Vakgroep Wiskunde:
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Characterisations and classifications in the theory of parapolar spaces

Anneleen De Schepper

Promotoren
Prof. dr. H. Van Maldeghem (Universiteit Gent)
Dr. J. Schillewaert (University of Auckland)

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The least one can expect from a thesis
is that it contains a quote.

(Anonymous mathematician)

PREFACE

Dear reader

I feel morally obliged to warn you that this thesis contains mathematics and might exceed the recommended daily intake of incidence geometry. As the title was so kind to suggest, the main ingredients are *parapolar spaces* (a certain type of geometry), several interesting subclasses of which will be characterised axiomatically in this thesis. Technically speaking, "geometries which are unions of polar spaces of rank at least 1, satisfying some axioms which are stronger than those used for parapolar spaces" would have been more appropriate instead of "parapolar spaces", but I take it that you forgive me my preference towards this slightly shorter title.

Axiomatic characterisations of certain collections of objects are very common in mathematics. The advantage is that such a characterisation tells you how the entire collection behaves – no need to study each object separately – and that you also know how special this behaviour is, judging by how large the obtained collection of objects is. Sometimes one starts from some properties which one wants to investigate, and sometimes one has a certain small class of special objects in mind, and then finding properties shared by these special objects and by no single other object is the challenge. It is the latter type of characterisation that you will find in here.

For those who actually intend on reading this thesis: more useful information can be found in the introduction, and a brief summary (both in English and in Dutch) is given at the very end. I hope you enjoy reading and I sincerely thank you for your interest, whichever reason brought you here.

Anneleen De Schepper
Gent, March 2019

DANKWOORD

Terugblikkend op de afgelopen vier jaar besef ik dat het werkelijk een voorrecht is om te kunnen doctoreren. De wondere wereld van de wiskunde is eindeloos, en ik heb er nog zo veel meer van kunnen ontdekken, mijn eigen grenzen verleggend. Onderweg heb ik vele interessante mensen ontmoet en ben ik naar vele landen gereisd, het ene al wat exotischer dan het andere. Dat allemaal terwijl ik gewoon kon doen wat ik altijd al graag gedaan heb en altijd graag zal doen: met wiskunde bezig zijn.

Niets van dat alles was geweest wat het was zonder Hendrik. Bedankt, om mij met veel enthousiasme wegwijs te maken in de gebouwentheorie. Bedankt, om zelfs op de drukste momenten tijd voor mij te maken. Bedankt, om de verpersoonlijking te zijn van het spreekwoord "waar een wil is, is een weg[1]". Je bent zonder twijfel een blijvende invloed.

Ook Jeroen, die gaandeweg co-promotor werd, kan ik niet genoeg bedanken. Vooreerst ben ik zeer blij dat ik welkom was bij de Veronese-projecten. Ook je hulp bij het indienen van aanvragen en je immer klare kijk op zaken worden ten zeerste geapprecieerd. Je bent een bron van inspiratie in de vermakelijke manier waarop jij jezelf en wiskunde kan presenteren.

Daarnaast wens ik ook mijn collega's te bedanken, zeker de "AAP"en, voor de leuke lunchpauzes en occasionele activiteiten buitenaf. In het bijzonder, dank aan mijn bureaugenootjes over de vier[2] burelen heen, voor de gezellige werkplek. Een speciale dankuwel gaat uit naar Karsten, Ana en Manuel, om mij zo goed op te vangen gedurende de donkere dagen in het eerste jaar van mijn doctoraat, en simpelweg om zo een plezante, warme, grappige mensen te zijn. Maarten, ook jou bedank ik graag, voor het fungeren als "wiskundige grote broer". Magali, jouw avontuurlijke ziel heeft al veel jolijt gebracht (van glijbanen tot eenwielers en wereldreizen), en ik kijk op naar jouw vrolijke en energieke manier van zijn. Johannes, jouw oog voor detail maakt van jou een heel attent persoon. Lins en Jozefien,

[1] Met wat geluk zelfs een fietspad.
[2] Je bent met gebouwen bezig, of je bent het niet.

de drijvende kracht achter vele initiatieven, zoals het kerstfeestje: doe zo verder! Samuel, hartelijk bedankt voor alle logistieke hulp. Aan al de rest die ik hier niet opnoem maar wel apprecieer: het was me een waar genoegen.

Doctoreren, dat gaat met ups en downs. Niets zo leuk als het moment waarop alles netjes in elkaar valt, niets zo frustrerend als wekenlang vastzitten op eenzelfde probleem. Het was dan ook bijzonder fijn dat een aantal van mijn vrienden tezelfdertijd in andere vakgebieden aan het ploeteren waren: Fien de statisticus, Marjorie de sterrenkundige en Silke de farmaceuticus, wat was het tof om al eens samen te klagen, en wat kijk ik er naar uit om jullie binnenkort ook te zien verdedigen. Ook alle andere vrienden, de twee Lisa's, huisgenoot Fien, Lieselotte, Micheline, Evy, ... die deze tijd nog zo veel beter maakten: merci!

Een dankuwel van formaat is voorbehouden aan mijn ouders. Om mij de kans te geven om te studeren, in alle vrijheid en zonder enige druk, om trots te zijn *no matter what*, en bovenal om mij te laten opgroeien in een warm nest. Deze laatste verdienste moeten jullie wel delen met onzen Tom, die altijd zorgt voor de grappige noot en waarop altijd gerekend kan worden[3]. Nauwe aanverwanten Elien en Joris, ook jullie worden enorm op prijs gesteld, net als de rest van de familie.

Last but not least, I want to express my gratitude towards the members of my jury. Thank you, Hans Cuypers, Bernhard Mühlherr, Bart De Bruyn and Tom De Medts, for reading this manuscript and for giving valuable remarks, and Marnix Van Daele for chairing the defenses.

[3]Niet vergeten, $a^2 + b^2 = c^2$.

CONTENTS

CHAPTER

1

INTRODUCTION

I always say that my thesis is about *buildings*, yet you will not find a precise definition of a building in this thesis. Let me explain why we can live without a precise definition, while at the same time placing my research in a general context.

1.1 Context

In the 1950s, Jacques Tits reversed Klein's Erlangen program by associating geometrical objects to semi-simple algebraic groups. For this work, he received the Abel prize in 2008. These geometrical objects, which were viewed as simplicial complexes, satisfy simple and natural incidence conditions. Around 1965, Tits' definition of a building was in terms of simplicial compleces, endowed with subcomplexes (apartments) and satisfying axioms based on the natural incidence conditions ([49]).

The apartments of a building are all isomorphic to a certain Coxeter complex (W,S), which is a "furnished" Cayley graph associated to a Coxeter group W (generated by the set S consisting of order 2 elements subject to relations). The rank of a building is the size of S.

1.1.1 (Exceptional) spherical and affine buildings

A building is called *spherical* if its Coxeter group W is finite. Irreducible spherical buildings of rank 2 coincide with generalized polygons. The thick, irreducible, spherical buildings of rank at least 3 were classified by Tits into three infinite classical classes of (Coxeter) types[1] A_n, B_n (equivalently: C_n) or D_n (the rank n is at least 3) and four exceptional classes of

[1]These types are often displayed by a Dynkin diagram and roughly correspond to the types of the simple algebraic groups.

(Coxeter) types F_4, E_6, E_7 and E_8, only existing for specific ranks. Opposed to the classical ones, the exceptional spherical buildings are definitely not well-understood. Their intricate behaviour has been studied widely.

1.1.2 Parapolar spaces

When given one particular building, the conceptual definition of a building does not always provide a good geometric intuition. The approach to buildings we take here, is via point-line geometries naturally associated to them (often referred to as *Lie incidence geometries*).

The *T-Grassmannian* of a building Δ of type X_n, for a subset T –often a singleton– of the type set of Δ, is the point-line geometry (denoted by $\mathsf{X}_{n,T}$, conform the Bourbaki labeling) whose points are the simplices of types in T and whose lines can be deduced by a well-known procedure[2]. E.g., the 1-Grassmannian of a building of type A_n gives an n-dimensional projective space; the 1-Grassmannian of a building of type $(\mathsf{B}/\mathsf{C})_n$ or D_n gives a polar space of rank n. Both projective spaces and polar spaces can axiomatically be defined in terms of their points and their lines.

The exceptional buildings on the other hand, all have projective and polar spaces as substructures. It was Cooperstein ([12]) who introduced *parapolar spaces*, a type of point-line geometries equipped with polar spaces (called *symplecta*) that occur as the convex closure of certain point pairs at distance 2, as a means to study the exceptional spherical buildings. In general, if T, with $|T| \geq 2$, is an independent vertex set of the Coxeter diagram of a spherical building Δ of type X_n of rank n (with $\mathsf{X} \in \{\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D},\mathsf{E},\mathsf{F}\}$), its T-Grassmannian $\mathsf{X}_{n,T}$ is a parapolar space (if not a projective or polar space).

1.1.3 The Freudenthal-Tits magic square

Many of the geometries that we will encounter in this thesis, are related to the *Freudenthal-Tits magic square* (FTMS). The FTMS originates from independent work of Jacques Tits and Hans Freudenthal.

Tits' geometric FTMS

The FTMS appeared for the first time in Tits' habilitation thesis ([47]). There, Tits defined 12 real classical geometries (the adjectives "real" and "classical" refer to the fact that these geometries are all defined as varieties in *real* projective spaces; yet one of these geometries is isomorphic to the *complex* projective plane). Based on their numeric properties (like dimension etc.), he arranged these geometries in a 3×4 array.

He also defined the "complexification" of these 12 geometries, yielding a second 3×4 table of geometries. Among these geometries are the ones he calls "R-espaces" arising from the complex semi-simple algebraic groups of types E_6 and E_7. Tits predicted that there was still one row missing, and for these missing geometries he alluded to those that Freudenthal would later on call "metasymplectic spaces" (see below), and whose complexifications are "R-espaces" related to complex semi-simple algebraic groups of types $\mathsf{E}_6, \mathsf{E}_7$ and E_8.

[2]For a line, one takes any type $t \in T$ and any flag F of cotype t; then the elements of the corresponding line are the simplices of type T that are incident with F.

Freudenthals algebraic FTMS

In a series of eleven papers ([24]), Freudenthal studied geometries related to the exceptional groups of types E_7 and E_8 and introduced the above mentioned *metasymplectic spaces*. His approach is algebraic, mainly using Lie algebras. He as well obtains a 4×4 table of representations of (forms of) Lie algebras exhibiting remarkable properties. One of the latter is that, although the table is defined row-by-row, the (absolute) types of the Lie algebras are symmetric with respect to the main diagonal.

Afterwards, Tits ([48]) gave a unified construction of the Lie algebras in this table using a pair $(\mathbb{A}_1, \mathbb{A}_2)$ of real quadratic alternative algebras. As there are four types of such algebras (the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$), this gives rise to 16 Lie algebras. In this construction, $\mathbb{A}_1$ and $\mathbb{A}_2$ do not play the same role: the Lie algebra in the entry of the square corresponding to the pair $(\mathbb{A}_1, \mathbb{A}_2)$ is determined by $\mathbb{A}_1$ together with the Jordan algebra of 3×3 Hermitian matrices over $\mathbb{A}_2$. By now, there are also constructions of this square which are symmetric, yet the surprising fact remains that this non-symmetric procedure yields a symmetric square.

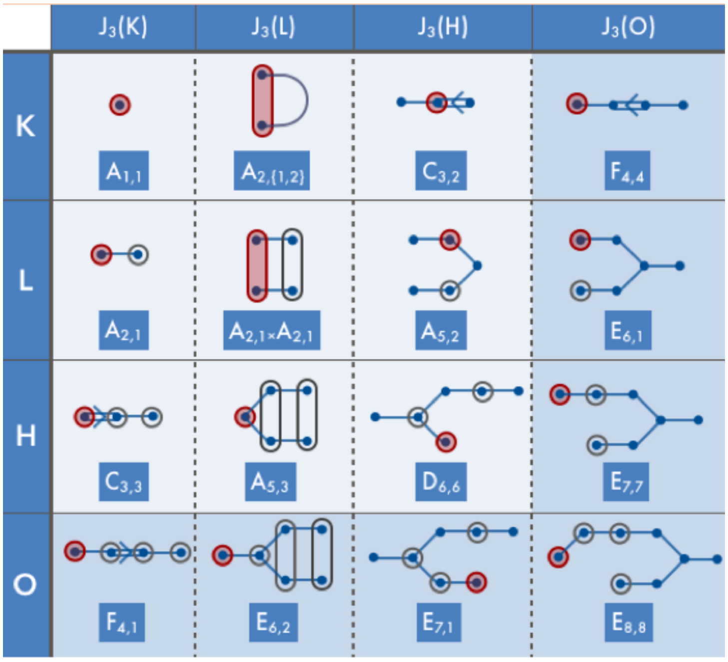


Figure 1.1: The Freudenthal-Tits magic square
defined over an arbitrary field $\mathbb{K}$

The modern FTMS

Nowadays one considers this FTMS over arbitrary fields rather than over the reals. Note however that not each field allows for a full non-split version (as, for instance, there are no division quaternions over finite fields). Tits' original, non-complexified version of geometries is now known as the **non-split** (geometric) version of the FTMS, whereas the table containing their complexifications is known as the **split** (geometric) version of the FTMS.

In the modern terminology, the geometries of the FTMS can all be seen as Grassmannians of certain spherical buildings. Figure 1.1 depicts this table, where the Lie algebras/geometries are represented by their absolute Dynkin types and are accompanied by the corresponding Grassmannians. Viewed as Tits diagrams, this is the non-split version of the square; to see the split version one takes the encircled nodes as the elements of the geometry – the red ones represent the points.

Remark 1.1.1. The non-split version of the FTMS can be obtained by letting $\mathbb{A}_2$ in the pair $(\mathbb{A}_1, \mathbb{A}_2)$ of the above construction being a split quadratic alternative algebra, whereas $\mathbb{A}_2$ is non-split, i.e., division; the split version can be obtained by letting both $\mathbb{A}_1$ and $\mathbb{A}_2$ be split quadratic alternative algebras. This explains why for each field, there is a full split version of the square. This also explains that for the first column of the square, the non-split and split version coincide. For more information, see for instance [2].

In the **non-split version** of the FTMS, it is very clear what connects the geometries that are on the same row:

- row 1 contains Moufang sets (rank 1 geometries);
- row 2 contains Moufang projective planes (rank 2 geometries);
- row 3 contains dual polar spaces of rank 3 (rank 3, obviously);
- row 4 contains metasymplectic spaces, i.e., parapolar spaces of type $\mathsf{F}_{4,1}$ (rank 4).

Moreover, the algebras over which these non-split geometries can be coordinatised are precisely the quadratic alternative algebras at the top of the column they are in; which hence increase in complexity going from left to right.

The **split versions** of these geometries then gives Grassmannians of spherical buildings of various ranks, and hence most of them are parapolar spaces (projective spaces and polar spaces are not considered to be parapolar spaces here).

One of the striking properties of the FTMS is that it contains all exceptional spherical buildings of rank at least 3, i.e., F_4, E_6, E_7, E_8. This makes the FTMS a highly interesting object to study, especially since it actually builds up to these exceptional buildings via classical geometries, along two directions: for each geometry in the square, the geometries above it and on its left, if any, are contained in it in some sense.

The second row of the FTMS

As the rows of the FTMS contain geometries with similar features, it makes sense to aim at a uniform axiomatic description of (the projective representations of) the geometries of each row. This has already been done for the second row, which is the easiest one to start with, containing "just" projective planes. In [39] and [29], all geometries and their projective representations of the second row of the FTMS (both the non-split and the split version) have been characterised geometrically by means of three simple axioms.

In the non-split case, the considered projective representations are the Veronese representations of the projective planes. In fact, also in the split case, the projective representation of these geometries can be seen as the Veronese representation of certain ring projective planes. Indeed, for each of these parapolar spaces, one can consider the point-line geometry where the points and lines are the points and symplecta of this parapolar space. For instance, for the fourth entry of the split second row, we get the parapolar space $\mathsf{E}_{6,1}$, and the abstract point-line geometry consisting of its points and its symplecta (which are polar spaces of type D_5) is the ring projective plane over the *split* octonions. An algebraic description of this ring projective plane is not easy (in his book "projective remoteness planes" [21], Faulkner gives such a description). Hence it is not trivial either to define its corresponding Veronese representation, but it can be done.

Although the starting point is an infinite family of geometries which also consist of points in a projective space equipped with a collection of quadrics (the Veronese representation turns lines into quadrics), no other geometries than those in the second row of the FTMS satisfy these three axioms. This characterisation puts forward the special properties shared by the geometries in this second row and the corresponding algebras.

1.2 This thesis

This thesis is divided into two parts, which can both be linked to the geometries of the FTMS. For now, I will just give the rough ideas and the motivation for the work conducted. Later on, at the start of each part, I will give a more precise introduction and I will indicate how exactly I contributed to this.

1.2.1 Part 1: A characterisation of the dualised version of the second row of the Magic Square

This part of the thesis is the core of my Ph.D. in the sense that it evolves around the project that was set up four years ago. It lead to two papers: [16] and [19].

In [55], Westbury suggests to extend the FTMS (which he considers as a square of complex semisimple Lie algebras) by adding a row/column between the third and the fourth one, related to a 6-dimensional subalgebra of the split octonions which contains the split quaternions. Around the same time, also Landsberg and Manivel consider this intermediate Lie algebra between $\mathfrak{e}_7$ and $\mathfrak{e}_8$. These sextonions are in fact *degenerate* quadratic alternative algebras, and in this part of the thesis we will consider a whole class of such degenerate quadratic alternative algebras which, in our interpretation, gives **a new dimension to the FTMS**: for the entire second row, we will consider a degenerate counterpart (both in the non-split and the split case) and characterise the corresponding Veronese varieties. It is in the split case that this new version of the second row also contains additional entries, corresponding to the sextonions and the ternions (a 3-dimensional subalgebra of the split quaternions).

The specific class of degenerate quadratic alternative algebras that we will consider, we named "*generalised dual numbers*", for the reason that they behave similar to the dual numbers $\mathsf{DN}(\mathbb{K}) := \mathbb{K} \oplus t\mathbb{K}$ (where $\mathbb{K}$ is a field and t an element with $t^2 = 0$); we will consider algebras which are set-wise given as $\mathbb{B} \oplus t\mathbb{B}$, where $\mathbb{B}$ is a non-degenerate quadratic alternative $\mathbb{K}$-algebra.

These generalised dual numbers "$\mathbb{B} \oplus t\mathbb{B}$" are introduced in **Chapter 4**; Veronese varieties associated to them are introduced in **Chapter 5**. These geometries are then the ones that belong to the "dualised" second row of the FTMS. Not entirely surprisingly, the Veronese representation is also composed of an "ordinary Veronese representation", namely the one that is associated to $\mathbb{B}$, and a singular part. What is more surprising, is the fact that there exists a duality between these two parts, which justifies the name "dualised version".

In **Chapters 6 and 7**, we then give an axiomatic characterisation of these Veronese representations both in the non-split and in the split case, respectively. I consider this characterisation to be the main achievement of this thesis, together with determining for which degenerate quadratic alternative algebras it makes sense to actually do this (in advance, this was not clear at all).

Apart from the fact that it feels rewarding to see that these characterisations work out neatly, these Veronese varieties might give further insight in the affine buildings associated to the second row of FTMS, in which they occur as certain spheres of radius 2. We will not explain this link further.

1.2.2 Part 2: Lacunary parapolar spaces

This part of the thesis is based on the paper "On exceptional Lie geometries" [18] that I obtained joint with J. Schillewaert, H. Van Maldeghem and M. Victoor.

In contrast to projective spaces and polar spaces of rank at least 3, there is no general classification result for parapolar spaces. Their general definition allows a priori for more examples than those related to (spherical) buildings. A great deal of work (e.g.: [10, 6, 43, 20]) has been done to find good additional properties with which one could reach a partial classification (preferably containing most exceptional geometries).

It turns out that many of the interesting exceptional Lie incidence geometries (some Grassmannians are more interesting than others, often those with the smallest diameter, etc.) have certain gaps in the spectrum of the dimensions of the singular subspaces that occur as intersections of two symplecta. Such parapolar spaces we will call *lacunary*; more precisely *k-lacunary*, if k is a dimension which cannot occur as the dimension of an intersection of two symplecta.

In [18], we classified k-lacunary parapolar spaces with minor additional assumptions: the symplecta should have rank at least $k+1$, and if there are symplecta of rank 2, then we assume that the parapolar space is strong. The resulting list of lacunary parapolar spaces extends several previously obtained results. Moreover, for each exceptional spherical building, T-Grassmannians appeared.

It is striking that the k-lacunary parapolar spaces have a rather big overlap with the FTMS: each of the split Lie incidence geometries that occur in the bottom right 3×3 square of the FTMS is a lacunary parapolar space. Moreover, the split Lie incidence geometries of the second row of the FTMS are all (-1)-lacunary parapolar spaces. Therefore I decided to incorporate the classification of the (-1)-lacunary parapolar spaces in this thesis.

As an additional effort, I present a proof of this classification which does not rely on other classification results on parapolar spaces, thus showing that it is independent of other results (in the paper, the use of other classification results allows for a short-cut at some point). As the classification of the k-lacunary parapolar spaces has an inductive nature, I also provide a classification of the k-lacunary parapolar spaces containing a (-1)-lacunary parapolar space as a residue (these are precisely the k-lacunary parapolar spaces of symplectic rank at least $k+3$).

CHAPTER

2

PRELIMINARIES

In this chapter, the background needed to read this thesis is provided. Apart from definitions and notational matters, some well-known properties which will be used many times are stated and proven in an elementary way. The reader is assumed to be familiar with basic concepts in geometry and linear algebra.

For the majority of **Part I**, it suffices to know projective spaces and some substructures (see **Sections 2.1.1 and 2.1.2**).

For **Part II**, **Section 2.1.3** on parapolar spaces contains the necessary background. From the definition, which does require familiarity with the definition of polar spaces, basic properties are deduced in elementary ways, for on the one hand this gives a feeling for this set of axioms, and on the other hand, these properties will be used frequently.

To fully appreciate the main results however, one should know something about spherical buildings (of exceptional type). As already explained in the introduction, I will restrict myself to giving a description of the Lie incidence geometries occurring in this thesis (see **Section 2.2**).

2.1 Point-line geometries

The following definitions are standard, see for instance [4, 6].

Although an incidence geometry can contain more than two types of objects, the geometry induced by its “points" and “lines" (whatever objects these may be) are often interesting to work with.

A **point-line geometry** Γ is a triple $(X, \mathscr{L}, *)$, where X is the set of points, $\mathscr{L}$ the set of lines, and $*$ a symmetric incidence relation.

Definition 2.1.1. Two point-line geometries $\Gamma = (X, \mathscr{L}, *)$ and $\Gamma' = (X', \mathscr{L}', *')$ are *isomorphic* if there is a bijection $\alpha : X \cup \mathscr{L} \to X' \cup \mathscr{L}'$ preserving the types (i.e., $\alpha(X) = X'$ and $\alpha(\mathscr{L}) = \mathscr{L}'$) and the incidence (i.e., if $x * L$ for $x \in X$ and $L \in \mathscr{L}$, then $\alpha(x) *' \alpha(L)$ and vice versa). If $\Gamma = \Gamma'$ then α is called an *automorphism*, if Γ' is the dual of Γ, i.e., if $\Gamma' = (\mathscr{L}, X, *)$, then α is called a *duality* and Γ is called *self-dual*.

We will not consider geometries with repeated lines, so henceforth we view $\mathscr{L}$ as a subset of the power set of X, and $*$, which is then just inclusion made symmetric, is not mentioned explicitly. If two points a and b are incident with a common line L (i.e., $a * L * b$), then we say that a and b are *collinear*, denoted $a \perp b$.

Definition 2.1.2. A *subspace* of Γ is a subset S of the point set such that, if two points a, b belong to S, then all lines containing both a and b are contained in S. A *singular* subspace is a subspace every two points of which are collinear.

Note that the empty set and a single point are legible singular subspaces. Two singular subspaces S_1 and S_2 are called collinear (also denoted $S_1 \perp S_2$) if each point of S_1 is collinear to each point of S_2. The set of points collinear to (each point of) a singular subspace S (possibly just a point) is denoted by $S^\perp$.

Definition 2.1.3. The subspace generated by a set S of subspaces, is the smallest subspace containing all members of S, and is denoted by $\langle S \rangle$.

If p and q are two collinear points, and if the line through them is unique, then $\langle p, q \rangle$ is denoted by pq.

There is a special kind of subspace worth mentioning separately:

Definition 2.1.4. A subspace H of Γ is called a *geometric hyperplane* if each line of Γ has either one or all its points contained in H.

Two (distinct) collinear points are at distance 1 from each other. In general, the distance between points is measured in the collinearity graph:

Definition 2.1.5. The *collinearity graph* is the graph on X with collinearity as adjacency relation. The *incidence graph* is the bipartite graph on $X \cup \mathscr{L}$ with incidence as adjacency relation.

Definition 2.1.6. The *distance* $\delta(p, q)$ between two points $p, q \in X$ is the distance between p and q in the collinearity graph.

If $\delta(p, q) = \infty$, then there is no path between p and q. If $\delta := \delta(p, q)$ is finite, then a *shortest path* between p and q is a path in the collinearity graph of length δ. The distance between a point p and a subspace S is given by the minimum distance, i.e., $\delta(p, S) = \min\{\delta(p, q) \mid q \in S\}$.

Definition 2.1.7. The *diameter* $\mathrm{Diam}\,\Gamma$ of Γ is the diameter of the collinearity graph. If $\mathrm{Diam}\,\Gamma < \infty$, then Γ is called *connected*.

Definition 2.1.8. A subspace S of Γ is called *convex* if, for any pair of points $\{p,q\} \subseteq S$, every point incident with a line occurring in a shortest path between p and q is contained in S.

We now give an overview of the most important classes of point-line geometries occurring in this thesis.

2.1.1 Projective spaces

The axioms of Veblen and Young ([52]) of projective spaces in terms of their points and lines go as follows.

A point-line geometry $\mathbb{P} = (X, \mathscr{L})$ is called a **projective space** if the following axioms holds.

(P1) Every line contains at least three points.

(P2) Through each pair of distinct points there is precisely one line.

(P3) The axiom of Pasch holds: if a, b, c, d are points, no three of which are on a line, then if the lines ab and cd have a point in common, then so do the lines ac and bd.

Note that each subspace itself is also a projective space, in particular, each subspace is singular. Axiom (P3) expresses the fact, if there are two intersecting lines, then the subspace generated by them is a *projective plane*: a point-line geometry satisfying (P1), (P2) in which not all points are on a line and in which each two lines have a point in common (which is unique by (P2)). The dimension of a projective plane is 2, that of a projective line (which is a set of at least 3 points) is 1, that of a point is 0 and that of the empty set is -1. In general, the dimension can be defined inductively:

Definition 2.1.9. The empty subspace $\emptyset$ has dimension -1. A non-empty subspace S of $\mathbb{P}$ has dimension $j \in \mathbb{N} \cup \{\infty\}$ if the dimension of a subspace $S' \subseteq S$ with the property that S has no subspace S'' with $S' \subsetneq S'' \subsetneq S$ is $j-1$.

We will not make a distinction between the kinds of infinity. Before continuing, we first give a standard example:

Example 2.1.10. Let V be a right vector space over a skew field $\mathbb{L}$, with $\dim(V) \geq 3$. Let X denote the set of all vector lines of V and $\mathscr{L}$ the set of all vector planes of V. Then $\mathbb{P} := (X, \mathscr{L})$, with containment made symmetric as incidence, is a projective space coordinatised over $\mathbb{L}$, which we denote by $\mathsf{PG}(V)$. If $\dim(V) = n+1 < \infty$, then $\mathbb{P}$ has dimension n and is also denoted by $\mathsf{PG}(n, \mathbb{L})$.

Definition 2.1.11. We say that two subspaces S and S' of $\mathbb{P}$ are complementary if $S \cap S' = \emptyset$ and $\langle S, S' \rangle = \mathbb{P}$.

If $\dim(\mathbb{P}) = n < \infty$, then $\dim(S) + \dim(S') + 1 = n$ for complementary subspaces S and S' of $\mathbb{P}$.

Definition 2.1.12. If S, U are subspaces of $\mathbb{P}$ with $S \subseteq U$, then the *codimension* of S w.r.t. U, denoted $\mathrm{codim}_U(S)$, is defined as $\dim(S') + 1$, where S' is any subspace of U such that S and V are complementary subspaces of U. If $U = \mathbb{P}$, then we simply speak of the codimension of S. A subspace of codimension 1 is called a *hyperplane*.

For any finite $j \geq 0$, we denote by $\mathscr{S}_j$ the set of subspaces of $\mathbb{P}$ of dimension j, and by $\mathscr{S}_j^d$ the set of subspaces of $\mathbb{P}$ of codimension j (so $\mathscr{S}_1^d$ is the set of hyperplanes of $\mathbb{P}$).

Definition 2.1.13. The *dual* of a projective space $\mathbb{P}$ is defined as $\mathbb{P}^d := (\mathscr{S}_1^d, \mathscr{S}_2^d)$, with natural incidence.

As can easily be verified, $\mathbb{P}^d$ is also a projective space. We say that $\mathbb{P}$ is *self-dual* if $\mathbb{P}$ and $\mathbb{P}^d$ are isomorphic point-line geometries. Other projective spaces that can be produced from $\mathbb{P}$ are their residues:

Definition 2.1.14. For any $k \in \{0, \ldots, n-2\}$, take an arbitrary $K \in \mathscr{S}_k$. We define the *K-residue* $\mathrm{Res}_{\mathbb{P}}(K)$ as the point-line geometry $(X_K, \mathscr{L}_K)$ where $X_K = \{S \in \mathscr{S}_{k+1} \mid K \subseteq S\}$, and $\mathscr{L}_K = \{\{L \in \mathscr{S}_{k+1} \mid K \subseteq L \subseteq M\} \mid K \subseteq M \in \mathscr{S}_{k+2}\}$.

It is easily verified that the above defined residue, for $k \in \{0, \ldots, n-2\}$, is a projective space of dimension $n-k-1$.

We shall only give a brief overview of some of the notions that play an important role at some point during this thesis. These notions are all assumed to be well-known and their proofs can be found in any standard work on projective spaces (for which we recommend, for instance the book of Cameron [9, 5]).

The Desargues property

A triangle Δxyz of a projective space $\mathbb{P}$ of dimension at least 2 consists of three points x, y, z which do not lie on one line. We say that two triangles Δxyz and $\Delta x'y'z'$ are in *central perspective* if the lines xx', yy' and zz' all contain one point c (called the *center*). We say that two triangles Δxyz and $\Delta x'y'z'$ are in *axial perspective* if the points $xy \cap x'y'$, $yz \cap y'z'$ and $zx \cap z'x'$ all lie on some line A (called the *axis*).

Definition 2.1.15. A projective space is called *Desarguesian* if two triangles Δxyz and $\Delta x'y'z'$ are in central perspective if and only if they are in axial perspective.

A projective space of dimension n can be coordinatised over a skew field (i.e., it is isomorphic to $\mathsf{PG}(n, \mathbb{L})$ for some skew field $\mathbb{L}$) if and only if it is Desarguesian which is always the case if $n \geq 3$ ([52]). An example of a non-Desarguesian projective plane is one which is coordinatised over an octonion division ring.

(Linear) automorphisms and dualities

Let $\mathbb{P}$ and $\mathbb{P}'$ be two projective spaces. A map $\alpha : \mathbb{P} \to \mathbb{P}'$ from $\mathbb{P}$ to $\mathbb{P}'$ is called a *collineation* if it induces an automorphism of the corresponding point-line geometries; a collineation $\alpha : \mathbb{P} \to \mathbb{P}'^d$ from $\mathbb{P}$ to the dual of $\mathbb{P}'$ is called a *duality* from $\mathbb{P}$ to $\mathbb{P}'$. Dualities are also called correlations. Note that, if $\mathbb{P}'$ is self-dual, then a duality from $\mathbb{P}$ to $\mathbb{P}'$ is just a collineation from $\mathbb{P}$ to $\mathbb{P}'$ composed with any duality between $\mathbb{P}'$ and $\mathbb{P}'^d$.

For our needs it suffices to know "linear dualities" between subspaces of $\mathsf{PG}(n,\mathbb{K})$ where $n \geq 2$ and where $\mathbb{K}$ is a (commutative) field. The *Fundamental theorem of projective geometry* states that each collineation/duality of $\mathsf{PG}(n,\mathbb{K})$ is induced by an invertible $(n+1)\times(n+1)$-matrix over $\mathbb{K}$ together with a field automorphism σ of $\mathbb{K}$. The collineations/dualities of $\mathsf{PG}(n,\mathbb{K})$ corresponding to the members of $\mathsf{PGL}(n+1,\mathbb{K})$ (i.e., the corresponding field automorphism is trivial), are called *linear*.

The cross-ratio

The linear collineations/correlations are precisely the collineations/correlations that preserve the *cross-ratio*. We will recall this definition:

Definition 2.1.16. Let $P_1,\ldots,P_4$ be four pairwise distinct points on a line L of a projective space $\mathsf{PG}(n,\mathbb{K})$, where $\mathbb{K}$ is a field and $n \geq 1$, with coordinates $P_i(1,x_i)$, $i = 1,2,3,4$, where we put $x_i = \infty$ if P_i has coordinates $(0,1)$. Then the *cross-ratio* $(P_0,P_1;P_2,P_3)$ is defined as

$$\frac{(x_3-x_1)(x_4-x_2)}{(x_3-x_2)(x_4-x_1)}.$$

If $\infty \in \{x_1,\ldots,x_4\}$, then we delete the two factors in which ∞ turns up.

Perspectivities and projectivities

Let S_1 and S_2 be two k-dimensional projective spaces contained in a projective space $\mathsf{PG}(n,\mathbb{K})$ (where $\mathbb{K}$ is still a field). Let c be a point of $\mathsf{PG}(n,\mathbb{K})$ outside $S_1 \cup S_2$.

Definition 2.1.17. A *perspectivity (with center c)* between S_1 and S_2 is an isomorphism φ between S_1 and S_2 such that φ is the identity on $S_1 \cap S_2$ and such that all lines $s_1\varphi(s_1)$ for $s_1 \in S_1 \setminus S_2$ contain c. A *projectivity* between S_1 and S_2 is the composition of a finite number of perspectivities (not necessarily with the same centers).

Each perspectivity can be extended to a linear automorphism of $\mathsf{PG}(n,\mathbb{K})$. Therefore perspectivities, and hence also projectivities, preserve the cross-ratio. Moreover, each isomorphism between S_1 and S_2 which preserves the cross-ratio, is a projectivity; if additionally $S_1 \cap S_2$ is fixed point-wise, it is a perspectivity.

Let U and F be complementary subspaces of $\mathsf{PG}(n,\mathbb{K})$ and let X be any collection of points of $\mathsf{PG}(n,\mathbb{K})$.

Definition 2.1.18. *The* projection ρ of X from U onto F *is defined as a map which takes* $x \in X \setminus U$ *to the unique point in* $\langle U,x\rangle \cap F$.

Note that ρ is the identity on $X \cap F$. Each projection taking S_1 to S_2 is a projectivity (as the name suggests). In particular, projections mapping S_1 to S_2 preserve the cross-ratio.

Definition 2.1.19. Two substructures $\mathscr{S}_1$ and $\mathscr{S}_2$ of $\mathsf{PG}(n,\mathbb{K})$ are called *projectively equivalent* if there is a linear collineation of $\mathsf{PG}(n,\mathbb{K})$ mapping $\mathscr{S}_1$ onto $\mathscr{S}_2$. A substructure $\mathscr{S}$ of $\mathsf{PG}(n,\mathbb{K})$ with certain properties is called *projectively unique* if each other substructure of $\mathsf{PG}(n,\mathbb{K})$ with these properties is projectively equivalent to $\mathscr{S}$.

Normal rational curves and scrolls

Now that we know the cross-ratio, we can define normal rational scrolls.

Definition 2.1.20. A *normal rational curve* in a projective space $\mathsf{PG}(m,\mathbb{K})$ over a field $\mathbb{K}$, with $m \in \mathbb{N}_{>0}$, is given by $\{(x_0^m, x_0^{m-1}x_1, \ldots, x_0x_1^{m-1}, x_1^m) \mid (x_0,x_1) \in (\mathbb{K}\times\mathbb{K}) \setminus (0,0)\}$.

Note that, if $m = 1$, then a normal rational curve is just a line; if $m = 2$, then a normal rational curve is a conic. If there is a point-set in $\mathsf{PG}(m,\mathbb{K})$ whose points have, with respect to a certain base B, coordinates of the shape $P(x)_B := (1,x,x^2,\ldots)_B$ (where $x = \infty$ is also included), and w.r.t. another base B' its coordinates have also such a shape, then the map which takes $(1,x)$ to $(1,y)$ if the point with coordinates $P(x)_B$ coincides with the point with coordinates $P(y)_{B'}$ is a linear automorphism of $\mathsf{PG}(1,\mathbb{K})$. This allows us to define the cross-ratio of four-tuples of a normal rational curve as the cross-ratio of the corresponding four points on the corresponding projective line.

Let Π_k and Π_ℓ be complementary subspaces of a projective space $\mathsf{PG}(n,\mathbb{K})$ where $\mathbb{K}$ is a field, of respective dimensions k and ℓ. In Π_k and Π_ℓ, respectively, we consider normal rational curves C_k and C_ℓ, between which φ is an isomorphism preserving the cross-ratio (which exists, as can be seen by noting that each such isomorphism is essentially induced by an element of $\mathsf{PGL}(2,\mathbb{K})$).

Definition 2.1.21. The union of all *transversal lines* $\langle p, \varphi(p)\rangle$ with $p \in C_k$ is called a *normal rational scroll* and is denoted by $\mathfrak{S}_{k,\ell}$.

We are mainly interested in $\mathfrak{S}_{1,2}$, which is called a normal rational *cubic* scroll. Its properties can be found in Section 6.6.5, together with a higher-dimensional version of this, using the notion of a regular spread instead of a line, which we now introduce.

Regular spreads

Definition 2.1.22. An *(r-)spread* $\mathscr{R}$ in $\mathsf{PG}(n,\mathbb{K})$ is a collection of r-dimensional subspaces of $\mathsf{PG}(n,\mathbb{K})$ such that each point of $\mathsf{PG}(n,\mathbb{K})$ is contained in exactly one such r-space.

Let R_1,R_2,R_3 be three pairwise disjoint r-dimensional subspaces in $\mathsf{PG}(2r+1,\mathbb{K})$. Then each point of R_3 is contained in a unique line meeting R_1 and R_2 in a point. A line intersecting each of R_1, R_2, R_3 in a point is called a *transversal*. For each point on such a transversal line, there is a unique r-space meeting all other transversals.

Definition 2.1.23. The collection of all r-spaces meeting each of the transversal lines of R_1,R_2,R_3 is called an (r-)regulus.

Note that the r-spaces in a regulus are pairwise disjoint.

Definition 2.1.24. An (r-)spread $\mathscr{R}$ of $\mathsf{PG}(2r+1,\mathbb{K})$ is called *regular* if the regulus determined by any three of its members belongs to $\mathscr{R}$. An (r-)spread $\mathscr{R}$ of $\mathsf{PG}(n,\mathbb{K})$ with $n \geq 2r+1$ is called *regular* if each two members R_1 and R_2 of $\mathscr{R}$ induce a regular r-spread on $\langle R_1,R_2\rangle$.

The cross-ratio of other objects

We list the other cross-ratios that we will encounter (assuming that we are inside a projective space $\mathsf{PG}(n,\mathbb{K})$ where $\mathbb{K}$ is a field and n is big enough to contain the mentioned objects):

- The cross-ratio of four hyperplanes sharing a subspace of codimension 2 is defined as the cross-ratio of the corresponding points in the dual of $\mathsf{PG}(n,\mathbb{K})$.
- The cross-ratio of four lines L_1,L_2,L_3,L_4 through a point P in a plane π is given by the cross-ratio of the points P_1,P_2,P_3,P_4 that arise by intersecting the lines L_1,L_2,L_3,L_4 with a line M in π not through P. This definition does not depend on the chosen line M: for each other such line M', there is a perspectivity (with center P) that maps M to M' and P_i to P_i', $i \in \{1,\dots,4\}$, and perspectivities preserve the cross-ratio.
- The cross ratio of four members $R_1,\dots,R_4$ of an r-regulus $\mathscr{R}$ is given by the cross-ratio of the points P_1,P_2,P_3,P_4 that arise by intersecting the members R_1,R_2,R_3,R_4 with any transversal line. The fact that this definition does not depend on the chosen line follows as these two lines would be on a 1-regulus (i.e., a grid), and a map taking a line of a 1-regulus to another line of it by taking each point to the unique other point on the same transversal, is a perspectivity.

2.1.2 Polar spaces

Polar spaces have been introduced by Veldkamp [53], later on included in the theory of buildings by Tits [49], and later on the axioms have been simplified by Buekenhout & Shult [7]. It is the latter point of view we take here. Standard references include [9, 15].

A point-line geometry $\Delta = (X,\mathscr{L})$ is called a **polar space** of finite rank if the following axioms holds.

