i'\right)'\right)' \vee \left(1 \vee \prod_{s\in S_2}\left(\prod_{i\in s} x_i \cdot \prod_{i\in \overline{s}} x_i'\right)'\right)', \tag{54}$$

for uniquely determined sets $S_1, S_2 \subseteq 2^{\{1,\dots,n\}}$ (here the domain of variables is the set B). This form is called the disjunctive normal form (DNF) of the super-Boolean function f (and it is unique for a given super-Boolean function). Note that here if $S_1 = S_2 = \emptyset$ then $f = 0$ and if $S_1 = S_2 = 2^{\{1,\dots,n\}}$ then $f = 1$.

Denote by $\mathcal{SB}_n$ the set of all super-Boolean functions of n variables. For any two functions $f, g \in \mathcal{SB}_n$ define $f + g$, $f \cdot g$, $f \vee g$, $f \wedge g$, f' in the standard way, i.e. $(f\lambda g)(x) = f(x)\lambda g(x)$, for all $\lambda \in \{+,\cdot,\vee,\wedge\}$, and $f'(x) = (f(x))'$, $x \in B^n$, where the operations on the right hand side are the operations on the set D defined above. Thus we get the algebras $(\mathcal{SB}_n; \{+,\vee,\ ',0,1\})$, $(\mathcal{SB}_n; \{+,\wedge,\ ',0,1\})$, $(\mathcal{SB}_n; \{\vee,\cdot,\ ',0,1\})$, $(\mathcal{SB}_n; \{\wedge,\cdot,\ ',0,1\})$ (here the nullary operations are the constant functions $0(x) = 0$ and $1(x) = 1$ for all $x \in B^n$), which are Boolean quasilattices (and they are isomorphic). Denote $\mathfrak{QB}_n = (\mathcal{SB}_n; \{\vee,\cdot,\ ',0,1\})$.

Consider the projection functions

$$\delta_n^i(x_1,\dots,x_n) = x_i,\ i = 1,\dots,n,$$

as super-Boolean functions $B^n \to D$. Clearly, (54) implies:

$$f = \left(0 \vee \prod_{s\in S_1}\left(\prod_{i\in s} \delta_n^i \cdot \prod_{i\in \overline{s}} (\delta_n^i)'\right)'\right)' \vee \left(1 \vee \prod_{s\in S_2}\left(\prod_{i\in s} \delta_n^i \cdot \prod_{i\in \overline{s}} (\delta_n^i)'\right)'\right)'.$$

Now let us formulate the following functional representation result, which relates to the Plotkin's problem, too (see [186]).

Theorem 43 (Functional representation theorem) ([193]) *The algebra $\mathfrak{QB}_n$ is the free super-Boolean algebra with two binary, one unary and two nullary operations (with the system of free generators $\Delta = \{\delta_n^1,\dots,\delta_n^n\}$). Hence every free n-generated super-Boolean algebra with two binary, one unary and two nullary operations is isomorphic to the super-Boolean algebra $\mathfrak{QB}_n$.*

Problem 7 *To develop the super-Boolean analogue of the theory of Boolean functions.*

10 Free super-De Morgan algebras with two binary and one unary operations, and super-De Morgan functions

Note that from the hyperidentity $X(x) = Y(x)$ of De Morgan algebras it follows that any super-De Morgan algebra has a unique unary operation, which we will denote by $^{-}$.

Denote by $\mathfrak{Q}\mathfrak{D}(2,2,1)$ the variety of all super-De Morgan algebras with two binary and one unary operations. So the variety $\mathfrak{Q}\mathfrak{D}(2,2,1)$ is a hypervariety.

The free algebras of the variety $\mathfrak{Q}\mathfrak{D}(2,2,1)$ are called the free super-De Morgan algebras with two binary and one unary operations. Our main result in this section is the characterization of the finitely generated free super-De Morgan algebras with two binary and one unary operations. Namely, in this section we introduce the concept of super-De Morgan function and give a functional representation of the free n-generated super-De Morgan algebras with two binary and one unary operations through super-De Morgan functions.

Denote $E = D \times D$, where $D = \{0, 1, a, b\}$. We will identify the diagonal subset $\Delta = \{(0,0), (1,1), (a,a), (b,b)\}$ with the set D (we identify (x,x) with x). And thus we will consider the set D as a subset of E. We define the operations $+, \cdot, \vee, \wedge, ^{-}$ on the set E as follows. For $u, v, u_1, v_1, u_2, v_2 \in D$ we set:

$$\overline{(u,v)} = (\overline{u}, \overline{v}),$$
$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2),\ (u_1, v_1) \cdot (u_2, v_2) = (u_1 \cdot u_2, v_1 \cdot v_2),$$
$$(u_1, v_1) \vee (u_2, v_2) = (u_1 + u_2, v_1 \cdot v_2),\ (u_1, v_1) \wedge (u_2, v_2) = (u_1 \cdot u_2, v_1 + v_2),$$

where the operations on the right hand side are the operations of the De Morgan algebra **4**.

For an element $\alpha \in E$ we denote by α^1 its first coordinate in D and by α^2 the second coordinate in D, i.e. $\alpha = (\alpha^1, \alpha^2)$. Obviously, $\alpha \in D$ if and only if $\alpha^1 = \alpha^2$.

For a function $f : D^n \to E$ we define its projection functions $f^1, f^2 : D^n \to D$ as follows:

$$f^i(x_1, \ldots, x_n) = (f(x_1, \ldots, x_n))^i,\ i = 1, 2.$$

So every function from D^n to E can be uniquely represented as a tuple of functions from D^n into D: $f = (f^1, f^2)$ (for simplicity we omit the variables).

Definition 6 A function $f : D^n \to E$ is called a *super-De Morgan function* of n variables, if its projection functions f^1, f^2 are nonconstant De Morgan functions.

Clearly, all nonconstant De Morgan functions are super-De Morgan functions (as we mentioned above, we consider the set D as a subset of E). And for a nonconstant De Morgan function f we have $f^1 = f^2 = f$.

If we define $c = (1,0), d = (0,1) \in E$ then we have: $f = (f^1, f^2) = (1,0) \cdot (f^1, f^1) + (0,1) \cdot (f^2, f^2) = c \cdot f^1 + d \cdot f^2$.

It follows immediately from Definition 6 that there are $(m_{2n} - 2)^2$ super-De Morgan functions of n variables.

Denote by $\mathcal{SD}_n$ the set of all super-De Morgan functions of n variables. For any two functions $f, g \in \mathcal{SD}_n$ define $f + g$, $f \cdot g$, $f \vee g$, $f \wedge g$, $\overline{f}$ in the standard way, i.e. $(f \lambda g)(x) = f(x) \lambda g(x)$, for all $\lambda \in \{+, \cdot, \vee, \wedge\}$, and $f'(x) = (f(x))'$, $x \in D^n$, where the operations on the right hand side are the operations on the set D defined above. Clearly, the set $\mathcal{SD}_n$ is closed under these operations. Thus we get the algebras $(\mathcal{SD}_n; \{+, \vee, {}^-\})$, $(\mathcal{SD}_n; \{+, \wedge, {}^-\})$, $(\mathcal{SD}_n; \{\vee, \cdot, {}^-\})$, $(\mathcal{SD}_n; \{\wedge, \cdot, {}^-\})$, which are super-De Morgan algebras (and they are isomorphic). Denote $\mathfrak{Q}\mathfrak{D}_n = (\mathcal{SD}_n; \{\vee, \cdot, {}^-\})$.