(PS1) Every line contains at least three points.

(PS2) No point is collinear to all other points.

(PS3) Every nested sequence of singular subspaces is finite.

(PS4) For any point x and any line L, either one or all points on L are collinear to x.

Let $\Delta = (X, \mathscr{L})$ be a polar space. We list some basic properties. First of all, through each two points, there is at most one line. Moreover, each singular subspace of Δ is a projective space, and its dimension can hence be defined as its projective dimension. There exists a natural number $r \geq 1$ such that each maximal singular subspace of Δ has dimension $r-1$. We call r the **rank** of Δ. We denote by $\mathscr{S}_j$ the set of singular subspaces of Δ of dimension j, with $j \in \{0, ..., n-1\}$.

Remark 2.1.25. Note that Axiom (PS3) implies that the rank is finite, which is strictly speaking not necessary, yet we will only consider polar spaces of finite rank, whence our preference for the above set of axioms. A polar space of rank 1 consists of a point set and an empty line set.

Axiom (PS4) implies that, if $r \geq 2$, the maximal distance between two points is 2. So two non-collinear points are at furthest distance and hence they are called *opposite*. Note that for a point p and a singular subspace S with $\dim(S) \geq 1$ and $p \notin S$ holds, by (PS4), that $p^{\perp} \cap S$ is a geometric hyperplane of S, i.e., it is either a hyperplane of S or it coincides with S.

Definition 2.1.26. Two singular subspaces S_1 and S_2 are *opposite* if for each point p_1 of S_1 there is a point $p_2 \in S_2$ opposite to it, and vice versa.

Opposite singular subspaces have the same dimension and are necessarily disjoint. For two maximal singular subspaces it suffices to be disjoint in order to be opposite. For each singular subspace S_1, there exists a singular subspace S_2 with S_1 and S_2 opposite.

Definition 2.1.27. For any $k \in \{0, ..., r-3\}$, take an arbitrary singular subspace $K \in \mathscr{S}_k$. We define the *K-residue* $\mathrm{Res}_{\Delta}(K)$ as the point-line geometry $(X_K, \mathscr{L}_K)$ where $X_K = \{S \in \mathscr{S}_{k+1} \mid K \subseteq S\}$, and $\mathscr{L}_K = \{\{L \in \mathscr{S}_{k+1} \mid K \subseteq L \subseteq M\} \mid K \subseteq M \in \mathscr{S}_{k+2}\}$.

It is easily verified that the above defined residue, for $k \in \{0, ..., r-3\}$, is a polar space of rank $r-k-1$. For a pair of opposite singular subspaces S_1 and S_2, the intersection $S_1^{\perp} \cap S_2^{\perp}$ forms a polar space of rank $n - \dim(S_1) - 1$ which is isomorphic to $\mathrm{Res}_{\Delta}(S_i)$, $i = 1, 2$.

If $k = r-2$ in the above definition, i.e., if K is a *submaximal singular subspace*, then $\mathscr{L}_K$ is empty and hence $\mathrm{Res}_{\Delta}(K)$ is a polar space of rank 1. In this case, $|X_K| \geq 2$ and this order does not depend on $K \in \mathscr{S}_{r-2}$.

Definition 2.1.28. We call Δ *hyperbolic* if through each submaximal singular subspace, there are exactly two maximal singular subspaces.

If Δ is not hyperbolic, there are at least 3 maximal singular subspaces through each submaximal one. For this reason, we call a hyperbolic polar space *non-thick* and a non-hyperbolic polar space *thick*.

If Δ is hyperbolic, there are two natural types of maximal singular subspaces–also called *generators* in this context–by stating that two generators sharing a submaximal singular subspace have different types. This is well-defined and implies that two generators M and

M' intersecting each other in a subspace S of odd codimension (i.e., $\dim(M)-\dim(S)$ is odd) have different types. The set of generators belonging to the same type form a *natural system of generators*. The following fact holds true for some other polar spaces as well (the embeddable ones associated to (id, id)-pseudo-quadratic forms), but we will only need it for hyperbolic polar spaces.

Fact 2.1.29. *If M_1 and M_2 are disjoint maximal singular subspaces of a hyperbolic polar space Δ, then the map $M_1 \to M_2 : x \mapsto x^\perp \cap S_2$ is a linear duality.*

We will need the following property concerning hyperbolic polar spaces.

Lemma 2.1.30. *Let Δ be a hyperbolic polar space. Given two generators, we can find a submaximal singular subspace disjoint from both generators.*

Proof. Let U and V be two generators. We proceed by induction. If $n=2$, it is clear that we can find a point disjoint from the lines U and V. For general n, consider non-collinear points p_U and p_V in U and V, respectively. In $p_U^\perp \cap p_V^\perp$, U and V correspond to maximal singular subspaces, so by induction there is a singular subspace Z in $p_U^\perp \cap p_V^\perp$ of dimension $n-3$ disjoint from U and V. As the residue of Z is a hyperbolic quadric, in which U and V correspond to lines, it contains a point disjoint from them, yielding a submaximal singular subspace of Δ disjoint from both U and V. $\square$

We will also need the following property.

Lemma 2.1.31. *Let $\Delta = (X, \mathscr{L})$ be a polar space and let $p \in X$ be arbitrary. Then $p^\perp$ is a geometric hyperplane of Δ and is not properly contained in another geometric hyperplane.*

Proof. That $p^\perp$ is a geometric hyperplane of Δ follows immediately from Axiom (PS4). By (PS2), there is a point q in Δ with $q \notin p^\perp$. We show that $p^\perp$ and q generate Δ. Firstly, $q^\perp$ belongs to $\langle p^\perp, q\rangle$ since for each line through q, p is collinear to one of its points. Likewise, for each point $q' \in q^\perp \setminus p^\perp$, also $q'^\perp$ belongs to $\langle p^\perp, q\rangle$. Clearly, q and q' play the same role. As such, we already obtained all points at distance 2 of p, except possibly those points r which are not collinear to any point of $q^\perp \setminus p^\perp$ (so they are collinear to $p^\perp \cap q^\perp$). Switching the roles of q and q' (noting that $q^\perp \cap p^\perp \neq q'^\perp \cap p^\perp$), we obtain $r \in \langle p^\perp, q\rangle$. The lemma follows. $\square$

If $p,q \in X$ are non-collinear, let $S_{p,q}$ denote the set $p^\perp \cap q^\perp$; if $p,q \in X$ are collinear, we use the notation $S_{p,q}$ for the points on pq.

Lemma 2.1.32. *Let $\Delta = (X, \mathscr{L})$ be a polar space and let $p,q \in X$ be non-collinear. Then the convex closure of p and q consists of all points in $S_{p,q} \cup \{S_{p',q'} \mid p',q' \in p^\perp \cup q^\perp\}$, and coincides with Δ.*

Proof. Clearly, the set $S_{p,q} \cup \{S_{p',q'} \mid p',q' \in S_{p,q} \cup \{p,q\}\}$ is contained in the convex closure of p and q. We now show that each point r of Δ belongs to it. If $r \in p^\perp$ then $r \in S_{p,q'}$ for some $q' \in S_{p,q}$. So suppose $r \notin p^\perp \cup q^\perp$. Consider two non-collinear points $p',q' \in p^\perp \cap q^\perp$. Then either $r \in S_{p',q'}$, or we may assume that r is not collinear to p' and then there are points $p'' \in \langle p,p'\rangle \setminus \{p,p'\}$ and $q'' \in \langle q,p'\rangle \setminus \{q,p'\}$ collinear to r, and hence $r \in S_{p'',q''}$. $\square$

2.1.3 Parapolar spaces

Parapolar spaces are point-line geometries introduced by Cooperstein ([12]) to capture the spherical buildings of exceptional type. As the name suggests, parapolar spaces are composed of polar spaces. The tandard reference is the book of Shult [42].

A point-line geometry $\Omega = (X, \mathscr{L})$ is called a **parapolar space** if the following axioms hold:

(PPS1) Ω is connected and, for each line L and each point $p \notin L$, p is collinear to either none, one or all of the points of L and there exists a pair $(p, L) \in X \times \mathscr{L}$ with $p \notin L$ such that p is collinear to no point of L.

(PPS2) For every pair of non-collinear points p and q in $\mathscr{P}$, one of the following holds:

(a) the convex closure of $\{p, q\}$ is a polar space, called a **symplecton**;

(b) $p^\perp \cap q^\perp$ is a single point;

(c) $p^\perp \cap q^\perp = \emptyset$.

(PPS3) Every line is contained in at least one symplecton.

Let $\Omega = (X, \mathscr{L})$ be a parapolar space. Note that, in contrast to polar spaces, it is now possible that Ω has singular subspaces which are not projective spaces[1]. Since lines of polar spaces have at least three points, (PPS3) implies that every line of a parapolar space is *thick*, i.e., has at least three points.

For two points p, q at distance 2, either possibility (a) or (b) in (PPS2) holds. In the first case we call $\{p, q\}$ a *symplectic pair* and its convex closure, the symplecton they determine, is denoted by $\xi(p, q)$ and briefly called a *symp*; in the second case we call $\{p, q\}$ a *special pair*. The set of symps is usually denoted by Ξ. By (PPS3) and (PPS1), we have $|\Xi| \geq 2$. It is possible that there are no special pairs though:

Definition 2.1.33. Parapolar spaces Ω containing no special pairs are called *strong*.

The rank of a symplecton is the rank of it as a polar space. Elements of Ξ need not be isomorphic, they do not even need to have the same rank.

Definition 2.1.34. A parapolar space Ω has *symplectic rank at least d* if all its symps have rank at least d; if additionally, rank d symps actually occur then Ω has *minimum symplectic rank d*. If all symps have the same rank d, then Ω is said to have *uniform* symplectic rank d. If there is an integer d^* such that all symps have rank at most d^*, we say that Ω has *bounded* symplectic rank.

[1]For instance, take as X the lines of the standard Hermitian quadrangle embedded in 4 dimensions, and as $\mathscr{L}$ its planar point pencils.

Since we assume polar spaces to have finite rank, uniform symplectic rank implies bounded symplectic rank.

Definition 2.1.35. The *singular rank(s)* of a parapolar space Ω is the set of dimensions of maximal singular subspaces which are projective (so that their dimension is well-defined).

We say that Ω has *bounded* singular rank if its singular subspaces are projective and if there is an upper bound on the dimension of these subspaces.

Basic properties

We now list some basic properties, the proofs of which can without doubt be found in the literature, but for completeness' sake, they are provided.

Lemma 2.1.36. *Given a symp ξ of Ω and a point $p \notin \xi$, $p^\perp \cap \xi$ is a singular subspace of ξ.*

Proof. Suppose for a contradiction that p_1 and p_2 are two non-collinear points in ξ collinear to p. Then, by (PPS2), p belongs to $\xi(p_1, p_2) = \xi$, contradicting $p \notin \xi$. □

Lemma 2.1.37. *Let L be a line of Ω contained in a symp ξ of rank at least 3. If p is collinear to L, then there is a symp containing $\langle p, L\rangle$ and hence $\langle p, L\rangle$ is a projective singular plane. Consequently, if the symplectic rank is at least 3, each singular subspace is projective.*

Proof. If $p \in \xi$, the first assertion follows. If not, take a point $q \in \xi$ collinear to all points of L and not contained in $p^\perp \cap \xi$ (which is a singular subspace of ξ by the previous lemma). Then p and q are at distance 2 and $L \subseteq p^\perp \cap q^\perp$. So, by Axiom (PPS2), there is a symp ξ' through p and q, which clearly contains L and p and hence $\langle L, p\rangle$ by convexity. Since ξ' is a polar space, $\langle L, p\rangle$ is a projective plane. By (PPS3), also the second statement of the lemma follows. □

In parapolar spaces of symplectic rank at least 3, each singular subspace now has a well-defined (projective) dimension. In parapolar spaces of minimum symplectic rank 2, we only know this for sure for points (0-dimensional) and lines (1-dimensional).

Sympthick subspaces

Each line is contained in a symp by Axiom (PPS3), and each point is contained in a line by connectivity, and hence it is also contained in some symp. We now show that for parapolar spaces of symplectic rank at least 3, this also holds for higher-dimensional singular subspaces.

Lemma 2.1.38. *Suppose Ω has symplectic rank at least d with $d \geq 3$. Then every singular subspace of dimension at most $d-1$ is contained in some symp.*

Proof. As noted above, this is trivial for points and lines. So let W be a singular subspace of Ω of dimension d^* with $2 \leq d^* \leq d-1$. Let $d' \leq d^*$ be the maximum number such that there exists a symp ξ with $\dim(\xi \cap W) = d'$ (note that $d' \geq 1$ by Axiom (PPS3)). Suppose for a contradiction that $d' < d^*$. Then we can pick a point $p \in W \setminus \xi$ and $q \in \xi \setminus p^\perp$ with q collinear to all points of $W \cap \xi$ (since the rank of ξ is at least $d > d' + 1$). Hence, p and q determine a symp by Axiom (PPS2); but then $\xi(p,q)$ contradicts the maximality of d'. We conclude that W is contained in some symp indeed. □

Being contained in one symp is one thing, being contained in multiple symps is a stronger notion which will be of importance to us, and we already discuss it here as it will also be needed when we get to the point-residue of a parapolar space (which is, as opposed in the case of projective and polar spaces, not automatically a parapolar space).

Definition 2.1.39. We say that a subspace of Ω is *sympthick* if it is contained at least two symps of rank at least 3. A symp containing no sympthick lines is called *isolated*.

For non-isolated symps, we can show that its singular subspaces up to a certain dimension are also sympthick:

Lemma 2.1.40. *Let $\Omega = (X, \mathscr{L})$ be a parapolar space of minimum symplectic rank d and suppose ξ is a non-isolated symp of rank d_1 (note that $d_1 \geq d$). If $d \geq 3$, then*

(i) *Every singular subspace S of ξ of dimension at most $d-2$ is sympthick, and,*

(ii) *if a singular subspace M of ξ of dimension $d-1$ is not sympthick, then each symp $\xi^* \neq \xi$ sharing a $(d-2)$-space with M is non-thick, has rank d and intersects ξ in a $(d-1)$-space (distinct from M).*

Moreover, if $d = 2$, then each line contained in a non-isolated symp ξ is sympthick.

Proof. By assumption, ξ contains a line L which is contained in a second symp of rank at least 3. Supposing that $d \geq 3$, we first show with singular subspaces S containing L of dimension at most $d-2$ are sympthick. Afterwards we show that this is not a restriction, by showing that for each symp of rank at least 3, all its lines are sympthick. So for now, suppose $d \geq 3$ and consider a singular subspace S of dimension at most $d-2$ with $L \subseteq S \subseteq \xi$.

Claim 1: There is a symp $\xi^* \neq \xi$ such that $S \subseteq \xi \cap \xi^*$.
Let U be a subspace of S through L, maximal with respect to the property that there exists a symp ξ^* with $U \subseteq \xi \cap \xi^*$ (this is well defined since at least L satisfies this requirement). Suppose for a contradiction that $U \subsetneq S$, so there is a point $p \in S \setminus U$. Then $p^\perp \cap \xi^*$ and $\xi \cap \xi^*$ are singular subspaces of ξ^*, both containing U. Since ξ^* is a symp of rank at least d and $\dim(U) < \dim(S)$, there is a point $q \in \xi^* \cap U^\perp$ not contained in $p^\perp \cup \xi$. Then q and p are non-collinear and $U \subseteq p^\perp \cap q^\perp$, so there is a symp ξ' through p and q, which is distinct from ξ since $q \notin \xi$. However, the fact that $\xi \cap \xi'$ contains $\langle p, U \rangle$ contradicts the maximality of U. This contradiction shows the claim.

Now suppose that the above found symp ξ^* is either thick, or has rank at least $d+1$ or is such that $\xi \cap \xi^* = S$. Let M be any singular subspace of ξ through S of dimension $d-1$.

Claim 2: Under the above assumptions on ξ^*, there is a symp ξ_M with $M \subseteq \xi \cap \xi_M$.
Take a point $p \in M \setminus S$. We may assume that $M \nsubseteq \xi \cap \xi^*$. If ξ^* contains a subspace M' of

dimension $d-1$ through S which is not contained in the singular subspace $p^\perp \cap \xi^*$ (clearly containing S) nor in the singular subspace $\xi \cap \xi^*$ (also containing S), then similarly as above, the unique symp ξ_M through p and a point $q \in M' \setminus S$ contains M. Now, if ξ^* is thick or if ξ^* has rank at least $d+1$, then it is easily seen that there is such a subspace M'. If ξ^* is non-thick and has rank d, but $\xi \cap \xi^* = S$, then we only need M' to be distinct from the subspace $p^\perp \cap \xi^*$, and since there two $(d-1)$-spaces through S in ξ^*, this is possible. This shows the claim.

If ξ^* does not satisfy those assumptions, then ξ^* is non-thick, has rank d and $S \subsetneq \xi \cap \xi^*$. Since ξ^* has rank d and $\dim(S) = d-2$, the latter implies that $\dim(\xi \cap \xi^*) = d-1$. We now complete the lemma by showing, for general $d \geq 2$, that each line in symp ξ of rank $d_1 \geq 3$ is sympthick.

Claim 3: Each line in ξ is sympthick.
Without loss of generality, we may consider a line K in ξ generating a plane π together with L. If $d_1 > 3$, π is contained in a $(d-2)$-space of ξ, so by Claim 1 we may assume that $d_1 = 3$. Let $\xi^* \neq \xi$ be a symp of rank at least 3 through L. By Claim 2 (even if $d = 2$ this applies since ξ^* has rank at least 3), we may assume that ξ^* is non-thick, has rank 3 and is such that $\xi \cap \xi^*$ is a plane π^* through L distinct from π. Let π' be the unique plane through L in ξ^* distinct from π^*. If $\pi \cup \pi'$ contains a pair of non-collinear points, these determine a symp containing $\pi \cup \pi'$, proving that K is sympthick. So suppose π and π' are collinear. Let q be a point of $\pi' \setminus L$ and note that $q^\perp \cap \xi = \pi$ since $d_1 = 3$. Hence a point $p \in \xi \cap K^\perp \setminus \pi$ is not collinear to q. Since $K \subseteq p^\perp \cap q^\perp$, there is a symp through p and q containing K, proving that K is sympthick, as required. $\square$

Locally connectivity and residues

We are ready to introduce residues. For a parapolar space Ω, its point-residue at a point p will only be a parapolar space if Ω is *locally connected at p*. We introduce this notion and define residues below.

Definition 2.1.41. We call a parapolar space $\Omega = (X, \mathscr{L})$ *locally connected at p*, for some $p \in X$, if each two lines through p are contained in a finite sequence of singular planes consecutively intersecting in lines through p. We say that Ω is *locally connected* if it is locally connected at p for all $p \in X$.

Definition 2.1.42. Let p be a point of a parapolar space $\Omega = (X, \mathscr{L})$. We define the *point-residual at p*, denoted $\Omega_p = (X_p, \mathscr{L}_p)$, as follows:

— X_p is the set of lines through p;
— $\mathscr{L}_p$ is the set of planar line-pencils with vertex p contained in singular planes through p which are contained in a symp of Ω (short: *symp-planes*).

Note that, if Ω_p is connected for a point p, then Ω is locally connected at p (the converse statement is not necessarily true). We list the possibilities for the connected components of Ω_p in the following proposition, which is based on Theorem 13.4.1 of [42] (we added the case where Ω is strong and has symps of rank 2).

Proposition 2.1.43. *Let $\Omega = (X, \mathscr{L})$ be a parapolar space, assumed to be strong if there are symps of rank 2, and let p be any of its points. Let C be a connected component of Ω_p. We have the following possibilities.*

— *C corresponds to a single line through p (such a line is contained in rank 2 symps only);*
— *C corresponds to the lines through a point in a symp ξ of Ω of rank at least 3 (which happens if ξ is isolated);*
— *C is a strong parapolar space. In this case, there is a bijective correspondence between the singular subspaces of Ω through p and the singular subspaces of Ω_p and between the symps of Ω through p of rank at least 3 and the symps of Ω_p (by taking the point-residue at p of each of those singular subspaces and symps, respectively).*

Proof. — Firstly, let $x \in X_p$ be a point through which there is no element of $\mathscr{L}_p$. Then the line L in Ω corresponding to x is not contained in a symp ξ of rank at least 3, for otherwise a plane through L in ξ would give us an element of $\mathscr{L}_p$ incident with x. Conversely, if L is a line of Ω through which there are only symps of rank 2, obviously there cannot be a singular plane through L contained in a symp of rank at least 3, so L corresponds to an isolated point of Ω_p.

— Next, suppose that the elements of C correspond to the set of lines through p in some symp ξ. If there would be a symp of rank at least 3 through ξ which intersects ξ in a line, then there is also a symp ξ' of rank at least 3 intersecting ξ in a line through p (cf. Lemma 2.1.40); in which case C also contains the lines through p in ξ', a contradiction. Conversely, suppose ξ is a symp such that no other symp intersects ξ in more than a point. If there would be an element of C corresponding to a line L not in ξ, then, by connectedness, C also contains a line L' through p such that, for some line L'' in ξ through p, the lines L' and L'' are contained in a singular plane π which is contained in some symp ξ'. But then ξ' is a symp of rank at least 3 intersecting ξ in a line, a contradiction.

— Thirdly, assume that the lines corresponding to the elements of C are contained in at least two symps of Ω. We verify the axioms of a parapolar space.

(PPS1) By assumption, C is connected. We consider an arbitrary non-incident point-line pair of C, i.e., a line L through p and a symp-plane π through p. Suppose that L is contained in a symp-plane with at least two lines L_1 and L_2 in π through p. If M is a line of π not through p, and p' a point on $L \setminus \{p\}$, then (PPS1) in Ω implies that p', being collinear to the distinct points $L_1 \cap M$ and $L_2 \cap M$ of M, is collinear to M. Again by (PPS1) in Ω, it then follows that L is contained in a plane with each line L' in π through p, and by Lemma 2.1.37, the planes $\langle L, L' \rangle$ are symp-planes.

We now show that there is such a pair (L, π) for which L is not contained in a symp-plane with any line of π through p. Since the elements of C do not correspond to the lines through a point of one symp, there is a pair (L, π) such that, for some symp ξ through π, the line L is not contained in ξ. Then, if L is collinear to a unique line L_1 through p of π, we let π' be a plane in ξ which intersects π in a line through p distint from L_1 and with π and π' not contained in a singular 3-space; if L is collinear to π, we let π' be a plane in ξ through p such that π and π' correspond to opposite lines in the polar space $\mathrm{Res}_p(\xi)$. In both cases, L is not collinear to any line of π' through p, for otherwise $L \subseteq \xi$, a contradiction.

(PPS2) Let L_1 and L_2 be two lines of Ω through p not contained in a symp-plane through p. Suppose that there is a line L contained in respective symp-planes through p with L_1 and L_2, respectively. Let p_i be a point on $L_i \setminus p$ for $i = 1, 2$. Then p_1 is not collinear to p_2, for otherwise there is a symp-plane through L_1 and L_2 after all (cf. Lemma 2.1.37). Since $L \subseteq p_1^{\perp} \cap p_2^{\perp}$, the pair $\{p_1, p_2\}$ is symplectic. The symp $\xi(p_1, p_2)$ then contains L_1, L_2 and L. Since L was an arbitrary element in $L_1^{\perp} \cap L_2^{\perp}$ (with $\perp$ viewed in Ω_p), we obtain that the convex closure of L_1 and L_2 in C corresponds to the point-residue $\mathrm{Res}_p(\xi(p_1, p_2))$, which is a polar space of rank at least 2. Note this also shows that there are no special pairs.

(PPS3) By definition, every line of Ω_p is a plane contained in at least one symp ξ of Ω, and hence it is contained in the symp $\mathrm{Res}_p(\xi)$.

We have shown that Ω_p is a parapolar space, which is moreover strong. Since the symps of Ω_p are point-residues of the symps of Ω, the minimum symplectic rank of Ω_p is $d-1$. □

In case Ω_p is a parapolar space, we can again consider its point-residual at some point p' of Ω_p, and $(\Omega_p)_{p'}$ is then denoted by $\Omega_{pp'}$. This inductively defines the *K-residual* Ω_K for any singular k-space K of Ω, with $k \in \mathbb{N}$, provided that the subsequent point-residuals yield parapolar spaces, i.e., provided that each intermediate parapolar space is locally connected.

To end this section, we give two consequences of the above.

Corollary 2.1.44. *If Ω is a locally connected parapolar space of symplectic rank at least 3, then each of its lines is sympthick.*

Proof. By Lemma 2.1.43, Ω_p is a strong parapolar space. As noted before, there is at least one symp ξ through each point L of Ω_p. By (PPS1), there is a point outside ξ and by connectedness, this yields two lines π, π' of Ω_p at distance 2 such that $L \in \pi \cup \pi' \nsubseteq \xi$. Since Ω_p is strong, there is a symp ξ' through π and π' which then contains L. Since there is a 1-1-correspondence between symps in Ω_p and Ω, the symps ξ and ξ' in Ω_p correspond to two symps in Ω through the line of Ω corresponding to L. □

Lemma 2.1.45. *Let $\Omega = (X, \mathscr{L})$ be a parapolar space with symplectic rank at least d. Then Ω is strong and there are no symps of rank 2 if and only if* $\mathrm{Diam}\,\Omega_p = 2$ *for all points p.*

Proof. Suppose first that Ω is strong and $d \geq 3$. Let L_1 and L_2 be two lines through any point p which are not contained in a singular plane (so they correspond to non-collinear points of Ω_p. Then strongness yields a symp ξ through L_1 and L_2. By assumption, the rank of ξ is at least 3 and hence there is a line L through p in ξ contained in symp planes with both L_1 and L_2, respectively. This shows that $\mathrm{Diam}\,\Omega_p = 2$ for any point p.

Conversely, suppose $\mathrm{Diam}\,\Omega_p = 2$ for all points p. Let L_1 and L_2 be two lines through any point p and suppose that they are not collinear in Ω_p. Since the diameter of Ω_p is 2, there is a line L in Ω such that, for $i = 1,2$, $\langle L, L_i \rangle$ is a singular plane contained in a symp ξ_i. If p_i is a point in $L_i \setminus p$, then p_1 and p_2 are not collinear (otherwise Lemma 2.1.37 implies that L_1 and L_2 are at distance 1) and hence, since $L \subseteq p_1^\perp \cap p_2^\perp$, there is a symp of rank at least 3 in Ω containing L_1 and L_2. Since p was arbitrary, Ω contains no special pairs indeed and there are no symps of rank 2 (all symps are of the form $\xi(L_1, L_2)$). □

2.2 Description of the parapolar spaces

We now give a description of the parapolar spaces occurring in this thesis. For more information on this, we again refer to the book of Shult [42].

2.2.1 Grassmannians of projective spaces

Let $\mathsf{PG}(n,\mathbb{L})$ be a projective space of dimension n (possibly infinite) over a skew field $\mathbb{L}$. Let ℓ be a natural number with $1 \le \ell \le (n+1)/2$. As before, we denote the set of $(\ell-1)$-dimensional subspaces of $\mathsf{PG}(n,\mathbb{K})$ by $\mathscr{S}_{\ell-1}$. Let $\mathscr{L}_{\ell-1}$ be the family of $(\ell-1)$-pencils, i.e., the family of subsets $L(W,U)$ of $\mathscr{S}_{\ell-1}$ consisting of all members of $\mathscr{S}_{\ell-1}$ containing a given $(\ell-2)$-space W and being contained in a given ℓ-space U, with $W \subseteq U$. The resulting point-line geometry $(\mathscr{S}_{\ell-1}, \mathscr{L}_{\ell-1})$ is called the **ℓ-Grassmann geometry** and it is denoted by $\mathsf{A}_{n,\ell}(\mathbb{L})$ (Bourbaki labeling).

Remark 2.2.1. The definition makes perfect sense for ℓ with $(n+1)/2 <\le \ell \le n$, but as $\mathsf{A}_{n,\ell}(\mathbb{L}) \cong \mathsf{A}_{n,n-\ell}(\mathbb{L}^d)$, there is no need to consider these Grassmannians.

In the below table, we list the nature of the point-line geometry $\mathsf{A}_{n,\ell}(\mathbb{L})$, depending on the parameters ℓ and n. If n is infinite, then $n-\ell+1$ should be read as: "there are infinite-dimensional subspaces".

Ω	type
$\mathsf{A}_{1,1}(\mathbb{L})$	the projective line $\mathsf{PG}(1,\mathbb{L})$
$\mathsf{A}_{n,1}(\mathbb{L}), n \ge 2$	point-line geometry of $\mathsf{PG}(n,\mathbb{L})$
$\mathsf{A}_{2,2}(\mathbb{L})$	dual of the point-line geometry of $\mathsf{PG}(2,\mathbb{L})$
$\mathsf{A}_{3,2}(\mathbb{L})$	hyperbolic polar space of rank 2
$\mathsf{A}_{n,\ell}(\mathbb{L}), n \ge 4, \ell > 1$	parapolar space of symplectic rank 2, singular rank $\{\ell-1, n-\ell+1\}$

Related notation–Projective spaces of dimension at most 2 are not necessarily coordinatised over skew fields. Arbitrary projective lines or planes are denoted by $\mathsf{A}_{1,1}(*)$ and $\mathsf{A}_{2,1}(*)$, respectively.

2.2.2 Grassmannians of thick polar spaces

Let $\Delta = (X, \mathsf{L})$ be a polar space of (finite) rank $n \ge 2$, which is not hyperbolic. Let ℓ be a natural number in $\{1,\dots,n\}$. Just like above, we denote the set of singular $(\ell-1)$-dimensional subspaces of $\mathsf{PG}(n,\mathbb{K})$ by $\mathscr{S}_{\ell-1}$. Let $\mathscr{L}_{\ell-1}$ be the family of $(\ell-1)$-pencils, i.e., the family of subsets $L(W,U)$ of $\mathscr{S}_{\ell-1}$ consisting of all members of $\mathscr{S}_{\ell-1}$ containing a given singular $(\ell-2)$-space W and, if $\ell < n$, being contained in a given singular ℓ-space U, with $W \subseteq U$. The resulting point-line geometry $(\mathscr{S}_{\ell-1}, \mathscr{L}_{\ell-1})$ is called the **polar ℓ-Grassmann geometry**. Since a polar space is usually not uniquely defined by the underlying skew field of the singular projective spaces, we denote this Grassmannian by $\mathsf{B}_{\mathsf{n},\ell}(*)$.

If $\ell = 1$, $\mathsf{B}_{\mathsf{n},\ell}(*)$ is the point-line geometry of Δ; if $\ell > 1$ then $\mathsf{B}_{\mathsf{n},\ell}(*)$ is always a parapolar space, whose symplectic rank (d) and singular rank (s) can be found in the following table.

Ω	n,ℓ	d	s
$\mathsf{B}_{n,\ell}(*)$	$2 \leq \ell \leq n-2$	$\{3, n-\ell+1\}$	$\{\ell-1, n-\ell\}$
$\mathsf{B}_{n,n-1}(*)$	$n > 2$	2	$n-2$
$\mathsf{B}_{n,n}(*)$	$n > 2$	2	1

The Grassmannians $\mathsf{B}_{n,n}(*)$ are also called *dual polar spaces* (of rank n), for obvious reasons. A dual polar space of rank n is a polar space precisely if $n = 2$ (as a polar space of rank 2 is just a generalised quadrangle).

2.2.3 Grassmannians of non-thick polar spaces

Let $\Delta = (X, \mathscr{L})$ be a hyperbolic polar space of rank $n \geq 4$ (if $n = 3$ then Δ is isomorphic to $\mathsf{A}_{3,2}(\mathbb{L})$ for some skew field $\mathbb{L}$, so its Grassmannians are already considered above; if $n = 2$ then there is not much too say). For $n \geq 4$, each hyperbolic polar space is such that its singular subspaces are defined over a field $\mathbb{K}$.

The fact that Δ is thin makes that it is associated to buildings of Coxeter type D_n. Therefore we consider, instead of Δ, its oriflamme-geometry: the geometry consisting of the m-dimensional singular subspaces of Δ with $m \in \{0, \ldots, n-3\}$ and its two natural families of maximal singular subspaces (which we say that have type n and $n-1$, in accordance to the Bourbaki labeling), with natural incidence except for the subspaces of types n and $n-1$, which are incident with each other precisely if their intersection has dimension $n-2$.

For any natural number $\ell \in \{1, \ldots, n-2\}$, the definition of the corresponding **polar Grassmannian** $\mathsf{D}_{n,\ell}(\mathbb{K})$ goes exactly the same as above. If $\ell = n$, then the corresponding polar Grassmannian $\mathsf{D}_{n,n}(\mathbb{K})$ is defined by taking one of the two natural families, say $\mathscr{M}$ of maximal singular subspaces of Δ as its point set, and as its line set the $\mathscr{M}$-pencils, i.e., the family of subsets $L(W)$ of $\mathscr{M}$ consisting of all members of $\mathscr{M}$ containing a given singular $(n-3)$-space W. Clearly, the two families of maximal singular subspaces play the same role. The Grassmannian $\mathsf{D}_{n,n}(\mathbb{K})$ is often referred to as the *half spin geometry* (of rank n).

Now, $\mathsf{D}_{\mathsf{n},1}(*)$ is the point-line geometry of Δ and $\mathsf{D}_{4,4}(*) \cong \mathsf{D}_{4,1}(*)$ by triality. If $\ell > 1$ and $\ell < n$ if $n = 4$, then $\mathsf{B}_{\mathsf{n},\ell}(*)$ is always a parapolar space, whose symplectic rank (d) and singular rank (s) can be found in the following table.

Ω	n,ℓ	d	s
$\mathsf{D}_{n,\ell}(\mathbb{K})$	$2 \leq \ell \leq n-4$	$\{3, n-\ell+1\}$	$\{\ell-1, n-\ell\}$
$\mathsf{D}_{n,n-3}(\mathbb{K})$	$n > 3$	2	$\{2, n-3\}$
$\mathsf{D}_{n,n}(\mathbb{K})$,	$n > 4$	3	$\{3, n-1\}$

2.2.4 Exceptional parapolar spaces of type E_i, $i = 6,7,8$.

Let Δ be a spherical building of type E_i, $i \in \{6,7,8\}$. As it contains a building of type D_4 as a substructure, we know that Δ is defined over some field $\mathbb{K}$, and by [49], it is uniquely determined by $\mathbb{K}$. There are certain choices for ℓ such that the ℓ-Grassmannian of Δ, denoted $\mathsf{E}_{i,\ell}(\mathbb{K})$, has small diameter and constant symplectic rank. These are precisely the ones that we will encounter in this thesis. We list them, together with their diameter, symplectic rank d and set S of dimensions of the maximal singular subspaces.

Ω	$\mathrm{Diam}\,\Omega$	d	S	strong
$\mathsf{E}_{6,1}(\mathbb{K})$	2	5	$\{4,5\}$	✓
$\mathsf{E}_{6,2}(\mathbb{K})$	3	4	$\{4\}$	
$\mathsf{E}_{7,7}(\mathbb{K})$	3	6	$\{5,6\}$	✓
$\mathsf{E}_{7,1}(\mathbb{K})$	3	5	$\{4,6\}$	
$\mathsf{E}_{8,8}(\mathbb{K})$	3	7	$\{6,7\}$	
$\mathsf{E}_{8,1}(\mathbb{K})$	5	5	$\{4,7\}$	

2.2.5 Homomorphic images

If the diameter of the parapolar space in one of the examples above is at least 5, then there exist homomorphic images that are again parapolar spaces. That such things exist is shown for example by the case $\mathsf{A}_{2n-1,n}(\mathbb{L})$, $n \geq 5$, where identifying n-subspaces that correspond to each other under a given polarity of Witt index at most $n-5$, produces a homomorphic image that is again a locally connected parapolar space with the same polar rank and singular rank, and the same diameter. We write a superscript h to indicate a homomorphic image (but it could also be an isomorphic image). In the above examples that will be relevant for us in Part II, homomorphic images are possible in the geometries $\mathsf{A}_{2n-1,n}(\mathbb{L})$, $n \geq 5$, $\mathsf{D}_{n,n}(\mathbb{K})$, $n \geq 9$, and $\mathsf{E}_{8,1}(\mathbb{K})$, as they are the ones whose diameter is at least 5.

Part I

A characterisation of the dualised version of the second row of the Freudenthal-Tits magic square

This part of the thesis is the core of my Ph.D. in the sense that it evolves around the project that was set up four years ago. Very roughly speaking, the idea was to geometrically characterise Veronese representations of ring projective planes over certain degenerate quadratic alternative algebras by means of axioms (preferably as general as possible, so without reference to the type of the underlying algebra), hereby aiming at a specific set of geometries related to the second row of the Freudenthal-Tits magic square. For a start, this required a study of these algebras and a description of the corresponding ring projective planes and their Veronese representations. As the story usually goes, things did not go as smoothly as planned: the algebras that we had in mind did not come with well-behaving Veronese representations at all.

However, the plot turn is not that bad. A better suited class of algebras presented itself as "the right one", bringing along Veronese representations which are associated to certain affine buildings (for the interested reader: these buildings are of absolute types $\tilde{\mathsf{A}}_2$, $\tilde{\mathsf{A}}_5$ and $\tilde{\mathsf{E}}_6$, respectively). I will describe this class of algebras and the corresponding Veronese representations in **Chapters 4 and 5** of this part. The desired class of algebras falls apart into two subclasses, and hence so does the class of corresponding Veronese representations; one class of which behaves as expected (almost surprisingly), the other one – how else could it be – not quite. The two respective geometric characterisations are the subject of **Chapters 6 and 7**, respectively.

Without exaggerating, it took me 6 times as long to find the right setting to study this second class of varieties than it took me to geometrically characterise them. Admittedly, I spent some time on side projects along the way (not all of which are incorporated in this thesis), and somehow I managed to prove and write down the characterisation of the second kind of varieties (this is the content of Chapter 7) in the three months prior to handing in this manuscript. Yet, what strikes me the most is how neat this characterisation turned out to be, in spite of the many times that I was not convinced that it would ever be possible to even come up with the right setting in which it could have the slightest chance to work out. I will later on try to point out some of the difficulties that had to be overcome, and I hope you appreciate the result that, in hindsight, I liked the most.

What's new? Except for the description of the algebras and the Veronese representations which have an overlap with the literature that exists on the non-degenerate case, this part is entirely new. Both geometric characterisations lead to a paper ([19] and [16], respectively), one being submitted and the other one being finished as we speak.

Let me be a bit more precise on this: Large parts of Chapter 4 are based on the chapter on quadratic alternative algebras in the book "A taste of Jordan algebras" by K. McCrimmon ([34]), especially Sections 4.2 and 4.4, and although McCrimmon only considers non-degenerate quadratic alternative algebras, most of the techniques used here can be extended to the degenerate case. In fact, even more useful were Chapters 2.2 and 2.3 of a pre-book ([35]) on alternative algebras, also by McCrimmon (which shows an overlap with his book on Jordan algebras). On the other hand, degenerate quadratic alternative algebras were studied, for example by R. A. Kunze and S. Scheinberg in [31], but they exclude the characteristic 2 case. In particular, Proposition 4.4.6 is, restricted to the case of odd characteristic, also proven in their paper. I could not find anything in the literature on quadratic alternative algebras that does not exclude at least one of “degenerate” and “characteristic 2”, but I will not claim that it has not been done yet. I mostly followed the approach of McCrimmon to extend the results to the degenerate case, though of course there are similarities with the approach of Kunze and Scheinberg.

CHAPTER

3

INTRODUCTION

Projective representations of the geometries occurring in the second row of the Freudenthal-Tits magic square (both the split and the non-split version) have been characterised axiomatically by O. Krauss (only involved in the non-split version), J. Schillewaert and H. Van Maldeghem ([29], [39]); providing a homogenous description of these geometries and proving that they are special among an infinite family of geometries. The geometric characterisation goes as follows.

Consider a pair (X,Ξ), where X is a spanning point set of a projective space $\mathsf{PG}(N,\mathbb{K})$ over some field $\mathbb{K}$ and with $N \in \mathbb{N}\cup\{\infty\}$, and where Ξ is a collection of $(d+1)$-dimensional subspaces of $\mathsf{PG}(N,\mathbb{K})$, where $d \geq 1$, such that each $\xi \in \Xi$ intersects X in a non-degenerate quadric which generates ξ, and such that for each pair $\xi,\xi' \in \Xi$, the quadrics $X\cap\xi$ and $X\cap\xi'$ are isomorphic. Suppose that (X,Ξ) satisfies the following three simple axioms:

(Ax1) Any pair of points $x_1,x_2 \in X$ lies in at least one element of Ξ;
(Ax2) if $\xi_1,\xi_2 \in \Xi$ are distinct, then $\xi_1\cap\xi_2 \subseteq X$;
(Ax3) for each $x \in X$, there are two members ξ_1,ξ_2 in Ξ through x such that, for each $\xi_3 \in \Xi$ through x holds that the tangent space[1] $T_x(X\cap\xi_3)$ is contained in what is generated by the tangent spaces $T_x(\xi_1)$ and $T_x(\xi_2)$.

The pair (X,Ξ) is called *non-trivial* if $|\Xi| \geq 2$.