Denote $I = \{(\{1, \ldots, n\}, \{1, \ldots, n\})\}$, $U = \{(\{i\}, \emptyset), (\emptyset, \{i\}) : 1 \leq i \leq n\}$. Clearly these sets are antichains. We denote the De Morgan functions corresponding to these antichains by f_I and f_U respectively, i.e.

$$f_I(x_1, \ldots, x_n) = x_1 \cdot \ldots \cdot x_n \cdot \overline{x}_1 \cdot \ldots \cdot \overline{x}_n,$$
$$f_U(x_1, \ldots, x_n) = x_1 + \ldots + x_n + \overline{x}_1 + \ldots + \overline{x}_n = \overline{f_I(x_1, \ldots, x_n)}.$$

It is easy to see that for any nonconstant De Morgan functions f the following equalities hold:

$$f + f_I = f, \; f \cdot f_I = f_I, \; f + f_U = f_U, \; f \cdot f_U = f.$$

And so we conclude that for any super-De Morgan function f the following equality is true:

$$f = (f^1, f^2) = (f^1, f_U) \vee (f_I, f^2) = \overline{(\overline{f^1}, f_I)} \vee \overline{(f_U, \overline{f^2})} =$$
$$\overline{(f_I, f_I) \vee \overline{(f^1, f^1)}} \vee \overline{(f_U, f_U) \vee \overline{(f^2, f^2)}} = \overline{f_I \vee \overline{f^1}} \vee \overline{f_U \vee \overline{f^2}}.$$

Now from the representation of De Morgan functions in DNF we conclude that for any super-De Morgan function $f : D^n \to E$ there exist two antichains $S_1, S_2 \subseteq 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$ (that are uniquely determined by the function f) with $S_1, S_2 \neq \emptyset$ and $S_1, S_2 \neq \{(\emptyset, \emptyset)\}$ such that:

$$f(x_1, \ldots, x_n) = \overline{\prod_{i=1}^{n} x_i \cdot \prod_{i=1}^{n} \overline{x}_i \vee \prod_{(s_1,s_2) \in S_1} \overline{\prod_{i \in s_1} x_i \cdot \prod_{i \in s_2} \overline{x}_i}} \vee$$
$$\vee \overline{\overline{\prod_{i=1}^{n} x_i \cdot \prod_{i=1}^{n} \overline{x}_i} \vee \prod_{(s_1,s_2) \in S_2} \overline{\prod_{i \in s_1} x_i \cdot \prod_{i \in s_2} \overline{x}_i}}. \quad (55)$$

Here the domain of the variables is the set D. This form is called the disjunctive normal form (DNF) of the super-De Morgan function f. DNF is unique for a given super-De Morgan function.

Consider the projection functions

$$\delta_n^i(x_1, \dots, x_n) = x_i, \ i = 1, \dots, n,$$

as super-De Morgan functions $D^n \to E$. Clearly, (55) implies:

$$\begin{aligned} f = {} & \overline{\prod_{i=1}^{n} \delta_n^i \cdot \prod_{i=1}^{n} \overline{\delta_n^i} \vee \prod_{(s_1,s_2)\in S_1} \overline{\prod_{i\in s_1} \delta_n^i \cdot \prod_{i\in s_2} \overline{\delta_n^i}}} \vee \\ & \vee \overline{\overline{\prod_{i=1}^{n} \delta_n^i \cdot \prod_{i=1}^{n} \overline{\delta_n^i}} \vee \prod_{(s_1,s_2)\in S_2} \overline{\prod_{i\in s_1} \delta_n^i \cdot \prod_{i\in s_2} \overline{\delta_n^i}}}. \end{aligned}$$

Now we arrive to the following result, which also relates to the Plotkin's problem (see [186]).

Theorem 44 (Functional representation theorem) ([194]) *The algebra $\mathfrak{Q}\mathfrak{D}_n$ is the free super-De Morgan algebra with two binary and one unary operations (with the system of free generators $\Delta = \{\delta_n^1, \dots, \delta_n^n\}$). Hence, every free n-generated super-De Morgan algebra with two binary and one unary operations is isomorphic to the super-De Morgan algebra $\mathfrak{Q}\mathfrak{D}_n$.*

Problem 8 *To develop the super-De Morgan analogue of the theory of Boolean functions.*

11 Termal hyperidentities of varieties

It's obvious that every variety of Ω-algebras is contained in the least solid variety of Ω-algebras, which is defined by all termal hyperidentities of the given variety and is called the solid variety of Ω-algebras, generated by the given variety.

Example ([285]). The variety of commutative rings doesn't satisfy any termal hyperidentity except $w = w$. Hence, this variety generates the solid variety of all Ω-algebras, where Ω is the signature of rings.

Theorem 45 ([174]) *Termal hyperidentities of the variety of Boolean algebras don't have a finite base of hyperidentities.*

Proof. The following hyperidentity is the termal hyperidentity of the variety of Boolean algebras:

$$X\left(x_1, \dots, x_{n-1}, X\left(x_1, \dots, x_{n-1}, X\left(x_1, \dots, x_n\right)\right)\right) = X\left(x_1, \dots, x_n\right) \quad (56)$$

for every natural $n \in \mathcal{N}$.

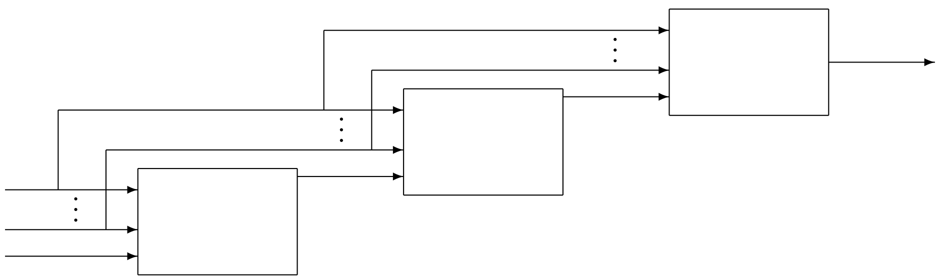
We shall denote by $H^{(m)}$ the system of hyperidentities, which are termally valid in the variety of Boolean algebras and have functional variables of the arity $\leqslant m$. Then for every m there exists a hyperidentity $w_1 = w_2$, which is termally satisfied in the variety of Boolean algebras and is not a consequence of $H^{(m)}$. As a $w_1 = w_2$ one can take the hyperidentity (56), under $n = 2^m$. Indeed, if $\mathfrak{A} = (Q;\, A)$ where Q is a free semilattice of *rank* n, generated by $\{a_1, \ldots, a_n\}$, and the n-ary operation A is defined by the rule

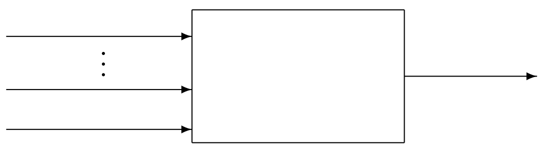
$$A(x_1, \ldots, x_n) = \begin{cases} x_1, & \text{if} \quad \{x_1, \ldots, x_n\} = \{a_1, \ldots, a_n\}, \\ x_1 + \ldots + x_n, & \text{in other cases,} \end{cases}$$

then in the termal algebra $\mathcal{F}(\mathfrak{A})$ the hyperidentity (56) does not hold, under $n = 2^m$, although any hyperidentity from $H^{(m)}$ is true in $\mathcal{F}(\mathfrak{A})$.

Corollary 10 *Termal hyperidentities of the variety of Boolean algebras don't have a finite t-base of hyperidentities.*

For the two-element Boolean algebra, the hyperidentity (56) means the equivalence of the following two switching circuits:





The hyperidentity (56) in case $n = 2$ considered in [256] and [57] as well.

Theorem 46 *Any unary termal hyperidentity of the variety of Boolean algebras is a t-consequence of the following hyperidentities:*

$$X(Y(X(X(x)))) = X(Y(x)), \tag{57}$$
$$X(Y(X(Y(x)))) = X(Y(Y(X(x)))). \tag{58}$$

Proof. First we note that $0, 1, x, x'$ are all unary term operation of every Boolean algebra.

We can shortly denote any unary term (i.e. a term with unary functional variables)

$$F(\cdots(G(\cdots(H(x))\cdots))\cdots)$$

by

$$(F \bullet \cdots \bullet G \bullet \cdots \bullet H)x.$$

Every unary term contains only one object variable and if the hyperidentity $\omega_1 = \omega_2$ termally satisfies the variety of Boolean algebras, then $[\omega_1] = [\omega_2] = \{x\}$, where $[\omega]$ is the set of the object variable of ω. So we can shortly denote any unary term without an object variable:

$$F \bullet \cdots \bullet G \bullet \cdots \bullet H;$$

The unary term ω is called k-normal form on fucntional variables $(F_1, \dots, F_k)$ if

$$\omega = F_1 \bullet \omega_1 \bullet F_2 \bullet \omega_2 \bullet \cdots \bullet F_k \bullet \omega_k,$$

where $\omega_i = F_{i_1} \bullet \cdots \bullet F_{i_m}$, $1 \leqslant i_1 < \cdots < i \leqslant i_m$, $i = 1, \dots, k$ or $\omega_i = \emptyset$. If functional variables $F_1, \dots, F_k$ are fixed, this k-normal form is presented as:

$$\omega = (\omega_1, \dots, \omega_k).$$

The following assertion is proved by induction: Let unary hyperidentity $u = v$ be termally satisfied in the variety of Boolean algebras. If k-normal forms of u, v are $u = (u_1, \dots, u_k)$ and $v = (v_1, \dots, v_n)$, then $u_1 \equiv v_1$, …, $u_k \equiv v_k$, i.e. u and v graphically coincide with $(u \equiv v)$.