In case the quadrics $\{X\cap\xi \mid \xi\in\Xi\}$ are either all of *minimal* Witt index (which means that there are no lines on them) or all of *maximal* Witt index (i.e., a parabolic or hyperbolic quadric), one gets that:

[1]For a quadric Q and one of its points x, the tangent space $T_x(Q)$ is the unique hyperplane of $\langle Q\rangle$ consisting of lines through x either fully contained in Q or only meeting Q in $\{x\}$.

- As an abstract point-line geometry, (X,Ξ) is isomorphic to a (ring) projective plane over a quadratic alternative $\mathbb{K}$-algebra $\mathbb{A}$ with $\dim_{\mathbb{K}}(\mathbb{A})=d$ (in case of minimal Witt index, these algebras are division algebras).
- The variety (X,Ξ) in $\mathsf{PG}(N,\mathbb{K})$ is projectively equivalent to the Veronese representation[2], denoted $\mathcal{V}_2(\mathbb{K},\mathbb{A})$, of the above ring projective plane over $\mathbb{A}$, and $d=\dim_{\mathbb{K}}(\mathbb{A})$.

The fact that the (ring) projective planes (X,Ξ) are coordinatised over quadratic alternative $\mathbb{K}$-algebras $\mathbb{A}$ shows that the options for non-trivial pairs (X,Ξ) are limited. Indeed, the following is a well-known fact (see for instance [26]):

Fact 3.0.1. *Let $\mathbb{A}$ be a non-degenerate quadratic alternative algebra over a field $\mathbb{K}$ and put $d=\dim_{\mathbb{K}}(\mathbb{A})$. Then $\mathbb{A}$ is one of the following.*

($d=1$) $\mathbb{A}=\mathbb{K}$;
($d=2$) *$\mathbb{A}$ is either a quadratic Galois extension $\mathbb{L}$ of $\mathbb{K}$ or $\mathbb{K}\times\mathbb{K}$;*
($d=4$) *$\mathbb{A}$ is either a quaternion division algebra $\mathbb{H}$ over $\mathbb{K}$ or the 2×2 matrices over $\mathbb{K}$;*
($d=8$) *$\mathbb{A}$ is a Cayley-Dickson algebra $\mathbb{O}$ with center $\mathbb{K}$ (either division or split);*
($insep$) *$\mathbb{A}$ is a purely inseparable extension of $\mathbb{K}$ with $\mathbb{A}^2\subseteq\mathbb{K}$, and if d is finite, it is a power of 2 (this case only occurs if $\mathrm{char}(\mathbb{K})=2$).*

If d is finite, $\mathbb{A}$ can be obtained by successively applying the Cayley-Dickson doubling process on the field $\mathbb{K}$. The division algebras among the above correspond to the case in which the quadrics of Ξ have minimal Witt index; the split algebras correspond to the quadrics of maximal Witt index. If $\mathrm{char}(\mathbb{K})\neq 2$, there are only seven types of such algebras and hence only seven types of corresponding Veronese representations $\mathcal{V}_2(\mathbb{K},\mathbb{A})$, which correspond precisely the seven geometries occurring at the second row of the Freudenthal-Tits magic square (considering both its non-split and split version).

The only known examples of pairs (X,Ξ) satisfying Axioms (Ax1), (Ax2) and (Ax3) are those in which the quadrics of Ξ either all have minimal Witt index, or all have maximal Witt index. Schillewaert and Van Maldeghem conjectured that, if the quadrics $\{X\cap\xi \mid \xi\in\Xi\}$ all have the same Witt index which is neither minimal nor maximal, then there are no non-trivial examples of such pairs (X,Ξ). However, this does not mean that there are no other options. Indeed, the quadrics could be degenerate.

In view of this conjecture, pairs (X,Ξ) were studied in which the members of Ξ were 3-dimensional quadrics. The above hints that one should start by studying Veronese representations over quadratic alternative algebras $\mathbb{A}$ with $\dim_K(\mathbb{A})=2$. Apart from the two above possibilities for such algebras (division and split), there is also third option: the dual numbers $\mathsf{DN}(\mathbb{K})$ over $\mathbb{K}$, i.e., the algebra $\mathbb{K}[t]/(t^2)$. The corresponding quadrics are cones with a point as vertex and an oval in a plane as base. The corresponding pair (X,Ξ) is such that:

- As an abstract point-line geometry, (X,Ξ) is isomorphic to a ring projective plane over $\mathsf{DN}(\mathbb{K})$ (more precisely: we get a Hjelsmlev projective plane).

[2]See Chapter 5

- The variety (X,Ξ) in $\mathsf{PG}(N,\mathbb{K})$ is projectively equivalent to the Veronese representation, denoted $\mathcal{V}_2(\mathbb{K},\mathsf{DN}(\mathbb{K}))$, of the above ring projective plane over $\mathsf{DN}(\mathbb{K})$.

An extension towards all degenerate quadratic alternative algebras is not realistic, as they do not behave as nice as do the non-degenerate ones: if $\mathbb{A}$ is a degenerate quadratic alternative $\mathbb{K}$-algebra, then $\mathbb{A}$ is the direct sum of a non-degenerate quadratic alternative subalgebra $\mathbb{B}$ and its radical R (see Section 4.4.1), and this radical can still be very large (see [31] for more details). So, this class of algebras being rather wild, there is no hope that the corresponding Veronese representations would fit in a homogenous axiomatic description.

The class of degenerate quadratic alternative algebras which turned out to be the sensible one to study in this context, consists of quadratic alternative algebras that are similar to the dual numbers in the sense that their radical is generated by a single element (for $\mathsf{DN}(\mathbb{K}) = \mathbb{K}[t]/(t^2)$, this radical is generated by the element t). It will turn out that these algebras are set-wise given by $\mathbb{B}\oplus t\mathbb{B}$, where $\mathbb{B}$ is a non-degenerate quadratic alternative algebra and t is an indeterminate with $t^2=0$. We will call these algebras *generalised dual numbers*. **Defining and characterising the corresponding Veronese representations is the main topic of this part of the thesis.** In both the non-split and the split case, the axioms that we will use for this are in the spirit of the axioms given above.

In the split case, there are two peculiar things. The first one being that for one of the split generalised dual numbers (the largest one, which means dimension 8), I could not capture the associated Veronese variety. Its behaviour is entirely different than that of the Veronese varieties corresponding to the smaller split generalised numbers. A serious issue is that fact that the convex closure of two points at distance 2 is not necessarily a polar space, ruining Axiom (Ax1) that we mentioned above, and no longer fitting in some “extended framework of parapolar spaces” (though there is not really a definition of parapolar spaces in case the symplecta are degenerate). After desperately looking for ways around, I decided to first classify the sets which do fit in the above axiomatic framework (at least, if you look at it in a slightly different way – see Chapter 7 for that), and I think the result is rather satisfactory seeing that each of the obtained structures essentially are the Veronese varieties we were aiming for, apart from that one annoying case. Moreover—this is the second peculiar fact, although you might consider it an answer to the first one—the obtained varieties are all subvarieties of the $\mathsf{E}_{6,1}(\mathbb{K})$-variety.

An overview of this part:

In **Chapter 4**, we will formally introduce these generalised dual numbers and discuss their (strong) link to the algebras resulting from the Cayley-Dickson process. There will be two kinds of algebras $\mathbb{A}$: ones which have a non-degenerate part which is a division (then $\mathbb{A}$ is called *non-split*), and ones whose non-degenerate part is not division, i.e., split (in which case $\mathbb{A}$ is called split too).

In **Chapter 5**, we can then introduce Veronese representations associated to the non-split and split generalised dual numbers.

In **Chapter 6**, we geometrically characterise the Veronese representations over the non-split generalised dual numbers.

In **Chapter 7**, we geometrically characterise the Veronese representations over the split generalised dual numbers.

CHAPTER

4

QUADRATIC ALTERNATIVE ALGEBRAS AND GENERALISED DUAL NUMBERS

As noted in the introduction, the algebras coordinatising the ring projective planes occurring in the second row of the Freudenthal-Tits magic square, are precisely the (non-degenerate) quadratic alternative algebras over some field $\mathbb{K}$. These algebras are precisely the ones that can be obtained by (successively) applying the (standard) Cayley-Dickson process on a field $\mathbb{K}$.

Initially, the idea was that the degenerate quadratic alternative algebras that can also be obtained by an extended version of the Cayley-Dickson process (allowing "0" as a primitive element, see later) could be used to define Veronese varieties which fit in a uniform axiomatic framework. However, studying those, it turned out that this is not the right setting, for some of these algebras are on the one hand "too degenerate" and on the other hand, there is no intrinsic reason to exclude the other degenerate quadratic alternative algebras.

The degenerate quadratic alternative algebras that we want to study should, above all, have a "small radical", and the smallest non-trivial possibility is that the radical is generated (as a ring) by a single non-zero element t. We will classify those algebras and connect them to the Cayley-Dickson algebras (see Theorem 4.4.1).

4.1 Quadratic alternative algebras over $\mathbb{K}$

Suppose $\mathbb{A}$ is a unital $\mathbb{K}$-algebra whose center contains $\mathbb{K}$ and which is:

- **alternative**: the associator $[a,b,c] := (ab)c - a(bc)$ is a trilinear alternating map;
- **quadratic**: each $a \in \mathbb{A}$ satisfies a quadratic equation $x^2 - \mathsf{T}(a)x + \mathsf{N}(a) = 0$ where the *trace* $\mathsf{T} : \mathbb{A} \to \mathbb{K} : a \mapsto \mathsf{T}(a)$ is a linear map with $\mathsf{T}(1) = 2$ and the *norm* $\mathsf{N} : \mathbb{A} \to \mathbb{K} : a \mapsto \mathsf{N}(a)$ is a quadratic map with $\mathsf{N}(1) = 1$.

As $\mathbb{A}$ is quadratic, the map $\mathbb{A} \to \mathbb{A} : x \mapsto \overline{x} := \mathsf{T}(x) - x$ is an involutive anti-automorphism ($\overline{xy} = \overline{y}\,\overline{x}$ for all $x, y \in \mathbb{A}$) which fixes $\mathbb{K}$. Note that $\mathsf{N}(a) = a\overline{a}$ for each $a \in \mathbb{A}$.

An element $a \in \mathbb{A}$ is *invertible* if and only if $\mathsf{N}(a) \neq 0$, and if invertible, its inverse is given by $\mathsf{N}(a)^{-1}\overline{a}$. The algebra $\mathbb{A}$ is a *division algebra* if all non-zero elements are invertible, which is equivalent with the norm form N being anisotropic, i.e., for all $x \in \mathbb{A}$, $\mathsf{N}(x) = 0 \Leftrightarrow x = 0$.

Remark 4.1.1. The above mentioned coefficients $\mathsf{T}(a)$ and $\mathsf{N}(a)$ are unique for each $a \in \mathbb{A} \setminus \mathbb{K}$. The fact that T needs to be linear (if so then N is automatically quadratic) is no real requirement, unless possibly if $\mathbb{K} = \mathbb{F}_2$, as then there could be elements $x, y \in \mathbb{A}$ for which $\mathsf{T}(x+y) \neq \mathsf{T}(x) + \mathsf{T}(y)$ on the condition that x, y commute and have $x^2 = x$, $y^2 = y$ and are such that $\{1, x, y\}$ is linearly independent. If $\dim_{\mathbb{K}}(\mathbb{A}) > 1$ then $\mathsf{T}(1) = 2$ and $\mathsf{N}(1) = 1$ also follow immediately.

4.1.1 General properties

The fact that the associator $[a,b,c] = (ab)c - a(bc)$ is an alternating map means that $[a,a,b] = [a,b,b] = 0$ for all $a, b \in \mathbb{A}$. These two identities are called the left and right alternative identities. From them, also the flexible alternative identity $[a,b,a] = 0$ follows. Artin's theorem says that a subalgebra of $\mathbb{A}$ generated by any two elements $a, b \in \mathbb{A}$ is associative.

There are also the *Moufang identities*: for all $x, y, a \in \mathbb{A}$ holds that

$$\begin{aligned} a(x(ay)) &= (axa)y, \\ ((xa)y)a &= x(aya), \\ (ax)(ya) &= a(xy)a. \end{aligned}$$

Since $\overline{a} = \mathsf{T}(a) - a$ and $[k,b,c] = 0$ for $k \in \mathbb{K}$, $b, c \in \mathbb{A}$ we have

$$[a,b,c] = -[\overline{a},b,c] \quad \forall a,b,c \in \mathbb{A}$$

and likewise for permutations of $\{a,b,c\}$.

4.1.2 The radical of $\mathbb{A}$

The bilinear form f associated to the quadratic form N is given by $f(x,y)=\mathsf{N}(x+y)-\mathsf{N}(x)-\mathsf{N}(y)=x\overline{y}+y\overline{x}$. Its *radical* is the set $\mathrm{rad}(f)=\{x\in\mathbb{A}\mid f(x,y)=0\,\forall y\in\mathbb{A}\}$. We call $\mathbb{A}$ *non-singular* if f is non-singular, i.e., if $\mathrm{rad}(f)=\{0\}$. We call $\mathbb{A}$ *non-degenerate* if its norm form N is non-degenerate, i.e., if N is anisotropic on $\mathrm{rad}(f)$.

Definition 4.1.2. The *radical* R of $\mathbb{A}$ is defined as the set $\{r\in\mathrm{rad}(f)\mid \mathsf{N}(r)=0\}$.

By the above, $\mathbb{A}$ is non-degenerate if and only if $R=\{0\}$, whence the definition.

Lemma 4.1.3. *Both* $\mathrm{rad}(f)$ *and* R *are 2-sided ideals of* $\mathbb{A}$.

Proof. By definition, $r\in\mathbb{A}$ belongs to $\mathrm{rad}(f)$ precisely if $a\overline{r}=-r\overline{a}$ for all $a\in\mathbb{A}$; so in particular $\overline{r}=-r$. Hence $r\in\mathrm{rad}(f)$ if and only if $\overline{r}=-r$ and $ar=r\overline{a}$ for all $a\in\mathbb{A}$. Now take any $r\in\mathrm{rad}(f)$. We verify that $ar\in\mathrm{rad}(f)$ for each $a\in\mathbb{A}$. Clearly, $\overline{ar}=\overline{r}\,\overline{a}=-r\overline{a}=-ar$ since $r\in\mathrm{rad}(f)$. Now let $c\in\mathbb{A}$ be arbitrary. Then $c(ar)=(ar)\overline{c}$ indeed:

$$\begin{aligned} c(ar) &= (ca)r+(r\overline{a})\overline{c}-r(\overline{a}\,\overline{c}) && ([c,a,r]=-[r,\overline{a},\overline{c}])\\ &= (r\overline{a})\overline{c} && ((ca)r=r(\overline{a}\,\overline{c}))\\ &= (ar)\overline{c} && (r\overline{a}=ar). \end{aligned}$$

Hence $ar=-r\overline{a}\in\mathrm{rad}(f)$ for all $a\in\mathbb{A}$, so $\mathrm{rad}(f)$ is a 2-sided ideal since $\{\overline{a}\mid a\in\mathbb{A}\}=\mathbb{A}$.

Also $R\subseteq\mathrm{rad}(f)$ is a 2-sided ideal, for if $r,r'\in R$ then $\mathsf{N}(r+r')=\mathsf{N}(r)+f(r,r')+\mathsf{N}(r')=0$ and for all $a\in\mathbb{A}$ and $r\in R$ we have $\mathsf{N}(ar)=\mathsf{N}(ra)=\mathsf{N}(r)\mathsf{N}(a)=0$. □

By definition, $R\subseteq\mathrm{rad}(f)$. Moreover, for each $r\in\mathrm{rad}(f)$ holds that $0=f(r,r)=2\mathsf{N}(r)$, so if $\mathrm{char}\,\mathbb{K}\neq 2$, then $R=\mathrm{rad}(f)$. We can make this statement even stronger:

Lemma 4.1.4. *Either* $R=\mathrm{rad}(f)$ *or* $\mathrm{rad}(f)=\mathbb{A}$ *and in the latter case,* $\mathrm{char}\,\mathbb{K}=2$*,* $\mathbb{A}$ *is a commutative associative ring and* $x\mapsto\overline{x}$ *is the identity.*

Proof. Suppose $R\subsetneq\mathrm{rad}(f)$. Then there is an element $r\in\mathrm{rad}(f)$ with $\mathsf{N}(r)\neq 0$. Recall that $\mathsf{N}(r)\neq 0$ implies that r is invertible, and consequently the ideal $\mathrm{rad}(f)$ coincides with $\mathbb{A}$. This has the following consequences: firstly, $0=f(1,1)=2$ implies that the characteristic of $\mathbb{K}$ is 2; secondly, $0=f(x,1)=x+\overline{x}$ implies, together with $\mathrm{char}\,\mathbb{K}=2$, that $\overline{x}=x$ for all $x\in\mathbb{A}$; thirdly, $0=f(x,y)=x\overline{y}+y\overline{x}$ implies, together with the previous two observations, that $\mathbb{A}$ is commutative: $xy=yx$ for all $x,y\in\mathbb{A}$; lastly we obtain that $\mathbb{A}$ is associative by using that $[a,b,c]=[b,a,c]$ and $ab=ba$: $(ab)c+a(bc)=(ba)c+b(ac)$, so $a(bc)=b(ac)$ for all $a,b,c\in\mathbb{A}$, and then it follows that $[a,b,c]=0$:

$$a(bc)=b(ac)=b(ca)=c(ba)=(ab)c.$$

This shows the lemma. □

Next, we show that the left and right ideal generated by an element $r \in \mathsf{rad}(f)$ is given by $\mathbb{A}r$ or $r\mathbb{A}$, respectively (this is not trivial in an alternative algebra for $a(a'r)$ need not be a multiple of r), and that $\mathbb{A}r = r\mathbb{A}$.

Lemma 4.1.5. *For each $r \in \mathsf{rad}(f)$, we have $\mathbb{A}(\mathbb{A}r) = \mathbb{A}r = r\mathbb{A} = (r\mathbb{A})\mathbb{A}$.*

Proof. Take $r \in \mathsf{rad}(f)$ and recall that this means that $\overline{r} = -r$ and $ar = r\overline{a}$ for each $a \in \mathbb{A}$. Using this, we obtain:

$$\begin{aligned}
(ra)b &= (\overline{a}r)b && (ra = \overline{a}r)\\
&= \overline{a}(br + rb) - (\overline{a}b)r && [\overline{a}, r, b] = -[\overline{a}, b, r]\\
&= \overline{a}((b + \overline{b})r) - (\overline{a}b)r && (rb = \overline{b}r)\\
&= (\overline{a}b + \overline{a}\,\overline{b} - \overline{a}b)r && (b + \overline{b} \in \mathbb{K})\\
&= (\overline{a}\,\overline{b})r\\
&= r(ba).
\end{aligned}$$

This shows $(r\mathbb{A})\mathbb{A} = r\mathbb{A}$. In a similar way, one can show that $\mathbb{A}(\mathbb{A}r) = \mathbb{A}r$. Since $\{\overline{a} \mid a \in \mathbb{A}\} = \mathbb{A}$, it follows from $ar = r\overline{a}$ that $\mathbb{A}r = r\mathbb{A}$. □

For completeness' sake, we want to show that the radical R can also be defined without using the bilinear form f.

Definition 4.1.6. A *nil ideal* of an algebra A is an ideal in which each element is nilpotent. The *nil radical* of an algebra A is defined as its maximal nil ideal.

We show that R coincides with the nil ideal of $\mathbb{A}$.

Lemma 4.1.7. *An element r belongs to R if and only if ar is nilpotent for each $a \in \mathbb{A}$.*

Proof. We claim that $x \in \mathbb{A}$ is nilpotent if and only if $\mathsf{T}(x) = \mathsf{N}(x) = 0$. Let $x \in \mathbb{A} \setminus \{0\}$ be arbitrary. As nilpotent elements cannot be invertible, we get $\mathsf{N}(x) = 0$. Because $\mathbb{A}$ is quadratic, this means $x^2 = \mathsf{T}(x)x$. Since x is nilpotent, there is a natural number $n \geq 1$ such that $x^n = \mathsf{T}(x)^{n-1}x = 0$, implying that $\mathsf{T}(x) = 0$ as $\mathsf{T}(x) \in \mathbb{K}$, but then $x = 0$, a contradiction. The converse is obvious, so the claim is shown. Observe that $\mathsf{T}(ax) = f(a, \overline{x}) = f(\overline{a}, x)$ for all $a, x \in \mathbb{A}$.

Let $x \in \mathbb{A}$ be such that ax is nilpotent for each $a \in \mathbb{A}$. As noted above, we have $0 = \mathsf{T}(ax) = f(\overline{a}, x)$ for all $a \in \mathbb{A}$, and for $a = 1$ we obtain $\mathsf{N}(x) = 0$, so $x \in R$ indeed. Conversely, take $r \in R$. Then $f(r, a) = \mathsf{T}(\overline{a}r) = 0$ for all $a \in \mathbb{A}$ and $\mathsf{N}(r) = 0$. It then follows immediately that $\mathsf{T}(ar) = 0$ for all $a \in \mathbb{A}$ and $\mathsf{N}(ar) = \mathsf{N}(a)\mathsf{N}(r) = 0$ by multiplicativity of N. So ar is nilpotent for each $a \in \mathbb{A}$. □

Corollary 4.1.8. *The radical R is the union of all nil ideals and as such it is the nil radical.*

Proof. Let x be an element contained in a nil ideal I. As $\mathbb{A}x \subseteq I$, we obtain that ax is nilpotent for each $a \in \mathbb{A}$ and hence by Lemma 4.1.7, $x \in R$. Hence $I \subseteq R$ for each nil ideal I. The same lemma also implies that each element $r \in R$ is nilpotent, and hence R is a nil ideal itself. By the foregoing, it is the maximal one. □

The non-degenerate quadratic alternative $\mathbb{K}$-algebras $\mathbb{A}$ with $\dim_{\mathbb{K}}(\mathbb{A}) < \infty$ can all be produced using the Cayley-Dickson doubling process. We will below explain an extended version of this process, extended in the sense that it also produces *degenerate* algebras.

4.2 The (extended) Cayley-Dickson doubling process

Let $\mathbb{A}$ be a quadratic alternative $\mathbb{K}$-algebra with associated involution $x \mapsto \overline{x}$ as before, and let ζ be some element in $\mathbb{K}$.

One application of the **Cayley-Dickson doubling process** on the algebra $\mathbb{A}$ using ζ as a **primitive element** results in a $\mathbb{K}$-algebra which set-wise equals $\mathbb{A} \times \mathbb{A}$, whose addition is defined component-wise too and whose multiplication is given by

$$(a,b) \times (c,d) = (ac + \zeta d\overline{b}, \overline{a}d + cb).$$

This resulting $\mathbb{K}$-algebra is denoted by $\mathsf{CD}(\mathbb{A},\zeta)$, is quadratic too and hence comes with an **involution**: $\overline{(a,b)} := (\overline{a}, -b)$ and a **norm** $\mathsf{N}(a,b) = (a,b) \cdot \overline{(a,b)} = (\mathsf{N}(a) - \zeta\mathsf{N}(b), 0)$.

Remark 4.2.1. The fact that we allow the primitive element ζ to be 0 is the point at which the above process extends the standard one.

Remark 4.2.2. We can also view $\mathsf{CD}(\mathbb{A},\zeta)$ as $\mathbb{A} \oplus t\mathbb{A} := \{a + tb \mid a, b \in \mathbb{A}\}$ for some element $t \in \mathsf{CD}(\mathbb{A},\zeta) \setminus \{0\}$ with $t^2 = \zeta$, where $a + tb$ corresponds to the pair (a,b) and in which the multiplication and involution are such that $(a,b) \mapsto a + tb$ preserves both the multiplication and the involution, which comes down to the following rules for all $a,b,c,d \in \mathbb{A}$ (we do not need that many letters but they correspond to the ones used in the formula above):

$$at = t\overline{a}, \tag{4.1}$$
$$a(td) = t(\overline{a}d), \tag{4.2}$$
$$(tb)c = t(cb), \tag{4.3}$$
$$(tb)(td) = t^2(d\overline{b}). \tag{4.4}$$

If for each $a \in \mathbb{A}$, $ta = 0$ implies $a = 0$, the morphism $(a,b) \mapsto a + tb$ is an isomorphism.

In fact, if property (4.1) holds in an alternative algebra $\mathbb{A}$, then automatically also properties (4.2) to (4.4) hold. We will show this statement later, we first list some properties of $\mathsf{CD}(\mathbb{A},\zeta)$.

4.2.1 Properties of $\mathsf{CD}(\mathbb{A},\zeta)$

Lemma 4.2.3. *The algebra $\mathsf{CD}(\mathbb{A},\zeta)$ is a division algebra if and only if $\zeta \notin \mathsf{N}(\mathbb{A})$ and $\mathbb{A}$ is division.*

Proof. An element $(a,b)\in\mathsf{CD}(\mathbb{A},\zeta)$ is not invertible if and only if $\mathsf{N}(a,b)\neq 0$. Recalling $\mathsf{N}(a,b)=(\mathsf{N}(a)-\zeta\mathsf{N}(b),0)$, we get that $\mathsf{N}(a,b)=0$ if and only if $\mathsf{N}(a)=\zeta\mathsf{N}(b)$. Now this equation has solutions precisely if either $\mathsf{N}(a)=\mathsf{N}(b)=0$ or if $\zeta=\mathsf{N}(ab^{-1})$ and $\mathsf{N}(b)\neq 0$. Hence only if $\mathbb{A}$ is a division algebra and if ζ is not a norm, there are no solutions other than $(a,b)=(0,0)$. □

Lemma 4.2.4. *The algebra $\mathsf{CD}(\mathbb{A},\zeta)$ is non-singular if and only if $\zeta\neq 0$ and $\mathbb{A}$ is non-singular.*

Proof. The radical $\mathsf{rad}(f)$ associated to $\mathsf{CD}(\mathbb{A},\zeta)$ consists of all elements r such that $\overline{r}=-r$ and $ar=r\overline{a}$ for all $a\in\mathsf{CD}(\mathbb{A},\zeta)$. A calculation shows that $\overline{(c,d)}=-(c,d)$ and $(a,b)(c,d)=(c,d)(\overline{a},-b)$ for all $a,b\in\mathbb{A}$ if and only if either $c,d\in\mathsf{rad}(f_{|\mathbb{A}})$ and $d=-d$, or if $\zeta=0$ and $c\in\mathsf{rad}(f_{|\mathbb{A}})$. So $\mathsf{rad}(f)$ is trivial if and only if $\zeta\neq 0$ and $\mathsf{rad}(f_{|\mathbb{A}})$ trivial. □

Lemma 4.2.5. *Suppose $\mathsf{CD}(\mathbb{A},\zeta)$ is singular. Then $\mathsf{CD}(\mathbb{A},\zeta)$ is non-degenerate if and only if $\mathbb{A}$ is non-degenerate and $\zeta\notin\mathsf{N}(\mathbb{A})=\mathbb{A}^2$.*

Proof. Since $\mathsf{CD}(\mathbb{A},\zeta)$ is singular, Lemma 4.2.4 implies that either $\mathbb{A}$ is also singular or $\zeta=0$. Then $\mathsf{CD}(\mathbb{A},\zeta)$ is non-degenerate if and only if $\mathsf{N}(a,b)=(\mathsf{N}(a)-\zeta\mathsf{N}(b),0)\neq(0,0)$ for all $(a,b)\neq(0,0)$. Clearly, this equation has no non-trivial solutions if and only if $\mathbb{A}$ is non-degenerate and $\zeta\notin\mathsf{N}(\mathbb{A})$. Since $\mathbb{A}$ is singular, we know by Lemma 4.1.4 that the involution is trivial, and hence $\mathsf{N}(\mathbb{A})=\mathbb{A}^2$. □

Lemma 4.2.6. *The algebra $\mathsf{CD}(\mathbb{A},\zeta)$ is*

- commutative *if and only if $\mathbb{A}$ is commutative and $\overline{a}=a$ for each $a\in\mathbb{A}$;*
- associative *if and only if $\mathbb{A}$ is commutative and associative;*
- alternative *if and only if $\mathbb{A}$ is associative.*

Proof. The commutator $[(a,b),(c,d)]$ equals $(ac-ca+\zeta(d\overline{b}+b\overline{d}),\overline{a}d+ad+cb-\overline{c}b)$, which is zero for all $a,b,c,d\in\mathbb{A}$ if and only if $ac=ca$ for all $a,c\in\mathbb{A}$ and $\overline{a}=a$ for all $a\in\mathbb{A}$. This shows the first assertion.

For $\mathsf{CD}(\mathbb{A},\zeta)$ to be associative, we must have, according to (4.3), that $t(bc)=t(cb)$ for all $b,c\in\mathbb{A}$, so $\mathbb{A}$ has to be commutative (recall that $ta=0$ for $a\in\mathbb{A}$ implies $a=0$, cf. Remark 4.2.2). Since $\mathbb{A}$ is a subalgebra of $\mathsf{CD}(\mathbb{A},\zeta)$, of course $\mathbb{A}$ needs to be associative too. Now suppose $\mathbb{A}$ is both commutative and associative. We only need to show that the braces occurring in (4.2), (4.3) and (4.4) do not matter, since then braces do not matter nor in $\mathbb{A}$, nor in interactions of $\mathbb{A}$ and t and hence they do not matter in the multiplication of $\mathsf{CD}(\mathbb{A},\zeta)$. For (4.3) this follows immediately from the commutativity of $\mathbb{A}$. For (4.2), we show that $a(td)=(at)d$ for all $a,d\in\mathbb{A}$: by definition of (4.2), $a(td)=t(\overline{a}d)$ and

$t(\overline{a}d) = t(d\overline{a}) = (t\overline{a})d = (at)d$ (by (4.1), (4.3) and commutativity of $\mathbb{A}$); and by (4.1) it then also follows that $t(\overline{a}d) = (t\overline{a})d$ for all $a, d \in \mathbb{A}$. Lastly, for the braces in the left hand side of (4.4), we note that $(tb)t = t(bt) = \zeta\overline{b}$ follows from (4.4) and (4.1), and $(bt)d = b(td)$ we have shown just before.

We check the left alternative identity in $\mathsf{CD}(\mathbb{A}, \zeta)$: $(a, b)^2(c, d) = (a, b)((a, b)(c, d))$ for all $a, b, c, d \in \mathbb{A}$. This comes down to checking, for all $a, b, c, d \in \mathbb{A}$, that

$$a^2c = a(ac), a^2(td) = a(a(td)), (tb)^2c = (tb)((tb)c), (tb)^2(td) = (tb)((tb)(td)), \quad (4.5)$$

$$((tb)a)c + (a(tb))c = (tb)(ac) + a((tb)c), \quad (4.6)$$

$$((tb)a)(td) + (a(tb))(td) = (tb)(a(td)) + a((tb)(td)). \quad (4.7)$$

Provided that $\mathbb{A}$ is alternative, the first four identities hold, as follows by a straightforward verification using properties (4.1) and (4.4). Using properties (4.1) and (4.4) to get t at the leftmost side, the two other equations are equivalent to

$$t(a(cb)) = t((ac)b) \quad \text{and} \quad \zeta(\overline{a}(d\overline{b})) = \zeta((\overline{a}d)\overline{b}). \quad (4.8)$$

It is clear that these identities hold if and only if $\mathbb{A}$ is associative. Likewise one obtains the right alternative identity. We conclude that $\mathsf{CD}(\mathbb{A}, \zeta)$ is alternative if and only if $\mathbb{A}$ is associative. □

After reaching an alternative algebra $\mathbb{A}$, we stop applying the Cayley-Dickson doubling process, for further applications would yield algebras which are no longer alternative, according to the previous lemma.

4.2.2 The Cayley-Dickson doubling process starting from $\mathbb{K}$

Suppose we apply the Cayley-Dickson doubling process on the field $\mathbb{K}$ (note that the associated involution is the identity then), using some $\zeta \in \mathbb{K}$ as the primitive element. Then the resulting algebra $\mathsf{CD}(\mathbb{K}, \zeta)$ is just the quotient of the polynomial ring $\mathbb{K}[x]$ by the ideal $(x^2 - \zeta)$, as the multiplication rules still coincide with ordinary multiplication. In particular, we obtain an commutative $\mathbb{K}$-algebra with involution $a + tb \mapsto a - tb$, where $t^2 = \zeta$. The involution is non-trivial precisely if $\mathrm{char}(\mathbb{K}) \neq 2$.

Suppose first that $\mathrm{char}(\mathbb{K}) \neq 2$. Another application of the Cayley-Dickson doubling process, using some primitive element $\zeta' \in \mathbb{K}$, gives us the algebra $\mathsf{CD}(\mathbb{K}, \zeta, \zeta')$, which is a quaternion algebra with center $\mathbb{K}$, in particular, no longer commutative. A third application, using $\zeta'' \in \mathbb{K}$ as primitive element, then yields an octonion algebra $\mathsf{CD}(\mathbb{K}, \zeta, \zeta', \zeta'')$ which is no longer associative, but alternative instead. By Lemma 4.2.6, we would not obtain an alternative algebra if we apply the process once more, so here it stops.

Next, suppose $\mathrm{char}(\mathbb{K}) = 2$. Since, for each $\zeta \in \mathbb{K}$, the algebra $\mathsf{CD}(\mathbb{K}, \zeta)$ is a commutative associative $\mathbb{K}$-algebra with a trivial involution (cf. Lemma 4.1.4), the same will hold after any finite number of applications of the process, the d-th of which gives a quadratic commutative associative $\mathbb{K}$-algebra $\mathsf{CD}(\mathbb{K}, \zeta_1, \ldots, \zeta_d)$, which equals $\mathbb{K}[x_1, \ldots, x_d]/(x_i^2 - \zeta_i)$, and is for short denoted by $\mathbb{A}_d$, and $\dim_{\mathbb{K}}(\mathbb{A}_d)=2^d$. In case $\mathbb{A}_d$ is a field, it is a purely inseparable field extension of $\mathbb{K}$. Note that, since the involution is trivial, the corresponding bilinear form f_d is the zero form and hence $\mathsf{rad}(f_d)=\mathbb{A}_d$.

4.2.3 An adapted version of the Cayley-Dickson process when $\operatorname{char}\mathbb{K}=2$

We can adapt the first step of the Cayley-Dickson doubling process in such a way that we obtain a non-trivial involution, by considering the quotient of $\mathbb{K}[x]$ by the ideal $(x^2+x+\zeta)$ (instead of $(x^2+\zeta)$) for some $\zeta\in\mathbb{K}$. The resulting (commutative and associative) $\mathbb{K}$-algebra will be denoted by $\mathbb{L}_\zeta$, to make a distinction with the general process. The multiplication and (non-trivial) involution go as follows:

$$(a,b)\cdot(c,d)=(ac+\zeta d\overline{b},\overline{a}d+cb+d\overline{b}),$$

$$\overline{(a,b)}=(a+b,b).$$

For the norm we obtain $\mathsf{N}(a,b)=(a^2+ab+\zeta b^2,0)$, in which we recognise the homogeneous version of $x^2+x+\zeta$. So $\mathbb{L}_\zeta$ is a division algebra if and only if $x^2+x+\zeta$ has no non-zero solutions over $\mathbb{K}$ (in which case $\mathbb{L}_\zeta$ is a Galois extension of $\mathbb{K}$). As we obtained a commutative associative $\mathbb{K}$-algebra with a non-trivial involution, we can reach strictly alternative algebras by successive applications of the Cayley-Dickson process on $\mathbb{L}_\zeta$.

Remark 4.2.7. Also here we have the morphism $(a,b)\mapsto a+tb$ for an element t for which $t^2=t+\zeta$ (note that then $\overline{t}=t+1$), and properties (4.1) to (4.4) from above are equivalent to the multiplication formula; again the morphism is an isomorphism if $tb=0$ implies $b=0$ for $b\in\mathbb{A}$.

4.3 Non-degenerate quadratic alternative algebras over $\mathbb{K}$

4.3.1 The classification in terms of the Cayley-Dickson doubling process

As already noted in Fact 3.0.1, there is a neat classification of the non-degenerate quadratic alternative algebras $\mathbb{A}$. The following lemma is the key to this classification (in the non-degenerate case it is sometimes called the Jacobson Necessity theorem, see 2.6.1 of [34]). We state it in general, i.e, for possibly degenerate quadratic alternative algebras, as it are in fact the degenerate algebras for which we needed this lemma.

Lemma 4.3.1. *If $\mathbb{B}$ is a non-singular finite-dimensional unital subalgebra of a (possibly degenerate) quadratic alternative algebra $\mathbb{A}$ and $t\in\mathbb{B}^\perp\setminus\{0\}$ (necessarily, $t^2=-\mathsf{N}(t)$ then), then $\mathbb{B}\oplus t\mathbb{B}$ is a subalgebra of $\mathbb{A}$ and the canonical map*

$$\sigma:\mathsf{CD}(\mathbb{B},-\mathsf{N}(t))\to\mathbb{B}\oplus t\mathbb{B}:a+tb\mapsto a+tb$$

is a morphism with $\mathsf{ker}(\sigma) \subseteq t\mathbb{B}$*. For all* $b \in \mathbb{B}$*,* $\sigma(tb) = 0$ *implies* $\mathsf{N}(b) = 0$*, and if* $\mathsf{N}(t) \neq 0$ *or if* $\mathbb{B}$ *is a division algebra, then* σ *is an isomorphism. Finally,* $\mathsf{CD}(\mathbb{B}, -\mathsf{N}(t))/\mathsf{ker}(\sigma)$ *is a subalgebra of* $\mathbb{A}$*.*

Proof. Observe that the fact that $\mathbb{B}$ is unital implies that for each $b \in \mathbb{B}$, also $\overline{b} = \mathsf{T}(b) - b$ belongs to $\mathbb{B}$. So also $\mathbb{B}$ is a quadratic alternative algebra, whose trace and norm are the restrictions of those of $\mathbb{A}$. Since $\mathbb{B}$ is finite-dimensional and non-singular, we have $\mathbb{A} = \mathbb{B} \oplus \mathbb{B}^{\perp}$. For $t \in \mathbb{B}^{\perp} \setminus \{0\}$, we have $\overline{t} = -t$ and $bt = t\overline{b}$ for all $b \in \mathbb{B}$. We show that properties (4.2) to (4.4) also hold for all $a, b, c, d \in \mathbb{B}$ this time. This will then also prove that $\mathbb{B} + t\mathbb{B}$ is a subalgebra of $\mathbb{A}$.

Since $[a, t, d] = [t, \overline{a}, d]$ holds for all $a, d \in \mathbb{B}$ by alternativity of $\mathbb{A}$, we obtain $0 = (at - t\overline{a})d = a(td) - t(\overline{a}d)$, and as such (4.2) follows. By using the involution, $\overline{t} = -t$ and $t\overline{b} = bt$ for all $b \in \mathbb{B}$, we see that $(tb)c = t(bc)$ is equivalent with $\overline{c}(tb) = t(cb)$, and so (4.3) follows from (4.2). Lastly, we see that $(tb)(td) = (tb)(\overline{d}t) = t(b\overline{d})t = t(t(d\overline{b})) = t^2(d\overline{b})$ for all $b, d \in \mathbb{B}$, using the Moufang identities. From this, it follows that $\mathbb{B} + t\mathbb{B}$ is a subalgebra of $\mathbb{A}$ whose multiplication is given by the Cayley-Dickson multiplication rule. Moreover, $\mathbb{B} \cap t\mathbb{B} = \{0\}$, since $t\mathbb{B} \subseteq \mathbb{B}^{\perp}$: for all $b, c \in \mathbb{B}$ we have $f(c, tb) = c(\overline{bt}) + (bt)\overline{c} = -c(t\overline{b}) + (t\overline{b})\overline{c} = -t(\overline{c}\,\overline{b}) + t(\overline{c}\,\overline{b}) = 0$. Hence $\mathbb{B} + t\mathbb{B} = \mathbb{B} \oplus t\mathbb{B}$.

Put $\zeta = t^2 = -\mathsf{N}(t)$. Since the multiplication of $\mathsf{CD}(\mathbb{B}, \zeta)$ is the same as the multiplication in $\mathbb{B} \oplus t\mathbb{B}$, the map σ is a morphism. An element $a + tb$ is mapped to 0 if $a = tb = 0$ (since $\mathbb{B} \cap t\mathbb{B} = \{0\}$). Now $tb = 0$ implies $b = 0$ if either $\zeta \neq 0$ (for then $t(tb) = \zeta b = 0$) or if $\mathbb{B}$ is a division algebra, since then, for a non-zero b, we would have $0 = (tb)b^{-1} = t \neq 0$, a contradiction. By definition, σ is surjective; and restricted to $\mathbb{B}$, σ is injective. The lemma follows. $\square$

A consequence is the following. We omit the proof, as it is highly similar to the proof of Proposition 4.4.7.

Corollary 4.3.2. *Let* $\mathbb{A}$ *be a non-degenerate quadratic alternative algebra over a field* $\mathbb{K}$ *and put* $d = \dim_{\mathbb{K}}(\mathbb{A})$*. If* $d < \infty$*,* $\mathbb{A}$ *can be obtained by successively applying the Cayley-Dickson process on* $\mathbb{K}$ *(possibly using the adapted form of the first application in case* $\mathrm{char}(\mathbb{K}) = 2$*); if* $d = \infty$*, then* $\mathbb{A}$ *is an inseparable field extension of* $\mathbb{K}$ *with* $\mathbb{A}^2 \subseteq \mathbb{K}$ *(in this case,* $\mathrm{char}(\mathbb{K}) = 2$*).*

4.3.2 The norm form and its Witt index

For (the proof of) the following fact, I refer to the pre-book [35] of McCrimmon, Chapter 2, Section 4, Theorem 5.

Fact 4.3.3. *Two non-degenerate quadratic algebras* $\mathbb{A}_1$ *and* $\mathbb{A}_2$ *over* $\mathbb{K}$ *with respective norm forms* N_1 *and* N_2 *are isomorphic if and only if* N_1 *and* N_2 *are equivalent quadratic forms.*

Definition 4.3.4. *A non-degenerate quadratic alternative* $\mathbb{K}$*-algebra* $\mathbb{A}$ *whose norm form* N *is isotropic is called* split.

Since non-invertible elements in a quadratic alternative algebra have norm 0, this is equivalent with the norm N being isotropic. It is also equivalent to $\mathbb{A}$ possessing proper idempotents (i.e., $e \in \mathbb{A} \setminus \{0,1\}$ with $e^2 = e$) and to $\mathbb{A}$ having zero divisors. The first of the following two facts expresses the fact that as soon as N is isotropic, it is hyperbolic (i.e., of maximal Witt index). In fact, this is a characteristic property of the more general notion of *n-fold Pfister forms* (the norm form of a non-degenerate quadratic alternative $\mathbb{K}$-algebra $\mathbb{A}$ with $\dim_{\mathbb{K}}(\mathbb{A}) = 2^n$ is a n-fold Pfister form, i.e., a quadratic form of dimension 2^n that can be written as a tensor product of quadratic forms of the form $q(x,y) = x^2 - ay^2$ for some $a \in \mathbb{K} \setminus \{0\}$).

Fact 4.3.5. (i) *The norm form of a non-degenerate quadratic alternative algebra over $\mathbb{K}$ is either anisotropic or has maximal Witt index.*

(ii) *All non-degenerate split quadratic alternative algebras over $\mathbb{K}$ with the same dimension over $\mathbb{K}$ are isomorphic.*

Proof. (i) Suppose that the norm form associated to $\mathbb{A}$ is isotropic. Then there is an $x \in \mathbb{A} \setminus \{0\}$ with $\mathsf{N}(x) = 0$. Since $\mathbb{A}$ is non-degenerate, $x \notin \mathsf{rad}(f)$, so there is a $y \in \mathbb{A}$ such that $f(x,y) = 1$. Hence $e := x\overline{y}$ is a proper idempotent: $\mathsf{N}(e) = \mathsf{N}(x)\mathsf{N}(\overline{y}) = 0$ and $\mathsf{T}(e) = f(x,y) = 1$, so $e^2 = e$ indeed (note that $e \neq 1$ as $\mathsf{N}(e) = 0$ and $e \neq 0$ as $\mathsf{T}(e) = 1$).

Then $\mathbb{A} = \mathbb{A}e + \mathbb{A}\overline{e}$ since $\overline{e} = 1 - e$. This is a direct sum because if $ae = b\overline{e}$ for $a, b \in \mathbb{A}$, then $ae = ae^2 = (ae)e = (b\overline{e})e = b(\overline{e}e) = 0$. The two subspaces $\mathbb{A}e$ and $\mathbb{A}\overline{e}$ are totally isotropic since $\mathsf{N}(ae) = \mathsf{N}(a)\mathsf{N}(e) = 0 = \mathsf{N}(a\overline{e})$ for each $a \in \mathbb{A}$. Hence N has maximal Witt index indeed.

(ii) Since any two quadratic forms of maximal Witt index in the same (even) dimension are equivalent, and given Fact 4.3.3, the second assertion follows immediately from the first one. □

The above allows us to speak of *the* non-degenerate split quadratic alternative algebra over $\mathbb{K}$, which we will refer to as $\mathbb{K}$, $\mathbb{L}'$, $\mathbb{H}'$ and $\mathbb{O}'$.

The non-degenerate split quadratic alternative algebras

Following the Cayley-Dickson process, we get, if $\mathrm{char}(\mathbb{K}) \neq 2$, that $\mathbb{L}' \cong \mathsf{CD}(\mathbb{K}, 1)$, that $\mathbb{H}' \cong \mathsf{CD}(\mathbb{K}, 1, 1)$ and $\mathbb{O}' \cong \mathsf{CD}(\mathbb{K}, 1, 1, 1)$. If $\mathrm{char}(\mathbb{K}) = 2$, then we denote by $\mathbb{L}_0$ the quotient $\mathbb{K}[x]/(x^2 + x)$ (cf. Subsection 4.2.3), and we obtain that $\mathbb{L}' \cong \mathbb{L}_0$, $\mathbb{H}' \cong \mathsf{CD}(\mathbb{L}_0, 1)$ and $\mathbb{O}' \cong \mathsf{CD}(\mathbb{L}_0, 1, 1)$.

A uniform characteristic-free description is given in the following lemma.

Lemma 4.3.6. *If $\mathbb{A}$ is a split non-degenerate quadratic alternative $\mathbb{K}$-algebra, then $\mathbb{A}$ is isomorphic to either $\mathbb{K}$, $\mathbb{K} \times \mathbb{K}$, the 2×2-matrices $\mathscr{M}_{2\times 2}(\mathbb{K})$ over $\mathbb{K}$ or $\mathsf{CD}(\mathscr{M}_{2\times 2}(\mathbb{K}), 1)$ (with the obvious multiplication rules).*

Proof. If $\mathrm{char}(\mathbb{K}) \neq 2$, then $(a,b) \mapsto (a+b, a-b)$ gives an isomorphism between $\mathsf{CD}(\mathbb{K}, 1)$ and $\mathbb{K} \times \mathbb{K}$; if $\mathrm{char}(\mathbb{K}) = 2$ then $(a,b) \mapsto (a+b, a)$ gives an isomorphism between $\mathbb{L}_0$ and $\mathbb{K} \times \mathbb{K}$.

This implies that, if $\mathrm{char}(\mathbb{K}) \neq 2$, then $\mathsf{CD}(\mathbb{K}\times\mathbb{K},1)$ and $\mathsf{CD}(\mathbb{K},1,1)$ are isomorphic, and if $\mathrm{char}(\mathbb{K}) = 2$, then $\mathsf{CD}(\mathbb{K}\times\mathbb{K},1)$ and $\mathsf{CD}(\mathbb{L}_0,1)$ are isomorphic. The isomorphism

$$((a,b),(c,d)) \mapsto \left(\begin{smallmatrix} a & d \\ c & b \end{smallmatrix}\right)$$

shows that $\mathsf{CD}(\mathbb{K}\times\mathbb{K},1)$ is isomorphic to $\mathscr{M}_{2\times 2}(\mathbb{K})$ indeed. The lemma then follows from Fact 4.3.5(ii). $\square$

The split octonions, being non-associative, cannot be given by ordinary matrices and their ordinary multiplication. Zorn's vector-matrices however are a special way of writing the split octonions as 2×2-matrices, the off-diagonal elements of which are vectors:

$$\mathbb{O}' = \left\{ \begin{pmatrix} a & \begin{bmatrix} b & x & y \end{bmatrix} \\ \begin{bmatrix} c \\ z \\ u \end{bmatrix} & d \end{pmatrix} : a,b,c,d,x,y,u,z \in \mathbb{K} \right\},$$

and the multiplication is given as follows (with the usual dot product and vector product):

$$\begin{pmatrix} a & \boldsymbol{v} \\ \boldsymbol{w} & d \end{pmatrix}\begin{pmatrix} a' & \boldsymbol{v'} \\ \boldsymbol{w'} & d' \end{pmatrix} = \begin{pmatrix} aa' + \boldsymbol{v}\cdot\boldsymbol{w'} & a\boldsymbol{v'} + d'\boldsymbol{v} + \boldsymbol{w}\times\boldsymbol{w'} \\ a'\boldsymbol{w} + d\boldsymbol{w'} - \boldsymbol{v}\times\boldsymbol{v'} & dd' + \boldsymbol{v'}\cdot\boldsymbol{w} \end{pmatrix}$$

For short, we denote the matrices in the above set by $M(a,b,c,d,x,y,z,u)$.

4.4 Degenerate quadratic alternative algebras

Let $\mathbb{A}$ be a degenerate quadratic alternative unital $\mathbb{K}$-algebra, with associated norm form N and corresponding bilinear form f, and radical R (which is assumed to be non-trivial).

We will show the following theorem.

Theorem 4.4.1. *Let $\mathbb{A}$ be a degenerate quadratic alternative $\mathbb{K}$-algebra whose radical R is generated (as a ring) by a single element $t \in \mathbb{A}\setminus\{0\}$. Then $\mathbb{A}$ has a non-degenerate quadratic associative unital algebra $\mathbb{B}$ such that $\mathbb{A} = \mathbb{B} \oplus t\mathbb{B}$. Moreover*

(i) *If $\mathbb{B}$ is division, then $\mathbb{A}$ is isomorphic to* $\mathsf{CD}(\mathbb{B},0)$*;*

(ii) *If $\mathbb{B}$ is split, then either $\mathbb{A}$ is isomorphic to* $\mathsf{CD}(\mathbb{B},0)$ *or* $\dim_{\mathbb{K}}(\mathbb{A}) \in \{3,6\}$ *and in the latter case, $\mathbb{A}$ is isomorphic to the following respective quotients of* $\mathsf{CD}(\mathbb{B},0)$*:*

(a) the upper triangular 2×2-matrices over $\mathbb{K}$ (the ternions *$\mathbb{T}'$);*
(b) $\{M(a,b,c,d,0,y,z,0) \mid a,b,c,d,y,z \in \mathbb{K}\}$ (the sextonions *$\mathbb{S}'$);*

Finally, if $\dim_{\mathbb{K}}(\mathbb{A}) < 8$, then $\mathbb{A}$ is isomorphic to a subalgebra of the split octonions $\mathbb{O}'$.

Remark 4.4.2. Usually, the ternions and the sextonions (see also [28]) are denoted by $\mathbb{T}$ and $\mathbb{S}$, respectively, but in this setting we prefer the accent to emphasise that these are split algebras.

Remark 4.4.3. By Lemmas 4.1.3 and 4.1.5, the fact that the radical R is generated by t as a ring means the same as that it is generated by t as an ideal.

Definition 4.4.4. We will refer to the algebras $\mathbb{A} = \mathbb{B} \oplus t\mathbb{B}$ of the above theorem as the (non-)split *generalised dual numbers* if $\mathbb{B}$ is (non-)split.

4.4.1 A decomposition

To start, we show in general that $\mathbb{A}$ can be written as the direct sum of a maximal non-degenerate quadratic unital subalgebra $\mathbb{B}$ and the radical. In case $\mathsf{rad}(f) = \mathbb{A}$, we can immediately do this.

Proposition 4.4.5. *Let $\mathbb{A}$ be a degenerate quadratic alternative unital $\mathbb{K}$-algebra for which* $\mathsf{rad}(f) = \mathbb{A}$. *Then* $\operatorname{char}\mathbb{K} = 2$ *and $\mathbb{A}$ is a commutative associative $\mathbb{K}$-algebra with trivial involution and with $\mathbb{A}^2 \subseteq \mathbb{K}$. Moreover, $\mathbb{A}$ contains a unital subalgebra $\mathbb{B}$ maximal with the property of being non-degenerate such that $\mathbb{A} = \mathbb{B} \oplus R$ (and $R = \{a \in \mathbb{A} : a^2 = 0\}$ in this case).*

Proof. By Lemma 4.1.4, $\mathsf{rad}(f) = \mathbb{A}$ implies that $\operatorname{char}\mathbb{K} = 2$ and that $\mathbb{A}$ is a commutative associative $\mathbb{K}$-algebra with trivial involution. Since $\mathbb{A}$ is quadratic, we then have $x^2 = \mathsf{N}(x) \in \mathbb{K}$ for each $x \in \mathbb{A}$, so $\mathbb{A}^2 \subseteq \mathbb{K}$. Using Zorn's Lemma, there exists a unital subalgebra $\mathbb{B}$ maximal with the property of being non-degenerate (here the latter is equivalent to $\mathsf{N}(b) = b^2 \neq 0$ for all $b \in \mathbb{B}$).

We claim that $\mathbb{B}^2 = \mathbb{A}^2$. Indeed, if there is an element $a \in \mathbb{A} \setminus \mathbb{B}$ with $a^2 \in \mathbb{A}^2 \setminus \mathbb{B}^2$, then the algebra generated by $\mathbb{B}$ and a is isomorphic to $\mathsf{CD}(\mathbb{B}, a^2)$, so according to Lemma 4.2.5, it is a non-degenerate unital subalgebra of $\mathbb{A}$ strictly containing $\mathbb{B}$, contradicting the latter's maximality. This shows the claim.

The elements r in R are those for which $\mathsf{N}(r) = r^2 = 0$. Let a be an arbitrary element of $\mathbb{A} \setminus \mathbb{B}$. Since $\mathbb{A}^2 = \mathbb{B}^2$, we have $a^2 = b^2$ for some $b \in \mathbb{B}$, so $a + b \in R$. We obtain $a = b + (a + b) \in \mathbb{B} + R$, from which we conclude that $\mathbb{A} = \mathbb{B} + R$. Since $\mathbb{B}$ is non-degenerate, $R \cap \mathbb{B} = \{0\}$ and hence $\mathbb{A} = \mathbb{B} \oplus R$. □

In case that $\mathsf{rad}(f) < \mathbb{A}$, we rely on Lemma 4.3.1.

Proposition 4.4.6. *Let $\mathbb{A}$ be a possibly degenerate quadratic alternative $\mathbb{K}$-algebra for which* $\mathsf{rad}(f) < \mathbb{A}$. *Then $\mathbb{A}$ contains a unital subalgebra $\mathbb{B}$ which is maximal with the property of being non-degenerate. Moreover, $\mathbb{B}^\perp = R$ and $\mathbb{A} = \mathbb{B} \oplus R$.*

Proof. Lemma 4.1.4 and $\mathsf{rad}(f) < \mathbb{A}$ imply that that $R = \mathsf{rad}(f)$. By assumption, R is non-trivial. Suppose first that there is a non-singular unital subalgebra $\mathbb{B}$ of $\mathbb{A}$. Then $\mathbb{B}$ is in particular a non-degenerate quadratic alternative algebra over $\mathbb{K}$, so it falls into one of the five classes described in Fact 3.0.1, in fact, since it is non-singular, the last possibility does not occur. Consequently, as $\mathbb{B}$ is finite-dimensional and non-singular, we obtain $\mathbb{A} = \mathbb{B} \oplus \mathbb{B}^\perp$ and we may apply Lemma 4.3.1, for any $t \in \mathbb{B}^\perp \setminus \{0\}$. If there is such a t with $\zeta := -\mathsf{N}(t) \neq 0$, then according to the lemma, $\mathsf{CD}(\mathbb{B}, \zeta) = \mathbb{B} \oplus t\mathbb{B}$ is a subalgebra of $\mathbb{A}$, which is moreover non-singular by Lemma 4.2.4.

We now show that such an algebra $\mathbb{B}$ exists. If $\mathbb{K}$ is non-singular, we put $\mathbb{B} = \mathbb{K}$. So suppose $\mathbb{K}$ is singular. Then $\mathsf{rad}(f|_{\mathbb{K}}) = \mathbb{K}$ since a field has no proper ideals, and consequently $\operatorname{char}\mathbb{K} = 2$ (cf. Lemma 4.1.4). Since $\mathsf{rad}(f) < \mathbb{A}$, there is an element $u \in \mathbb{A}$ such that $f(1,u) \neq 0$ and by rescaling u we may assume that $f(1,u) = 1$. Clearly, $u \notin \mathbb{K}$ and hence $\mathbb{K} \cap u\mathbb{K} = \{0\}$. For this u, we have $\overline{u} = u + 1$ and we claim that $\mathbb{K} \oplus u\mathbb{K}$ is a non-singular subalgebra of $\mathbb{A}$. Indeed, the fact that it is a subalgebra follows easily since $\mathbb{K}$ is contained in the center of $\mathbb{A}$. As for the non-singularity, if there are $x, y \in \mathbb{K}$ such that $f(x + uy, x' + uy') = 0$ for all $x', y' \in \mathbb{K}$, then a calculation shows that $xy' = yx'$ for all $x', y' \in \mathbb{K}$, from which we conclude that $x = y = 0$. This shows the claim. So if $\mathbb{K}$ is singular, there is a $u \in \mathbb{A}$ such that $\mathbb{K} \oplus u\mathbb{K}$ is not, and hence this serves as $\mathbb{B}$. Note that $\mathbb{K} \oplus u\mathbb{K}$ is the result of one application of the Cayley-Dickson doubling process adapted to characteristic 2.

Now, applying the reasoning of the first paragraph, we obtain an element $t \in \mathbb{A}$ such that $\mathsf{CD}(\mathbb{B}, -\mathsf{N}(t))$ is a unital subalgebra of $\mathbb{A}$. Continuing like this (noting that this process ends after at most three applications, by Lemma 4.2.6 and the fact that $\mathbb{A}$ is alternative), we obtain a unital subalgebra of $\mathbb{A}$, which we also denote by $\mathbb{B}$, maximal with the property of being non-singular. By construction, $\mathbb{B}$ is a non-degenerate quadratic alternative algebra (as listed in Fact 3.0.1).

We then claim that $R = \mathbb{B}^\perp$. Clearly, $R = \mathsf{rad}(f) = \mathbb{A}^\perp \subseteq \mathbb{B}^\perp$. For the reverse inclusion, we first note that for each $t \in \mathbb{B}^\perp$ holds that $\overline{t} = -t$ (since $f(1,t) = 0$) and $\mathsf{N}(t) = -t^2 = 0$ (if not then the above implies that $\mathsf{CD}(\mathbb{B}, -\mathsf{N}(t))$ is a non-singular algebra strictly containing $\mathbb{B}$, a contradiction). In particular, for all $t, v \in \mathbb{B}^\perp$ we have $(t+v)^2 = tv + vt = 0$, so $tv = -vt$, or $t\overline{v} + v\overline{t} = f(v,t) = 0$. Now take any $t \in \mathbb{B}^\perp$ and let $a \in \mathbb{A}$ be general. Then $a = b + v$ for unique $b \in \mathbb{B}$ and $v \in \mathbb{B}^\perp$. We get $f(a,t) = (b+v)\overline{t} + (\overline{b} + \overline{v})t = (b\overline{t} + \overline{b}t) + (v\overline{t} + \overline{t}v) = 0$: the first term is 0 since $t \in \mathbb{B}^\perp$ and $b \in \mathbb{B}$, the second one is 0 by the foregoing ($t, v \in \mathbb{B}^\perp$). Hence $t \in R$ indeed. We conclude that $\mathbb{A} = \mathbb{B} \oplus R$.

Since $\mathbb{A} = \mathbb{B} \oplus R$, this implies that $\mathbb{B}$ is also maximal with the property of being non-degenerate: indeed, $\mathbb{B}$ is non-degenerate as it is even non-singular, and any subalgebra of $\mathbb{A}$ strictly containing $\mathbb{B}$ would intersect R non-trivially and hence be degenerate.

This shows the proposition. □

Note that the proofs of Proposition 4.4.5 and 4.4.6 also yield a proof of Fact 3.0.1.

4.4.2 Degenerate quadratic alternative algebras whose radical is a principal ideal

We are especially interested in the quadratic alternative algebras $\mathbb{A}$ where R is a principal ideal of $\mathbb{A}$, i.e., generated by a single element t. By Lemma 4.1.5, this notion is well-defined.

Proposition 4.4.7. *Let $\mathbb{A}$ be a degenerate quadratic alternative $\mathbb{K}$-algebra whose radical R is generated by a single element $t \in \mathbb{A} \setminus \{0\}$. Then $\mathbb{A}$ has a non-degenerate associative unital subalgebra $\mathbb{B}$ such that $\mathbb{A} = \mathbb{B} \oplus t\mathbb{B}$ and the map*

$$\sigma : \mathsf{CD}(\mathbb{B}, 0) \to \mathbb{B} \oplus t\mathbb{B} : (a, b) \mapsto a + tb$$

is a morphism. If $\mathbb{B}$ is a division algebra then σ is an isomorphism. In particular, $\dim_{\mathbb{K}}(\mathbb{B}) < \dim_{\mathbb{K}}(\mathbb{A}) \leq 2\dim_{\mathbb{K}}(\mathbb{B})$.

Proof. By Proposition 4.4.6 and Lemma 4.4.5, we have $\mathbb{A} = \mathbb{B} \oplus R$, where $\mathbb{B}$ is a maximal non-degenerate unital subalgebra of $\mathbb{A}$. Our assumption implies that R is generated by a single element $t \neq 0$, and by Lemma 4.1.5, the ideal generated by t is equal to $t\mathbb{A}$ (which is equal to $\mathbb{A}t$). Note that $t \in R$ implies $t^2 = 0$. Since $\mathbb{A} = \mathbb{B} \oplus t\mathbb{A}$, it follows that $t\mathbb{A} = t\mathbb{B} \oplus t(t\mathbb{A}) = t\mathbb{B}$ (as $t(ta) = t^2 a = 0$ for all $a \in \mathbb{A}$). So $\mathbb{A} = \mathbb{B} \oplus t\mathbb{B}$.

If $\mathsf{rad}(f) = \mathbb{A}$, it follows that $\mathbb{A} = \mathbb{B}[t]/(t^2)$, and hence $\mathbb{A} = \mathsf{CD}(\mathbb{B}, 0)$. So suppose $\mathsf{rad}(f) < \mathbb{A}$. By Lemma 4.3.1, $\mathbb{B} \oplus t\mathbb{B}$ is isomorphic to $\mathsf{CD}(\mathbb{B}, 0)/\ker(\sigma)$, and $\ker(\sigma) = \{0\}$ if $\mathbb{B}$ is a division algebra. If $\ker(\sigma) = \{0\}$, it follows immediately by Lemma 4.2.6 that $\mathbb{A} = \mathsf{CD}(\mathbb{B}, 0)$ cannot be alternative if $\mathbb{B}$ is not associative. If $\mathbb{B}$ is not a division algebra, we need to do some more work.

So suppose $\mathbb{B}$ is strictly alternative and split (so $\mathbb{B} = \mathbb{O}'$). We show that $\mathbb{A}$ cannot be alternative. Now, $\mathsf{CD}(\mathbb{B}, 0)/\ker(\sigma)$ is alternative if and only if the equations in 4.8 hold for all $a, b, c, d \in \mathbb{B}$ (following exactly the same reasoning as in the proof of Lemma 4.2.6, using the fact that $\mathbb{B}$ is alternative). Since $\zeta = 0$, the equations in 4.8 are equivalent to $t(a(cb) - (ac)b) = 0$ for all $a, b, c \in \mathbb{B}$. Moreover, Lemma 4.3.1 tells us that $tx = 0$ for $x \in \mathbb{B}$ implies that $\mathsf{N}(x) = 0$. So, if there is an associator $[a, b, c]$ with $a, b, c \in \mathbb{B}$ and $\mathsf{N}([a, b, c]) \neq 0$, then $\mathsf{CD}(\mathbb{B}, 0)/\ker(\sigma)$ is not alternative. Now, if $\operatorname{char}\mathbb{K} \neq 2$, then $\mathbb{B}$ results from three applications of the Cayley-Dickson process with primitive element 1, where the "new" elements are u, v, w, respectively; i.e., $u^2 = v^2 = w^= 1$ and $u \in \mathbb{K}^{\perp}$, $v \in (\mathbb{K} \oplus u\mathbb{K})^{\perp}$ and $w \in (\mathbb{L}' \oplus v\mathbb{L}')^{\perp}$ for $\mathbb{L}' = \mathbb{K} \oplus u\mathbb{K}$, and u, v, w anti-commute and anti-associate as one can check (by applying properties (4.1) to (4.4) again). So $[u, v, w] = (uv)w - u(vw) = 2(uv)w$, and hence $\mathsf{N}(2(uv)w) = 4\mathsf{N}(u)\mathsf{N}(v)\mathsf{N}(w) = 4 \neq 0$, as required. Next, if $\operatorname{char}(\mathbb{K}) = 2$, then the first (adapted) step of the Cayley-Dickson process takes $\mathbb{K}$ to $\mathbb{K} \oplus u\mathbb{K}$ where $\overline{u} = 1 + u$; in the later steps we again use 1 as a primitive element and hence get v and w with$v^2 = w^2 = 1$ and $v \in (\mathbb{K} \oplus u\mathbb{K})^{\perp}$ and $w \in (\mathbb{L}' \oplus v\mathbb{L}')^{\perp}$ for $\mathbb{L}' = \mathbb{K} \oplus u\mathbb{K}$. This time $[u, v] = v$ and $[u, v, w] = wv$, so $\mathsf{N}([u, v, w]) = \mathsf{N}(w)\mathsf{N}(v) = 1 \neq 0$. We conclude that $\mathsf{CD}(\mathbb{B}, 0)/\ker(\sigma)$ cannot be alternative if $\mathbb{B}$ is not associative. □

Proposition 4.4.8. *Let $\mathbb{A}$ be a degenerate quadratic alternative $\mathbb{K}$-algebra whose radical R is (as a ring) generated by a single element $t \in \mathbb{A} \setminus \{0\}$ and suppose that $\mathbb{A}$ does not result from the Cayley-Dickson doubling process on $\mathbb{K}$. Then either*

- (i) $\dim_{\mathbb{K}}(\mathbb{A}) = 3$*, and then $\mathbb{A}$ is isomorphic to the upper triangular 2×2-matrices over $\mathbb{K}$, i.e., to the ternions $\mathbb{T}'$;*
- (ii) $\dim_{\mathbb{K}}(\mathbb{A}) = 6$ *and then $\mathbb{A}$ is isomorphic to $\{M(a, b, c, d, 0, y, z, 0) \mid a, b, c, d, y, z \in \mathbb{K}\}$, i.e., to the (non-associative) sextonions $\mathbb{S}'$.*

Proof. By Proposition 4.4.7, $\mathbb{A} \cong \mathsf{CD}(\mathbb{B},0)/\ker(\sigma)$ for a non-degenerate unital subalgebra $\mathbb{B}$ of $\mathbb{A}$, and our assumption means that $\ker(\sigma)$ is non-trivial. The same proposition implies that $\mathbb{B}$ is a quadratic associative algebra which is split, and moreover $\dim(\mathbb{B}) < \dim(\mathbb{A}) < 2\dim(\mathbb{B})$, which in particular means that $\dim(\mathbb{B}) \in \{2,4\}$. By Lemma 4.3.1 we have that $\ker(\sigma) \subseteq t\mathbb{B}$ and that for $b \in \mathbb{B}$, $tb \in \ker(\sigma)$ mplies $\mathsf{N}(b) = 0$.

(i) Suppose first that $\dim(\mathbb{B}) = 2$. The dimension restriction then yields $\dim(\ker(\sigma)) = 1$. By Lemma 4.3.6, $\mathbb{B}$ is isomorphic to $\mathbb{K} \times \mathbb{K}$ (regardless of $\mathrm{char}(\mathbb{K})$). Suppose that $t(c,d) = 0$ for $(c,d) \in \mathbb{K} \times \mathbb{K}$. Then by the above, $\mathsf{N}(c,d) = 0$, so either $c = 0$ or $d = 0$ (and only one of the two, for otherwise $\dim(\ker(\sigma)) = 2$). Noting that we can represent the elements $(a,b) + t(c,d)$ of $\mathsf{CD}(\mathbb{K} \times \mathbb{K}, 0)$ as 2×2-matrices $\left(\begin{smallmatrix} a & td \\ tc & b \end{smallmatrix}\right)$ (with ordinary matrix multiplication), the two possibilities for $\ker(\sigma)$ are $\left(\begin{smallmatrix} 0 & td \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ tc & 0 \end{smallmatrix}\right)$; and the corresponding quotients $\mathsf{CD}(\mathbb{B},0)/\ker(\sigma)$ are isomorphic to the following two 3-dimensional $\mathbb{K}$-algebras, respectively:

$$\left\{\left(\begin{smallmatrix} a & 0 \\ tc & b \end{smallmatrix}\right) \mid a,b,c \in \mathbb{K}\right\} \quad \text{and} \quad \left\{\left(\begin{smallmatrix} a & td \\ 0 & b \end{smallmatrix}\right) \mid a,b,c \in \mathbb{K}\right\}.$$

In these matrices, we can leave out the "t" without harming the multiplication rules (which we could not do for $\mathsf{CD}(\mathbb{K} \times \mathbb{K}, 0)$), and hence we obtain the upper and lower triangular matrices (which are of course isomorphic to each other).

(ii) Next, suppose that $\dim(\mathbb{B}) = 4$. By Lemma 4.3.6, $\mathbb{B}$ is isomorphic to $\mathscr{M}_{2\times 2}(\mathbb{K})$. The dimension restriction now yields $1 \leq \dim(\ker(\sigma)) \leq 3$. Put $M(e,f,g,h) := \left(\begin{smallmatrix} e & h \\ g & f \end{smallmatrix}\right) \in \mathscr{M}_{2\times 2}(\mathbb{K})$. As above, $tM(e,f,g,h) = 0$ implies $\mathsf{N}(M(e,f,g,h)) = ef - gh = 0$. Clearly, $\ker(\sigma)$ is closed under left multiplication: if $tM = 0$ for some $M \in \mathscr{M}_{2\times 2}(\mathbb{K})$, then also $t(XM) = (tM)X = 0$ for each $X \in \mathscr{M}_{2\times 2}(\mathbb{K})$ (cf. 4.3). It then follows that there are $e, f\, g \in \mathbb{K}$ such that

$$\ker(\sigma) = t\left\{K_{r,s} := \left(\begin{smallmatrix} re & rg \\ se & sg \end{smallmatrix}\right) \mid r,s \in \mathbb{K}\right\}.$$

Let $M \in \mathscr{M}_{2\times 2}(\mathbb{K})$ be arbitrary. If $e \neq 0$, then we can choose $r,s \in \mathbb{K}$ such that the first column of $M + K_{r,s}$ is zero; if $g \neq 0$ then we can likewise choose $r,s \in \mathbb{K}$ such that the last column of $M + K_{r,s}$ is zero. We obtain that the quotient $\mathsf{CD}(\mathbb{B},0)/\ker(\sigma)$ is isomorphic to one of the following two 6-dimensional $\mathbb{K}$-algebras:

$$\left\{\left(\begin{smallmatrix} a & d \\ c & b \end{smallmatrix}\right) + t\left(\begin{smallmatrix} 0 & h \\ 0 & f \end{smallmatrix}\right) \mid a,b,c,d,f,h \in \mathbb{K}\right\} \quad \text{and} \quad \left\{\left(\begin{smallmatrix} a & d \\ c & b \end{smallmatrix}\right) + t\left(\begin{smallmatrix} e & 0 \\ g & 0 \end{smallmatrix}\right) \mid a,b,c,d,e,g \in \mathbb{K}\right\}. \tag{4.9}$$

Clearly, both options are isomorphic to each other.

It is a straightforward verification that the algebra $\mathbb{S}' = \{M(a,b,c,d,0,y,z,0) \mid a,b,c,d,y,z \in \mathbb{K}\}$ is a 6-dimensional subalgebra of $\mathbb{O}'$ (i.e., one should verify that it is closed under multiplication), and that its radical is a principal ideal (generated by $M(0,0,0,0,0,1,1,0)$, for instance). Hence $\mathbb{S}'$ is isomorphic to the algebras in 4.9 and hence to $\mathbb{A}$.

This shows the proposition. □

Remark 4.4.9. A shortcut in the above proof (which we did not take for reasons of transparency but is worth mentioning) is the following. In the second case, as soon as we know that $\dim(\ker(\sigma)) > 1$, it follows from Theorem 5 of [37] that $\dim(\ker(\sigma)) = 2$ and that $\mathsf{CD}(\mathbb{B},0)/\ker(\sigma)$ is isomorphic to $\mathbb{S}'$, for that theorem says that the a maximal subalgebra of $\mathbb{O}'$ is either a division quaternion algebra or $\mathbb{S}'$.

In order to complete the proof of Theorem 4.4.1, we need one more thing.

Lemma 4.4.10. *The split quadratic alternative algebras $\mathbb{A}$ whose radical R is generated by a single element $t \in \mathbb{A} \setminus \{0\}$ and with $\dim_{\mathbb{K}}(\mathbb{A}) < 8$ are isomorphic to subalgebras of $\mathbb{O}'$.*

Proof. By definition, $\mathbb{S}'$ is a subalgebra of $\mathbb{O}' \cong \{M(a,b,c,d,x,y,z,u) \mid a,b,c,d,x,y,z,u \in \mathbb{K}\}$. Furthermore one can verify that

$$\mathsf{CD}(\mathbb{L}',0) \cong \{M(a,0,0,d,0,y,z,0) \mid a,d,y,z \in \mathbb{K}\},$$

$$\mathbb{T}' \cong \{M(a,0,0,d,0,y,0,0) \mid a,d,y \in \mathbb{K}\}$$

$$\mathsf{CD}(\mathbb{K},0) \cong \{M(a,0,0,0,0,y,0,0) \mid a,y \in \mathbb{K}\},$$

from which the assertion follows. □

We have shown Theorem 4.4.1.

Remark 4.4.11. To make the story complete, we add that $\mathbb{S}'$ is isomorphic to $\mathsf{CD}(\mathbb{T}',1)$.

CHAPTER

5

VERONESE VARIETIES ASSOCIATED TO GENERALISED DUAL NUMBERS

In this chapter, we associate Veronese varieties to the quadratic alternative $\mathbb{K}$-algebras $\mathbb{A}$ whose radical R is generated by a single element $t \in \mathbb{A} \setminus \{0\}$, and where $\mathbb{K}$ is an arbitrary field. Recall from Theorem 4.4.1 that these algebras are one of the following:

(i) $\mathsf{CD}(\mathbb{B},0)$, where $\mathbb{B}$ is a quadratic associative division algebra;

(ii) $\mathbb{T}'$, $\mathsf{CD}(\mathbb{L}',0)$, $\mathbb{S}'$, $\mathsf{CD}(\mathbb{H}',0)$.

We call the algebras in (i) *non-split generalised dual numbers* and the algebras in (ii) *split generalised dual numbers*. Seeing their different nature, we treat these two cases separately.

Remark 5.0.1. We do not consider the algebra $\mathsf{CD}(\mathbb{K},0)$ twice, instead we categorise it among the non-split generalised dual numbers.

5.1 Geometries over the non-split generalised dual numbers $\mathbb{A}$

Let $\mathbb{A}$ be $\mathsf{CD}(\mathbb{B},0) \cong \mathbb{B} \oplus t\mathbb{B}$, where $\mathbb{B}$ is a quadratic associative division algebra and t an element generating the radical R of $\mathbb{A}$. Since $\mathbb{B}$ is associative and $t^2 = 0$, we have $t((xy)z) = t(x(yz))$, which we can hence write as $t(xyz)$, for all $x, y, z \in \mathbb{A}$. For each element $a = a_0 + ta_1 \in \mathsf{CD}(\mathbb{B},0)$, with $a_0, a_1 \in \mathbb{B}$, we write $\widetilde{a} = a_0$. Put $d = \dim_{\mathbb{K}}(\mathbb{A})$.

5.1.1 The ring projective plane $\mathsf{G}_2(\mathbb{K},\mathbb{A})$

We define a ring projective plane coordinatised over $\mathbb{A}$, equipped with a *neighbouring relation*.

Definition 5.1.1. The point-line geometry $\mathsf{G}_2(\mathbb{K},\mathbb{A}) := (\mathscr{P},\mathscr{L})$ is defined as follows:

- $\mathscr{P} = \{(x,y,1) \mid x,y \in \mathbb{A}\} \cup \{(1,y,tz_1) \mid z_1 \in \mathbb{B}, y \in \mathbb{A}\} \cup \{(tx_1,1,tz_1) \mid x_1,z_1 \in \mathbb{B}\}$
- $\mathscr{L} = \{[a,1,c] \mid a,c \in \mathbb{A}\} \cup \{[1,tb_1,c] \mid b_1 \in \mathbb{B}, c \in \mathbb{A}\} \cup \{[ta_1,tb_1,1] \mid a_1,b_1 \in \mathbb{B}\}$
- A point (x,y,z) is *incident* with a line $[a,b,c]$ if and only if $ax+by+cz=0$.

The neighbouring relation $\approx$ on $(\mathscr{P}\cup\mathscr{L})\times(\mathscr{P}\cup\mathscr{L})$ is defined as follows. Two points (x,y,z) and (x',y',z') are neighbouring if $(\widetilde{x},\widetilde{y},\widetilde{z}) = (\widetilde{x'},\widetilde{y'},\widetilde{z'})$. Likewise for two lines. A point (x,y,z) and a line $[a,b,c]$ are called neighbouring if $ax+by+cz \in t\mathbb{B}$.

One can verify that a point P and a line L are not neighbouring if and only if P and Q are not neighbouring for each point Q on L.

Remark 5.1.2. If $\mathbb{A}$ is associative, we could also use the homogenous point set $\{(x,y,z)r \mid x,y,z,r \in \mathbb{A}$ with $\neg(n(x)=n(y)=n(z)=0)$ and $n(r)\neq 0\}$ and dually, the homogenous line set $\{s[a,b,c] \mid a,b,c,s \in \mathbb{A}$ with $\neg(n(a)=n(b)=n(c)=0)$ and $n(s)\neq 0\}$. The lack of associativity prevents us from doing this, as scalar multiples are not well-defined.

The point-line geometry $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ is a projective Hjelmslev plane of level 2, with above neighbouring relations. The canonical epimorphism to a projective plane is given by the map $\mathbb{A}\to\mathbb{B} : a \mapsto \widetilde{a}$. Indeed, if we set $t=0$ in the above, then $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ becomes $\mathsf{G}_2(\mathbb{K},\mathbb{B}) \cong \mathsf{PG}(2,\mathbb{B})$ (this should be clear from the homogenous description above; the neighbouring relation then coincides with "equality" between elements of the same type and with "incidence" between points and lines).

5.1.2 The Veronese representation of $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ of $\mathsf{G}_2(\mathbb{K},\mathbb{A})$

Definition 5.1.3. The Veronese representation $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ of $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ is the point-subspace structure (X,Ξ) defined by means of the Veronese map

$$\rho : \mathsf{G}_2(\mathbb{K},\mathbb{A}) \to \mathsf{PG}(3d+2,\mathbb{K}) : (x,y,z) \mapsto \mathbb{K}(x\overline{x}, y\overline{y}, z\overline{z}; y\overline{z}, z\overline{x}, x\overline{y})$$

by setting $X = \{\rho(p) \mid p \in \mathscr{P}\}$ and $\Xi = \{\langle\rho(L)\rangle \mid L \in \mathscr{L}\}$, where $\rho(L)$ is defined as $\{\rho(p) \mid p \in L\}$ and incidence is given by containment made symmetric.

Note that, for $p \in \mathscr{P}$, $\rho(p) \in \mathsf{PG}(3d+2,\mathbb{K})$ indeed: the six entries correspond to d-tuples over $\mathbb{K}$ and the first three first belong to $\mathbb{K}$, being norms. In the next section we discuss the geometric structure of a line and the features of this geometry, after having studies transitivity properties.

5.1.3 Properties of the Veronese variety $\mathscr{V}_2(\mathbb{K},\mathbb{A})$

We continue with the same notation as in the previous section.

The induced action of the collineation group of $\mathscr{V}_2(\mathbb{K},\mathbb{A})$—The geometry $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ is a Moufang projective plane if $\mathbb{A}$ is division, and we assert that it is a *Moufang Hjelmslev plane of level* 2 if $\mathbb{A}$ is not division. However, most Hjelmslev planes with the Moufang property studied in the literature are *commutative* extensions of Moufang projective planes, i.e., if the underlying projective plane is defined over the not necessarily associative alternative division ring $\mathbb{D}$, then the Hjelmslev plane is defined over the ring $\mathbb{D}[t]/(t^n=0)$, where t is an indeterminate that commutes with each element of $\mathbb{D}$. Hence we provide a full proof of the above stated assertion (suppressing tedious calculations), using standard methods (we need the explicit forms of certain collineations anyway in the proof of Proposition 5.1.5). The fact that $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ is a Hjelmslev plane of level 2 if $\mathbb{A}$ is not division, is proved in Section 6.4.3, where one also can find the precise definition of that notion. We now concentrate on the Moufang property.