Indeed, if $k = 1$, then there exist two 1-normal forms on (F_1): F_1 and $F_1 \bullet F_1$. If $F_1(x) = \bar{x}$, then $F_1(x) \neq (F_1 \bullet F_1)(x) = F_1(F_1(x))$.

Let us prove that if the assertion is valid for $k - 1$, then it is valid for k. Namely, if $u_\ell \neq v_\ell$ for some $\ell = 1, \dots, k$, then the following two cases are possible.

1) $\ell < k$. Let ℓ be the minimal number with this condition. By condition of induction, the following hyperidentity is not termally valid in the variety of Boolean algebras:

$$F_1 \bullet u_1 \bullet \cdots \bullet F_\ell \bullet u_\ell = F_1 \bullet v_1 \bullet \cdots \bullet F_\ell \bullet v_\ell.$$

Since $u_i \equiv v_i$ for $i < \ell$, then substituting 0 or 1 for some F_i, where $i \leqslant \ell$, we obtain a valid equality. So we obtain a wrong equality if we substitute x or x' for any F_i, where $i = 1, \dots, \ell$, i.e. we obtain the equality $x = x'$. Thus, if in equality $u = v$, we substitute the same values for F_i, $i = 1, \dots, \ell$, and $F_{l+1} = 0$, we obtain $0 = 1$. Contradiction!

2) $u_i \equiv v_i$, where $i < k$, and $u_k \not\equiv v_k$. Thus there exists F_r in u_k which is not in v_k. From the equality $u_i \equiv v_i$, where $i < k$, it follows that in the

words u and v, we have different evenness of F_r. Now if in the equality $u = v$ we substitute x' for F_r and x for F_i, $i < k$, we obtain $x = x'$. Contradiction!

In the next step we prove that one can reduce any unary term to the k-normal form (cf.[174]) using the following unary termal hyperidentities of the variety of Boolean algebras: (57), (58),

$$F = F \bullet F \bullet F, \tag{59}$$

$$F \bullet H \bullet G \bullet K \bullet F \bullet G = F \bullet H \bullet G \bullet K \bullet G \bullet F; \tag{60}$$

Besides, if we substitute x for Y in (57), we obtain hyperidentity (59), i.e. (57)$\Rightarrow_t$(59), but the hyperidentity (60) follows from (57), (58):

$$\begin{aligned} F \bullet H \bullet G \bullet K \bullet F \bullet G &\overset{(57)}{=} F \bullet H \bullet F \bullet F \bullet G \bullet K \bullet F \bullet G \overset{(57)}{=} \\ &= F \bullet H \bullet F \bullet F \bullet G \bullet K \bullet F \bullet G \bullet F \bullet G \bullet F \bullet G \overset{(58)}{=} \\ &= F \bullet H \bullet F \bullet F \bullet G \bullet K \bullet F \bullet G \bullet F \bullet G \bullet G \bullet F \overset{(57)}{=} \\ &= F \bullet H \bullet G \bullet K \bullet G \bullet F. \end{aligned}$$

□

Problem 9 *Characterize binary termal hyperidentities of the variety of Boolean algebras.*

Problem 10 *Characterize algebras with termal hyperidentities (binary termal hyperidentities) of the variety of Boolean algebras.*

Theorem 47 ([218]) *Any non-trivial variety of lattices and the variety of all semi-lattices don't have a finite t-basis of termal hyperidentities.*

Theorem 48 ([219]) *Any binary termal hyperidentity of the variety of lattices is a t-consequence of the following hyperidentities: (39), (41), (42) and*

$$X(X(x, y), X(u, v)) = X(X(x, u), X(y, v)), \tag{61}$$

$$X(x, Y(y, X(z, X(u, v)))) = X(x, Y(y, X(Y(y, u), X(z, X(u, v))))). \tag{62}$$

Theorem 49 ([219]). *Any binary termal hyperidentity of the variety of distributive lattices is a t-consequence of the following hyperidentities: (39), (41), (61), (17).*

Theorem 50 ([25]) *The variety of all groups (metabelian groups, monoids) doesn't have a finite t-base of termal hyperidentities. Every such hyperidentity is a consequence of one objective variable hyperidentities of this variety.*

Let $\mathcal{L}$ be some non-empty set of hyperidentities. We shall denote by $\mathcal{K}_{\mathcal{L}}$ the class of all varieties of algebras, in every one of which, any hyperidentity from $\mathcal{L}$ is termally valid. The class of varieties of algebras is called the variety of varieties, if there exists a system of hyperidentities $\mathcal{L}$, such that

$$\mathcal{K} = \mathcal{K}_{\mathcal{L}}.$$

In that case we say that the variety of varieties $\mathcal{K}$ is definable by the system of hyperidentities $\mathcal{L}$.

The intersection of varieties of varieties is the variety of varieties, namely:

$$\bigcap_{i \in I} \mathcal{K}_{\mathcal{L}_i} = \mathcal{K}_{\bigcup_{i \in I} \mathcal{L}_i},$$

that's why for every class $\mathcal{K}$ of the varieties of algebras, there exists a smallest (relative to set-theoretic inclusion) variety of varieties $\mathcal{K}^* \supseteq \mathcal{K}$ called the variety of varieties generated by $\mathcal{K}$. Obviously $\mathcal{K}^*$ is defined by the system of all termal hyperidentities of the class of varieties $\mathcal{K}$.

Two varieties of algebras are called equivalent if their clones are isomorphic. If the class of varieties $\{V_i \mid i \in J\}$ is a variety of varieties, then the class of the corresponding clones $\{Cl(V_i) \mid i \in J\}$ is a variety of clones. One easily sees, that subvarieties correspond exactly to homomorphic image of their clones, subclones correspond exactly to forming reduct varieties, products of varieties correspond to direct products of their clones. A variety of varieties is closed under the formation of equivalent varieties, products of varieties, reducts of varieties and subvarieties.

Theorem 51 ([285]) *If a collection of varieties is closed under these operators, then it is a variety of varieties.*

Sometimes the variety of varieties is called a hypervariety of varieties ([285]).

The class of varieties $\{V_i \mid i \in J\}$ is called a quasivariety of varieties, if the class of their clones $\{Cl(V_i) \mid i \in J\}$ forms a quasivariety of clones. About the characterization of quasivarieties of varieties see [173].

12 Algebraic foundation of logic programming structures

In this section we shall consider bilattices as algebras with two separate bounded lattices satisfying the connecting identities. In [15, 16] Belnap introduced a logic, which is based on the algebraic structure called *FOUR*, having four truth values. In the papers [75, 76, 77, 78] M.L. Ginsberg proposed algebraic structures called bilattices that naturally generalize Belnap's *FOUR*.

Definition 7 *An algebra* $\mathfrak{B} = (Q; \{+, \cdot, \vee, \wedge, ^{-}, 0_1, 1_1, 0_2, 1_2\})$ *with four binary, one unary and four nullary operations is called a bilattice if* $(Q; \{+, \cdot, 0_1, 1_1\})$ *and* $(Q; \{\vee, \wedge, 0_2, 1_2\})$ *are bounded lattices and* $\mathfrak{B}$ *satisfies the following identities:*

$$\begin{aligned} \overline{\overline{x}} &= x, \\ \overline{x+y} &= \overline{x} \cdot \overline{y}, \\ \overline{x \cdot y} &= \overline{x} + \overline{y}, \\ \overline{x \vee y} &= \overline{x} \vee \overline{y}, \\ \overline{x \wedge y} &= \overline{x} \wedge \overline{y}. \end{aligned}$$

Recent developments in logic programming are related to the bilattices, their applications in logic programming semantics and Ginsberg-Fitting theorem for characterization of (bounded) distributive bilattices ([78], [70]). See ([246], [12], [160], [161],[73], [211], [212], [184], [185], [201], [203],[34],[237]) about analogous results for interlaced, modular and Boolean bilattices.