A *collineation* of $\mathsf{G}_2(\mathbb{K},\mathbb{A})=(\mathscr{P},\mathscr{L})$ is a permutation of $\mathscr{P}\cup\mathscr{L}$ preserving both $\mathscr{P}$ and $\mathscr{L}$ and preserving the incidence relation. An *elation* of $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ is a collineation that fixes all points on a certain line L—called the axis—and all lines incident with a certain point P—called the center—with $P*L$ (the pair $\{P,L\}$ is called a *flag*). Such an elation is, with this notation, sometimes also called a (P,L)-*elation*. The geometry $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ is called (P,L)-*transitive*, for $P\in\mathscr{P}$ and $L\in\mathscr{L}$, with $P*L$, if for some line $M*P$, with $M\not\approx L$, the group of (P,L)-elations acts transitively on the set of points of M not neighbouring P. Then $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ has *the Moufang property*, or $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ *is a Moufang (projective or Hjelmslev) plane*, if for every point P and every line L incident with P the plane is (P,L)-transitive. It is well known and easy to see that this is equivalent with the existence of a triangle $P_0*L_1*P_2*L_0*P_1*L_2*P_0$, with $P_i\not\approx L_i$, $i=0,1,2$, such that $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ is (P_i,L_j)-transitive for $i\neq j$ and $i,j\in\{0,1,2\}$, because the collineation group generated by the (P_i,L_j)-elations, $i\neq j$, $\{i,j\}\subseteq\{0,1,2\}$, acts transitively on the set of flags.

The collineation group of $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ generated by all elations is called its *little projective group* and shall be denoted by $\mathsf{PSL}_3(\mathbb{A})$.

Lemma 5.1.4. *The plane $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ is Moufang.*

Proof. Indeed, the mappings (using the notation as above, and with $X,Y\in\mathbb{A}$ arbitrarily)

$$varphi_{23}(Y):(\mathscr{P},\mathscr{L})\longrightarrow(\mathscr{P},\mathscr{L}):\left\{\begin{array}{rcl}(x,y,1)&\mapsto&(x,y+Y,1),\\(1,y,tz_1)&\mapsto&(1,y-t\overline{Y}z_1,tz_1),\\(tx_1,1,tz_1)&\mapsto&(tx_1,1,tz_1),\\\hline [a,1,c]&\mapsto&[a,1,c-Y],\\ [1,tb_1,c]&\mapsto&[1,tb_1,c-tYb],\\ [ta_1,tb_1,1]&\mapsto&[ta_1,tb_1,1]\end{array}\right.$$

and

$$\varphi_{13}(X):(\mathscr{P},\mathscr{L})\longrightarrow(\mathscr{P},\mathscr{L}):\left\{\begin{array}{rcl}(x,y,1)&\mapsto&(x+X,y,1),\\(1,y,tz_1)&\mapsto&(1,y-t\overline{X}\overline{y}z_1,tz_1),\\(tx_1,1,tz_1)&\mapsto&(tx_1+t\overline{X}z_1,1,tz_1),\\\hline [a,1,c]&\mapsto&[a,1,c-aX],\\ [1,tb_1,c]&\mapsto&[1,tb_1,c-X],\\ [ta_1,tb_1,1]&\mapsto&[ta_1,tb_1,1]\end{array}\right.$$

are $((0,1,0),[0,0,1])$-elations and $((1,0,0),[0,0,1])$-elations, respectively, and by varying Y and X we obtain $((0,1,0),[0,0,1])$-transitivity and $((1,0,0),[0,0,1])$-transitivity, respectively. Moreover, the triality map

$$\tau:(\mathscr{P},\mathscr{L})\longrightarrow(\mathscr{P},\mathscr{L}):\left\{\begin{array}{rcll}(x,y,1)&\mapsto&(y^{-1},xy^{-1},1),&\text{if } y\in\mathbb{A}\setminus t\mathbb{B},\\(x,y,1)&\mapsto&(1,x,ty_1),&\text{if } t\mathbb{B}\ni y=ty_1,y_1\in\mathbb{B},\\(1,y,tz_1)&\mapsto&(t(y^{-1}z_1),y^{-1},1),&\text{if } y\in\mathbb{A}\setminus t\mathbb{B},\\(1,y,tz_1)&\mapsto&(tz_1,1,ty_1),&\text{if } t\mathbb{B}\ni y=ty_1,y_1\in\mathbb{B},\\(tx_1,1,tz_1)&\mapsto&(tz_1,tx_1,1),&\\\hline [a,1,c]&\mapsto&[a^{-1}c,1,a^{-1}]&\text{if } a\in\mathbb{A}\setminus t\mathbb{B},\\ [a,1,c]&\mapsto&[1,t(\overline{c}^{-1}a_1),c^{-1}],&\text{if } c\in\mathbb{A}\setminus t\mathbb{B}\text{ and } t\mathbb{B}\ni a=ta_1,\\ [a,1,c]&\mapsto&[tc_1,ta_1,1],&\text{if } t\mathbb{B}\ni a=ta_1\text{ and } t\mathbb{B}\ni c=tc_1\\ [1,tb_1,c]&\mapsto&[c,1,tb_1],&\\ [ta_1,tb_1,1]&\mapsto&[1,ta_1,tb_1]&\end{array}\right.$$

preserves incidence, as one can easily check, and is bijective with inverse τ^2. Conjugating $\varphi_{23}(Y)$ and $\varphi_{13}(X)$ with τ and τ^2 shows that $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ is $((0,0,1),[1,0,0])$-transitive, $((0,1,0),[1,0,0])$-transitive, $((1,0,0),[0,1,0])$-transitive and $((0,0,1),[0,1,0])$-transitive. Hence $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ is a Moufang Hjelmslev plane, as claimed. □

Proposition 5.1.5. *The action of the little projective group* $\mathsf{PSL}_3(\mathbb{A})$ *of* $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ *on* $\mathscr{P}$ *is induced by the action on* X *of the stabiliser in* $\mathsf{PSL}_{3d+3}(\mathbb{K})$ *of the point set* X *of* $\mathscr{V}_2(\mathbb{K},\mathbb{A})$.

Proof. It suffices to show that the maps $\varphi_{23}(Y)$, $\varphi_{13}(X)$ and τ are induced in this manner. We label a generic point of $\mathsf{PG}(3d+2,\mathbb{K})$ with $(x,y,z;\xi,\upsilon,\zeta)$, where $x,y,z\in\mathbb{K}$ and $\xi,\upsilon,\zeta\in\mathbb{A}$. Then one calculates that the following $\mathbb{K}$-linear map $\varphi(X,Y)$ induces $\varphi_{23}(Y)$ on $\mathscr{P}$ if $X=0$ and $\varphi_{13}(X)$ if $Y=0$.

$$\varphi(X,Y):\mathsf{PG}(3d+2,\mathbb{K})\longrightarrow\mathsf{PG}(3d+2,\mathbb{K}):(x,y,z;\xi,\upsilon,\zeta)\mapsto(x',y',z';\xi',\upsilon',\zeta'),$$

with

$$\left\{\begin{array}{rcl}x'&=&x+\overline{\upsilon}\overline{X}+X\upsilon+X\overline{X}z,\\y'&=&y+\xi\overline{Y}+Y\overline{\xi}+Y\overline{Y}z,\\z'&=&z,\\\hline \xi'&=&\xi+Yz,\\\upsilon'&=&\upsilon+\overline{X}z,\\\zeta'&=&\zeta+\overline{\upsilon}\overline{Y}+X\overline{\xi}+X\overline{Y}z.\end{array}\right.$$

Also, the triality map τ is induced in $\mathscr{P}$ by the $\mathbb{K}$-linear map

$$\mathsf{PG}(3d+2,\mathbb{K}) \longrightarrow \mathsf{PG}(3d+2,\mathbb{K}) : (x,y,z;\xi,\upsilon,\zeta) \mapsto (z,x,y;\zeta,\xi,\upsilon),$$

which can again be verified by an elementary but tedious calculation.

Hence, noting that the above maps belong to $\mathsf{PSL}_{3d+3}(\mathbb{K})$ and stabilise the point set of $\mathscr{V}_2(\mathbb{K},\mathbb{A})$, this concludes the proof. $\square$

Corollary 5.1.6. *The little projective group is transitive on the set of triangles $P_0 * L_1 * P_2 * L_0 * P_1 * L_2 * P_0$, with $P_i \not\approx L_i$, $i = 0,1,2$ and transitive on the set of pairs of points and on the set of pairs of lines which are either neighbouring or not.*

Proof. By Proposition 5.1.4 and the discussion preceding that proposition, $\mathsf{PSL}_3(\mathbb{A})$ is transitive on the set of flags, and so $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ is (P,L)-transitive for each point P and each line L incident with P. This implies easily that $\mathsf{PSL}_3(\mathbb{A})$ is transitive on the set of triangles $P_0 * L_1 * P_2 * L_0 * P_1 * L_2 * P_0$, with $P_i \not\approx L_i$, $i = 0,1,2$.

Since $\mathsf{G}(2,\mathbb{A})$ is (P,L)-transitive for all flags (P,L), $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ is clearly transitive on the pairs of points (Q,R) which are far from each other (note that the neighbouring relation is preserved by all (P,L)-elations). By transitivity on points, it suffices to show that two points neighbouring $(1,0,0)$, but different from $(1,0,0)$, can be mapped to each other while fixing $(1,0,0)$. The elations $\varphi_{13}(X)$ and their conjugates under the triality map τ take care of this. The statement for the lines follows by duality. $\square$

The geometric structure of a line—By Corollary 5.1.6, each line behaves as does the line $[1,0,0]$, whose points (X,Y,Z) satisfy $X = 0$, so they are given by $(0,1,z)$ with $z \in \mathbb{A}$ and $(0,ty_1,1)$ with $y_1 \in \mathbb{B}$. Their images under ρ are $(0,1,z\overline{z};\overline{z},0,0)$ and $(0,0,1;ty_1,0,0)$, respectively. These are exactly the points $(K_0,K_1,K_2;A_0,A_1,A_2)$ of $\rho(\mathscr{P})$ satisfying $K_1K_2 = n(A_0)$ and $K_0 = A_1 = A_2 = 0$. Recalling $A_i = (B_{i0},B_{i1}) = (K_{i0},...,K_{id})$, we can write these as equations over $\mathbb{K}$, where n' is the (anisotropic) norm form associated with $\mathbb{B}$.

$$K_1K_2 = n'(B_{00}) \tag{1}$$

$$K_0 = K_{10} = \cdots = K_{1d} = K_{20} = \cdots = K_{2d} = 0 \tag{2}$$

We conclude that the corresponding element of Ξ, spanned by the points of $\rho([1,0,0])$, is the $(d+1)$-dimensional subspace of $\mathsf{PG}(3d+2,\mathbb{K})$ satisfying equation (2), and the points of $\rho([1,0,0])$ it contains are the ones that additionally satisfy the quadratic equation (1). Moreover, ξ contains no other points of $\rho(\mathscr{P})$ than those of $\rho([1,0,0])$: suppose $(x,y,z) \in \mathscr{P}$ is such that $x \neq 0$ and $x\overline{x} = z\overline{x} = x\overline{y} = 0$. Then an easy calculation shows that $x_0 = y_0 = z_0 = 0$ and hence $(x,y,z) \notin \mathscr{P}$, so $x = 0$ and hence (x,y,z) belongs to the line $X = 0$ indeed.

So $X(\xi)$ is a cone with base a quadric of Witt index 1 in $\mathsf{PG}(d'+1,\mathbb{K})$, where $d' = \frac{d}{2} - 1$, and a d'-dimensional vertex which is omitted.

We now show some properties of $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ (which we will later on use as their characterising properties).

Proposition 5.1.7. *The Veronese representation $\mathcal{V}_2(\mathbb{K},\mathbb{A}) = (X,\Xi)$ of $\mathsf{G}_2(\mathbb{K},\mathbb{A})$, where $\mathbb{A} = \mathsf{CD}(\mathbb{B},0)$ for a quadratic associative division algebra $\mathbb{B}$ over a field $\mathbb{K}$ satisfies the following two properties:*

(H1) *Any two distinct points x_1 and x_2 of X lie in at least one element of Ξ.*

(H2*) *Any two distinct elements ξ_1 and ξ_2 of Ξ intersect in points which belong to the degenerate quadric uniquely determined by $\xi_1 \cap X$ (i.e., including its vertex). Moreover, $\xi_1 \cap \xi_2 \cap X$ is non-empty.*

Proof. We first verify property (H1). This follows from the fact that $\mathsf{G}(2,\mathbb{A})$ is a Hjelmslev plane. An explicit proof goes as follows. By transitivity (cf. Corollary 5.1.6), we may assume that x_1 is $\rho((1,0,0))$ and x_2 is either $\rho((0,1,0))$ (if $\rho^{-1}(x_1) \not\approx \rho^{-1}(x_2)$) or $\rho((1,t,0))$ (if $\rho^{-1}(x_1) \approx \rho^{-1}(x_2)$). Either way, $\rho([0,0,1])$ is an element of Ξ containing both x_1 and x_2. This shows property (H1).

For the second property, transitivity again implies that we may assume that ξ_1 corresponds to $\rho([1,0,0])$ and ξ_2 to either $\rho([0,1,0])$ (if they come from non-neighbouring lines) or to $\rho([1,t,0])$ (if they come from neighbouring lines). In the first case, ξ_1 is as described in the geometric structure of a line above, and ξ_2 is completely analogous. It follows that the intersection of ξ_1 and ξ_2 is the unique point $(0,0,1;0,0,0)$ which is exactly $\rho((0,0,1)) \in X$, which hence belongs to $X \cap \xi_1$ indeed. In the second case, ξ_2 is spanned by $\rho((-ty_0,y,1)) = (0,n(y_0),1;y,ty_0,-tn(y_0))$ for $y = y_0 + ty_1 \in \mathbb{A}$ and $\rho((-t,1,tz_1)) = (0,1,0;-tz_1,0,-t)$ for $z_1 \in \mathbb{B}$, so ξ_2 is given by $K_0 = B_{10} = B_{20} = B_{00} - B_{11} = B_{21} - K_1 = 0$. The subspace $\xi_1 \cap \xi_2$ is then given by $K_0 = K_1 = B_{00} = A_1 = A_2 = 0$, so we get the points $(0,0,k_2;tb_{01},0,0) = \rho((0,tb_{01},1))$, for $k_2 \in \mathbb{K}$ and $b_{01} \in \mathbb{B}$, if $k_2 \neq 0$. If $k_2 = 0$, we get precisely the vertex of the quadric determined by $\xi_i \cap X$, $i = 1,2$. This shows the claim. $\square$

5.2 Geometries over the split generalised dual numbers

Now let $\mathbb{A}$ be one of the following algebras: $\mathbb{T}'$, $\mathsf{CD}(\mathbb{L}',0)$, $\mathbb{S}'$, $\mathsf{CD}(\mathbb{H}',0)$, i.e., $\mathbb{A} = \mathbb{B} \oplus t\mathbb{B}$ where $\mathbb{B} \in \{\mathbb{L}',\mathbb{H}'\}$ (and $t\mathbb{B}$ is not necessarily isomorphic to $\mathbb{B}$). In this case, the algebra $\mathbb{B}$ is split and hence contains non-invertible elements. As such, it is not advisable to attempt to list all triples in an affine way, as above. Already for the split octonions $\mathbb{O}'$ (which are still non-degenerate), it is not so obvious how to define a geometry similar to what has been done in 5.1.1 for the non-split case, let alone for non-associative *degenerate* algebras $\mathbb{S}'$ and $\mathsf{CD}(\mathbb{H}',0)$. Instead of defining a ring projective plane first and then taking the image under the Veronese map, we will define it as a geometry embedded in projective space, by using only an affine part of the abstract geometry (i.e., triples $(1,y,z)$ with $y,z \in \mathbb{A}$), and considering the projective closure (see below) of the image under the Veronese map of the set of these triples.

Remark 5.2.1. Concerning the abstract ring geometry: There are other, group-theoretical constructions for the ring projective plane over the split octonions $\mathbb{O}'$, for instance due to Faulkner in his book Projective remoteness planes ([21]). More generally, Mühlherr and

Weiss construct ring projective planes over an arbitrary ring of stable range 2—in particular, all (possibly degenerate) quadratic alternative algebras—using root group sequences ([36]).

So we need a new approach. Instead of defining the abstract geometry and then considering its Veronese representation, we just construct the geometry inside a projective space via a "partial" Veronese map:

Definition 5.2.2. Let $\mathbb{A}$ be one of $\mathbb{L}'$, $\mathbb{H}'$, $\mathbb{O}'$, $\mathbb{T}'$, $\mathsf{CD}(\mathbb{L}',0)$, $\mathbb{S}'$,$\mathsf{CD}(\mathbb{H}',0)$ and put $d := \dim_{\mathbb{K}}(\mathbb{A})$. Then we define the following map ρ, called the *partial Veronese map*,

$$\rho : \mathbb{A} \times \mathbb{A} \to \mathsf{PG}(3d+2,\mathbb{K}) : (A,B) \mapsto (1,A\overline{A},B\overline{B},AB,B,A)$$

and define $\mathcal{V}_2(\mathbb{K},\mathbb{A})$ as the *projective closure* of $\mathrm{im}(\rho)$, i.e., for each affine subspace in $\mathcal{V}_2(\mathbb{K},\mathbb{A})$, also its projective subspace at infinity belongs to $\mathcal{V}_2(\mathbb{K},\mathbb{A})$.

The above procedure of taking the projective closure is well-defined if $|\mathbb{K}| > 2$. However, if $|\mathbb{K}| = 2$ one would have to decide which pairs of points should be singular lines (and then we should add a third point) and which are secants (and then we should not add a point). This can be done, but we do not have to worry about the details for we will characterise the Veronese varieties $\mathcal{V}_2(\mathbb{K},\mathbb{A})$ under the assumption that $|\mathbb{K}| > 2$ anyway.

Remark 5.2.3. The usual definition of the Veronese map takes (A,B) to $(1,A\overline{A},B\overline{B},A\overline{B},B,\overline{A})$, but we can change B to $\overline{B}$, and then obtain $(1,A\overline{A},B\overline{B},AB,\overline{B},\overline{A})$, which linearly transforms into the above definition.

Moreover, if $A = M(a,b,c,d,x,y,z,u) \in \mathbb{O}'$, then $\overline{A} = M(d,-b,-c,a,-x,-y,-z,-u)$ and hence $A\overline{A} = ad - bc - xz - yu$.

Remark 5.2.4. The variety $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$ (for $\mathbb{K} = \mathbb{R}$) has also been discussed in Section 8 of [32] by Landsberg and Manivel.

The following proposition is proven in the paper [51] of H. Van Maldeghem and M. Victoor.

Proposition 5.2.5. *The geometry $\mathcal{V}_2(\mathbb{K},\mathbb{L}')$ is isomorphic to the Segre variety $S_{2,2}(\mathbb{K})$; the geometry $\mathcal{V}_2(\mathbb{K},\mathbb{H}')$ is isomorphic to the line Grassmannian $\mathsf{G}_{5,1}(\mathbb{K})$, the geometry $\mathcal{V}_2(\mathbb{K},\mathbb{O}')$ is isomorphic to the $\mathsf{E}_{6,1}(\mathbb{K})$ variety.*

We now analyse the geometry $\mathcal{V}_2(\mathbb{K},\mathbb{A})$ for each $\mathbb{A} \in \{\mathbb{T}',\mathsf{CD}(\mathbb{L}',0),\mathbb{S}'\}$. There is one major restriction: we will not do this for $\mathsf{CD}(\mathbb{H}',0)$ (as it turns out not to fit in our general framework, as alluded to before). This restriction makes that all algebras that we are considering are actually isomorphic to sub-algebras of the split octonions $\mathbb{O}'$. We start with the most difficult case, being that of $\mathbb{S}'$; as then the others can be treated more or less analogously.

5.2.1 A Veronese variety $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$ associated to $\mathbb{S}' < \mathbb{O}'$

Recall that $\mathbb{S}' \cong \{M(a,b,c,d,0,y,z,0) \mid a,b,c,d,y,z \in \mathbb{K}\}$. We would like to see $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$ as a subgeometry of $\mathcal{V}_2(\mathbb{K},\mathbb{O}')$. We achieve this by just restricting $\rho(\mathbb{O}'\times\mathbb{O}')$ to $\mathbb{S}'\times\mathbb{S}'$. Then $\rho(M(a,b,c,d,0,y,z,0),M(a',b',c',d',0,y',z',0)) = (x_0,\ldots,x_{26})$, where:

$$(x_0,x_1,x_2) = (1, ad-bc, a'd'-b'c'),$$

$$(x_3,\ldots,x_{10}) = (aa'+bc', d'b+ab', a'c+dc', dd'+b'c, 0, d'y+ay'+cz'-c'z, a'z+dz'+by'-b'y, 0)$$

$$(x_{11},\ldots,x_{18}) = (a',b',c',d',0,y',z',0),$$

$$(x_{19},\ldots,x_{26}) = (a,b,c,d,0,y,z,0).$$

For the readability, we will always write the 27-tuples in the image as composed of a triple and three 8-tuples. Let $(e_0,\ldots,e_{26})$ be the standard basis of $\mathsf{PG}(26,\mathbb{K})$. Set $I = \{0,\ldots,26\}\setminus\{7,10,15,18,23,26\}$. Put $Y := \langle e_8,e_9,e_{16},e_{17},e_{24},e_{25}\rangle$ and $B := \langle\{e_i : i\in I\setminus\{8,9,16,17,24,25\}\rangle$. Clearly, $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$ is contained in the 20-dimensional subspace $\langle B,Y\rangle = \langle e_i \mid i\in I\rangle$.

Lemma 5.2.6. *The subspace Y is a singular 5-space of $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$. The subspace B is such that $\mathcal{V}_2(\mathbb{K},\mathbb{S}')\cap B$ contains $G := \mathsf{G}_{5,1}(\mathbb{K}) = \mathcal{V}_2(\mathbb{K},\mathbb{H}')$.*

Proof. Considering the images of the pairs $(M(a,0,0,d,0,\ell y,\ell z,0),M(0,0,0,0,0,\ell,\ell z',0))$, where $a,b,y,z,z',\ell\in\mathbb{K}$, we get points of $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$ with coordinates

$$((1,0,0),(0,0,0,0,0,\ell a,\ell dz',0),(0,0,0,0,0,\ell,\ell z',0),(a,b,0,0,0,\ell y,\ell z,0)).$$

Since this belongs to $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$ for each $\ell\in\mathbb{K}$, and since $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$ is projectively closed, also the points corresponding to $\ell=\infty$ belong to $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$. This gives us a point set

$$\big\{((0,0,0),(0,0,0,0,0,a,dz',0),(0,0,0,0,0,1,z',0),(0,0,0,0,0,y,z,0)) \mid a,b,y,z,z'\in\mathbb{K}\big\}.$$

The projective closure of this set is precisely the subspace Y. Again, as $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$ is projectively closed, this shows that Y is a singular 5-space of $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$. This shows the first assertion.

Next, note that $\rho(M(a,b,c,d,0,0,0,0),M(a',b',c',d',0,0,0,0))$, for all $a,\ldots,d'\in\mathbb{K}$, belongs to B and that $\{M(a,b,c,d,0,0,0,0)\mid a,b,c,d\in\mathbb{K}\}$ is isomorphic to $\mathbb{H}'$. Since $\mathcal{V}_2(\mathbb{K},\mathbb{H}')$ (which then also belongs to B) is isomorphic to $\mathsf{G}_{5,1}(\mathbb{K})$ by Proposition 5.2.5, the second assertion follows. Note that it could still be that there are more points in B, arising from the projective closure of points of $\mathrm{im}(\rho)$. We will later on show that this is not the case. $\square$

Noting that $\langle Y,B\rangle = \langle e_i \mid i\in I\rangle$, we obtain that $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$ is contained in $\mathcal{V}_2(\mathbb{K},\mathbb{O}')\cap\langle Y,B\rangle$. Knowing this, we determine some more structure of $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$. We start with transitivity properties.

Lemma 5.2.7. *Let $N,N'\in\mathbb{S}'$ be arbitrary. Then the mapping $\rho(M,M')\mapsto\rho(M+N,M'+N')$ extends uniquely to a linear collineation φ of $\mathsf{PG}(20,\mathbb{K}) = \langle Y,B\rangle$.*

Proof. By symmetry it suffices to show that $\rho(M,M')\mapsto\rho(M+N,M')$ extends uniquely to a linear collineation of $\mathsf{PG}(20,\mathbb{K})$. Setting $N=M(\alpha,\beta,\gamma,\delta,0,\upsilon,\zeta,0)$, one can check that the linear collineation φ given by

$$\begin{array}{rcl}
x_0' &=& x_0,\\
x_1' &=& x_1+\alpha x_{22}-\beta x_{21}-\gamma x_{20}+\delta x_{19}+(\alpha\delta-\beta\gamma)x_0,\\
x_2' &=& x_2,\\
x_3' &=& x_3+\alpha x_{11}+\beta x_{13},\\
x_4' &=& x_4+\alpha x_{12}+\beta x_{14},\\
x_5' &=& x_5+\gamma x_{11}+\delta x_{13},\\
x_6' &=& x_6+\gamma x_{12}+\delta x_{14},\\
x_8' &=& x_8+\alpha x_{16}+\upsilon x_{14}+\gamma x_{17}-\zeta x_{13},\\
x_9' &=& x_9+\zeta x_{11}+\delta x_{17}+\beta x_{16}-\upsilon x_{12},\\
x_i' &=& x_i,\ \ i=11,12,13,14,16,17\\
x_{19}' &=& x_{19}+\alpha x_0,\\
x_{20}' &=& x_{20}+\beta x_0,\\
x_{21}' &=& x_{21}+\gamma x_0,\\
x_{22}' &=& x_{22}+\delta x_0,\\
x_{24}' &=& x_{24}+\upsilon x_0,\\
x_{25}' &=& x_{25}+\zeta x_0
\end{array}$$

takes $\rho(M,M')$ to $\rho(M+N,M')$. Uniqueness follows from the fact that $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$ contains a skeleton. $\square$

At the one hand, restricting N and N' to $\mathbb{H}'$, the previous lemma implies:

Corollary 5.2.8. *The full little projective group of G, i.e., the group $\mathsf{PSL}_6(\mathbb{K})$, acting on G in the standard way, is induced by the stabiliser in $\mathsf{PSL}_{21}(\mathbb{K})$ of $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$ and G.*

On the other hand, we also have:

Corollary 5.2.9. *The stabiliser in $\mathsf{PSL}_{21}(\mathbb{K})$ of $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$ acts transitively on the set $\mathcal{V}_2(\mathbb{K},\mathbb{S}')\setminus Y$.*

Lemma 5.2.10. *Let p be a point of $\langle Y,B\rangle$ not in $Y\cup B$. Then $p\in\mathcal{V}_2(\mathbb{K},\mathbb{S}')$ if and only if the unique line $L=\langle p,Y\rangle\cap\langle p,B\rangle$ belongs to $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$. In particular, $\mathcal{V}_2(\mathbb{K},\mathbb{S}')\cap B=\mathcal{V}_2(\mathbb{K},\mathbb{H}')=G$, and the projection of $\mathcal{V}_2(\mathbb{K},\mathbb{S}')\setminus Y$ from Y onto B coincides with G.*

Proof. The assertion is trivial when $p\notin\mathcal{V}_2(\mathbb{K},\mathbb{S}')$. So we assume that $p\in\mathcal{V}_2(\mathbb{K},\mathbb{S}')$. By Corollary 5.2.8, we may assume that $L\cap B$ has its first coordinate distinct from 0. Then let p be be given by the coordinates $(x_i)_{i\in I}$, with $x_0=1$. Then clearly

$$p=\rho(M(x_{19},x_{20},x_{21},x_{22},0,x_{24},x_{25},0),M(x_{11},x_{12},x_{13},x_{14},0,x_{16},x_{17},0)),$$

and it is easily verified that $L\cap B$ is given by

$$\rho(M(x_{19},x_{20},x_{21},x_{22},0,0,0,0),M(x_{11},x_{12},x_{13},x_{14},0,0,0,0)).$$

Since the three points $p, L \cap B$ and $L \cap Y$ belong to $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$, all points of L belong to $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$. $\square$

Now we determine the possible structures of the intersection of $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$ with a symp of $\mathcal{V}_2(\mathbb{K},\mathbb{O}')$ (the latter viewed as a parapolar space with symplecta isomorphic to hyperbolic quadrics in $\mathsf{PG}(9,\mathbb{K})$).

Proposition 5.2.11. *Let Σ be a symp of $\mathcal{V}_2(\mathbb{K},\mathbb{O}')$ such that $\zeta := \Sigma \cap \mathcal{V}_2(\mathbb{K},\mathbb{S}')$ contains at least two non-collinear points of $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$. Then either*

- (i) *$Y \cap \Sigma$ is a line L, in which case ζ is a cone with 1-dimensional vertex L and base isomorphic to a Klein quadric over $\mathbb{K}$ (so $\zeta = V^{\perp} \cap \Sigma$);*
- (ii) *$Y \cap \Sigma$ is a 4-space, in which case $\zeta = \Sigma$.*

Proof. We show this proposition with a series of claims.

Claim 1. *For each point $p \in \mathcal{V}_2(\mathbb{K},\mathbb{S}') \setminus Y$, $p^{\perp} \cap Y$ is a 3-space.*
Indeed, by Corollary 5.2.9, we may assume that $p = \rho((0,0))$ (where 0 denotes the zero matrix). An arbitrary point in $\langle p, Y \rangle \setminus Y$ has coordinates

$$((1,0,0),(0,0,0,0,0,x_8,x_9,0),(0,0,0,0,0,y',z',0),(0,0,0,0,0,y,z,0)),$$

with $x_8, x_9, y, y', z, z' \in \mathbb{K}$. This point belongs to $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$ if and only if $x_8 = x_9 = 0$. Hence Claim 1. Standard arguments then readily imply the following claim:

Claim 2. *The mapping $G \to Y : p \mapsto p^{\perp} \cap Y$ is an isomorphism from G to $\mathsf{G}_{5,3}(\mathbb{K})$ defined by the 5-space Y.*

Claim 3. *$\Sigma \cap Y$ is either a line or a 4-space.*
In $\mathcal{V}_2(\mathbb{K},\mathbb{O}')$, which is a variety of type $\mathsf{E}_{6,1}(\mathbb{K})$, a 5-space and a symp either have a point, a line or a singular 4-space in common (see for instance Fact 4.2.10 of [17]). Let p, p' be two non-collinear points of ζ. Then $p^{\perp} \cap p'^{\perp} \cap Y$ is at least 1-dimensional by Claim 1. By convexity of the symps, $p^{\perp} \cap p'^{\perp} \cap Y \subseteq \zeta$, from which Claim 3 follows.

Claim 4. *If a line $L \subseteq Y \cap \Sigma$, then $L^{\perp} \cap \Sigma \subseteq \zeta$.*
Indeed, Claim 2 implies that L is collinear to all points of a Klein quadric Q in G. Since L is also collinear to all points of Y, this means that $L^{\perp} \cap \mathcal{V}_2(\mathbb{K},\mathbb{S}')$ has at least dimension 11. But in the whole of $\mathcal{V}_2(\mathbb{K},\mathbb{O}')$, the dimension of $L^{\perp}$ is exactly 11 (since the residue at L is a $\mathsf{G}_{4,1}(\mathbb{K})$). Hence $L^{\perp} \subseteq \mathcal{V}_2(\mathbb{K},\mathbb{S}')$ and Claim 4 follows.

Claim 5. *If $\Sigma \cap Y$ is a line L, then $\zeta = L^{\perp} \cap \Sigma$.*
By Claim 4, $L^{\perp} \cap \Sigma \subseteq \zeta$. Suppose for a contradiction that there exists a $p \in \zeta$ and a $q \in L$ such that p and q are not collinear. But then $p^{\perp} \cap q^{\perp} \cap Y = p^{\perp} \cap Y \subseteq \Sigma \cap Y$, whereas $\dim(p^{\perp} \cap Y) = 3$ by Claim 1 and $\dim(\Sigma \cap Y) = 1$ by our assumption. Claim 5 is proved.

Claim 6. *If $\Sigma \cap Y$ is a 4-space, then $\zeta = \Sigma$.*
Indeed, for an arbitrary point $p \in \Sigma$ we have that $p^{\perp} \cap Y$ contains a line L, and hence $p \in \zeta$ by Claim 4.

This completes the proof of the proposition. $\square$

To distinguish between these two types of symps, we define:

$$\Xi := \{\Sigma \cap \mathcal{V}_2(\mathbb{K},\mathbb{S}') \mid \dim(Y \cap \Sigma) = 1\},$$

$$\Theta := \{\Sigma \cap \mathcal{V}_2(\mathbb{K},\mathbb{S}') \mid \dim(Y \cap \Sigma) = 4\}.$$

We now determine the dimension of the subspace generated by all lines through a fixed point.

Lemma 5.2.12. *Let $p \in \mathcal{V}_2(\mathbb{K},\mathbb{S}') \setminus Y$. Then* $\dim\langle p^\perp \cap \mathcal{V}_2(\mathbb{K},\mathbb{S}')\rangle = 12$.

Proof. As before, Corollary 5.2.9 allows us to assume $p = \rho(0,0)$. It is then an easy exercise to verify that $p^\perp \cap \mathcal{V}_2(\mathbb{K},\mathbb{S}') = \langle e_0, e_{11}, e_{12}, e_{13}, e_{14}, e_{16}, e_{17}, e_{19}, e_{20}, e_{21}, e_{22}, e_{24}, e_{25}\rangle$, which is indeed 12-dimensional. $\square$

Remark 5.2.13. Note that, if $p \in G$, then this subspace is spanned by $p^\perp \cap Y$ (dimension 3 by Claim 1) and by $p^\perp \cap \mathcal{V}_2(\mathbb{K},\mathbb{H}')$ (dimension 8 as the residue of a point in a $\mathsf{G}_{5,1}(\mathbb{K})$ is a $\mathsf{S}_{1,3}(\mathbb{K})$).

Finally we show that through each point p of $\mathcal{V}_2(\mathbb{K},\mathbb{S}') \setminus Y$ we can find two members of Ξ intersecting Y in respective lines, and mutually only intersecting in p.

Lemma 5.2.14. *For each point p of $\mathcal{V}_2(\mathbb{K},\mathbb{S}') \setminus Y$, there exist $\xi_1, \xi_2 \in \Xi$ with $\xi_1 \cap \xi_2 = \{p\}$.*

Proof. By Corollary 5.2.9 we may assume $p \in G$. We can consider two Klein quadrics (symps of G viewed as a parapolar space) Q_1 and Q_1 in G with $Q_1 \cap Q_2 = \{p\}$. By Claim 2, there are L_1, L_2 in Y with $L_i \perp Q_i$, $i = 1,2$, and they are disjoint. For $i = 1,2$, let Σ_i be the unique symp of $\mathcal{V}_2(\mathbb{K},\mathbb{O}')$ containing Q_i; which then also contains L_i and for which $\xi_i := \Sigma_i \cap \mathcal{V}_2(\mathbb{K},\mathbb{S}') = \langle L_i, Q_i\rangle \cap \mathcal{V}_2(\mathbb{K},\mathbb{S}')$. As such, it is clear that $\xi_1 \cap \xi_2 = \{p\}$. $\square$

5.2.2 A Veronese variety $\mathcal{V}_2(\mathbb{K}, \mathsf{CD}(\mathbb{L}', 0))$ associated to $\mathsf{CD}(\mathbb{L}', 0) < \mathbb{O}'$

Put $\mathbb{H}'' := \mathsf{CD}(\mathbb{L}', 0)$ and recall that $\mathbb{H}'' \cong \{M(a,0,0,d,0,y,z,0) \mid a,d,y,z \in \mathbb{K}\}$. Clearly, $\mathbb{H}''$ is a subalgebra of $\mathbb{S}'$ and hence the corresponding split Veronese set $\mathcal{V}_2(\mathbb{K},\mathbb{H}'')$ is a subset of $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$, contained in the 14- space M given by $\langle e_i \mid i \in I \setminus \{4,5,12,13,20,21\}\rangle$. Following (the proof of) Lemma 5.2.6, we see that the 5-space Y defined in the previous subsection belongs to $\mathcal{V}_2(\mathbb{K},\mathbb{H}'')$ and that $M \cap B$ contains $\mathcal{V}_2(\mathbb{K},\mathbb{L}')$, which is isomorphic to a Segre variety $S := \mathsf{S}_{2,2}(\mathbb{K})$ (which is contained in $\mathcal{V}_2(\mathbb{K},\mathbb{H}') = G$). As such, the analogues of Corollaries 5.2.8 and 5.2.9 and Lemma 5.2.10 also hold in this case.

Now, let $U \cong \mathsf{PG}(5,\mathbb{K})$ be (an abstract) projective space whose line Grassmannian gives G. Then the Segre variety S, as sub-variety of G, arises as the set of lines of U intersecting two given disjoint planes π_1 and π_2 each non-trivially. Claims 1 and 2 of Proposition 5.2.11 then imply that Y is isomorphic to the dual of U, and hence the lines of U intersecting π_1 and π_2 non-trivially correspond to 3-spaces having a line in common with two planes Z_1 and Z_2 in Y, and these 3-spaces all arise as $p^\perp \cap Y$, with $p \in S$.

We now show the following analogue of Proposition 5.2.11.

Proposition 5.2.15. *Let Σ be a symp of $\mathscr{V}_2(\mathbb{K},\mathbb{O}')$ such that $\zeta := \Sigma \cap \mathscr{V}_2(\mathbb{K},\mathbb{H}'')$ contains at least two non-collinear points of $(\mathscr{V}_2(\mathbb{K},\mathbb{H}'') \setminus Y) \cup Z_1 \cup Z_2$. Then either*

(i) *$Y \cap \Sigma$ is a line V, in which case ζ is a cone with vertex V and base isomorphic to a grid quadric over $\mathbb{K}$ (so $\zeta \subseteq L^\perp \cap \Sigma$); or,*

(ii) *$Y \cap \Sigma$ is a 4-space W generated by a line V_i in Z_i and the plane Z_j, with $\{i,j\} = \{1,2\}$, in which case ζ is a cone with vertex V_i and base isomorphic to a Klein quadric over $\mathbb{K}$.*

Proof. We already know from Proposition 5.2.11 that $\Sigma \cap Y$ is either a line V or a 4-space W. In the former case, we obtained in that proposition that the projection of $\Sigma \cap \mathscr{V}_2(\mathbb{K},\mathbb{S}')$ on G is isomorphic to a Klein quadric, which then intersects S either in at most a grid quadric or at most a singular plane. As we assume that $\Sigma \cap \mathscr{V}_2(\mathbb{K},\mathbb{H}'')$ contains at least two non-collinear points, so does the projection on B (seeing that all points in the projection are collinear to V), as such, it must be a grid quadric. That takes care of the case where $\Sigma \cap Y$ is a line.

Now suppose $\Sigma \cap Y$ is a 4-space W. The projection of Σ onto B is, as follows from Claim 2, a maximal singular 4-space W' of G. Then there are two possibilities: W' intersects S either in a unique point p, or in a singular plane α. The correspondence between G and Y then implies that, in the former case, W is a 4-space intersecting both Z_1 and Z_2 in unique lines (which generate $p^\perp \cap Y$); in the second case, W is a 4-space containing Z_i and sharing a line with Z_j, for $\{i,j\} = \{1,2\}$ (the line Z_j being collinear to all points of the plane α). In the first case, Σ contains no two non-collinear points of $\mathscr{V}_2(\mathbb{K},\mathbb{H}'') \setminus Y) \cup Z_1 \cup Z_2$; in the latter case we obtain possibility (ii).

The proposition is proved. □

Again, we define

$$\Xi := \{\Sigma \cap \mathscr{V}_2(\mathbb{K},\mathbb{H}'') \mid \dim(Y \cap \Sigma) = 1\},$$

$$\Theta := \{\Sigma \cap \mathscr{V}_2(\mathbb{K},\mathbb{H}'') \mid Y \cap \Sigma \text{ is a 4-space containing } Z_1 \text{ or } Z_2\}.$$

The following lemma has approximately the same proof as Lemma 5.2.12.

Lemma 5.2.16. *Let $p \in \mathscr{V}_2(\mathbb{K},\mathbb{H}'') \setminus Y$. Then* $\dim\langle p^\perp \cap \mathscr{V}_2(\mathbb{K},\mathbb{H}'')\rangle = 8$.

We note that the map φ sending a point p of $\mathscr{V}_2(\mathbb{K},\mathbb{H}'') \setminus Y$ to the pair $(p^\perp \cap Z_1, p^\perp \cap Z_2)$ is an isomorphism of S to $Z_1^* \times Z_2^*$, where the star means "dual".

Lemma 5.2.17. *For each point p of $\mathscr{V}_2(\mathbb{K},\mathbb{B}) \setminus Y$, there exist $\xi_1, \xi_2 \in \Xi$ with $\xi_1 \cap \xi_2 = \{p\}$.*

Proof. By the above observation, a symp of S (viewed as parapolar space; so symps are grid quadrics) is collinear to a symp of $Z_1^* \times Z_2^*$, which is a pair of points $(p_1, p_2) \in Z_1 \times Z_2$. The union of such a symp with the lines joining it to the line p_1p_2 yields a member of Ξ. As we can choose disjoint lines p_1p_2 and $p_1'p_2'$ in the 3-space $p^\perp \cap Y$, the corresponding members of Ξ only share the point p. □

5.2.3 A Veronese variety $\mathcal{V}_2(\mathbb{K},\mathbb{T}')$ associated to $\mathbb{T}' < \mathbb{O}'$

Finally, if $\mathbb{T}' \cong \{M(a,0,0,d,0,y,0,0) \mid a,d,y \in \mathbb{K}\}$, then the image through ρ of $\mathbb{T}' \times \mathbb{T}'$ is a variety spanning an 11-space. It is the intersection of $\mathcal{V}_2(\mathbb{K},\mathbb{H}'')$ with the 11-space generated by the Segre-variety S in B and by the plane Z_1 in Y. The following assertions follow immediately from the previous lemmas.

Proposition 5.2.18. *Let Σ be a symp of $\mathcal{V}_2(\mathbb{K},\mathbb{O}')$ such that $\zeta := \Sigma \cap \mathcal{V}_2(\mathbb{K},\mathbb{T}')$ contains at least two non-collinear points of $\mathcal{V}_2(\mathbb{K},\mathbb{T}')$. Then either*

- (i) *$\Sigma \cap Z_1$ is a point V, in which case ζ is a cone with vertex V and base isomorphic to a grid quadric over $\mathbb{K}$;*
- (ii) *$\Sigma \cap Z_1 = Z_1$, in which case ζ is isomorphic to a Klein quadric over $\mathbb{K}$,.*

Also here, we define:

$$\Xi := \{\Sigma \cap \mathcal{V}_2(\mathbb{K},\mathbb{T}') \mid \dim(Z_1 \cap \Sigma) = 0\},$$
$$\Theta := \{\Sigma \cap \mathcal{V}_2(\mathbb{K},\mathbb{T}') \mid Z_1 \subseteq \Sigma\}.$$

Lemma 5.2.19. *Let $p \in \mathcal{V}_2(\mathbb{K},\mathbb{T}') \setminus Z_1$. Then* $\dim\langle p^\perp \cap \mathcal{V}_2(\mathbb{K},\mathbb{T}')\rangle = 6$.

Lemma 5.2.20. *For each point p of $\mathcal{V}_2(\mathbb{K},\mathbb{T}') \setminus \pi_1$, there exist symps $\xi_1, \xi_2 \in \Xi$ with $\xi_1 \cap \xi_2 = \{p\}$.*

5.2.4 Properties of the Veronese varieties $\mathcal{V}_2(\mathbb{K},\mathbb{T}')$, $\mathcal{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{L}',0))$ and $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$

We now show some properties satisfied by each of the varieties $\mathcal{V}_2(\mathbb{K},\mathbb{T}')$, $\mathcal{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{L}',0))$ and $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$ (which we will later on use as their characterising properties).

In the case of $\mathcal{V}_2(\mathbb{K},\mathbb{T}')$ and $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$, we define Z as the points of the subspace Y; in $\mathcal{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{L}',0))$ we define Z as the union of the two subspaces Z_1 and Z_2 and Y as $\langle Z_1, Z_2\rangle$. In the three varieties $\mathcal{V}_2(\mathbb{K},\cdot)$, we set X equal to the points in $\mathcal{V}_2(\mathbb{K},\cdot) \setminus Y$. Recall the definitions of Ξ and Θ.

Proposition 5.2.21. *The Veronese varieties $\mathcal{V}_2(\mathbb{K},\mathbb{T}')$, $\mathcal{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{L}',0))$ and $\mathcal{V}_2(\mathbb{K},\mathbb{S}')$ satisfy the following three properties:*

- (S1) *Each pair of distinct points $p_1, p_2 \in X \cup Z$ is contained in a member of $\Xi \cup \Theta$;*
- (S2) *for each pair of distinct members $\zeta_1, \zeta_2 \in \Xi \cup \Theta$, the intersection $\zeta_1 \cap \zeta_2$ is a singular subspace;*
- (S3) *for each point $x \in X$, there exists ξ_1, ξ_2 in Ξ such that $T_x = \langle T_x(\xi_1), T_x(\xi_2)\rangle$.*

Proof. Consider $\mathcal{V}_2(\mathbb{K},(\mathbb{A})$, where $\mathbb{A} \in \{\mathbb{T}', \mathsf{CD}(\mathbb{L}',0), \mathbb{S}'\}$.

- (S1) If p_1 and p_2 are non-collinear, then they determine a unique symp Σ of $\mathcal{V}_2(\mathbb{K},\mathbb{O}')$, which by assumption intersects $\mathcal{V}_2(\mathbb{K},\mathbb{B})$ in two non-collinear points, so $\Sigma \cap \mathcal{V}_2(\mathbb{K},\mathbb{O}') \in \Xi \cup \Theta$ by Propositions 5.2.18, 5.2.15 and 5.2.11.

If p_1 and p_2 are on a line, then we can always find a point p_3 in $X \cup Z$ which is collinear to p_1 and not to p_2. Then the symp of $\mathscr{V}_2(\mathbb{K},\mathbb{O}')$ containing p_2 and p_3 also contains p_1 and the same argument as above applies.

(S2) This is immediate as each member of $\Xi \cup \Theta$ is contained in a symp of $\mathscr{V}_2(\mathbb{K},\mathbb{O}')$ and two symps of the latter intersect in a singular subspace, which at its turn will intersect $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ in a singular subspace.

(S3) This follows by combining Lemmas 5.2.19 and 5.2.20 in case $\mathbb{A} = \mathbb{T}'$; by combining Lemmas 5.2.16 and 5.2.17 in case $\mathbb{A} = \mathsf{CD}(\mathbb{L}',0)$ and by combining Lemma 5.2.16 and 5.2.14 in case $\mathbb{A} = \mathbb{S}'$.

$\square$

CHAPTER

6

HJELMSLEVEAN VERONESE SETS

We will now use the two properties occurring in Proposition 5.2.21 to axiomatically define *Hjelmslevean Veronese sets*. We can then formally state our main results. Doing so, we also provide a neat geometric description and construction of the Veronese varieties over non-split dual numbers: they fall apart, just like the corresponding algebras, in two isomorphic (in fact, dual) parts, one non-degenerate and one degenerate, in the sense that after projecting from the degenerate part, one obtains a non-degenerate Veronese variety consisting of points and non-degenerate quadrics off minimal Witt index, and there exists a duality between these two parts.

> Throughout, $\mathbb{K}$ denotes an arbitrary (commutative) field, unless explicitly mentioned otherwise.

6.1 Main results

Let d, v be elements of $\mathbb{N} \cup \{-1, \infty\}$ with $d \geq 1$.

6.1.1 Definitions

Recall that an *ovoid* O in $\mathsf{PG}(d+1, \mathbb{K})$ is a set of points spanning $\mathsf{PG}(d+1, \mathbb{K})$ such that for each point $x \in O$ the union of the set of lines intersecting O in x is a hyperplane of $\mathsf{PG}(d+1, \mathbb{K})$, called the *tangent hyperplane at* x, and each other line through x intersects O in exactly one more point (hence O does not contain triples of collinear points). For short we call this a Q_d^0-quadric, where the 0 indicates the (projective) dimension of the maximal subspaces lying on O and d the (projective) dimension of its tangent hyperplanes. Examples are given by quadrics of Witt index 1 (or, equivalently, projective index 0), explaining our notation. Moreover, in the setting we will consider, the ovoids will turn out to be quadrics, see Corollary 6.4.9.

Definition 6.1.1. In a projective space $\mathsf{PG}(d+v+2,\mathbb{K})$, we consider a v-space V and a Q_d^0-quadric in a $(d+1)$-space complementary to V. The union of lines joining all points of V with all points of Q_d^0 is called a (d,v)-*cone* with *base* Q_d^0 and *vertex* V. The cone without its vertex is called a (d,v)-*tube* (with base Q_d^0).

A (d,v)-tube with base Q_d^0 will be referred to as a *tube* whenever (d,v) and Q_d^0 are clear from the context.

Let C be a (d,v)-tube. The unique (d,v)-cone containing C is denoted by $\overline{C}$, so $\overline{C}=C\cup V$ for a certain v-space V. Even though V is not contained in C, we also call V the *vertex* of C. A *tangent line* to C is a line which has either one or all its points in $\overline{C}$. Let x be any point in C. All lines through c entirely contained in $\overline{C}$ are those contained in $\langle x,V\rangle$. The latter subspace is called a *generator* of $\overline{C}$ and of C. The union of the set of tangent lines to C through x is a hyperplane of $\langle C\rangle$, denoted $T_x(C)$, which intersects $\overline{C}$ in the generator through x.

6.1.2 A characterisation of Hjelmslevean and ordinary Veronesean sets

Consider a spanning point set X of $\mathsf{PG}(N,\mathbb{K})$, $N>d+v+2$, together with a collection Ξ of $(d+v+2)$-dimensional projective subspaces of $\mathsf{PG}(N,\mathbb{K})$, called the *tubic spaces* of X, such that, for any $\xi\in\Xi$, the intersection $\xi\cap X$ is a (d,v)-tube $X(\xi)$ in ξ with base Q_d^0. For $\xi\in\Xi$ and $C=X(\xi)$, we define $\Xi(C)=\xi$. The union of all vertices of those tubes is denoted by Y. The unique (d,v)-cone containing $X(\xi)$ is denoted by $\overline{X(\xi)}$ as before. Note that $X\cap Y=\emptyset$: if a point $x\in X$ belongs to the vertex V of some tubic space ξ, then $X\cap\xi$ would strictly contain a (d,v)-tube instead of being one. We often denote $T_x(X(\xi))$ by $T_x(\xi)$ and we define the *tangent space* T_x *of* x as the subspace spanned by all tangent spaces through x to all tubes through x, i.e., $T_x=\langle T_x(\xi)\mid x\in\xi\in\Xi\rangle$.

The pair (X,Ξ), or simply X, is called a **Hjelmslevean Veronesean set (of type (d,v))** if the following properties hold.

(H1) Any two distinct points x_1 and x_2 of X lie in at least one element of Ξ,

(H2*) Any two distinct elements ξ_1 and ξ_2 of Ξ intersect in points of $X\cup Y$, i.e., $\xi_1\cap\xi_2=\overline{X(\xi_1)}\cap\overline{X(\xi_2)}$. Moreover, $\xi_1\cap\xi_2\cap X$ is non-empty.

Our main theorem states that the geometries satisfying those axioms are essentially those we introduced before:

Main Theorem 6.1.2. *Suppose (X,Ξ) is a Hjelmslevean Veronesean set of type (d,v) such that X generates $\mathsf{PG}(N,\mathbb{K})$, where $\mathbb{K}$ is a field with $|\mathbb{K}|>2$. Then d is a power of 2, with $d\leq 8$ if $\mathrm{char}(\mathbb{K})\neq 2$, and one of the following holds.*

(i) There is only one vertex V and projected from V, the resulting geometry (X',Ξ') is projectively equivalent to $\mathscr{V}_2(\mathbb{K},\mathbb{B})$, where $\mathbb{B}$ is a quadratic alternative division algebra over $\mathbb{K}$ and, in particular, $N=3d+v+1$ and $d=\dim_{\mathbb{K}}(\mathbb{B})$;

(ii) There is a quadratic associative division algebra $\mathbb{B}$ over $\mathbb{K}$ and two complementary subspaces U and W of $\mathsf{PG}(N,\mathbb{K})$, where U is possibly empty and $\dim W=6d+2$, with $d=v-\dim(U)=2\dim_{\mathbb{K}}(\mathbb{B})$, such that the intersection of every pair of distinct vertices is U, and the structure of (X,Ξ) induced in W is projectively equivalent to $\mathscr{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{B},0))$.

In particular, the basis of the tube $X\cap\xi$, for each $\xi\in\Xi$, is always a quadric.

The proof of the theorem will reveal the geometric structure of the Hjelmslevean Veroneseans:

Corollary 6.1.3. *Let $\mathscr{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{B},0))=(X,\Xi)$ be the Hjelmslevean Veronesean, where $\mathbb{B}$ is a quadratic associative division algebra over $\mathbb{K}$ with $\dim_{\mathbb{K}}(\mathbb{B})=d$. Then X spans a projective space $\mathbb{P}=\mathsf{PG}(6d+2,\mathbb{K})$, each vertex of a quadric in a member of Ξ has dimension $d-1$ and the vertices form a regular spread $\mathscr{S}$ of a $(3v+2)$-space in $\mathbb{P}$, and there exists a complementary space F of $\mathbb{P}$ such that $(X\cap F,\{\xi\cap F\mid\xi\in\Xi\})=:(X',\Xi')$ is projectively equivalent to $\mathscr{V}_2(\mathbb{K},\mathbb{B})$. Moreover, if $\mathscr{B}$ is the set of $(2v+1)$-spaces each spanned by two distinct vertices, then $(\mathscr{S},\mathscr{B})$, with natural incidence, is a projective plane isomorphic to $\mathsf{PG}(2,\mathbb{B})$ and there is a linear duality χ between (X',Ξ') and $(\mathscr{S},\mathscr{B})$ such that X is the union of the subspaces $\langle x',\chi(x')\rangle$, for x' ranging over X'.*

A special case of Main Result 6.1.2 is the case $v=-1$. Since this is interesting in its own right, we phrase it explicitly:

Main Theorem 6.1.4. *Suppose (X,Ξ) is a Hjelmslevean Veronesean set of type $(d,-1)$ such that X generates $\mathsf{PG}(N,\mathbb{K})$, where $\mathbb{K}$ is a field. Then, as a point-line geometry, (X,Ξ) (with natural incidence) is isomorphic to $\mathsf{PG}(2,\mathbb{A})$ where $\mathbb{A}$ is a quadratic alternative division algebra over $\mathbb{K}$ with $\dim_{\mathbb{K}}(\mathbb{A})=d$. Moreover,*

- *If $|\mathbb{K}|>2$, (X,Ξ) is projectively equivalent to $\mathscr{V}_2(\mathbb{K},\mathbb{A})$, so $N=3d+2$;*
- *If $|\mathbb{K}|=2$, then either $d=1$ or $d=2$.*
  - *If $d=1$, then $N\in\{5,6\}$. If $N=5$, there are two projectively non-isomorphic examples, among which $\mathscr{V}_2(\mathbb{F}_2,\mathbb{F}_2)$; if $N=6$, there is a unique possibility.*
  - *If $d=2$, then $N\in\{8,9,10\}$. If $N=10$, then there is precisely one example; in the other two cases there are precisely two projectively unique examples, among which is $\mathscr{V}_2(\mathbb{F}_2,\mathbb{F}_4)$, if $N=8$.*

6.1.3 A note on the results

The geometric characterisation of the Hjelmslevean Veronese sets is especially remarkable since the axioms we use are a straightforward extension of the elementary axioms used by Mazzocca and Melone [33] to characterise the Veronese representation of all conics of a projective plane over a finite field of odd order (the simplest case in a finite setting), and those axioms also describe and characterise the exceptional Veronesean map related to an octonion division algebra, owing its existence to the Tits index $\mathsf{E}_{6,2}^{28}$ of simple algebraic groups. The following two facts demonstrate the strength of our geometric approach:

- All our results hold over arbitrary fields and in arbitrary dimension (even infinite), except that we exclude the field of order 2 in our main result (as will be explained later on).
- Our approach is uniform in the sense that we also capture the *non-degenerate* Veronese varieties, i.e., the Veronese representations of the Moufang projective planes over the quadratic alternative division algebras.

In the non-degenerate case, we can even lift the assumption $|\mathbb{K}| > 2$ and although more examples pop up when $|\mathbb{K}| = 2$, they can still be classified. These extra examples are *pseudo-embeddings* (for more information on those, see [14]). Interestingly, one of these is strongly related to the large Witt design $S(24,5,8)$ and the sporadic Mathieu group M_{24}. For the structure of the additional examples in case $|\mathbb{K}| = 2$, and their relation with the Witt design $S(24,5,8)$, we refer to Subsection 6.3.2.

6.1.4 A note on the axioms

In [40], J. Schillewaert and H. Van Maldeghem showed a similar theorem in the specific case of $(d,v) = (1,0)$, though using slightly different axioms: their first axiom is the same, but their second is weaker and a third axiom was used.

(H2) Any two distinct elements ξ_1 and ξ_2 of Ξ intersect in points of $X \cup Y$, i.e., $\xi_1 \cap \xi_2 = \overline{X}(\xi_1) \cap \overline{X}(\xi_2)$, and $\xi_1 \cap \xi_2 \cap Y$ is either empty or a subspace of $\xi_1 \cap \xi_2$ of codimension 1.

(H3) For each $x \in X$, $\dim(T_x) \leq 4$.

The difference in the second axiom lies in our requirement that the intersection of two tubic spaces always contains at least one point of X. I will now explain our motivation for this. First note that, for arbitrary but finite d, v, (H3) can be generalised to $\dim(T_x) \leq 2(d+v+1)$.

To develop a feeling for these Hjelmslevean Veronesean sets, I started by classifying those of type $(d,0)$, in which case the vertices of the quadrics are just points, and d is arbitrary but finite. For this I used (H2) and (H3) as above (but with $\dim(T_x) \leq 2(d+2)$ – which explains why I for now require that d is finite). This worked out well (though the methods used in [40] cannot just be extended to general d), but I choose not to include this partial result in my thesis as I do not consider it to be that much of an added value.

The next test case on my list was $v=1$, as I expected this to behave differently than $v=0$ but more or less similar to any $v>0$, as the main difference is that vertices can now intersect each other non-trivially. Not quite as expected, I found an example satisfying Axioms (H1), (H2) and (H3), but which did not fit in our framework (i.e., it was not one of the Veronese representations that we had in mind). This example (which I will briefly describe below) concerns a pair (X,Ξ) consisting of $(1,1)$-tubes.

Example 6.1.5. Inside $\mathsf{PG}(13,\mathbb{K})$, we take a 3-space Π_Y and a 9-space F complementary to it. Inside F, we consider the Veronese representation $\mathscr{V}_3(\mathbb{K},\mathbb{K})$ of the projective space $\mathsf{PG}(3,\mathbb{K})$ (defined analogously as $\mathscr{V}_2(\mathbb{K},\mathbb{K})$). Let χ be a linear duality between $\mathscr{V}_3(\mathbb{K},\mathbb{K})$ and Π_Y, i.e., χ takes points of $\mathscr{V}_3(\mathbb{K},\mathbb{K})$ to planes of Π_Y and conversely. We define X as the points on the affine 3-spaces $\langle c,\chi(c)\rangle\setminus\chi(c)$, where c is a point of $\mathscr{V}_3(\mathbb{K},\mathbb{K})$. It is clear that X is a spanning point set of $\mathsf{PG}(13,\mathbb{K})$. Now, for each conic C in $\mathscr{V}_3(\mathbb{K},\mathbb{K})$, we want to have the 4-space $\langle C,\chi(C)\rangle$ as a member Ξ (they intersect X in the tube with vertex $\chi(C)$ and basis C). However, we need more members of Ξ. To exactly know which ones, one would have to know “normal rational cubic scrolls” (see Section 6.6.5). Roughly and in short, the tubes with vertex $\chi(C)$ are precisely those that have a generator in common with each subspace $\langle c,\chi(c)\rangle$, where c varies over all points of C. As can be verified, the thus obtained pair (X,Ξ) consists of $(1,1)$-tubes and satisfies axioms (H1), (H2) and (H3) (the latter adapted to $(d,v)=(1,1)$, so $\dim(T_x)\leq 6$).

A similar example consisting of $(2,3)$-tubes can be given, this time using $\mathscr{V}_3(\mathbb{K},\mathbb{L})$ instead of $\mathscr{V}_3(\mathbb{K},\mathbb{K})$, where $\mathbb{L}$ is a quadratic Galois extension of $\mathbb{K}$. At the time it did not look feasible to classify the extra examples too, hence our need to change the axioms slightly as to avoid the above examples. The feature these examples share is that there exist pairs of disjoint tubic spaces, which is not the case in the Veronese representations $\mathscr{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{B},0))$ we are aiming at. Therefore, we opted to strengthen Axiom (H2) by requiring that each two tubic spaces share at least one point of X. The good thing about this choice is that (H2*) is strong enough to, together with (H1), classify the Hjelsmslevean Veronese sets of type (d,v), with d,v arbitrary, without needing a third axiom.

As such, it was no longer necessary to rephrase Axiom (H3) in such a way that it would also make sense when d or v is infinite. The way this was done for the characterisation of the non-degenerate Veronese representations of Moufang projective planes ([29]), so in the case where $v=-1$, but d possibly infinite, is by putting it as follows:

(H3') for each $x\in X$ and for each pair $\xi_1,\xi_2\in\Xi$ containing x, T_x is generated by $T_x(\xi_1)$ and $T_x(\xi_2)$.

This is however not true in the Veronese representations $\mathscr{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{B},0))$, where $v\geq 0$, as two tubic spaces ξ_1,ξ_2 through x sometimes have the same vertex, in which case they do not generate T_x.

In hindsight, I have to admit that, instead of replacing (H2) by (H2*), the better option might have been to replace (H3) by the following axiom:

(H3*) for each $x \in X$, there exist two tubic spaces ξ_1 and ξ_2 containing x, for which T_x is generated by $T_x(\xi_1)$ and $T_x(\xi_2)$.

This axiom holds true in the varieties that we want to obtain, and it does not hold true in the above described "counter-examples". I did not think of this at the time, and even then I might not have been convinced why I should prefer this over using (H2*), and even if so, there is no guarantee that it would have been doable.

Only after studying the split case (which is discussed in the next chapter), it became clear to me that using (H3*) would have been the more uniform option (uniform with respect to both the split case and the non-degenerate case). Indeed, in the split case, Axiom (H2*) does not hold; and it is possible to use Axioms (H1), (H2) and (H3*)[1]. It might be worth it to investigate whether the sets of axioms {(H1),(H2*)} and {(H1),(H2),(H3*)} are equivalent (this does not seem trivial to me though). One good thing about working with the former set of axioms is that it brings along an equivalent set of axioms for the non-degenerate case.

6.1.5 A note on the fields

The restriction $|\mathbb{K}| > 2$ is not necessary in Case (i) of Main Result 6.1.2, since Main Result 6.1.4 also deals with fields with two elements and no more is needed that depends on $|\mathbb{K}|$. So in this case, the varieties that one obtains after projecting from the single vertex V are in fact as listed in Main Result 6.1.4.

However, in Case (ii) of Main Result 6.1.2, when there is more than one vertex, our method breaks down very soon. Even the fact that the quadrics $X(\xi)$, with $\xi \in \Xi$ are convex, which is normally easily shown, now could not be proven. Though we did not succeed in finding counter examples, we do not believe a proof of this case is within reach (noting that, in principle, the vertices could also have infinite dimension). Chapter 6.5 contains a test case with some additional assumptions, and should give an idea of why this case is so troublesome.

6.1.6 Structure of the proof

In **Section 6.2**, we start the proof of our main results by reducing the situation to two separate cases: the case that there are no degenerate quadrics (in which case the corresponding algebras are division) and the case that all quadrics are degenerate (here the corresponding algebras possess a non-trivial radical).

In **Section 6.3**, we deal with the first of the two above cases and as such we provide an alternative approach to the Veronese representation of projective planes over quadratic

[1] Notwithstanding the fact that the setting is at first sight totally different there. More on that in the next chapter.

alternative division algebras. A large part is devoted to the case of the field of order 2 and although it is not essential for the rest of the chapter and is of a different flavour (being strictly finite), it does reveal a beautiful link with the large Witt design.

In **Section 6.4**, we treat the case that the quadrics are degenerate. Basically our approach amounts to a study of the structure induced on the set of vertices (which forms a subspace, say Y); the structure of the points of X after projecting from Y and the relation between X and Y. The regular scrolls alluded to before play a crucial role in this, in the sense that they are a restriction of the entire structure to all the quadrics having the same vertex.

In **Section 6.5**, we run a test case for the situation in which $|\mathbb{K}|=2$, which is more difficult than one would expect.

In **Section 6.6**, we discuss an important geometric substructure of the Hjelmslevean Veroneseans: *regular scrolls*. A regular scroll is a generalisation of a normal rational cubic scroll. Essentially, a regular scroll consists of a quadric Q in a $(d+1)$-space with points as maximal singular subspaces, and a regular spread of $(d-1)$-spaces in $(2d-1)$-space, between which there is a projectivity φ; and then the scroll is given as the set $\{\langle x,\varphi(x)\rangle \mid x\in Q\}$. It is not at all clear for which quadrics (it should depend on the field of definition) such a projectivity exists, though apparently, in the situations we consider, they do. In this section, we define these scrolls and show some of their properties.

6.2 Preliminaries and vertex-reduction

We reduce the proof to two essential cases, namely $v=-1$, i.e., vertices are empty (Case (i)); or there is more than one vertex and distinct vertices are pairwise disjoint (Case (ii)). In Section 6.3, we deal with Case (i), also allowing $\mathbb{K}=\mathbb{F}_2$, and in fact covering Main Result 6.1.4 and Main Result 6.1.2(i). In Section 6.4 we then treat Case (ii) above, covering Main Result 6.1.2(ii).

We now start with the reduction.

Definition 6.2.1 (Singular subspaces)**.** We define a *singular line* L as a line of $\mathsf{PG}(N,\mathbb{K})$ that has all its points in $X\cup Y$. Two (distinct) points z,z' of $X\cup Y$ are called *collinear* if they are on a singular line. A subspace Π of $\mathsf{PG}(N,\mathbb{K})$ will be called *singular* if it belongs to $X\cup Y$ and each pair of its points is collinear. If a singular subspace Π intersects Y in a hyperplane of Π, then $\Pi\cap X$ is called a *singular affine subspace*; in particular, if Π is a singular line containing a unique point in Y, then $\Pi\cap X$ is called a singular affine line.

Lemma 6.2.2. *Let L be a line of $\mathsf{PG}(N,\mathbb{K})$ containing two points x_1 and x_2 in X. Then either L is a singular line having a unique point in Y (hence $L\cap X$ is a singular affine line) or $L\cap(X\cup Y)=\{x_1,x_2\}$. In particular, each singular line contains at least one point of Y.*

Proof. By (H1), there is a tube C through x_1 and x_2. Suppose that L contains a third point $z\in X\cup Y$. As $\Xi(C)\cap(X\cup Y)=\overline{C}$, the line L belongs to a generator of C and hence it is a singular line containing a unique point in the vertex of $\overline{C}$, all its other points clearly belonging to X. It follows that there are no singular lines entirely contained in X. □

Lemma 6.2.3. *Let L and L' be distinct singular lines containing unique points y and y' in Y, respectively, with $y = y'$. Then either L and L' belong to a unique common tube, or the plane $\langle L, L'\rangle$ is a singular plane and $\langle L, L'\rangle \cap X$ is a singular affine plane.*

Proof. If $L \cup L'$ belongs to at least two tubes, then (H2) implies that $L \cup L'$ is contained in a generator of those tubes, and then $\langle L, L'\rangle$ is a singular plane with a unique line in Y indeed. So suppose $L \cup L'$ does not belong to any tube.

Take any point p in the plane π spanned by L and L', but not on $L \cup L'$. We consider two lines M, M' in π through p not incident with y. We consider tubes C_M and $C_{M'}$ through M and M', respectively. If $C_M = C_{M'}$, then C_M contains $L \cup L'$, contradicting our assumption. So C_M and $C_{M'}$ are distinct and hence, $p \in C_M \cap C_{M'} \subseteq X \cup Y$ by (H2*). This already shows that π is a singular plane, i.e., $\pi \subseteq X \cup Y$.

Now π contains a unique line in Y since, by Lemma 6.2.2 and the fact that $\pi \cap X \neq \emptyset$, $\pi \cap Y$ is a geometric hyperplane of π. □

Corollary 6.2.4. *Let x_1 and x_2 be non-collinear points of X. Then there is a unique v-space V in Y collinear to both of them. In particular, there is a unique tube through x_1 and x_2 (denoted $[x_1, x_2]$), and its vertex is V.*

Proof. By (H1), there is at least one tube C through x_1 and x_2 and hence the vertex of C is a v-space V collinear to both x_1, x_2. If there would be a point $y \in Y \setminus V$ collinear with both x_1 and x_2, then Lemma 6.2.3 implies that the lines x_1y and yx_2 are in a tube C'. Since $y \notin V$, the tubes C and C' are distinct, but then their intersection contains two non-collinear points, contradicting (H2)*. □

Already at this point, the need for $|\mathbb{K}| > 2$ arises. If $|\mathbb{K}| = 2$, we would only be able to prove that collinearity is an equivalence relation.

Lemma 6.2.5. *Two singular affine subspaces Π and Π' intersecting in at least one point $x \in X$ generate a singular subspace and $\langle \Pi, \Pi'\rangle \cap X$ is a singular affine subspace.*

Proof. Let $x \in X$ and suppose that L and L' are distinct singular lines through x, having unique points y and y' in Y, respectively (note that $y \neq y'$). Take a point p on $yy' \setminus \{y, y'\}$. Since $|\mathbb{K}| > 2$, there are lines M and M' through p, each of which meets both $L \setminus \{y\}$ and $L' \setminus \{y'\}$ in distinct points of X. If M and M' are contained in the same tubic subspace, then, since $X \ni x \notin M \cup M'$, the plane spanned by the lines M, M' is singular, with a unique line in Y. Hence we may assume that M and M' are contained in distinct tubic subspaces, and so $p \in X \cup Y$ by (H2*). As p was arbitrary on $yy' \setminus \{y, y'\}$, it follows that the line yy' is contained in $X \cup Y$. So by Lemma 6.2.2 and $|\mathbb{K}| > 2$, we may assume $p \in Y$. Applying Lemma 6.2.3 on the lines M and M', we again obtain that $\langle L, L'\rangle$ is a singular plane with a unique line (namely yy') in Y.

Now let Π and Π' be general singular affine subspaces with $x \in \Pi \cap \Pi'$. Repeated use of the previous argument for all affine singular lines in $\langle \Pi, \Pi'\rangle$ sharing x shows the lemma. □

Corollary 6.2.6. *The set Y is a subspace.*

Proof. Let $y_1, y_2 \in Y$, $y_1 \neq y_2$. Let y_i be contained in a tube C_i, $= 1,2$. If C_1 equals C_2, then the line $y_1 y_2$ joining y_1 and y_2 is contained in its vertex, so in particular in Y. If $C_1 \neq C_2$, then their intersection contains a point $x \in X$ by (H2*). Applying Lemma 6.2.5 on the lines $L = xy_1$ and $L' = xy_2$, we obtain that $y_1 y_2 \subseteq Y$. □

Later on, we will project X from Y and show that the obtained structure is a Hjelmslevean Veronese set with $v = -1$. However, we need much more structure before it makes sense to do this.

Definition 6.2.7 (Maximal singular subspaces)**.** Let x be any point in X. By the previous lemma, we can define Π_x as the unique maximal singular affine subspace containing x, and we denote its projective completion (i.e., $\langle \Pi_x \rangle$) by $\overline{\Pi}_x$. Finally, we define $\Pi_x^Y = \overline{\Pi}_x \setminus \Pi_x = \overline{\Pi}_x \cap Y$.

We have the following corollary.

Corollary 6.2.8. *For $x, x' \in X$, $\Pi_x \cap \Pi_{x'}$ is non-empty if and only if $\Pi_x = \Pi_{x'}$ if and only if x and x' are collinear. If x and x' are not collinear, then $\overline{\Pi}_x \cap \overline{\Pi}_{x'}$ is the vertex of $[x, x']$.*

Proof. Suppose $\Pi_x \cap \Pi_{x'}$ contains a point of X. It follows from Lemma 6.2.5 that Π_x and $\Pi_{x'}$ generate a singular subspace Π. As both were the maximal ones containing x and x', $\Pi_x = \Pi = \Pi_{x'}$. Clearly, $\Pi_x = \Pi_{x'}$ implies $x' \in \Pi_x$, so x and x' are collinear. For collinear points x and x', we have $x, x' \subseteq \Pi_x \cap \Pi_{x'}$, hence the "if and only if"-statements follow.

If x and x' are non-collinear points, then Corollary 6.2.4 implies that the vertex of $[x, x']$ coincides with $\overline{\Pi}_x \cap \overline{\Pi}_{x'}$. □

Lemma 6.2.9. *Let C_1 and C_2 be tubes with respective vertices V_1 and V_2. Set $V^* = V_1 \cap V_2$. Then the vertex V of each tube C contains V^*. Hence each point of X is collinear with V^*. Moreover, the intersection of any pair of distinct vertices is precisely V^*.*

Proof. By (H2*), the tubes C_1 and C_2 share a point $x \in X$, so $\overline{C_1} \cap \overline{C_2} = \langle x, V^* \rangle$. By the same axiom, $C \cap C_i$ contains a point $z_i \in X$, $i = 1,2$.

We claim that $z_1 \perp z_2$ if and only if $z_1 \in \langle x, V_1 \rangle$ and $z_2 \in \langle x, V_2 \rangle$. Indeed, suppose z_1 does not belong to $\langle x, V_1 \rangle$ (which is equivalent to z_1 not being collinear to x) and suppose $z_1 \perp z_2$. The first fact means $C_1 = [x, z_1]$, so by Corollary 6.2.8 we have $\Pi_x^Y \cap \Pi_{z_1}^Y = V_1$. By the same corollary, the second fact means $V_2 \subseteq \Pi_{z_1}^Y$ and hence $V_2 \subseteq \Pi_x^Y \cap \Pi_{z_1}^Y = V_1$, a contradiction. The other implication is clear since $\langle x, V_1, V_2 \rangle$ belongs to the singular subspace Π_x. This shows the claim.

Now, if z_1 is not collinear with z_2, then Corollary 6.2.8 readily implies $V^* \subseteq V$. So suppose $z_1, z_2 \in x^\perp$ (and hence z_1 and z_2 are equal or collinear). For $i = 1,2$, let z_i' be a point of C_i not collinear to x (and hence neither to z_i). Then by the above, the vertex V' of the tube

C' through z_1' and z_2' contains V^*. For $i = 1,2$, we now consider C_i and C' instead of C_1 and C_2. Then again, since z_i and z_i' are not collinear, the previous cases reveal that $V \cap V_i$ contains V^*.

As C was arbitrary, we conclude that each tube's vertex V contains V^*. It immediately follows that each point x is collinear with V^*. Assume that there would be two tubes C_1' and C_2' whose respective distinct vertices V_1' and V_2' would intersect in more than V^*. Repeating the above argument, we would obtain that C_1 and C_2 both contain $V_1' \cap V_2'$, a contradiction. □

For an arbitrary subspace F of $\mathsf{PG}(N,\mathbb{K})$ complementary to V^*, we now consider the map $\rho : X \to F : x \mapsto \langle x, V^* \rangle \cap F$. The pair $(\rho(X), \rho(\Xi))$ is well defined then and consists of (d,v')-tubes with base Q_d^0, where $v' = \mathrm{codim}_V(V^*)$, for any vertex V.

Proposition 6.2.10. *Let C_1 and C_2 be tubes with respective vertices V and V' that intersect in a subspace V^* of V. Then $(\rho(X), \rho(\Xi)) = (X \cap F, \{\xi \cap F \mid \xi \in \Xi\})$ is a Hjelmslevean Veronese set with (d,v')-tubes for $v' = \mathrm{codim}_V(V^*)$. If $v' \geq 0$ then two vertices either coincide or are disjoint and both cases occur.*

Proof. By Lemma 6.2.9, each point of $x \in X$ is collinear with V^* and all elements of Ξ contain V^*. Hence $(\rho(X), \rho(\Xi)) = (X \cap F, \{\xi \cap F \mid \xi \in \Xi\})$. Clearly, $\rho(\xi) \cap \rho(X)$ is a (d,v')-tube for $v' = \mathrm{codim}_V(V^*)$ for each $\xi \in \Xi$. We show that (H1) and (H2*) are satisfied.

• *Axiom* (H1). Let x and x' be points of $\rho(X)$. Then Axiom (H1) in (X,Ξ) implies that there is a tube C containing x and x'. By Lemma 6.2.9, the vertex of C contains V^* and hence $\rho(C)$ is a tube through x and x'.

• *Axiom* (H2*). Let ξ and ξ' be distinct members of $\rho(\Xi)$. Then $\langle \xi, V^* \rangle \cap \langle \xi', V^* \rangle$ belongs to $X \cup Y$ and contains at least one point $x \in X$ by Axiom (H2*) in (X,Ξ). It is clear that $\xi \cap \xi'$ belongs to $X \cup Y$ and that it contains $\rho(x) \in \rho(X)$.

The rest from the statement follows immediately from Lemma 6.2.9. □

Consider the following property.

(V) Two vertices either coincide or have empty intersection, and both cases occur.

If (V) is not valid, then either all tubes have the same vertex V or there are tubes C and C' such that their respective vertices V and V' are neither disjoint nor equal. In both cases, we have shown above that the projection from V or from $V \cap V'$, respectively, yields a point-quadric variety with (d,v')-tubes with base Q_d^0 with $v' = -1$ or with $0 \leq v' \leq v$, respectively, which satisfies (H1), (H2*) and, if $v' \geq 0$, also (V) applies. We deal with those two possibilities separately.

6.3 The case $\nu = -1$: Ordinary Veroneseans

This section contains the proof of Main Result 6.1.4, as we now deal with the case where members of Ξ are non-degenerate, i.e., $\nu = -1$. At the same time, by Proposition 6.2.10, this provides a proof of Main Result 6.1.2(i).

6.3.1 The general set-up

Let X be a spanning point set of $\mathsf{PG}(N,\mathbb{K})$, $N > d+1$ (possibly infinite), and let Ξ be a collection of $(d+1)$-dimensional projective subspaces of $\mathsf{PG}(N,\mathbb{K})$ (called the *elliptic spaces*) such that, for any $\xi \in \Xi$, the intersection $\xi \cap X$ is a Q_d^0-quadric $X(\xi)$ whose points span ξ. The tangent spaces $T_x(X(\xi))$ are also denoted by $T_x(\xi)$ as before and the subspace spanned by all such tangent spaces through x is again called the *tangent space T_x of x*, i.e., $T_x = \langle T_x(\xi) \mid x \in \xi \in \Xi \rangle$. For such a quadric Q, we denote by $\Xi(Q)$ the unique member of Ξ containing Q. We assume that (X', Ξ) satisfies the following two properties:

(MM1) each pair of distinct points $x_1', x_2' \in X'$ is contained in some element of Ξ', and

(MM2*) the intersection of each pair of distinct elements of Ξ' is precisely a point of X'.

Seeing the fact that Main Theorem 6.1.4 has a different outcome when $|\mathbb{K}| = 2$, we will divide the proof into two cases accordingly. We start with the "regular" case where $|\mathbb{K}| > 2$.

6.3.2 The case $|\mathbb{K}| > 2$

Our aim is to show that (X, Ξ) satisfies the following properties.

(MM1) Any two distinct points x_1 and x_2 of X lie in a element of Ξ;

(MM2) for any two distinct members ξ_1 and ξ_2 of Ξ, the intersection $\xi_1 \cap \xi_2$ belongs to X;

(MM3) for any $x \in X$ and any three distinct members ξ_1, ξ_2 and ξ_3 of Ξ with $x \in \xi_1 \cap \xi_2 \cap \xi_3$ we have $T_x(\xi_3) \subseteq \langle T_x(\xi_1), T_x(\xi_2) \rangle$.

Indeed, if (X, Ξ) satisfies (MM1) up to (MM3), then (X, Ξ) is projectively equivalent to $\mathscr{V}_2(\mathbb{K}, \mathbb{B})$, with $\mathbb{B}$ a quadratic alternative division algebra $\mathbb{B}$ over $\mathbb{K}$ with $\dim_{\mathbb{K}}(\mathbb{B}) = d$, as follows from a characterisation of the ordinary Veronese representations $\mathscr{V}_2(\mathbb{K}, \mathbb{B})$ of a projective plane $\mathsf{PG}(2, \mathbb{B})$ over such an algebra $\mathbb{B}$, by means of Mazzocca-Melone axioms by O. Krauss, J. Schillewaert and H. Van Maldeghem in [29]. Conversely, then it is easily verified that, for each such algebra $\mathbb{B}$, the corresponding Veronese representation $\mathscr{V}_2(\mathbb{K}, \mathbb{B})$ satisfies our axiom (MM2*). This then shows Main Theorem 6.1.4 in case $|\mathbb{K}| > 2$ (if $|\mathbb{K}| = 2$, (MM3) not necessarily holds).

Fix an elliptic space $\xi \in \Xi$ and put $Q = X(\xi)$; let F be a subspace of $\mathsf{PG}(N, \mathbb{K})$ complementary to ξ. We denote by $\rho : \mathsf{PG}(N, \mathbb{K}) \to F$ the projection operator that projects from ξ onto F.

Lemma 6.3.1. *The projection ρ is injective on $X \setminus Q$.*

Proof. Take two (distinct) points $p,q \in X \setminus Q$ and suppose for a contradiction that $\rho(p) = \rho(q)$. Then $\xi = \langle Q \rangle$ is a hyperplane of $\langle Q,p,q \rangle$, implying that the line pq intersects ξ in a point z. On the other hand, by (MM1*), p and q are contained in a quadric Q' which intersects ξ, by (MM2*), in a point of Q. Since $z \in \xi \cap \Xi(Q')$, it follows from (MM2*) that $z \in X$. Clearly, the points p,q,z are all distinct, and hence the line pq in $\Xi(Q')$ contains three points of X, a contradiction. The assertion follows. $\square$

Lemma 6.3.2. *The image $\rho(X \setminus Q)$ is an affine subspace A whose projective completion equals F.*

Proof. We first prove that the image is an affine space. Let Q' be any quadric distinct from Q. By (MM2*), $Q \cap Q' = \xi \cap \Xi(Q')$ contains exactly one point, say z. So $\rho(Q')$ is given by projecting Q' from z and as such it is an affine d-space, whose projective $(d-1)$-space at infinity corresponds to $\rho(T_z(Q'))$.