The concept of hyperidentity offers a general approach and the general point of view.

A bilattice is called *interlaced* if it (as a binary algebra) satisfies the hyperidentity (42). An interlaced bilattice is called *modular* (distributive), if it satisfies the hyperidentity (43) (accordingly hyperidentity (17)). A distributive bilattice is called *Boolean*, if it admits a unary operation $x \to x'$ such that

$$(x')' = x, 0_i' = 1_i, 1_i' = 0_i, i = 1, 2,$$

and the hyperidentity

$$X(x, X(y, y')') = x \tag{63}$$

is satisfied. Thus, a Boolean bilattice is an algebra with one unary, four binary and four nullary operations. Observe that each of the introduced classes of bilattices is a variety with nullary operations for the bounds. So every modular bilattice is an algebra with two bounded modular lattices, every distributive bilattice is an algebra with two bounded distributive lattices and every Boolean bilattice is an algebra with two Boolean lattices which have the same unary operation $'$.

Theorem 52 *([246], [12], [161], [239], [211], [212]) A bilattice is interlaced iff it is isomorphic to the superproduct of two bounded lattices.*

Theorem 53 *([211], [212]) A bilattice is modular iff it is isomorphic to the superproduct of two bounded modular lattices.*

Theorem 54 *([78], [70], [103], [10], [160], [211], [212]) A bilattice is distributive iff it is isomorphic to the superproduct of two bounded distributive lattices.*

Theorem 55 *([211], [212]) A bilattice is Boolean iff it is isomorphic to the superproduct of two Boolean lattices.*

Analogous results are valid for bilattices without bounds, and for bilattices with negations [212].

In these theorems the term superproduct (in particular supersquare) refers to the product in the category of algebras with bihomomorphisms $(\varphi, \tilde{\psi})$ as morphisms. In particular, the superproduct of two lattices (modular, distributive, Boolean lattices) is a interlaced bilattice (modular, distributive, Boolean bilattice). If $\mathfrak{B}_1$ is a distributive (Boolean) lattice of all subsets of the set I, and $\mathfrak{B}_2$ is a distributive (Boolean) lattice of all subsets of the set J, then every subalgebra of the superproduct of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is called a natural distributive (Boolean) bilattice of sets I, J. If $\mathfrak{B}_1 = \mathfrak{B}_2$, we obtain the concept of natural distributive (Boolean) bilattice of the set $I = J$.

Theorem 56 *Every distributive bilattice is isomorphic to a natural distributive bilattice of some sets I, J.*

Theorem 57 *Every Boolean bilattice is isomorphic to a natural Boolean bilattice of some sets I, J.*

Analogous results are valid for distributive (and Boolean) bilattices with negations.

Problem 11 *For which varieties (axiomatizable classes) of algebras are the Ginsberg-Fitting type theorems valid ?*

Hyperidentities (42), (43) and (17) are in the Theorems 31, 32 and 33 in which hyperidentities of lattices, modular lattices and distributive lattices are characterized. The hyperidentity (63) is a consequence of the hyperidentities from Theorem 34 in which hyperidentities of Boolean algebras are characterized.

Corollary 11 *A bilattice is interlaced iff it satisfies all hyperidentities of the variety of lattices.*

Corollary 12 *A bilattice is modular iff it satisfies all hyperidentities of the variety of modular lattices.*

Corollary 13 *A bilattice is distributive iff it satisfies all hyperidentities of the variety of distributive lattices.*

Corollary 14 *A bilattice is Boolean iff it satisfies all hyperidentities of the variety of Boolean algebras.*

The following results reducts with two binary operations of corresponding bilattices are characterized (for the Plonka sum see [229]).

Corollary 15 *Any reduct with two binary operations of interlaced bilattice is either a lattice or the Plonka sum of lattices.*

Corollary 16 *Any reduct with two binary operations of modular bilattice is either a modular lattice or the Plonka sum of modular lattices.*

Corollary 17 *Any reduct with two binary operations of distributive bilattice is either a distributive lattice or the Plonka sum of distributive lattices.*

Corollary 18 *Any reduct with one unary and two binary operations of Boolean bilattice is either a Boolean algebra or the Boolean sum of Boolean algebras (see the section* 18*).*

An algebra $Q(+_1, \circ_1, 0_1, 1_1, +_2, \circ_2, 0_2, 1_2,', ^-)$ with four binary, two unary and four nullary operations is called a Boolean bilattice with negation, if $Q(+_1, \circ_1, 0_1, 1_1, +_2, \circ_2, 0_2, 1_2,')$ is a Boolean bilattice , $Q(+_1, \circ_1, 0_1, 1_1, +_2, \circ_2, 0_2, 1_2, ^-)$ is a distributive bilattice with negation $^-$ and the unary operations are commutative. By the equality

$$\overline{(x, y)} = (y, x)$$

we convert the supersquare of Boolean algebra $\mathfrak{B}$ into a Boolean bilattice $\mathfrak{B}^{\mathfrak{bool}}$ with negation. In particular, if $\mathfrak{B}$ is a Boolean algebra of all subsets of the set I, then every subalgebra of Boolean bilattice $\mathfrak{B}^{\mathfrak{bool}}$ with negation is called a natural Boolean bilattice with negation of the set I.

Theorem 58 *Any Boolean bilattice with negation is isomorphic to* $\mathfrak{B}^{\mathfrak{bool}}$ *for some Boolean algebra* $\mathfrak{B}$.

Theorem 59 *Any Boolean bilattice with negation is isomorphic to a natural Boolean bilattice with negation of some set* I.

Theorem 60 *Any Boole-De Morgan algebra is a reduct for some Boolean bilattice with negation.*

13 Free distributive bilattices and bi-De Morgan functions

A bilattice $\mathfrak{B} = (Q; \{+, \cdot, \vee, \wedge, ^-, 0_1, 1_1, 0_2, 1_2\})$ is a *distributive bilattice* if any two of its binary operations distribute over each other. In this case $(Q; \{+, \cdot, ^-, 0_1, 1_1)$ is a De Morgan algebra. Thus, as mentioned above, the distributive bilattices form a variety, the free algebras of which are called free distributive bilattices. In this section we give characterization of the finitely generated free distributive bilattices.

The superproduct of any two bounded distributive lattices is a distributive bilattice. And conversely, any distributive bilattice can be represented as a superproduct of two bounded distributive lattices.

If $D = \{0, a, b, 1\}$ and $\mathbf{4} = \{D; \{+, \cdot, ^-, 0, 1\}$ be the four-element De Morgan algebra with two fixed points (it is unique up to isomorphism) then $\mathfrak{F}_4 = (D; \{+, \cdot, \vee, \wedge, ^-, 0, 1, b, a\})$ is the four-element distributive bilattice (it is unique up to isomorphism).

Let us remind the definition of the De Morgan function in terms of clone theory.

A function $f : D^n \to D$ is called a *De Morgan function* if the following conditions hold:

(1) the function f preserves the unary relation $\{0, 1\} \subseteq D$,

(2) the function f preserves the binary relation $\{(0, 0), (a, b), (b, a), (1, 1)\} \subseteq D^2$,

(3) the function f preserves the order relation $\rho = \{(b, b), (b, 0), (b, 1), (b, a), (0, 0), (0, a), (1, 1), (1, a), (a, a)\} \subseteq D^2$.

Definition 8 (Bi-De Morgan function) A function $f : D^n \to D$ is called a *bi-De Morgan function* of n variables if it preserves the order relation ρ.

Examples

The constant functions $f = 1$ and $f = 0$ are De Morgan functions, but the constant functions $f = a$ and $f = b$ are not. Instead, the latter two functions are bi-De Morgan functions. The functions $f(x) = x,\ g(x) = \overline{x},\ h(x, y) = x \cdot y,\ q(x, y) = x + y$, where the operations on the right hand side are the operations of the De Morgan algebra $\mathbf{4}$, are De Morgan functions. The functions $a \cdot x,\ x \vee y,\ x \wedge y$ are bi-De Morgan functions, but not De Morgan functions.