Consequently, (MM1*) implies that each two points $\rho(p),\rho(q)$, with $p,q \in X \setminus Q$, are contained in an affine line L_{pq} of $\rho(X([p,q]))$. Let y be the point on $\rho(p)\rho(q)$ not contained in $\rho(X([p,q]))$, i.e., $L_{pq} \cup \{y\} = \rho(p)\rho(q)$. If $y = \rho(r)$ for some $r \in X \setminus Q$, then $\rho(p)\rho(q) = \rho(p)\rho(r)$, and as $|\mathbb{K}| > 2$, this yields at least two points in $L_{pq} \cap L_{pr}$, and by injectivity, their inverses belong to $X[p,q] \cap X[p,r]$, which only contains one point by (MM1*), a contradiction. This shows $y \notin \rho(X \setminus Q)$. The set $Y := \{\rho(p)\rho(q) \setminus L_{pq} \mid p,q \in X \setminus Q, p \neq q\}$ thus belongs to $F \setminus \rho(X \setminus Q)$.

Now take three points p,q,r in $\rho(X \setminus Q)$ which are not on a line. We claim that $\langle p,q,r \rangle \cap \rho(X \setminus Q)$ is an affine plane (whose projective completion equals $\langle p,q,r \rangle$). Let y_q be the unique point of Y on pq and y_r the unique point of Y on pr. By the previous paragraph we already know that a line containing two points of $\rho(X \setminus Q)$ has all but one points in $\rho(X \setminus Q)$, the remaining point being contained in Y. In particular, the line $y_q y_r$ contains at most one point in $\rho(X \setminus Q)$. Suppose $y_q y_r$ contains a unique point $x \in \rho(X \setminus Q)$. Then px contains a unique point $y \in Y$ through which there is a line L intersecting $pq \setminus \{p, y_q\}$ and $pr \setminus \{p, y_r\}$, since $|\mathbb{K}| > 2$. Clearly, $L \neq px$, so L intersects $y_q y_r$ in a point distinct from x and hence L contains at least two points of $\rho(X \setminus Q)$ and two points of Y, a contradiction. Hence all points of $y_q y_r$ belong to Y. Now each point $v \in \langle p,q,r \rangle \setminus y_q y_r$ is on a line containing at least two points of $(pq \cup qr \cup rp) \cap X$, implying $v \in \rho(X \setminus Q)$. The claim is proved.

It follows that $\rho(X \setminus Q)$ is an affine subspace of F and, as X generates $\mathsf{PG}(N,\mathbb{K})$, we have $\rho(X \setminus Q) \cup Y = F$. $\square$

We keep referring to the projective space at infinity of $\rho(X \setminus Q)$ in F as Y.

Lemma 6.3.3. *Let x be a point of $X \setminus Q$. For distinct points p,q of Q, $\rho(T_p([p,x])) \cap \rho(T_q([q,x]))$ is empty and $\bigcup_{p \in Q} \rho(T_p([p,x])) = Y$.*

Proof. Put $Q_p = X([p,x])$ and $F_p := \rho([p,x])$, i.e., $F_p = \rho(Q_p) \cup \rho(T_p([p,x]))$; likewise $Q_q = X([q,x])$ and $F_q := \rho([q,x])$. As ρ is injective by Lemma 6.3.1, (MM2*) implies that $\rho(Q_p) \cap \rho(Q_q)$ is exactly $\rho(x)$. Moreover, this also implies that $F_p \cap F_q = \rho(x)$, as otherwise

$F_p \cap F_q$ contains a line through $\rho(x)$ and then $\rho(Q_p)\cap\rho(Q_q)$ would be an affine line through $\rho(x)$, a contradiction. It follows that $\rho(T_p(Q_p))\cap\rho(T_q(Q_q))$ is empty.

Now take $y \in Y$ arbitrary. Let r be a point of $F \setminus Y$ on the line $\rho(x)y$ and put $r' = \rho^{-1}(r)$ (which is well defined by Lemmas 6.3.1 and 6.3.2). Then $[x, r'] \cap Q$ is a point r'' and we obtain that $y \in \rho(T_{r''}[r'', x])$. Note that any point r' on $\rho(x)y$ would yield the same point r'' by the previous paragraph. $\square$

Lemma 6.3.4. *Let p be a point in Q. Then $\rho(T_p(Q_1)) = \rho(T_p(Q_2))$ for all quadrics Q_1 and Q_2 distinct from Q and with $p \in Q_1 \cap Q_2$.*

Proof. Suppose for a contradiction that $\rho(T_p(Q_1)) \neq \rho(T_p(Q_2))$ for two quadrics Q_1 and Q_2 distinct from Q with $p \in Q_1 \cap Q_2$. Then there is a point $y \in \rho(T_p(Q_1)) \setminus \rho(T_p(Q_2))$. Let x_2 be a point in $Q_2 \setminus \{p\}$. By Lemma 6.3.3, y belongs to $\rho(T_{p'}([p', x_2]))$ for some $p' \in Q$ with $p' \neq p$. By (MM2*), $Q_3 = X([p', x_2])$ intersects Q_1 in a point x_1, and $x_1 \neq p$ since $x_2 \notin Q$. Then Q_1 and Q_3 are two different quadrics through x_1, and $y \in \rho(T_p(Q_1)) \cap \rho(T_{p'}([x_2, p']))$, whereas this intersection should be empty according to Lemma 6.3.3. This contradiction shows the lemma. $\square$

Lemma 6.3.5. *Let p be any point in Q. For any member $\xi' \in \Xi \setminus \{\xi\}$ with $p \in \xi \cap \xi'$, $T_p = \langle T_p(\xi), T_p(\xi')\rangle$.*

Proof. By Lemma 6.3.4, $\rho(T_p(Q')) = \rho(T_p(Q''))$ for all quadrics Q', Q'' distinct from Q. Now fix any quadric $Q' \neq Q$ through p. By definition of T_p we have $\rho(T_p) = \rho(T_p(Q'))$. We obtain $\langle T_p(Q), T_p(Q')\rangle \subseteq T_p \subseteq \langle Q, T_p(Q'),\rangle$. Since $\langle T_p(Q), T_p(Q')\rangle$ is a hyperplane of $\langle Q, T_p(Q')\rangle$, we have that either $T_p = \langle T_p(Q), T_p(Q')\rangle$, in which case the lemma is proven, or $T_p = \langle Q, T_p(Q')\rangle$. So suppose we are in the latter case, in which $Q \subseteq T_p$. Then no quadric $Q'' \neq Q$ through p can be contained in T_p, for otherwise $\Xi(Q) \cap \Xi(Q'')$ contains at least a line, a contradiction. Switching the roles of Q and Q', we obtain that $\langle T_p(Q), T_p(Q')\rangle \subseteq T_p \subseteq \langle Q', T_p(Q)\rangle$, and the latter situation cannot occur since $Q \nsubseteq \langle Q', T_p(Q)\rangle$. The lemma is proven. $\square$

Since Q and $p \in Q$ were arbitrary, it follows from Lemma 6.3.5 that (X, Ξ) satisfies Axiom (MM3), finishing the proof of Theorem 6.1.4 in case $|\mathbb{K}| > 2$.

6.3.3 The case $|\mathbb{K}| = 2$

When there are only 3 points on a line, the above techniques fail and for a very good reason: We get more examples. As the field is finite, Q^0_d-quadrics only exist when $d = 1, 2$. We deal with those cases separately. We will sometimes write $\mathbb{F}_2$ instead of $\mathbb{K}$ to emphasise the size of the field.

Since we are working here in projective spaces of order 2, we can add points together: The sum of two points is the third point on the line determined by those two points. This additive structure, with additional neutral element $\emptyset$, where $a + a = \emptyset$, for each point a, is an elementary abelian 2-group (the additive group of the underlying vector space).

The case $d = 1$

Axioms (MM1) and (MM2*) imply that (X, Ξ), if existing, is as a point-line geometry isomorphic to a projective plane of order 2 (i.e., $\mathsf{PG}(2, \mathbb{F}_2)$) and hence contains seven points in total.

Proposition 6.3.6. *For any pair* (X, Ξ) *satisfying* (MM1) *and* (MM2*), *with* X *a spanning point set of* $\mathsf{PG}(N, 2)$, $N > 2$, *and* Ξ *a set of planes, we have* $N \in \{5, 6\}$. *If* $N = 5$ *there are, up to projectivity, two possibilities—among which* $\mathcal{V}_2(\mathbb{F}_2, \mathbb{F}_2)$; *if* $N = 6$ *then* X *is any basis of* $\mathsf{PG}(6, \mathbb{F}_2)$.

Proof. Since there are only seven points, we readily obtain $N \leq 6$. Now by (MM2*), we see that $N \geq 4$ and moreover this axiom implies that each plane of $\mathsf{PG}(N, 2)$ contains at most three points of X and each 3-space of $\mathsf{PG}(N, 2)$ at most four points of X (indeed, any set of five points of a projective plane of order 2 forms exactly the set of points on two lines and hence spans a 4-space of $\mathsf{PG}(N, 2)$).

First suppose $N = 4$. We choose five points of X, which, by the above, form a basis of $\mathsf{PG}(4, 2)$. Now the two remaining points of X are not contained in any 3-space spanned by four points of the basis. But there is only one such point in $\mathsf{PG}(4, 2)$. This contradiction rules out $N = 4$.

Next, suppose $N = 5$. Since no line contains three points of X, no plane contains four points of X and no 3-space contains five points of X, there are only two options. Firstly, it could be that no 4-space contains six points of X, in which case we obtain that the seven points of X form a frame. Then (X, Ξ) is projectively equivalent to $\mathcal{V}_2(\mathbb{F}_2, \mathbb{F}_2)$. Secondly, if there is a 4-space S containing six points of X (seven is impossible by the previous paragraph), then these six points form a frame of S and the seventh point of X is a point outside S forming a basis with any 5 points of $S \cap X$. One easily checks that such a set satisfies the axioms (MM1) and (MM*), no matter how we choose the elliptic spaces.

Finally, suppose $N = 6$. Then X generates $\mathsf{PG}(N, \mathbb{K})$ and hence is any basis of it. Also in this case, any choice of the elliptic spaces will do. □

The case $d = 2$

As each ovoid in $\mathsf{PG}(3, 2)$ contains five points, it follows as before that the pair (X, Ξ), as a point-line geometry, is a projective plane of order 4, hence containing 21 points and as such isomorphic to $\mathsf{PG}(2, 4)$. Clearly, each set of four points on an ovoid O in $\mathsf{PG}(3, 2)$ determines a basis of $\mathsf{PG}(3, 2)$. Note that there is a unique frame of $\mathsf{PG}(3, 2)$ containing this basis, which then coincides with O. More precisely, if we let e_0, e_1, e_2, e_3 be any four of its points, then the fifth point is $e_0 + e_1 + e_2 + e_3$. This will be the key observation to show the following proposition.

Proposition 6.3.7. *For any pair (X,Ξ), where X is a spanning point set of $\mathsf{PG}(N,2)$ with $N>3$ and Ξ a family of 3-spaces, satisfying* (MM1) *and* (MM2*), *we have $8\le N\le 10$. If $N=10$ then (X,Ξ) is projectively unique (and denoted by $\mathscr{M}^{10}(\mathbb{F}_2)$); if $N=9$ or $N=8$, then (X,Ξ) results from projecting $\mathscr{M}^{10}(\mathbb{K})$ from a suitable point or line, respectively, and there is a unique such line that gives $\mathscr{V}_2(\mathbb{F}_2,\mathbb{F}_4)$. In both cases, (X,Ξ) is projectively unique.*

We prove this proposition in a small series of lemmas. In the first lemma (Lemma 6.3.8) we consider all representations of $\mathsf{PG}(2,4)$ as point-block geometries in $\mathsf{PG}(N,2)$, such that blocks of $\mathsf{PG}(2,4)$ correspond to ovoids in 3-dimensional subspaces of $\mathsf{PG}(N,2)$. Noting that an ovoid in $\mathsf{PG}(3,2)$ is a *frame* (in general this is a set of $n+2$ points of an n-dimensional projective space such that each $n+1$ among them generate the space), the lemma is in fact about pseudo embeddings of $\mathsf{PG}(2,4)$. *Pseudo embedding* of point-line geometries have been introduced and studied by De Bruyn [13, 14]. In Proposition 4.1 of [14], he obtained that the universal pseudo-embeddings of $\mathsf{PG}(2,4)$ lives in $\mathsf{PG}(10,2)$ and an explicit (coordinate) construction has been given by him in Theorem 1.1 of [13]. Nevertheless we include our construction, which is in terms of a basis of $\mathsf{PG}(10,2)$ because we will rely on it in the lemmas thereafter to prove results in our more specific setting (in which (MM2) also holds).

Lemma 6.3.8. *Let (X,Φ) be a pair with X a spanning point set of $\mathsf{PG}(N,2)$, $N>3$ and Φ a family of ovoids in 3-spaces, such that, with the natural incidence, (X,Φ) is a projective plane of order 4. Then*

(i) *if $N=10$, then (X,Φ) is projectively unique and denoted by $\mathscr{M}^{10}(\mathbb{F}_2)$; and*
(ii) *each such structure is the projection of $\mathscr{M}^{10}(\mathbb{F}_2)$.*

In particular $N\le 10$. Moreover, the stabiliser of $\mathscr{M}^{10}(\mathbb{F}_2)$ in $\mathsf{PSL}(11,2)$ a group isomorphic to $\mathsf{P\Gamma L}(3,4)$.

Proof. For convenience, we shall call a member of Φ a *block*. So a block is a line of the projective plane $(X,\Phi)\cong\mathsf{PG}(2,4)$, and at the same an ovoid in some 3-space of $\mathsf{PG}(N,2)$. The unique block through two distinct points a,b will be denoted by $[a,b]$, since the notation ab will mean something else (namely, $a+b$).

Let $\circ$ and $*$ be any two (distinct) elements of X. Take arbitrarily three blocks ξ_1^*, ξ_2^* and ξ_3^* through $*$ and not through $\circ$, and three arbitrary blocks ξ_1°, ξ_2° and ξ_3° through $\circ$ but not through $*$, in such a way that the points $\xi_i^*\cap\xi_i^\circ$, $i=1,2,3$, are on a block (this can be achieved by possibly just interchanging ξ_2° and ξ_3°). Then we claim that the nine intersection points of these blocks, together with $\circ$ and $*$, fully determine the pair (X,Φ), as a substructure of $\mathsf{PG}(N,2)$, and X is contained in the span of these eleven points. In particular, $N\le 10$.

Let us label the nine intersection points of ξ_i^* and ξ_j°, $i,j\in\{1,2,3\}$, by the digits 1 up to 9 according to the picture below. Set $I=\{1,2,\dots,9\}$.

Since each of those six blocks now contains exactly four points of the set $I\cup\{*,\circ\}$, its fifth point is uniquely determined by their sum (and we denote $\circ+1+2+3$ as $\circ123$ and

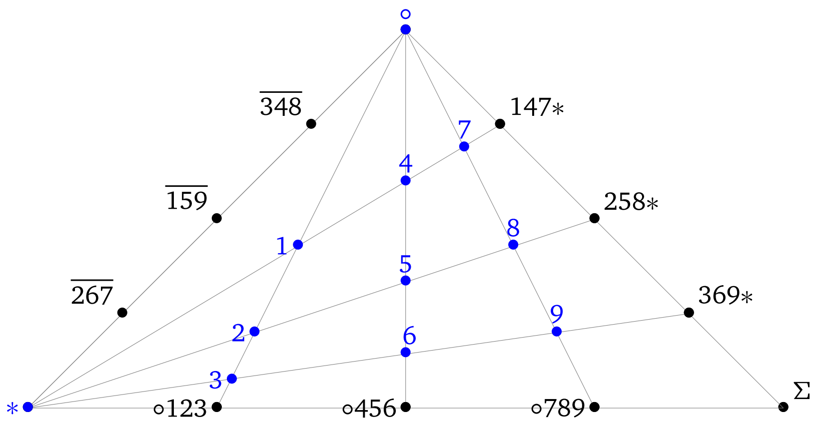


Figure 6.1: The projective plane (X,Ξ)

$1+4+7+*$ as $147*$—be aware that we use the elements of I as mere symbols; in general we shall denote the sum of elements of $I\cup\{*\circ\}$ by juxtaposition; we overline a string of elements of $I\cup\{*,\circ\}$ if we mean the sum of the complement of the elements in the string). We obtain six additional points: $147*$, $258*$ and $369*$ (called $*$-*triples*) on the blocks ξ_1^*, ξ_2^* and ξ_3^*, respectively; and $\circ123$, $\circ456$ and $\circ789$ (called $\circ$-*triples*) on ξ_1°, ξ_2° and ξ_3°. The $*$-triples are on a block ξ_4° through $\circ$ and the $\circ$-triples are on a block ξ_4^* through $*$. From each of the blocks ξ_4° and ξ_4^*, four points are determined, and hence the remaining point is determined as well. For both blocks, this remaining point is $\circ123456789* =: \Sigma$.

To define the three remaining points of X (those in $[\circ,*]\setminus\{\circ,*\}$), we consider the blocks $[\Sigma,7]$, $[\Sigma,8]$ and $[\Sigma,9]$. By our assumption that $1,5$ and 9 are on a block, the block $[\Sigma,7]$ contains the points $2,6,7$, the block $[\Sigma,8]$ contains the points $3,4,8$ and, lastly, the block $[\Sigma,9]$ contains the points $1,5,9$. The fifth points on these blocks are $\overline{267}$, $\overline{348}$ and $\overline{159}$, respectively (these three are called the Σ-triples; as introduced above, $\overline{abc}$ denotes the sum of the complement of $\{a,b,c\}$ in the set $I\cup\{\circ,*\}$). The points $\overline{267}$, $\overline{159}$ and $\overline{348}$ lie on a block together with $\circ$ and $*$ and they do sum up to zero indeed.

We need the nine remaining blocks of the projective plane (X,Ψ) to conclude that this is well defined. This could be done by using coordinates, though we prefer to give the remaining blocks by reasoning as follows.

For each point in I, we need two more blocks through it. Taking $1\in I$ as an example, we note that the blocks through 1 and $*$, $\circ$ and Σ, respectively, are given as follows: $[1,*]=\{1,4,7,*,147*\}$, $[1,\circ]=\{\circ,1,2,3,\circ123\}$, $[1,\Sigma]=\{1,5,9,\Sigma,\overline{159}\}$, so for each $\times\in\{*,\circ,\Sigma\}$, the $\times$-triple containing 1 reveals which points are on the block $[1,\times]$. Since $6,8$ do not occur in any such triple, the remaining blocks are $[1,6]$ and $[1,8]$, and they need to be distinct (there is no $*$-triple neither containing 1 nor 6 nor 8). Hence the block $[1,6]$ has to contain $\times$-triples not containing 1 and 6, but there are exactly three such. Consequently, there is only one possibility: $[1,6]=\{1,6,258*,\circ789,\overline{348}\}$. Likewise $[1,8]=\{1,8,369*,\circ456,\overline{267}\}$. These indeed have sum zero. In general, let $\{a,b\}\subseteq I$ be any pair that is, just like $\{1,6\}$ and $\{1,8\}$, not contained in any triple. Then for each $\times\in\{*,\circ,\Sigma\}$, there is a unique triple not containing a, nor b, which we denote by $T^\times(ab)$. For those pairs $\{a,b\}$ (for the record, these are all pairs occurring in $\{1,6,8\}$, in $\{2,4,9\}$ and in $\{3,5,7\}$) we define

$$[a,b] := \{a,b,T^*(ab),T^\circ(ab),T^\Sigma(ab)\}.$$

A straightforward verification shows that each such block sums up to zero.

We now have 21 blocks, 5 through each point and one through each pair of distinct points, confirming that the above defined set of points and blocks indeed is the projective plane of order 4. Hence the set $I \cup \{*,\circ\}$ defines (X,Ξ) entirely. In particular, $N \leq 10$.

Now let $N = 10$. Then we can take for $I \cup \{*,\circ\}$ any basis of $\mathsf{PG}(10,2)$ and we obtain a unique example $\mathscr{M}^{10}(\mathbb{F}_2)$. Now, there are $21 \cdot 20 \cdot (4 \cdot 3 \cdot 2) \cdot (4 \cdot 3 \cdot 2)/2 = |\mathsf{P\Gamma L}(3,4)|$ choices for the set $I \cup \{*,\circ\}$ in X. All these produce X by the above algorithm in a unique way. Since a base change boils down to an element of $\mathsf{PGL}(11,2)$, this implies that the stabiliser of X in $\mathsf{PGL}(11,2)$ has size at least $|\mathsf{P\Gamma L}(3,4)|$, and since the point-wise stabiliser must be trivial (as X contains the frame $I \cup \{*,\circ,\Sigma\}$), we conclude that this stabiliser is isomorphic to $\mathsf{P\Gamma L}(3,4)$.

Now define Ξ as the family of 3-spaces spanned by the members of Φ, and still denote by $\mathscr{M}^{10}(\mathbb{F}_2)$ the pair $(X\Xi)$. It is easy to verify that $\mathscr{M}^{10}(\mathbb{F}_2)$ satisfies (MM1) and (MM2*): one only needs to verify (MM1) for one particular block, e.g., ξ_1^* and (MM2*) for two particular blocks, e.g., ξ_1^* and ξ_2^*.

Now let $N < 10$. Then the 11 points $I \cup \{*,\circ\}$ are not linearly independent, and they are the projection of a base of $\mathsf{PG}(10,2)$ into $\mathsf{PG}(N,2)$, say from the subspace U. Since the rest of X is determined uniquely by these eleven points by consecutively summing up sets of four already obtained points, the whole of X is the projection from U of $\mathscr{M}^{10}(\mathbb{F}_2)$.

This completes the proof of the lemma. □

The projective plane $(\mathscr{P},\mathscr{L}) \cong \mathsf{PG}(2,4)$—For future reference, we give a brief description of the projective plane $(\mathscr{P},\mathscr{L})$ that emerged in the above proof. Put $I = \{1,2,3,4,5,6,7,8,9\}$, $T^\circ = \{123,456,789\}$, $T^* = \{147,258,369\}$, $T^\Sigma = \{159,267,348\}$ and $T = \{168,249,357\}$. For each pair a,b occurring in a triple of T, and for each $\times \in \{\circ,*,\Sigma\}$, we let $T^\times(ab)$ be the unique element of $T^\times$ neither containing a, nor b. Then we have:

$$\mathscr{P} = I \cup \{\circ,*,\Sigma\} \cup \{\circ abc \mid \forall abc \in T^\circ\} \cup \{abc* \mid \forall abc \in T^*\} \cup \{\overline{abc} \mid \forall abc \in T^\Sigma\}$$

$$\begin{aligned}\mathscr{L} = \{\{\circ abc,a,b,c,\circ\} \mid \forall \circ abc \in T^\circ\} \cup \{\{abc*,a,b,c,*\} \mid \forall abc* \in T^*\}\\ \cup \{\{\overline{abc},a,b,c,\Sigma\} \mid abc \in T^\Sigma\} \cup \{\{a,b,T^\circ(ab),T^*(ab),T^\Sigma(ab)\} \mid \forall abc \in T\}\}\end{aligned}$$

Now that the $N = 10$ case is settled, we look at the lower dimensional cases. By the previous lemma these arise as projections of $\mathscr{M}^{10}(\mathbb{F}_2)$. So we search for subspaces of $\mathsf{PG}(10,2)$ from which to project $\mathscr{M}^{10}(\mathbb{F}_2)$. We call a subspace S *admissible* when $S \cap \langle \xi_1, \xi_2 \rangle$ is empty for all blocks $\xi_1, \xi_2 \in \Phi$ of $\mathscr{M}^{10}(\mathbb{F}_2)$. Projecting from an admissible subspace yields a pair

(X_S, Ξ_S) (with obvious meaning) which still satisfies Axioms (MM1) and (MM2*). Conversely, if these axioms are still satisfied after projecting from a subspace S, it means that S is admissible.

Lemma 6.3.9. *Consider* $(X, \Xi) = \mathscr{M}^{10}(\mathbb{F}_2)$ *in* $\mathsf{PG}(10,2)$. *Then there is a unique line* M *in* $\mathsf{PG}(10,\mathbb{K})$ *from which the projection* (X_M, Ξ_M) *of* $\mathscr{M}^{10}(\mathbb{F}_2)$ *is projectively equivalent to* $\mathscr{V}_2(\mathbb{F}_2, \mathbb{F}_4)$. *In this case,* $M = T_x \cap T_y \cap T_z$ *for any three points* $x, y, z \in X$ *not contained in a common elliptic space.*

Proof. By Lemma 6.3.8 and the existence of $\mathscr{V}_2(\mathbb{F}_2, \mathbb{F}_4)$ in $\mathsf{PG}(8,2)$ (which we view as an 8-dimensional subspace of $\mathsf{PG}(10,2)$), we know that there is at least one such line M.

Now, for each point p in $\mathscr{V}_2(\mathbb{F}_2, \mathbb{F}_4)$, the tangent space T_p has dimension 4. We claim that, for each $x \in X$, $\dim(T_x) \geq 6$. Indeed, since in Lemma 6.3.8, the point $\circ$ was arbitrary, it suffices to look at $T_\circ$, where we see that $T_\circ([\circ,1]) = \langle \circ, 12, 23\rangle$, $T_\circ([\circ,4]) = \langle \circ, 45, 56\rangle$ and $T_\circ([\circ,7]) = \langle \circ, 78, 89\rangle$. Hence $T_\circ$ contains the 6-space $\langle \circ, 12, 23, 45, 56, 78, 89\rangle$, showing the claim. Since the projection from M onto $\mathsf{PG}(8,2)$ maps tangent spaces of $\mathscr{M}^{10}(\mathbb{F}_2)$ to tangent spaces of $\mathscr{V}_2(\mathbb{F}_2, \mathbb{F}_4)$, this implies that M is contained in every tangent spaces of $\mathscr{M}^{10}(\mathbb{F}_2)$, and every such tangent space has dimension 6.

We now establish uniqueness. It suffices to show the last assertion of the lemma. As above, we deduce that $T_* = \langle *, 14, 47, 25, 58, 36, 69\rangle$ and $T_\Sigma = \langle \Sigma, 95, 51, 62, 27, 84, 43\rangle$. A straightforward calculation shows that $\{124689, 135678, 234579\} = T_* \cap T_\circ \cap T_\Sigma$. Since any three points $x, y, z \in X$ not contained in an elliptic space can play the role of $\circ$, $*$ and Σ, the last assertion follows. $\square$

Remark 6.3.10. The line M could also be found as the intersection of all tangent hyperplanes: For each ξ in Ξ, there is a hyperplane H_ξ of $\mathsf{PG}(10,2)$, called a *tangent hyperplane*, with the property $H_\xi \cap X = \xi \cap X$.

We now determine all admissible subspaces. First a seemingly unrelated lemma.

Lemma 6.3.11. *Let* $(X, \Xi) \cong \mathscr{M}^{10}(\mathbb{F}_2) \subseteq \mathsf{PG}(10,2)$. *Let* $S \subseteq X$ *with* $1 \leq |S| \leq 8$. *If the sum of* S *is* $\overline{0}$, *then either* S *is the set of points on a line or* S *is the symmetric difference of two distinct lines.*

Proof. The assumption is equivalent with saying that S is the union of disjoint frames of subspaces. Since no four points of X are contained in a plane, and no three are collinear, a frame inside X has at least five points. Since $|S| \leq 8$, S has to be a frame itself. So $|S| \geq 5$. Suppose $|S| = 5$ and assume for a contradiction that S is not a block. If no triple of points of S are contained in a common line, then S is a non-degenerate conic. All such conics are projectively equivalent, and so we may assume $S = \{\circ, *, 1, 6, 8\}$. As this is clearly not a frame, we may assume that three points of S are on a common block ξ. But then the elliptic space spanned by the block ξ' defined by the remaining pair $\{a, b\}$ intersects $\langle \xi \rangle$ in a point c, with a, b, c collinear. By (MM2*), $c \in X$, a contradiction. Hence S is a block if $|S| = 5$.

Now assume $|S| = 6$. If no three points of S are on a common line, then S is a hyperoval, and all such things are projectively equivalent, hence we may take $S = \{\circ, *, \Sigma, 1, 6, 8\}$, which is not a frame. Hence S is not a hyperoval and there exist three points $a,b,c \in S$ on a common block ξ. Let d,e be the remaining pair of points on ξ (hence $\xi = \{a,b,c,d,e\}$). Then $a+b+c+d+e = \overline{0}$, and we can replace $\{a,b,c\}$ with $\{d,e\}$ in S, cancel double occurrences (which sum up to $\overline{0}$ already) to obtain a set S' of either 5, 3 or 1 point(s) that sum up to $\overline{0}$. From the foregoing, $|S'| = 5$ and S' coincides with the block defined by d,e; hence $S' = \xi$, contradicting the fact that $a,b,c \notin S'$. Consequently $|S| \neq 6$.

Now assume $|S| = 7$. Then S contains three points on a common line ξ, say a,b,c. We again replace these with the two remaining points of ξ, cancel double occurrences, and obtain a set S' of either 6, 4 or 2 points whose sum is $\overline{0}$. However, such set does not exist by the foregoing.

Now assume $|S| = 8$. The same procedure as in the previous paragraph produces a set S' of either 7, 5 or 3 points whose sum is $\overline{0}$. By the foregoing, S' is a block ξ, and we cancelled exactly one double occurrence. This means that S contains exactly four points of a certain block ξ', and also four points of ξ.

The lemma is proved. □

Lemma 6.3.12. *Let M be the intersection of all T_x, for $x \in X$, where $(X,\Xi) \cong \mathscr{M}^{10}(\mathbb{F}_2) \subseteq$ $\mathsf{PG}(10,2)$. Then there are no admissible subspaces of dimension greater than* 1 *and all admissible points and lines are contained in $\bigcup_{x \in X}(\langle M,x\rangle \setminus \{x\})$.*

Proof. We determine the admissible points by counting the non-admissible ones. To that end, we introduce *X-triangles* and *X-quadrangles*: These are sets of three or four points of X, respectively, no three of which are contained in a common elliptic space. The *center* of an X-triangle or X-quadrangle is the sum of its points.

Note that X does not contain a set of four coplanar points. Indeed, such a set is clearly not contained in a common elliptic space, and intersecting the elliptic spaces determined by two disjoint pairs of points produces a line contained in X, a contradiction.

Now, the projection of an X-triangle from its center is a line; the projection of an X-quadrangle from its center is a set of four coplanar points. As in the previous paragraph, these sets cannot be contained in a structure that satisfies (MM1) and (MM2*). Hence no center of an X-triangle or X-quadrangle is admissible. We now show the converse statement.

Claim 1: *Each non-admissible point which is not contained in an elliptical space is either the midpoint of an X-triangle or the midpoint of at least two X-quadrangles.*
Let $p \in \mathsf{PG}(10,\mathbb{K})$ be non-admissible. Recall that this means $p \in \langle \xi_1,\xi_2\rangle$ for some $\xi_1,\xi_2 \in \Xi$ with $\xi_1 \neq \xi_2$. Put $x = \xi_1 \cap \xi_2$. If $p \notin \xi_1 \cup \xi_2$, then there are unique lines $L_1 \subseteq \xi_1$ and $L_2 \subseteq \xi_2$ through x such that $p \in \langle L_1,L_2\rangle$. Let $i \in \{1,2\}$. If L_i is a secant of $X(\xi_i)$, then we denote by y_i the unique point on $L_i \cap X \setminus \{x\}$; if L_i is tangent to $X(\xi_i)$ and then there are two planes, say Z_i and $\overline{Z}_i$ through L_i not tangent to $X(\xi_i)$, and we denote by z'_i and z''_i the points of $Z_i \cap X \setminus \{x\}$ and by z_i the intersection point $L_i \cap \langle z'_i,z''_i\rangle$ (clearly, $z_i \neq x$), likewise for $\overline{Z}_i$ (note that $z_i \neq \overline{z}_i$ since X does not contain a set of four coplanar points). There are four possibilities.

1. *Both L_1 and L_2 are secants and $p \in \langle y_1, y_2 \rangle$.* Then p belongs to $[y_1, y_2]$.
2. *Both L_1 and L_2 are secants and $p \notin \langle y_1, y_2 \rangle$.* In this case, p is the center of the X-triangle $\{x, y_1, y_2\}$.
3. *The line L_1 is a secant whereas L_2 is a tangent (possibly switching $\{1,2\}$).* Without loss, the plane Z_2 is such that $z_2 \in \langle y_1, p \rangle$. Then p is the center of the X-triangle $\{y_1, z_2', z_2''\}$, since $z_1' + z_1'' = z_1$ and $z_1 + y_1 = p$.
4. *Both L_1 and L_2 are tangents.* Now we have $L_i = \{x, z_i, \overline{z}_i\}$, $i = 1,2$ and we can choose notation so that $\{p, z_1, z_2\}$ and $\{p, \overline{z}_1, \overline{z}_2\}$ are lines. Hence $p = z_1 + z_2 = z_1' + z_1'' + z_2' + z_2''$ and $p = \overline{z}_1 + \overline{z}_2 = \overline{z}_1' + \overline{z}_1'' + \overline{z}_2' + \overline{z}_2''$. Hence p is the midpoint of the two X-quadrangles $\{z_1', z_1'', z_2', z_2''\}$ and $\{\overline{z}_1', \overline{z}_1'', \overline{z}_2', \overline{z}_2''\}$ (called *complementary* X-quadrangles).

Claim 1 is proved. In order to be able to count the number of non-admissible points, it now suffices to determine how many times a point can arise as center of an X-triangle or X-quadrangle. This is the content of the next two claims.

Claim 2: *If p is the center of an X-triangle $\{x, y, z\}$, then p cannot be the center of a second X-triangle $\{x', y', z'\}$.*
Indeed, suppose for a contradiction that $p = x + y + z = x' + y' + z'$, with $x, y, z, x', y', z' \in X$. Then $x + x' + y + x' + z + z' = \overline{0}$ (and $|\{x, y, z, x', y', z'\}| \in \{2,4,6\}$), contradicting Lemma 6.3.11.

Claim 3: *If p is the center of an X-triangle $\{x, y, z\}$, then p cannot be the center of an X-quadrangle $\{a, b, c, d\}$.*
Indeed, as above we obtain $x + y + z + a + b + c + d = \overline{0}$, and hence, by Lemma 6.3.11, $(\{x, y, z\} \cup \{a, b, c, d\}) \setminus (\{x, y, z\} \cap \{a, b, c, d\})$ is a block. So we may assume $d = z$ and $\{a, b, c, x, y\}$ is a block. This contradicts the fact that an X-quadrangle does not contain three collinear points by definition.

Claim 4: *If p is the center of an X-quadrangle $\{a, b, c, d\}$, then p is the center of precisely three other X-quadrangles.*
Let $\{a', b', c', d'\}$ be a second X-quadrangle with center p. For $x, y \in \{a, b, c, d\}$ denote by $\xi_{a,b}$ the block containing a, b. Now $a + b + c + d + a' + b' + c' + d' = \overline{0}$. By Lemma 6.3.11, $\{a, b, c, d, a', b', c', d'\}$ is the symmetric difference of two distinct lines. Since $\{a, b, c, d\}$ intersects each of these lines in exactly two points, a', b', c', d' are either the points of $\xi_{a,b}$ and $\xi_{c,d}$ distinct from a, b, c, d and $\xi_{a,b} \cap \xi_{c,d}$, or the points of $\xi_{a,c}$ and $\xi_{b,d}$ distinct from a, b, c, d and $\xi_{a,c} \cap \xi_{b,d}$, or the points of $\xi_{a,d}$ and $\xi_{b,c}$ distinct from a, b, c, d and $\xi_{a,d} \cap \xi_{b,c}$. Conversely, all of these possibilities give rise to an X-quadrangle with center p. The claim follows.

A straightforward count now reveals that there are $\frac{21 \cdot 20 \cdot 16}{3 \cdot 2 \cdot 1} = 1120$ X-triangles and $\frac{1120 \cdot 9}{4} = 630 \cdot 4$ X-quadrangles. Lastly, there are $21 \cdot 10 = 210$ points contained in elliptical spaces but not in X and 21 points in X. This amount to 1981 non-admissible points, so we miss exactly 66 of the 2047 points of $\mathsf{PG}(10, \mathbb{K})$. This is exactly the number of points contained in the union of $\langle M, x \rangle \setminus \{x\}$ with x varying in X. Moreover, all admissible points should be contained in such planes, as they have to "disappear" after projecting from M, since $\mathcal{V}_2(\mathbb{F}_2, \mathbb{F}_4)$ contains no admissible points. This shows the lemma. $\square$

Action of $\mathsf{PGL}(3,4)$ **and** $\mathsf{PSL}(3,4)$ **on** $\mathscr{M}^{10}(\mathbb{K})$—Recall that the line

$$M = \{124689, 135678, 234579\}$$

of $\mathsf{PG}(10,2)$ is fixed under the action of $\mathsf{PGL}(3,4)$ on $\mathsf{PG}(10,2)$ stabilising X. Note that the point sets $\{1,2,4,6,8,9\}$, $\{1,3,5,6,7,8\}$ and $\{2,3,4,5,7,9\}$ are disjoint hyperovals of $\mathsf{PG}(2,4)$. Each of them spans a subspace intersecting M in a different point. Since $\mathsf{PGL}(3,4)$ is transitive on the hyperovals, it is also transitive on the points of M. Then the stabiliser of a point of M in $\mathsf{PGL}(3,4)$ is a subgroup of $\mathsf{PGL}(3,4)$ of index 3 and as such a copy of $\mathsf{PSL}(3,4)$ (which indeed has three orbits on the set of hyperovals). Since $\mathsf{PSL}(3,4)$ has no index 2 subgroups, it also fixes the two other points of M. Since each point stabiliser in $\mathsf{PGL}(3,4)$ contains elements of $\mathsf{PGL}(3,4) \setminus \mathsf{PSL}(3,4)$, the group $\mathsf{PGL}(3,4)$ has exactly two orbits on the set of admissible points, namely M and the set of the other 63 admissible points. It is now also easy to see that it has two orbits on set of admissible lines: $\{M\}$, and the set of 63 other lines.

Moreover, if we project $\mathscr{M}^{10}(\mathbb{K})$ from a point on M, then all tangent spaces T_x, $x \in X$, get mapped into 5-dimensional spaces, whereas this is only true for the tangent space T_y if we project from a point of $\langle y, M\rangle \setminus (M \cup \{y\})$, with $y \in X$. Hence these two projections cannot be isomorphic. A similar argument shows that the projection from M is projectively inequivalent to the projection from any other admissible line.

Conclusion—For any pair (X,Ξ), where X is a spanning point set of $\mathsf{PG}(N,\mathbb{K})$, $N > 3$, satisfying (MM1) and (MM2*), Lemmas 6.3.8, 6.3.9 and 6.3.12 and the previous discussion show that $8 \le N \le 10$ and, more precisely:

($N = 10$) $(X,\Xi) \cong \mathscr{M}^{10}(\mathbb{K})$ if $N = 10$;

($N = 9$) (X,Ξ) is the projection of $\mathscr{M}^{10}(\mathbb{K})$ from one of its 66 admissible points p, and there are two non-isomorphic projections, depending on $p \in M$ or $p \notin M$;

($N = 8$) (X,Ξ) is either the projection from M and then we obtain $\mathscr{V}_2(\mathbb{F}_2,\mathbb{F}_4)$, or it is the projection of $\mathscr{M}^{10}(\mathbb{K})$ from one of the 63 admissible lines distinct from M.

This shows Proposition 6.3.7.

Remark 6.3.13. In fact, one can show that $\mathscr{M}^{10}(\mathbb{K})$ is the projection in $\mathsf{PG}(11,2)$ from the sum of three points chosen arbitrarily in a set of 24 points, of the 21 remaining points onto a complementary hyperplane. These 24 points, together with all the frames of 6-subspaces formed by 8-subsets, form the Witt design $S(5,8,24)$. The stabiliser of that 24-set in $\mathsf{PGL}(12,2)$ is exactly the Mathieu group M_{24}. It follows that the Veronesean cap $\mathscr{V}_2(\mathbb{F}_2,\mathbb{F}_4)$ is the projection of this representation of the Witt design from the subspace generated by any three of its points.

This set of 24 points can also be obtained as follows: Let V be the 24-dimensional free $\mathbb{Z}_2$ module on the points of the Witt design $S(5,8,24)$. We factor out the submodule generated by all the characteristic vectors of blocks (octads). Since the (extended) binary Golay code had dimension 12, the factor module has dimension $24-12=12$, and the standard basis of

V gets mapped onto a set of 24 points on which the group M_{24} acts naturally. Projectively, we obtain an 11-dimensional projective space with an action of M_{24} on a set of 24 points structured as $S(5,8,24)$ where the octads are contained in 6-spaces.