Denote the set of all bi-De Morgan functions of n variables by $\mathcal{B}_n$. For any two functions $f, g : D^n \to D$ define $f + g,\ f \cdot g,\ f \vee g,\ f \wedge g,\ \overline{f}$ in the standard way, i.e. $(f \lambda g)(x) = f(x) \lambda g(x)$, for all $\lambda \in \{+, \cdot, \vee, \wedge\}$, and $\overline{f}(x) = \overline{f(x)}$, where the operations on the right hand side are the operations on the set D. The set $\mathcal{B}_n$ is closed under the operations $+, \cdot, \vee, \wedge, ^-$.

We get the following algebra: $\mathfrak{B}_n = (\mathcal{B}_n, \{+, \cdot, \vee, \wedge, ^-, 0, 1, b, a\})$. Evidently, the algebra $\mathfrak{B}_n$ is a distributive bilattice.

Let us consider the projection functions

$$\delta_n^i(x_1, \ldots, x_n) = x_i,\ i = 1, \ldots, n,$$

as functions $D^n \to D$.

Theorem 61 *A function $f : D^n \to D$ is a bi-De Morgan function if and only if f can be represented in the form:*

$$f = (a \cdot f_1) + (b \cdot f_2), \tag{64}$$

where f_1 and f_2 are De Morgan functions of n variables (here the operations on the right hand side are the operations of the algebra $\mathfrak{F}_4$). In that case the representation is unique.

Corollary 19 *There are m_{2n}^2 bi-De Morgan functions of n variables.*

Corollary 20 *For every bi-De Morgan function f of n variables there exists a unique pair of antichains (S_1, S_2) $(S_1, S_2 \subseteq 2^{\{1,\dots,n\}} \times 2^{\{1,\dots,n\}})$ such that:*

$$f(x_1, \dots, x_n) = \\ a \cdot \sum_{(s_1,s_2)\in S_1} \left(\prod_{i\in s_1} x_i \cdot \prod_{i\in s_2} \overline{x}_i \right) + b \cdot \sum_{(s_1,s_2)\in S_2} \left(\prod_{i\in s_1} x_i \cdot \prod_{i\in s_2} \overline{x}_i \right), \tag{65}$$

where the operations on the right hand side are the operations of the algebra $\mathfrak{F}_4$.

The form (65) is called the disjunctive normal form of bi-De Morgan function f.

The next result is related to the Plotkin's problem, too (see [186]).

Theorem 62 (Functional representation theorem) ([198]) *The algebra $\mathfrak{B}_n$ is a free distributive bilattice with the system of free generators $\Delta = \{\delta_n^1, \dots, \delta_n^n\}$. Hence every free n-generated distributive bilattice is isomorphic to the distributive bilattice $\mathfrak{B}_n$.*

Problem 12 *To develop the bi-De Morgan analogue of the theory of Boolean functions.*

14 On the Schauffler type theorems

During the second World War, while working for the German cryptography service R. Schauffler developed a method of error detection based on the usage of $\forall\exists(\forall)$-identity of associativity [253], [254], [255] (cf. [84], [257],[258],[270],[271]). Schauffler's main result is:

Theorem 63 ([255]) *The following conditions are equivalent for any non-empty set Q:*

1) For every quasigroups $Q(X)$ and $Q(Y)$ there exist quasigroups $Q(X')$ and $Q(Y')$ with identity:

$$X(x, Y(y, z)) = X'(Y'(x, y), z); \tag{66}$$

2) For every quasigroups $Q(X)$ and $Q(Y)$ there exist quasigroups $Q(X')$ and $Q(Y')$ with identity:

$$X(Y(x, y), z)) = X'(x, Y'(y, z)); \tag{67}$$

3) Cardinality $|Q| \leqslant 3$.

The following results are extensions of the Theorem 63.

Theorem 64 ([175]) *The following conditions are equivalent for any non-empty set Q:*

4) For every loops $Q(X)$ and $Q(Y)$ there exist quasigroups $Q(X')$ and $Q(Y')$ with identity (66);

5) For every loops $Q(X)$ and $Q(Y)$ there exist quasigroups $Q(X')$ and $Q(Y')$ with identity (67);

6) For every loops $Q(X)$ and $Q(Y)$ there exist loops $Q(X')$ and $Q(Y')$ with identity (66);

7) For every loops $Q(X)$ and $Q(Y)$ there exist loops $Q(X')$ and $Q(Y')$ with identity (67);

8) In the algebra $(Q; \mathcal{L}_Q)$ the hyperidentity

$$X(x, Y(y, z)) = Y(X(x, y), z)$$

is valid, where $\mathcal{L}_Q$ is the set of all loop operations on Q;

9) In the algebra $(Q; \mathcal{L}_Q)$ the hyperidentity

$$X(x, Y(y, z)) = X(Y(x, y), z)$$

is valid;

10) For every quasigroup $Q(X)$ there exist quasigroups $Q(X')$ and $Q(Y')$ with identity:

$$X(x, X(y, z)) = X'(Y'(x, y), z); \tag{68}$$

11) For every quasigroup $Q(X)$ there exist quasigroups $Q(X')$ and $Q(Y')$ with identity:

$$X(X(x, y), z)) = X'(x, Y'(y, z)); \tag{69}$$

12) Cardinality $|Q| \leqslant 3$.

Theorem 65 ([120]) *The following conditions are equivalent for any non-empty set Q:*

13) For every groupoids $Q(X)$ and $Q(Y)$ there exist groupoids $Q(X')$ and $Q(Y')$ with identity (66);

14) For every groupoids $Q(X)$ and $Q(Y)$ there exist groupoids $Q(X')$ and $Q(Y')$ with identity (67);

15) The set Q is infinite or one-element.

The following result is general than Theorems 63, 64 and 65.

Theorem 66 ([175]) *The following conditions are equivalent for any non-empty set Q:*

16) For every quasigroups $Q(X)$ and $Q(Y)$ there exist groupoids $Q(X')$ and $Q(Y')$ with identity (66);

17) For every quasigroups $Q(X)$ and $Q(Y)$ there exist groupoids $Q(X')$ and $Q(Y')$ with identity (67);

18) For every loops $Q(X)$ and $Q(Y)$ there exist groupoids $Q(X')$ and $Q(Y')$ with identity (66);

19) For every loops $Q(X)$ and $Q(Y)$ there exist groupoids $Q(X')$ and $Q(Y')$ with identity (67);

20) For every quasigroup $Q(X)$ there exist groupoids $Q(X')$ and $Q(Y')$ with identity (68);

21)For every quasigroup $Q(X)$ there exist groupoids $Q(X')$ and $Q(Y')$ with identity (69);

22) The set Q is infinite or cardinality $|Q| \leqslant 3$.

Corollary 21 *Non associative loop $Q(\circ)$ is infinite iff the following functional equation has a solution in the set Q:*

$$x \circ (y \circ z) = A(B(x, y), z);$$

Corollary 22 *Non associative loop $Q(\circ)$ is infinite iff the following functional equation has a solution in the set Q:*

$$(x \circ y) \circ z = A(x, B(y, z)).$$

Problem 13 *Characterize the semigroups $Q(\circ)$ with the following condition: for any two binary term operations A, B of $Q(\circ)$ there exist two binary term operations C, D of $Q(\circ)$ such that the identity*

$$A(x, B(y, z)) = C(D(x, y), z)$$

is valid.

Problem 14 *Characterize the semigroups $Q(\circ)$ with the following condition: for any two binary term operations A, B of $Q(\circ)$ there exist two binary term operations C, D of $Q(\circ)$ such that the identity*

$$A(B(x,y),z)) = C(x, D(y,z))$$

is valid.

Problem 15 *Characterize the semigroups $Q(\circ)$ with the following condition: for any binary term operation A of $Q(\circ)$ there exist two binary term operations C, D of $Q(\circ)$ such that the identity*

$$A(x, A(y,z)) = C(D(x,y),z)$$

is valid.

Problem 16 *Characterize the semigroups $Q(\circ)$ with the following condition: for any binary term operation A of $Q(\circ)$ there exist two binary term operations C, D of $Q(\circ)$ such that the identity*

$$A(A(x,y),z)) = C(x, D(y,z))$$

is valid.

Problem 17 *Characterize the semigroups $Q(\circ)$ with the following condition: for any two binary term operations A, B of $Q(\circ)$ there exist two binary term operations C, D of $Q(\circ)$ such that the identity*

$$A(D(x,y),z)) = C(x, B(y,z))$$

is valid.

Problem 18 *Characterize the semigroups $Q(\circ)$ with the following condition: for any two binary term operations A, B of $Q(\circ)$ there exist two binary term operations C, D of $Q(\circ)$ such that the identity*

$$A(D(x,y),z)) = B(x, C(y,z))$$

is valid.

Problem 19 *Characterize the semigroups $Q(\circ)$ with the following condition: for any two binary term operations A, B of $Q(\circ)$ there exist two binary term operations C, D of $Q(\circ)$ such that the identity*

$$D(A(x,y),z)) = C(x, B(y,z))$$

is valid.

Problem 20 *Characterize the semigroups $Q(\circ)$ with the following condition: for any two binary term operations A, B of $Q(\circ)$ there exist two binary term operations C, D of $Q(\circ)$ such that the identity*

$$D(A(x,y),z)) = B(x, C(y,z))$$

is valid.

15 $\forall\exists^*(\forall)$-identity of distributivity and hyperidentities

For functional equations in algebra, logics, real analysis and topology see [2, 3, 5, 18, 55, 67, 119, 120, 121, 122, 123, 148, 149, 150].

However the solution of general functional equation of left (right) distributivity with quasigroup operations is open [2, 3, 5, 119]. The well known Belousov's Theorem [17] stated that every distributive quasigroup is isotopic to a certain commutative Moufang loop (cf. [104],[146],[273]).

Lemma 6 *If the binary algebra $Q(A,B,H,K)$ with four operations satisfies the following identity*

$$A(x,B(y,z)) = H(K(x,y),K(x,z)),$$

where A and K are quasigroup operations, then quasigroups $Q(A)$ and $Q(K)$ are isotopic to some quasigroup $Q(A_0)$, and groupoids $Q(B)$ and $Q(H)$ are isotopic to some idempotent groupoid $Q(B_0)$ such that the operations A_0 and B_0 satisfy the identity of left distributivity:

$$A_0(x,B_0(y,z)) = B_0(A_0(x,y),A_0(x,z)).$$

Besides,

$$A(x,B_0(y,z)) = B_0^{L_A^{-1}}(A(x,y),A(x,z)),$$

where $L_A(x) = A(0,x)$ and 0 is a fix element in Q and

$$B_0^{\alpha}(x,y) = \alpha^{-1}B_0(\alpha x,\alpha y).$$

In particular, if the operation B is idempotent, then

$$A(x,B(y,z)) = B^{L_A^{-1}}(A(x,y),A(x,z)).$$

If $Q(B)$ is a quasigroup, then $Q(H)$ and the idempotent groupoid $Q(B_0)$ are also quasigroups.

We say that the binary algebra $(Q;\Sigma)$ with quasigroup operations satisfies the $\forall\exists^*(\forall)$-identity of left distributivity, if for every quasigroup operations $X,Y\in\Sigma$ there exist quasigroup operations X',Y' on Q with identity:

$$X(x,Y(y,z)) = X'(Y'(x,y),Y'(x,z));$$

A binary algebra $(Q;\Sigma)$ with quasigroup operations is called D_r-algebra, if it satisfies the $\forall\exists^*(\forall)$-identity of left distributivity and hyperidentity of distributivity (16). D_l-algebra is defined in the dual way, i.e. by hyperidentity of distributivity (18) and $\forall\exists^*(\forall)$-identity of right distributivity:

$$X(Y(x,y),z)) = X'(Y'(x,z),Y'(y,z));$$

Theorem 67 *If* $(Q;\,\Sigma)$ *is a* D_l*-algebra, then all quasigroup operations from* Σ *are distributive and consequently isotopic to commutative Moufang loops. Every* D_l*-algebra satisfies the following hyperidentity:*

$$X(Y(X(y,z),z),x) = X(Y(X(y,x),X(z,x)),X(z,x)).$$

Theorem 68 *If* $(Q;\,\Sigma)$ *is a* D_r*-algebra, then all quasigroup operations from* Σ *are distributive and consequently isotopic to commutative Moufang loops. Every* D_r*-algebra satisfies the following hyperidentity:*

$$X(x,Y(y,X(y,z))) = X(X(x,y),Y(X(x,y),X(x,z))).$$

Corollary 23 *If the quasigroups* $Q(A)$*,* $Q(K)$ *and a groupoid* $Q(H)$ *satisfies the following identities:*

$$A(x,A(y,z)) = A(A(x,y),A(x,z)),$$

$$A(A(y,z),x) = H(K(y,x),K(z,x)),$$

then $Q(A)$ *and* $Q(K)$ *are isotopic to a commutative Moufang loop.*

Corollary 24 *If the quasigroups* $Q(A)$*,* $Q(K)$ *and a groupoid* $Q(H)$ *satisfies the following identities:*

$$A(A(y,z),x) = A(A(y,x),A(z,x)),$$

$$A(x,A(y,z)) = H(K(x,y),K(x,z)),$$

then $Q(A)$ *and* $Q(K)$ *are isotopic to a commutative Moufang loop.*

Corollary 25 *If the quasigroups* $Q(A)$*,* $Q(K)$*,* $Q(K')$ *and groupoids* $Q(H)$*,* $Q(H')$ *satisfies the identities:*

$$A(x,x) = x,$$

$$A(A(y,z),x) = H(K(y,x),K(z,x)),$$
$$A(x,A(y,z)) = H'(K'(x,y),K'(x,z)),$$

then $Q(A)$*,* $Q(K)$ *and* $Q(K')$ *are isotopic to a commutative Moufang loop.*

Let Σ_l be the set of loop operations corresponding to the quasigroup operations from D_l-algebra $(Q;\Sigma)$ according to the previous Theorem. We obtain a new algebra $(Q;\Sigma_l)$.

Let Σ_r be the set of loop operations corresponding to the quasigroup operations from D_r-algebra $(Q;\Sigma)$ according to the previous Theorem, too. We obtain an algebra $(Q;\Sigma_r)$.

Our next result shows the connection (through the hyperidentity) between the loop-operations from Σ_l and Σ_r.

Theorem 69 *If $(Q;\Sigma)$ is a D_ℓ-algebra (D_r-algebra), then the algebra $(Q;\Sigma_l)$ (algebra $(Q;\Sigma_r)$) satisfies the following non-trivial hyperidentity:*

$$X(x, Y(x, X(y, z))) = X(Y(x, y), Y(x, z)). \tag{70}$$

Problem 21 *Characterize the semigroups $Q(\circ)$ with the following condition: for any two binary term operations A, B of $Q(\circ)$ there exist two binary term operations C, D of $Q(\circ)$ such that the identity*

$$A(x, B(y, z)) = C(D(x, y), D(x, z))$$

is valid.

Problem 22 *Characterize the semigroups $Q(\circ)$ with the following condition: for any two binary term operations A, B of $Q(\circ)$ there exist three binary term operations C, D, H of $Q(\circ)$ such that the identity*

$$A(x, B(y, z)) = C(D(x, y), H(x, z))$$

is valid.

Problem 23 *Characterize the semigroups $Q(\circ)$ with the following condition: for any two binary term operations A, B of $Q(\circ)$ there exist two binary term operations C, D of $Q(\circ)$ such that the identity*

$$A(B(x, y), z)) = C(D(x, z), D(y, z))$$

is valid.

Problem 24 *Characterize the semigroups $Q(\circ)$ with the following condition: for any two binary term operations A, B of $Q(\circ)$ there exist three binary term operations C, D, H of $Q(\circ)$ such that the identity*

$$A(B(x, y), z)) = C(D(x, z), H(y, z))$$

is valid.

16 Binary representations of groups and semigroups. Binary G-sets, their hyperidentities

Let us consider the monoid $O_p^{(2)}Q$ of binary operations on Q under the multiplication:

$$f \cdot g(x, y) = f(x, g(x, y)).$$

Let G be an arbitrary semigroup, which in general has no connection with Q; Every homomorphism of semigroups $\varphi : G \to O_p^{(2)}Q$ is called the binary representation of G on Q. If G is a monoid with identity element e, then the homomorphism φ is considered as a homomorphism of semigroups with nullary operations e and δ_2^2, i.e. $\varphi(e) = \delta_2^2$. If homomorphism φ is a monomorphism, then the binary representation is called faithful. Binary representation $\varphi : G \to O_p^{(2)}Q$ of monoid G is called symmetric, if $(\varphi\alpha)^* \in \varphi(G)$ for every $\alpha \in G, \alpha \neq e$, where $A^*(x, y) = A(y, x)$ for all $x, y \in Q$.

A set Q with a mapping

$$G \times Q^2 \to Q$$

is called a (left) binary G-set, if it associates with each $\alpha \in G$ and every pair $(x, y) \in Q^2$ with an element $\alpha(x, y) \in Q$ such that the following condition is valid:

$$\alpha \cdot \beta(x, y) = \alpha(x, \beta(x, y))$$

for all $x, y \in Q$ and all $\alpha, \beta \in G$. If G is a monoid with an identity element $e \in G$, we add the condition $e(x, y) = y$ to the definition of binary G-set. So the elements of G act on Q as binary operations $\alpha : (x, y) \to \alpha(x, y)$. Consequently, Q becomes a binary G-algebra satisfying the identity:

$$\alpha \cdot \beta(x, y) = \alpha(x, \beta(x, y)).$$

Hence if Q is a binary G-set, then the mapping $\varphi : \alpha \to \alpha(x, y)$ defines a binary representation of G on Q, and vice versa.

Examples. 1)If $G(\circ)$ is a semigroup, then equality

$$\alpha(x, y) = \alpha \circ y,$$

where $x, y, \alpha \in G$, converts $Q = G$ into a binary G-set ;

2)If $G(\circ)$ is a group, then the equality

$$\alpha(x, y) = y \circ x^{-1} \circ \alpha^{-1} \circ x,$$

where $x, y, \alpha \in G$, converts $Q = G$ into a binary G-set.

We say that the binary G-set Q or the binary representation of G on Q satisfies the hyperidentity $w_1 = w_2$ (or the other formula), if the corresponding binary G-algebra satisfies this hyperidentity (or the given formula). A binary representation is said to be right invertible if the corresponding binary G-algebra Q is right invertible, that is for arbitrary $\alpha \in G$ and $a, b \in Q$ the equation $\alpha(a, x) = b$ has a unique solution x in Q. A binary representation is said to be orthogonal, if any two operations $\alpha \neq \beta$ of corresponding binary G-algebra is orthogonal ([42, 69, 186]). A binary representation of G on Q is said to be transitive if Q is a singleton or else for all a, b, c in Q with $b \neq a \neq c$ there exists an element α in G with equality $\alpha(a, b) = c$.

Lemma 7 *Every binary representation of a group is a right invertible.*

Proposition 4 *(Binary Cayley theorem for semigroups). Every semigroup has a faithful binary representation satisfying the hyperidentities: (18), (61).*

Proposition 5 *(Binary Cayley theorem for idempotent semigroups). Every idempotent semigroup has a faithful binary representation satisfying the hyperidentities: (18), (61), (41).*

Proposition 6 *(Binary Cayley theorem for commutative semigroups). Every commutative semigroup has a faithful binary representation satisfying the hyperidentities: (17), (20).*

Theorem 70 *(*[172], [178]*: Binary Cayley theorem for multiplicative groups of fields). A monoid is the multiplicative group of a field iff it has a faithful transitive right invertible binary representation satisfying the hyperidentities: (17), (39).*

Theorem 71 *(Binary Cayley theorem for multiplicative groups of Grätzer algebras). A monoid is the multiplicative group of a Grätzer algebra iff it has a faithful transitive right invertible symmetric and orthogonal binary representation satisfying the hyperidentity (39).*

Theorem 72 *The multiplicative groups of finite Grätzer algebras and finite near-fields are the same (for definitions see [186]).*

The investigation of topological binary representations for topological semigroups and topological groups leads to the solution of Pontryagin's problem on the characterization of topological multiplicative groups of topological fields.

17 Other Open Problems

Along with the problems above, naturally there arise a set of other problems about the characterization of hyperidentities and termal hyperidentities of varieties related to classical varieties of groups, rings, lattices and Boolean algebras. The solutions of these problems could serve for the development of the next steps of the results and concepts involved in the current survey.

1. Characterize $\{1,2\}$-algebras with hyperidentities of commutativity (40), associativity (41) and Robbins's hyperidentity (46). Can every $\{1,2\}$-algebra with these three hyperidentities and one unary operation be extended to an algebra with hyperidentities of Boolean algebras? Characterize subdirectly irreducible $T = \{1,2\}$-algebras with these three hyperidentities.

2. Let Gr be the class of all Grätzer algebras, and let qGr be the class of all Grätzer q-algebras (see [186]).

   Characterize hyperidentities and termal hyperidentities of Gr.

   Characterize hyperidentities and termal hyperidentities of qGr.

   Characterize algebras with hyperidentities (termal hyperidentities) of Gr.

   Characterize algebras with hyperidentities (termal hyperidentities) of qGr.

3. Characterize hyperidentities and termal hyperidentities of the variety of De Morgan bisemigroups.

   Characterize algebras with hyperidentities (termal hyperidentities) of the variety of De Morgan bisemigroups.

4. Characterize hyperidentities and termal hyperidentities of the variety of Boole-De Morgan algebras.

   Characterize algebras with hyperidentities (termal hyperidentities) of the variety of Boole-De Morgan algebras.

5. Characterize hyperidentities of the variety of bilattices with negations;

   Characterize hyperidentities of the variety of interlaced bilattices with negations;

   Characterize hyperidentities of the variety of modular bilattices with negations;

   Characterize hyperidentities of the variety of distributive bilattices with negations;

   Characterize hyperidentities of the variety of Boolean bilattices with negations .

6. A lattice ordered group (briefly an l-group) is an algebra $\mathfrak{A} = Q(+, \cdot, \circ)$ with tree binary operations such that $Q(+, \cdot)$ is a lattice, $A(\circ)$ is a group, and the group multiplication is isotone in each of its argument. Actually, the lattice reduct $Q(+, \cdot)$ of an l-group in always distributive, and the group operation $\circ$ distributes over lattice joins and meets. Indeed, the maps $x \to a \circ x$ and $x \to x \circ a$ are ordinary automorphisms of $Q(+, \cdot)$. Let L_p be the variety of all l-groups.

   Characterize hyperidentities and termal hyperidentities of the variety L_g. Is every hyperidentity of the variety L_g the consequence of hyperidentities with functional $rank \leqslant 2$ of this variety? (posed by V. Garbunov).

Characterize algebras with hyperidentities (termal hyperidentities) of the variety L_g.

7. An algebra $Q(+, \cdot, \to)$ with three binary operations is called Heyting algebra (see H. Rasiowa [241], who calls them pseudo-Boolean algebras), if it satisfies the following conditions:

$Q(+, \cdot)$ is a distributive lattice,

$(x \to y) \cdot y = y,$

$x \cdot (x \to y) = x \cdot y,$

$x \to (y \cdot z) = (x \to y) \cdot (x \to z),$

$(x + y) \to z = (x \to z) \cdot (y \to z).$

These algebras were introduced by G. Birkhoff under a different name, Brouwerian algebras, and with a different notation ($v : u$ for $u \to v$). For example, if $Q(+, \cdot, ')$ is a Boolean algebra and $x \to y = x' + y$, then $Q(+, \cdot, \to)$ is a Heyting algebra. Let He be the variety of all Heyting algebras.

Characterize hyperidentities and termal hyperidentities of the variety He. Is every hyperidentity of the variety He the consequence of hyperidentities with functional $rank \leqslant 2$ of this variety?

Characterize algebras with hyperidentities (termal hyperidentities) of the variety He.

Characterize hyperidentities and termal hyperidentities of polyadic Heyting algebras.

Characterize hyperidentities and termal hyperidentities of cylindric Heyting algebras.

8. An $\mathcal{O}$ckham algebra ([26], [106]) is an algebra $\mathfrak{A} = Q(+, \cdot, f)$ such that $Q(+, \cdot)$ is a distributive lattice and f is an antiendomorphism of $Q(+, \cdot)$, i.e. for every $x, y \in Q$:

$$f(x + y) = f(x) \cdot f(y),$$

$$f(x \cdot y) = f(x) + f(y).$$

Let $\mathcal{O}$ be the variety of all $\mathcal{O}$ckham algebras.

Characterize hyperidentities and termal hyperidentities of the variety $\mathcal{O}$.

Characterize algebras with hyperidentities (termal hyperidentities) of the variety $\mathcal{O}$.

The equation

$$f^2(x) = x$$

defines the subvariety of De Morgan algebras.

Characterize termal hyperidentities of the variety of De Morgan algebras.

Characterize algebras with termal hyperidentities of the variety of De Morgan algebras.

9. A modal algebra ([32]) is an algebra $\mathfrak{A} = Q(+, \cdot, ', f, 0, 1)$ such that $Q(+, \cdot, ', 0, 1)$ is a Boolean algebra and f is a unary operation (the modal operator) that satisfies the identity $f(1) = 1$ and

$$f(x \cdot y) = f(x) \cdot f(y).$$

Modal algebras have been extensively investigated because of their connection with the modal logic. Let $\mathcal{M}$ be the variety of all modal algebras.

Characterize hyperidentities and termal hyperidentities of the variety $\mathcal{M}$.

Characterize algebras with hyperidentities (termal hyperidentities) of the variety $\mathcal{M}$.

Several varieties of modal algebras have received a great deal of attention in the investigations. We mention the varieties of interior algebras, monadic algebras, diagonalizable algebras, etc.

10. A relation algebra ([281], [107], [155], [156], [131], [105], [132], [133], [134], [135], [136], [137], [138]) is an algebra $\mathfrak{A} = Q(+, \cdot, ', \circ, ^{-}, 0, 1)$ such that $Q(+, \cdot, ', 0, 1)$ is a Boolean algebra and $Q(\circ)$ is a monoid, and that the identities

$$(x + y) \circ z = (x \circ z) + (y \circ z),$$
$$\overline{x + y} = \overline{x} + \overline{y},$$
$$\overline{x \circ y} = \overline{y} \circ \overline{x},$$
$$\overline{\overline{x}} = x,$$
$$\overline{x} \circ (x \circ y)' \leqslant y'$$

hold. Let Re be the variety of all relation algebras.

Characterize hyperidentities and termal hyperidentities of the variety Re. Is every hyperidentity of the variety Re the consequence of hyperidentities with functional $rank \leqslant 2$ of this variety?

Characterize algebras with hyperidentities (termal hyperidentities) of the variety Re.

Characterize hyperidentities and termal hyperidentities of all symmetric relation algebras ($\overline{x} = x$).

Characterize hyperidentities and termal hyperidentities of all commutative relation algebras ($x \circ y = y \circ x$).

Characterize hyperidentities and termal hyperidentities of all (full) relation algebras of binary relations.

Characterize hyperidentities and termal hyperidentities of all representable relation algebras.

Clearly, every symmetric relation algebra is commutative and every Boolean algebra is a relation algebra with

$x \circ y = x \cdot y$ and $\overline{x} = x$.

11. Characterize hyperidentities of the variety of all rings. Are all hyperidentities, satisfied by the variety of all rings, the consequences of one object variable hyperidentities of this variety? (posed by B.I.Plotkin).

    Characterize binary algebras with hyperidentities of the variety of all rings.

    Characterize hyperidentities of all fields.

    Characterize hyperidentities of all fields with a fix characteristic.

    Characterize hyperidentities of all finite fields.

12. The loop $Q(\cdot)$ is called a Moufang loop ([164], [38], [20], [125], [170], [274]) if one of the following equivalent identities

$$(zx \cdot y)x = z(x \cdot yx),$$

$$x(y \cdot xz) = (xy \cdot x)z,$$

$$xy \cdot zx = x(yz \cdot x)$$

is valid in it. Commutative Moufang loops are characterized by one identity:

$$x^2 \cdot yz = xy \cdot xz.$$

Moufang's theorem([164], [38], [20]). *If the relation $a \cdot bc = ab \cdot c$ is valid in a Moufang loop for its some three elements a, b, c, then the subloop, generated by these three elements, is a group.*

Characterize termal hyperidentities of the variety of all Moufang loops (commutative Moufang loops) (posed by V.D.Belousov).

Characterize algebras with termal hyperidentities of the variety of all Moufang loops (commutative Moufang loops).

Characterize termal hyperidentities of G-loops (see G.M.Bergman ([25])).

13. Let P be a linearly ordered field. The interval $(0,\,1) \subseteq P$ is a semigroup under multiplication of field P. The binary algebra $\mathfrak{A} = (Q,\,\Sigma)$ is called a stochastic algebra over P (in other terminology is a convexor ([266]) or a barycentric algebra ([244])), if $\mathfrak{A}$ is a Ω-algebra, under $\Omega = (0,\,1)$, and the following equations are true:

    S1. $\alpha(x,\,x) = x$,

    S2. $\alpha(x,\,y) = (1-\alpha)(y,\,x)$,

    S3. $\alpha(x,\,\beta(y,\,z)) = \alpha\beta(\frac{\alpha(1-\beta)}{1-\alpha\beta}(x,\,y),\,z)$

    for any $x,\ y,\ z \in Q$ and $\alpha,\ \beta \in (0,\,1)$. The first axiom is a hyperidentity of idempotency, while the second and third equations are $\forall\exists(\forall)$-identities of commutativity and associativity. Non-trivial hyperidentities (17) and (19) of left and right distributivity are satisfied in every stochastic algebra:

$$\beta\left(\alpha(x,\,y),\,\alpha(x,\,z)\right) = \beta\alpha\left(\frac{\beta(1-\alpha)}{1-\beta\alpha}\left(\alpha(x,\,y),\,x\right),\,z\right) =$$

$$= \beta\alpha\left(\frac{1-\beta\alpha-\beta+\beta\alpha}{1-\beta\alpha}\left(x,\,\alpha(x,\,y)\right),\,z\right) =$$

$$= \beta\alpha\left(\frac{(1-\beta)\alpha}{1-\beta\alpha}(x,\,y),\,z\right) = \alpha\left(x,\,\beta(y,\,z)\right),$$

    while from the hyperidentity (17) and axiom S.2 implies the hyperidentity (19).

    In addition, the solution of L.A.Skornyakov's problem about the characterization of ideals' lattices of stochastic algebras is obtained taking into account these facts.

    Theorem 73 *A lattice of ideals of every stochastic algebra is isomorphic to a lattice of ideals of some semilattice.*

    Theorem 74 *Every ideal of a stochastic algebra is an intersection of its simple ideals.*

    Let Stoch be the variety of all stochastic algebras.

    Characterize hyperidentities (termal hyperidentities) of the variety Stoch (posed by L.A.Skornyakov).

    Characterize algebras with hyperidentities (termal hyperidentities) of the variety Stoch.

14. Characterize termal hyperidentities of the class of BCK-algebras (BCH-algebras, BCC-algebras) ([101], [102]) (posed by K.Iseki). Characterize algebras with termal hyperidentities of BCK-algebras (BCH-algebras, BCC-algebras).

15. Obtain categorical characterization of

    a) hypervarieties of Ω-algebras;

    b) hypervarieties of T-algebras ;

    c) solid varieties of Ω-algebras;

    d) solid hypervatieties of T-algebras;

    (posed by J.D.H.Smith).

    See [127], [231] and [291].

16. a) Is elementary theory of any De-Morgan algebra decidable?

    b) Is elementary theory of the variety of De-Morgan algebras decidable?

    c) Is elementary theory of any super-De Morgan algebra with one unary and one binary operations is decidable?

    d) Is elementary theory of the variety of super-De Morgan algebras with one unary and one binary operations is decidable?

    e) Is elementary theory of any super-De Morgan algebra with one unary and two binary operations is decidable?

    f) Is elementary theory of the variety of super-De Morgan algebras with one unary and two binary operations is decidable?

    See [252], [281] and [301].

17. In the paper [308] free algebras of the variety of algebras with two binary operations satisfying the hyperidentity of associativity $(ass)_1$ were characterized.

    a) Characterize free algebras of the variety of algebras with two binary operations satisfying the hyperidentity of associativity $(ass)_2$.

    b) Characterize free algebras of the variety of algebras with two binary operations satisfying the hyperidentity of associativity $(ass)_3$.

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Yuri Movsisyan
Yerevan State University,
Alex Manoogian 1, 0025 Yerevan, Armenia.
movsisyan@ysu.am

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