We conclude that $\mathscr{M}^{10}(\mathbb{K})$, i.e., the universal pseudo embedding of $\mathsf{PG}(4,2)$, arises from a projection of the universal pseudo embedding of $S(5,8,24)$ from the sum of three arbitrary points of $S(5,8,24)$.

6.4 Vertex-reduced Hjelmslevean Veronese sets

Henceforth, we assume that (X,Ξ) in $\mathsf{PG}(N,\mathbb{K})$ has $v \geq 0$ and satisfies property (V). The latter for instance implies that two distinct tubes intersect in either a unique point or a generator:

Lemma 6.4.1. *Let C, C' be two distinct tubes. Then $C \cap C'$ is either a point of X or a generator of both C and C'.*

Proof. By (H2*), there is a point $x \in X$ contained in both $C \cap C'$. If $C \cap C' = \{x\}$ we are done, so suppose $\overline{C} \cap \overline{C'}$ contains a point $y \in Y$. So y is contained in the respective vertices V and V' of C and C', which means by Property (V) that $V = V'$. Hence C and C' share the generator determined by x and V in this case. Since no point of $C \setminus \langle x, V\rangle$ is collinear with x, it is clear by (H2*) that $\Xi(C) \cap \Xi(C') = \langle x, V\rangle$. □

6.4.1 Local properties and the structure of Y

We investigate the singular subspaces Π_x for $x \in X$ (see Definition 6.2.7) and the set of tubes $\mathscr{C}_V$ going through a fixed vertex V. This brings along the structure of the vertex set Y.

Lemma 6.4.2. *For each $x \in X$, there are tubes C and C' with $C \cap C' = \{x\}$ and, if V and V' are the vertices of C and C' respectively, then $\Pi_x = \langle x, V, V'\rangle$. In particular,* $\dim(\Pi_x) = 2v+2$.

Proof. Let C be a tube through x and let V be its vertex. Suppose for a contradiction that no tube through x intersects C in $\{x\}$ only. Then Lemma 6.4.1 implies that all tubes through x contain V. But then, for each point $x' \in X$ distinct from x, the tube through x and x' also has V as its vertex, so x' is also collinear with V. Consequently, by Corollary 6.2.8, all tubes have V as their vertex. This contradiction to (V) implies that there is a tube C' with $C \cap C' = \{x\}$. Denote its vertex by V'.

We now show that $\Pi_x = \langle x, V, V'\rangle$. If not, there is a tube C'' through x with vertex $V'' \nsubseteq \langle V, V'\rangle$. By (V), $C \cap C' = C \cap C'' = \{x\}$. Take a point $z \in C$ not collinear to x (so not contained in Π_x) and a point $z' \in C' \setminus \{x\}$ collinear to x (so contained in $\langle x, V'\rangle \subseteq \Pi_x$), chosen in such a way that $\langle z', V\rangle$ is disjoint from $\langle x, V'' \cap \langle V, V'\rangle\rangle$ (note that V and $\langle x, V'\rangle$ span $\langle x, V, V'\rangle$ whereas $V'' \cap \langle V, V'\rangle$ and $\langle x, V'\rangle$ do not, by assumption on V''). By Corollary 6.2.8, z and z' are not collinear and $\overline{\Pi}_z \cap \overline{\Pi}_{z'} = V$ is the vertex of the tube $[z, z']$. Axiom (H2*) implies that $[z, z']$ intersects C'' in a point $z'' \in X$. If $z'' \perp z'$, then z'' belongs to the generator $\langle z', V\rangle$ of

$[z,z']$. But then $z'' \in \langle z',V\rangle \cap C''$, or more precisely, z'' belongs to $\langle z',V\rangle \cap \langle x, V'' \cap \langle V,V'\rangle\rangle$, which is empty by the choice of z'. Hence z'' is not collinear to z' and hence by Corollary 6.2.8, z'' is not collinear to x, so $C'' = [x,z'']$ and $V'' = \overline{\Pi}_x \cap \overline{\Pi}_{z''} = V$, a contradiction. The lemma is proven. □

Notation—Given a tube C, we denote $\Pi_C^Y = \{\Pi_x^Y \mid x \in C\}$ and $\Pi_C = \{\Pi_x \mid x \in C\}$. For a given vertex V, the set of all tubes with vertex V is $\mathscr{C}_V$; its structure is described in the following lemma.

Lemma 6.4.3. *Let C be a tube with vertex V. Then each point of X collinear to V is contained in a unique member of Π_C and hence $\langle \mathscr{C}_V\rangle = \langle \Pi_C\rangle = \langle C, \Pi_C^Y\rangle$. This has the following consequences:*

- (i) *For each tube C' with vertex V, containment gives a bijection between the set of generators of C' and the set Π_C. Consequently, $\Pi_C^Y = \Pi_{C'}^Y$ and collinearity gives a bijection between the generators of C and C'.*
- (ii) *For each point z not collinear to V; V and Π_z^Y are complementary subspaces in $\langle \Pi_C^Y\rangle$; in particular, $\dim(\langle \Pi_C^Y\rangle) = 3v+2$;*
- (iii) *For all non-collinear points $x, x' \in C$, the subspace $\langle \Pi_C^Y\rangle$ is spanned by Π_x^Y and $\Pi_{x'}^Y$;*
- (iv) *Each point $y \in Y$ belongs to Π_c^Y for some $c \in C$.*
- (v) *$\langle \Pi_C^Y\rangle \cap \langle C\rangle = V$ and the dimension of $\langle \mathscr{C}_V\rangle$ is $3v+d+4$.*

Proof. Let $p \in X$ be collinear to V and suppose p is not contained in any member of Π_C. Take any point $q \in X \setminus (C \cup \Pi_p)$. By (H2*), the tube $[p,q]$ then intersects C in a point c, and since $p \notin \Pi_c$ we have $[p,q] = [p,c]$, which implies that V is the vertex of $[p,q]$. In particular, q is collinear with V. We obtain that all points of X are collinear to V, and hence all tubes have V as their vertex, contradicting (V). We conclude that $p \in \Pi_c$ for some $c \in C$, uniqueness follows from the fact that the sets Π_c are mutually disjoint.

Generated by all points of X collinear to V, $\langle \mathscr{C}_V\rangle$ equals $\langle \Pi_C\rangle$. Since for each $c \in C$, $\langle \Pi_c\rangle = \overline{\Pi}_c = \langle c, \Pi_c^Y\rangle$, we obtain $\langle \mathscr{C}_V\rangle = \langle C, \Pi_C^Y\rangle$.

(i) It immediately follows from the above that each point $c' \in C'$ is contained in a unique member of Π_C. Observing that collinear points of C' need to be contained in the same such subspace and that points in the same such subspace are necessarily collinear, each generator of C' is contained in a unique member of Π_C.

Interchanging the roles of C and C', we see that each generator of C is contained in a member of $\Pi_{C'}$ and so $\Pi_C = \Pi_{C'}$. Hence also $\Pi_C^Y = \Pi_{C'}^Y$.

We have shown that each tube through V has exactly one generator in each member of Π_C, and all members of Π_C contain a generator of that tube. It follows that the map from C to C' taking a generator of C to the unique generator of C' contained in the same member of Π_C (i.e., the generator of C' collinear with it) is a bijection.

(ii) Let z be a point not-collinear with V (note that this exists by Property (V)). Such a point is not collinear with any point $c \in C$ and hence, for each $c \in C$, the intersection $\Pi_c^Y \cap \Pi_z^Y$ is a v-space V_c. Again by (V), $V_c \cap V = \emptyset$; so also $\Pi_z^Y \cap V = \emptyset$. Moreover, Lemma 6.4.2 implies

$\Pi_c^Y = \langle V_c, V\rangle$ for each $c \in C$, and also $\Pi_z^Y = \langle V_{c_1}, V_{c_2}\rangle = \langle V_c \mid c \in C\rangle$. So $\langle \Pi_C^Y\rangle = \langle \Pi_c^Y \mid c \in C\rangle = \langle V, V_c \mid c \in C\rangle = \langle V, \Pi_z^Y\rangle$. In particular, $\dim(\langle \Pi_C^Y\rangle) = 3v+2$.

(iii) This follows immediately from the previous item since $\langle V, \Pi_z^Y\rangle = \langle V, V_{c_1}, V_{c_2}\rangle = \langle \Pi_{c_1}^Y, \Pi_{c_2}^Y\rangle$.

(iv) For an arbitrary point $y \in Y$, we have $y \in \Pi_z^Y$ for some $z \in X$. If z is collinear with V then $\Pi_z^Y = \Pi_c^Y$ for some $c \in C$. If z is not collinear with V, then in the previous paragraph, we showed that $\langle \Pi_C^Y\rangle = \langle V, \Pi_z^Y\rangle$. Hence $y \in \langle \Pi_C^Y\rangle$ and this already implies $Y = \langle \Pi_C^Y\rangle$. Now take a tube C' through zy. By (H2*), $C \cap C'$ contains a point $c \in X$ and hence $y \in \Pi_c^Y$.

(v) As $\langle \Pi_C^Y\rangle$ only contains points of Y, the intersection $\langle C\rangle \cap \langle \Pi_C^Y\rangle$ coincides with V. Hence $\langle \mathscr{C}_V\rangle$, generated by a quadric Q on C complimentary to V and by Π_C^Y, has dimension $3v+d+4$. □

Notation—Point (i) of the previous lemma implies that Π_C^Y could, and shall, more accurately be denoted by Π_V^Y, with V the vertex of C, as it does not depend on the element of $\mathscr{C}_V$.

We consider the following point-line geometry.

Definition 6.4.4 (The point-line geometry $\mathscr{G}_V$). We define $\mathscr{G}_V$ as a geometry having as point set the set $\mathscr{P}_V$ of singular affine $(v+1)$-spaces having V as their v-space at infinity and as line set the set $\mathscr{C}_V$ of tubes with vertex V, with containment made symmetric as incidence relation.

Corollary 6.4.5. *The point-line geometry $\mathscr{G}_V = (\mathscr{P}_V, \mathscr{C}_V, I)$ is a dual affine plane.*

Proof. Any two tubes through V intersect in an element of $\mathscr{P}_V$ as they need to share a point of X by (H2*). Let $C \in \mathscr{C}_V$ be arbitrary. Each singular affine $(v+1)$-space $W \in \mathscr{P}_V$, not contained in C, is collinear with a unique generator of C by Lemma 6.5.15. That generator corresponds to the unique element of $\mathscr{P}_V$ in C that is not contained in a member of $\mathscr{C}_V$ together with W. Clearly, no element of $\mathscr{P}_V$ is contained in all tubes through V. We verified all axioms of a dual affine plane. □

6.4.2 Connecting X and Y

Recall that Y is a subspace (cf. Corollary 6.5.2). The connection between the X-points and the Y-points is a crucial step towards understanding the structure of (X, Ξ). To this end, we consider the projection ρ of X from Y onto a subspace F of $\mathsf{PG}(N, \mathbb{K})$ complementary to Y, i.e.,

$$\rho : X \to F : x \mapsto \langle Y, x\rangle \cap F.$$

We show that this projection gives us a well-defined point-quadric set (the quadrics $\rho([x,z])$ and $\rho([x',z'])$ with $\rho(x) = \rho(x')$ and $\rho(z) = \rho(z')$ have to coincide). For that we need one more general lemma.

Lemma 6.4.6. *Let C and C' be tubes sharing only one point $x \in X$. Then two points $z \in C$ and $z' \in C'$ are collinear if and only if $\langle z, z'\rangle$ belongs to Π_x.*

Proof. Denote by V and V' the respective vertices of C and C'. By Lemma 6.4.2, $\Pi_x = \langle x, V, V'\rangle$. Consider two distinct points $z \in C \setminus \langle x, V\rangle$ and $z' \in C' \setminus \langle x, V'\rangle$. If z and z' would be collinear, then $\Pi_z^Y = \Pi_{z'}^Y$ by Corollary 6.2.8, in particular this means that V' belongs to Π_z^Y. This contradicts the fact that, also by Corollary 6.2.8, $\Pi_z^Y \cap \Pi_x^Y = V$. We conclude that z and z' are not collinear. The converse statement is clear. □

Lemma 6.4.7. *The projection ρ is such that $\rho(x) = \rho(x')$, for $x, x' \in X$, if and only if x and x' are equal or collinear. In particular, for each $x \in X$, we have $\rho^{-1}(\rho(x)) = \Pi_x$ and for any tube C with vertex V, $\rho(C)$ is a Q_d^0-quadric. For any two tubes C, C' we have that $\rho(C) = \rho(C')$ if and only if C and C' have the same vertex; and if the vertices are distinct, then $\rho(C) \cap \rho(C') = \rho(C \cap C')$. In particular, $\rho^{-1}(\rho(C)) = \mathscr{C}_V$.*

Proof. Let x and x' be two points of X. Then $\rho(x) = \rho(x')$ (or equivalently, $\langle Y, x\rangle = \langle Y, x'\rangle$) if and only if xx' contains a point of Y, which on its turn is equivalent with x and x' being collinear. It is then clear that $\rho^{-1}(\rho(x))$ equals the set of points collinear with x, so Π_x. Now let C be a tube with vertex V. Since $\langle C\rangle \cap Y = V$, we obtain that $\rho(C)$ is a quadric of type Q_d^0. It follows from Lemma 6.5.15(i) that all tubes in $\mathscr{C}_V$ have the same image, as collinear generators are mapped onto the same point. If C and C' have distinct vertices V and V' (hence $V \cap V' = \emptyset$ by (V)) then $C \cap C'$ is a unique point x by (H2*) and hence it follows from Lemma 6.4.6 that $\rho(C) \cap \rho(C') = \rho(x)$. □

We now show that $(\rho(X), \rho(\Xi))$, as a pair of points and Q_d^0-quadrics in F, satisfies the Axioms (MM1) and (MM2*) introduced in Section 6.3.

Proposition 6.4.8. *The pair $(\rho(X), \rho(\Xi))$ satisfies Axioms* (MM1) *and* (MM2*). *As a point-line geometry, $(\rho(X), \rho(\Xi))$ is hence isomorphic to* $\mathsf{PG}(2, \mathbb{B})$, *where $\mathbb{B}$ is a quadratic alternative division algebra with* $\dim_{\mathbb{K}}(\mathbb{B}) = d$. *Consequently, d is a power of* 2, *with $d \le 8$ if* $\mathrm{char}(\mathbb{K}) \neq 2$, *and $N = 6d + 2$.*

Proof. By Lemma 6.4.7, $\rho(\Xi)$ is a well-defined family of $(d+1)$-dimensional subspaces in F (called the *elliptic spaces*) such that for each $\xi \in \Xi$, $\rho(\xi) \cap \rho(X)$ contains $\rho(X(\xi))$ (equality will be shown once (MM2*) is established). We prove that the pair $(\rho(X), \rho(\Xi))$ satisfies Axioms (MM1) and (MM2*) and as such is a Veronese variety (cf. Theorem 6.1.4).

- *Axiom* (MM1). Let z and z' be distinct points of $\rho(X)$. Then there are points $x, x' \in X$ with $z = \rho(x)$ and $z' = \rho(x')$. By the above, x and x' are not collinear, so by (H1), they are contained in a unique tubic space ξ. Hence z and z' are contained in the elliptic space $\rho(\xi)$ and Axiom (MM1) follows.

- *Axiom* (MM2*). Let $\xi, \xi' \in \Xi$ be distinct tubic spaces and put $\rho(\xi) = \zeta$, $\rho(\xi') = \zeta'$ (note that $\zeta = \zeta'$ is a priori not impossible), and put $C = X(\xi)$ and $C' = X(\xi')$. If the respective vertices V and V' of C and C' coincide, then Lemma 6.5.15(i) implies that $\rho(C) = \rho(C')$, and hence there is nothing to show. So suppose that V and V' are distinct, and hence disjoint by (V). Axiom (H2*) implies that $C \cap C'$ is a unique point $x \in X$ and by Lemma 6.4.7 we obtain $\rho(C) \cap \rho(C') = \{\rho(x)\}$. For (MM2*) to hold, we have to show that $\zeta \cap \zeta' = \{\rho(x)\}$

too. So it suffices to show that $\langle C, C'\rangle \cap Y = \langle V, V'\rangle$: in this case, the projection of $\langle C, C'\rangle$ from Y is then isomorphic to the projection of $\langle C, C'\rangle$ from $\langle V, V'\rangle$, and hence (MM2*) follows from (H2*).

Suppose for a contradiction that $\langle C, C'\rangle \cap Y$ contains a point y which does not belong to $\langle V, V'\rangle$. By (iii) of Lemma 6.5.15, y is collinear to unique generators $\langle z, V\rangle \subseteq C$ and $\langle z', V'\rangle \subseteq C'$ for some points $z \in C$ and $z' \in C'$. Note that x is not contained in those generators as $y \notin \langle V, V'\rangle = \Pi_x^Y$. Since $y \notin \langle C'\rangle$, it is clear that $\langle C', y\rangle$ intersects $\langle C\rangle$ in a line L through x. Moreover, L is disjoint from the singular line $\langle z', y\rangle$, for no point of C is collinear to z' by Lemma 6.4.6. So, $\langle L, y, z'\rangle$ is a 3-space in $\langle C', y\rangle$, which thus has a plane α in common with $\langle C'\rangle$ (note that L and $\langle z', y\rangle$ do not belong to $\langle C'\rangle$, so neither to α).

The plane α contains x and z' and hence $\alpha \cap C'$ is either a conic through x and z', or it is the union of two lines through x and z' respectively, having a point v' of V' in common. In both cases, there is only one line in α through x which does not contain a unique second point of C' (in the first case, the tangent line through x to $\alpha \cap C'$; in the second case, the line $\langle x, v'\rangle$). Take any line L' in α through x having a unique second point r' in common with C'. The plane $\langle L, L'\rangle$ intersects the singular line $\langle y, z'\rangle$ in a point s. There are at least three valid choices for L' since $|\mathbb{K}| > 2$, each yielding another point $s \in \langle y, z'\rangle$. So we can choose L' such that $s \notin \{y, z'\}$. In particular, $s \neq r'$ since $s \in L'$ only occurs if $s = z'$ (as $\langle y, z'\rangle \cap \alpha = \{z'\}$).

The lines $\langle s, r'\rangle$ and L, contained in the plane $\langle L, L'\rangle$, share a point r. As L and $\langle y, z'\rangle$ were disjoint, $r \neq s$, and as $r' \neq x$, we also have $r \neq r'$. Note that r is contained in the intersection of ξ and any tubic space containing r' and s; hence $r \in C \subseteq X$. So the line $\langle s, r'\rangle$ contains three points in X and is hence singular. But then $\langle r, r'\rangle$ needs to be contained in Π_x by Lemma 6.4.6, implying that also $s \in \Pi_x$ and hence $y \in \Pi_x^Y$ as well, a contradiction.

- *Claim: The intersection $\rho(\xi) \cap \rho(X)$ equals $\rho(X(\xi))$ for each $\xi \in \Xi$.*

Put $C = X(\xi)$. Suppose for a contradiction that $\rho(\xi)$ contains a point $\rho(z)$ with $z \notin \rho^{-1}(\rho(C))$, i.e., $z \notin \mathscr{P}_V$. Take any point $x \in X$ with $\rho(x) \in \rho(C)$. Then $[z, x]$ is a tube with vertex V' distinct (and hence disjoint) from V. So by (H2*), $[z, x] \cap \xi = \{x\}$. By Lemma 6.4.7 and Axiom (MM2*), $\rho([z, x]) \cap \rho(\xi) = \{\rho(x)\}$; yet this intersection contains the line $\langle \rho(z), \rho(x)\rangle$ by construction. This contradiction shows the claim.

Knowing this, it follows from Theorem 6.1.4 that, as a point-line geometry, $(\rho(X), \rho(\Xi))$ is isomorphic to a projective plane $\mathsf{PG}(2, \mathbb{B})$, where $\mathbb{B}$ is a quadratic alternative division algebra with $\dim_{\mathbb{K}}(\mathbb{B}) = d$. Consequently, d is a power of 2, smaller or equal than 8 if $\mathrm{char}(\mathbb{K}) \neq 2$. Since $\rho(X)$ spans F (because X spans $\mathsf{PG}(N, \mathbb{K}) = \langle F, Y\rangle$), this result also implies that $\dim(F) = 3d + 2$. Together with $\dim(Y) = 3v + 2 = 3d - 1$, it follows that $N = 6d + 2$. $\square$

The next corollary now immediately follows from Theorem 6.1.4.

Corollary 6.4.9. *All Q_d^0-quadrics are quadrics of Witt index* 1.

Definition 6.4.10 (The connection map)**.** We define χ as the map from $\rho(X)$ to Y, taking a point $z=\rho(x)$ to the subspace Π_x^Y (recall Definition 6.2.7 at infinity, i.e.,

$$\chi : \rho(X) \to Y : \rho(x) \mapsto \Pi_x^Y.$$

Note that Lemma 6.4.7 assures that this map is well defined: points with the same image under ρ are collinear and hence determine the same subspace Π_x^Y. The following proposition contains an important local property of χ. The proof uses the notion of regular d-scrolls, the definition and properties of which we have recorded in Section 6.6 (see Definition 6.6.5 and Lemma 6.6.6), preceded by some auxiliary properties.

Notation. Let V be some fixed vertex and C a fixed tube belonging to $\mathscr{C}_V$. We now choose the subspace F (complementary to Y) such that is contains a Q_d^0-quadric Q of C (and so $\langle Q,V\rangle = \langle C\rangle$ and $\rho(C)=Q$). We define ρ_V as the projection from V onto a complementary subspace $\widetilde{F}$ containing F. Then $\rho_V(C)=Q$. Let X_V be the set of points of X collinear to V. By Lemma 6.4.7, $\rho(X_V)$ is a Q_d^0-quadric, and it obviously coincides with Q. For any point $x \in X_V$, we denote $\check{x} := \rho(x) \in Q$ and $\widetilde{x} := \rho_V(x) \in \widetilde{F}$. Also, we denote $\widetilde{Y} := \rho_V(Y) = Y \cap \widetilde{F}$.

Proposition 6.4.11. *Let V be a vertex. Then, firstly, the set $\{\rho_V(\Pi_x^Y) \mid x \in \mathscr{P}_V\}$ induces a regular spread $\mathscr{R}_V$ of v-spaces on $\widetilde{Y}$ and the (well-defined) map*

$$\chi_V : \rho(X_V) \to \mathscr{R}_V : \check{x} \mapsto R_{\check{x}} := \rho_V(\Pi_x^Y)$$

takes a conic of Q onto a regulus of $\mathscr{R}_V$ and its restriction to such a conic preserves the cross-ratio (i.e., χ_V is a projectivity between Q and $\mathscr{R}_V$). Secondly, the regular spread $\mathscr{R}_V$, the quadric Q and the map χ_V determine a regular d-scroll $\mathfrak{R}_d(\mathbb{K})$ in $\widetilde{F}$ and for each tube $C' \in \mathscr{C}_V$, we have that $\rho_V(C')$ is an $\mathfrak{R}_d(\mathbb{K})$-quadric and vice versa. Thirdly, $v = d-1$.

Proof. By Lemma 6.4.7, the map χ_V is indeed well defined since $\rho^{-1}(\widetilde{x}) = \Pi_x$. We proceed in three steps.

Part 1: *The set $\mathscr{R}_V$ is a spread.*
Recall that, by Lemma 6.4.2 and Corollary 6.2.8, the set $\Pi_V^Y = \{\Pi_x^Y \mid x \in X_V\}$ is a set of $(2v+1)$-spaces pairwise intersecting each other in V, and, by Lemma 6.5.15(iii), each point of $Y \setminus V$ is contained in a member of Π_V^Y. So $\{R_{\check{x}} \mid x \in X_V\}$ indeed defines a spread $\mathscr{R}_V$ of v-spaces on $\widetilde{Y}$.

For the sequel, let C' be an arbitrary member of $\mathscr{C}_V$ distinct from C. We know that C and C' share a generator $\langle x_0, V\rangle$. So Q and $Q' := \rho_V(C')$ are quadrics sharing the point $\widetilde{x}_0$. Let $x \in V \setminus \langle x_0, V\rangle$. By Lemma 6.5.15($i$), there is a unique generator, say $\langle x', V\rangle$ of C' collinear to $\langle x, V\rangle$, i.e., $\check{x} = \check{x}'$. This implies that the mapping $f : \widetilde{x} \mapsto f(\widetilde{x}) := \widetilde{x}'$ (with $f(\widetilde{x}_0) = \widetilde{x}_0$ by definition) is a projectivity. Note that the points $\widetilde{x}$ and $\widetilde{x}'$ are collinear and the line joining them intersects $R_{\check{x}}$ in some point (since $\langle x, x'\rangle$ intersects $\Pi_x^Y = \Pi_{x'}^Y$). Note also that, since $\widetilde{Y}$ and F are complementary subspaces of $\widetilde{Y}$, Q is the projection of Q' from $\widetilde{Y}$ onto F.

Part 2: *There is an affine d-space $\alpha \subseteq \widetilde{Y}$ intersecting all transversals $\langle \widetilde{x}, f(\widetilde{x})\rangle$. The subspace $R_{\check{x}_0}$ is the $(d-1)$-space at infinity of α. Consequently, $v = d-1$.*
Lemma 6.6.4 yields an affine d-space α intersecting each transversal $\langle \widetilde{x}, f(\widetilde{x})\rangle$ with $\widetilde{x} \in$

$Q\setminus\{\widetilde{x}_0\}$ in a point $\widetilde{x}^\alpha$. By the same lemma, the induced map $\varphi: Q\setminus\{\widetilde{x}_0\}\to\alpha:\widetilde{x}\mapsto\widetilde{x}^\alpha$ is such that for any conic K on Q through $\widetilde{x}_0$, $\varphi(K\setminus\{\widetilde{x}_0\})$ is an affine line L and vice versa; moreover, the induced map $\overline{\varphi}_K$ taking $\widetilde{x}\in K\setminus\{\widetilde{x}_0\}$ to $\widetilde{x}^\alpha$ and $\widetilde{x}_0$ to $\langle L\rangle\setminus L$ preserves the cross-ratio.

We now show that α belongs to $\widetilde{Y}$. Note that each line $\langle\widetilde{x},f(\widetilde{x})\rangle$, with $\widetilde{x}\in Q\setminus\{\widetilde{x}_0\}$, is contained in $\langle\widetilde{x},R_{\check{x}}\rangle$ (which belongs to $\rho_V(\Pi_x)$) and as such is a singular line having a unique point in $\widetilde{Y}$. Consequently, $\alpha\subseteq X\cup Y$, and as $|\mathbb{K}|>2$, $\langle\alpha\rangle$ is a singular d-space. Suppose that $\widetilde{x}^\alpha$ and $\widetilde{z}^\alpha$ belong to X, for two distinct points $\widetilde{x},\widetilde{z}\in Q\setminus\{\widetilde{x}_0\}$. By Corollary 6.2.8, $\widetilde{x}^\alpha$ and $\widetilde{z}^\alpha$ are not collinear (since $\Pi_{\widetilde{x}}\neq\Pi_{\widetilde{z}}$), contradicting the fact the line $\langle\widetilde{x}^\alpha,\widetilde{z}^\alpha\rangle$ is singular, as it lies in $\langle\alpha\rangle$. Again relying on $|\mathbb{K}|>2$, this reveals that each line in the affine space α contains at least two points in $\widetilde{Y}$ and as such, $\alpha\subseteq\widetilde{Y}$.

As a consequence, $\widetilde{x}^\alpha\in R_{\check{x}}$. Moreover, the above implies that collinearity is a bijection between $Q\setminus\{\widetilde{x}_0\}$ and α and as such, each member $R_{\check{x}}$ of $\mathscr{R}_V\setminus\{R_{\check{x}_0}\}$ intersects α in precisely $\widetilde{x}^\alpha$. As $\mathscr{R}_V$ is a spread, also the points of $\langle\alpha\rangle\setminus\alpha$ need to be contained in a member of $\mathscr{R}_V$ too, and the only possibility left is $\langle\alpha\rangle\setminus\alpha\subseteq R_{\check{x}_0}$. We claim that actually $\langle\alpha\rangle\setminus\alpha=R_{\check{x}_0}$. Indeed, suppose for a contradiction that $\langle\alpha\rangle\setminus\alpha\subsetneq R_{\check{x}_0}$. Since $R_{\check{x}_0}\cap\alpha$ is empty, $R_{\check{x}_0}$ is a hyperplane of $\langle R_{\check{x}_0},\alpha\rangle$. Any point $y\in\langle R_{\check{x}_0},\alpha\rangle\setminus(R_{\check{x}_0}\cup\alpha)$ has to be contained in $R_{\check{x}}$ for some $\widetilde{x}\neq\widetilde{x}_0$. But then the line $\langle y,\widetilde{x}^\alpha\rangle\subseteq R_{\check{x}}$ has to intersect $R_{\check{x}_0}$ in a point, whereas $R_{\check{x}}\cap R_{\check{x}_0}=\emptyset$. This contradiction shows the claim. As a consequence, since α is an affine d-space, $v=d-1$.

Part 3: *$\mathscr{R}_V$ is regular; χ_V is a projectivity between Q and $\mathscr{R}_V$; Q, $\mathscr{R}_V$ and χ_V define a regular d-scroll $\mathfrak{R}_d(\mathbb{K})$.*
Consider three distinct members $R_{\check{x}_1}$, $R_{\check{x}_2}$ and $R_{\check{x}_3}$ of $\mathscr{R}_V$. Denote by K the conic $Q\cap\langle\check{x}_1,\check{x}_2,\check{x}_3\rangle$. We claim that the regulus determined by $R_{\check{x}_1}$, $R_{\check{x}_2}$ and $R_{\check{x}_3}$ is $\{R_{\check{x}}\mid\check{x}\in K\}$ and as such belongs to $\mathscr{R}_V$, showing that the latter is regular indeed and that a regulus of it corresponds with a conic of Q and vice versa.

Let z_1 be any point in $R_{\check{x}_1}$. We view Q as $\rho_V(C)$. Choose auxiliary points $\widetilde{x}_0\in K\setminus\{\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3\}$ and $\widetilde{x}_1'\in\langle\widetilde{x}_1,z_1\rangle\setminus\{\widetilde{x}_1,z_1\}$, and denote the quadric $\rho_V(X[\widetilde{x}_0,\widetilde{x}_1'])$ by Q'' (the tube $X[\widetilde{x}_0,\widetilde{x}_1']$ indeed has vertex V). Like before, there is a projectivity f between Q and Q'' which induces a projectivity $\overline{\varphi}_K$ between K and some line L in $\widetilde{Y}$ that takes each $\widetilde{x}\in K\setminus\{\widetilde{x}_0\}$ to $\langle\widetilde{x},f(\widetilde{x})\rangle\cap L$, or equivalently, $\overline{\varphi}_K(\widetilde{x})=R_{\check{x}}\cap L$, and which maps $\widetilde{x}_0$ to $R_{\check{x}_0}\cap L$ (this follows from the previous paragraph). Moreover, as $\overline{\varphi}_K(\widetilde{x}_1)=z_1$ (recall that z_1 is on $\langle\widetilde{x}_1,\widetilde{x}_1'\rangle$, and $\widetilde{x}_1'=f(\widetilde{x}_1)$), the line L is the unique line through z_1 intersecting $R_{\check{x}_2}$ and $R_{\check{x}_3}$. Since z_1 in $R_{\check{x}_1}$ was arbitrary, the claim follows: each transversal of $R_{\check{x}_1}$, $R_{\check{x}_2}$ and $R_{\check{x}_3}$ is intersected by $R_{\check{x}}$ for each $\widetilde{x}\in K$ and no other member of $\mathscr{R}_V$.

This implies that we indeed have a regular d-scroll $\mathfrak{R}_d(\mathbb{K})$ defined by $Q=\rho(\mathscr{C}_V)$ and $\mathscr{R}_V$. Since this is independent of C', each tube C'' of $\mathscr{C}_V$ is such that the quadric $\rho_V(C'')$ intersects each transversal subspace $\langle\widetilde{x},R_{\check{x}}\rangle$ with $x\in\mathscr{P}_V$ in a unique point. Moreover, since any two points $\widetilde{x}$ and $\widetilde{z}$ on distinct transversal subspaces determine a unique such tube $X[x,z]$ by (H2*) and each two points of $\mathfrak{R}_d(\mathbb{K})$ determine a unique $\mathfrak{R}_d(\mathbb{K})$-quadric (cf. Lemma 6.6.6), the set $\{\rho_V(C'')\mid C''\in\mathscr{C}_V\}$ coincides with the set of $\mathfrak{R}_d(\mathbb{K})$-quadrics. □

The combination of Propositions 6.4.11 and 6.4.8 now gives us the relation between the point-quadric variety in F and the set Y of vertices. Our next aim is to show that we can choose F in such a way that $F \cap X = \rho(X)$. But first we deduce something useful from the above proof.

Lemma 6.4.12. *For each point $x \in X$, $T_x \cap Y = \Pi_x^Y$. If C^* is any tube whose vertex V^* is not collinear to x, then T_x and $\langle C^* \rangle$ are complementary subspaces of* $\mathsf{PG}(N,\mathbb{K})$.

Proof. The tangent space T_x is generated by all tangent spaces $T_x(C)$ where C varies over the set of tubes through x. The vertices of such tubes are these contained in Π_x. Take such a vertex V. Then each tube C_x through x with vertex V corresponds, projected from V, to a quadric Q_x on the scroll $\mathfrak{R}_d(\mathbb{K})$. The subspace generated by all tangent spaces through x at these quadrics is precisely $\langle T_x(Q_x), R_x \rangle$, for some fixed arbitrarily chosen quadric Q_x (using the notation of the above proposition), as follows from the properties of scrolls (cf. last assertion of Lemma 6.6.4). We obtain that the subspace generated by the tangent spaces at x of tubes through $\langle x, V \rangle$ intersects Y precisely in Π_x^Y. Since V was an arbitrary vertex collinear to x, we conclude that $T_x \cap Y = \Pi_x^Y$ indeed.

Now consider the tube C^* with vertex V^*. Since V^* is not collinear to x, V^* and Π_x^Y are complementary subspaces of Y by Lemma 6.5.15(ii) and (iv). In the Veronese variety $(\rho(X), \rho(\Xi))$, the point $\rho(x)$ is not contained in $\rho(C^*)$ (since x is not collinear to V^*), so $\rho(T_x) = T_{\rho(x)}$ and $\rho(C^*)$ are also complementary subspaces by the properties of Veronese varieties (this can be verified algebraically but it has also been proven in Proposition 4.5 of [29]). Since $T_x \cap Y$ and $\langle C^* \rangle \cap Y$ are complementary subspaces of Y and since the projections $\rho(T_x)$ and $\rho(\langle C^* \rangle)$ from Y onto F are complementary in F, we obtain that T_x and $\langle C^* \rangle$ are complementary in $\langle Y, F \rangle = \mathsf{PG}(N,\mathbb{K})$. □

Lemma 6.4.13. *There exists a subspace F^* of* $\mathsf{PG}(N,\mathbb{K})$ *complementary to Y such that the projection of X from Y onto F^* is precisely the intersection of F^* with X.*

Proof. As before, we denote the projection operator from Y onto F by ρ (and F is an arbitrary subspace of $\mathsf{PG}(N,\mathbb{K})$ complementary to Y). Let C_1, C_2, C_3 be three tubes of X such that $\rho(C_1)$, $\rho(C_2)$ and $\rho(C_3)$ correspond to the sides of a triangle in the projective plane $(\rho(X), \rho(\Xi))$. Let x_i be the unique intersection point $C_j \cap C_k$, for all $\{i,j,k\} = \{1,2,3\}$ and denote the vertex of C_i by V_i, $i = 1,2,3$. In $\langle C_i \rangle$, we choose an arbitrary subspace W_i containing $\{x_j, x_k\}$ complementary to V_i, with $\{i,j,k\} = \{1,2,3\}$.

Claim: $\langle W_1, W_2, W_3 \rangle$ and Y are complementary subspaces of $\mathsf{PG}(N,\mathbb{K})$.
Firstly, $\langle W_1, W_2, W_3, Y \rangle = \mathsf{PG}(N,\mathbb{K})$, since $\langle F, Y \rangle = \mathsf{PG}(N,\mathbb{K})$ and F is generated by the projections of W_1, W_2, W_3. If d is finite, a dimension argument shows that $\langle W_1, W_2, W_3 \rangle \cap Y$ is empty, but since $d = \infty$ is possible, we need a more general argument. First, we show that $\langle W_2, W_3 \rangle \cap Y = \emptyset$. Indeed, assume for a contradiction that $p \in \langle W_2, W_3 \rangle \cap Y$. Since $p \notin W_2 \cup W_3$, this implies that $\langle p, W_2 \rangle \cap W_3$ contains a line L. Since $p \notin L$, the projection of L under ρ is contained in $\rho(W_2) \cap \rho(W_3)$, a contradiction to (MM2*) proved in Proposition 6.4.8. Next, we show that $\langle W_1, W_2, W_3 \rangle \cap Y = \emptyset$. Assume for a contradiction that $p \in \langle W_1, W_2, W_3 \rangle \cap Y$. Since $p \notin W_1 \cup \langle W_2, W_3 \rangle$, the subspace $\langle p, W_1 \rangle$ intersects $\langle W_2, W_3 \rangle$ in at least a plane. But then, as $p \in Y$, the spaces $\rho(W_1)$ and $\langle \rho(W_2), \rho(W_3) \rangle$ also share at least a plane, a contradiction. This shows the claim.

Put $F^* := \langle W_1, W_2, W_3 \rangle$ and denote the projection of X from Y onto F^* by ρ^* (this projection makes sense by the above claim). If for each $x \in X$, the intersection $\Pi_x \cap F^*$ is a point of X, say $p^*(x)$, then $\rho^*(x) = p^*(x)$ and hence $F^* \cap X$ is isomorphic to $\mathscr{V}_2(\mathbb{K},\mathbb{B})$ (by Proposition 6.4.8 and with the same notation).

Claim: We can choose W_1, W_2 and W_3 such that $\Pi_x \cap F^*$ is non-empty for each point $x \in X$; equivalently, $\rho^*(x) \in X$, for all $x \in X$.
We keep the points x_1, x_2, x_3 and the subspace W_2 as above; and we will determine W_1 and W_3 in such a way that, for each pair of points $c_1 \in (W_1 \cap X) \setminus \{x_3, x_2\}$ and $c_2 \in (W_2 \cap X) \setminus \{x_3, x_1\}$ holds that $[c_1, c_2] \cap C_3 \in W_3$. To that end, take a point x_1' on $C_1 \setminus (\langle x_3, V \rangle \cup \langle x_2, V \rangle)$ and a point x_2' on $W_2 \cap X \setminus \{x_1, x_3\}$. We define W_3 as $\langle [x_1', c_2] \cap C_3 \mid c_2 \in W_2 \cap X \rangle$ and W_1 as $\langle [x_2', c_3] \cap C_1 \mid c_3 \in W_3 \cap X \rangle$. We first show that W_3 is indeed a subspace of $\langle C_3 \rangle$ complementary to V_3; and in exactly the same way, then also W_1 is a subspace of $\langle C_1 \rangle$ complementary to V_1.

Consider the projection of $X \cup Y$ from $T_{x_1'}$ onto $\langle C_3 \rangle$ (by Lemma 6.4.12, these are complementary subspaces). Note that, for each tube C through x_1', C is mapped to the unique point $C \cap C_3$ since C shares the hyperplane $T_{x_1'}(C)$ with $T_{x_1'}$. This means that each point x_2' of C_2 is mapped to $[x_1', x_2'] \cap C_3$. Moreover, the vertex V_2 of C_2 is mapped to V_3 since $T_{x_1'} \cap Y = \Pi^Y_{x_1'}$ and V_3 are complementary subspaces of Y. As such, the map $C_2 \to C_3 : x_2' \mapsto [x_1', x_2'] \cap C_3$ is the restriction of a projection that takes W_2 to W_3 (by definition of the latter) and V_2 to V_3. Since W_2 and V_2 are complementary in $\langle C_2 \rangle$, the same holds for their images W_3 and V_3 in $\langle C_3 \rangle$. Note also that the points x_1 and x_2 are fixed, so W_3 contains these; likewise, W_1 contains x_2 and x_3. By definition of W_1, also each tube $[c_1, x_2']$ with $c_1 \in (W_1 \cap X)$ intersects C_3 in a point of W_3.

Now let $c_1 \in (W_1 \cap X) \setminus \{x_3, x_2\}$ and $c_2 \in (W_2 \cap X) \setminus \{x_3, x_1\}$ be arbitrary. If $c_1 = x_1'$ or $c_2 = x_2'$ then, by definition, $[c_1, c_2] \cap C_3 \in W_3$, so suppose $c_1 \neq x_1'$ and $c_2 \neq x_2'$. Then the four points x_1', c_1, x_2', c_2 determine a unique $\mathbb{K}$-subplane π of $(\rho^*(X), \rho^*(\Xi))$, which on F^* corresponds to a copy $\mathscr{V}$ of $\mathscr{V}_2(\mathbb{K},\mathbb{K})$ (see Section 5.2 of [?]). Let c_3, c_3' and c_3'' denote the points of C_3 obtained by the intersection with $[x_1', x_2']$, $[x_1', c_2]$ and $[c_1, x_2']$, respectively. Then these belong to a conic C_3 on W_3 by the above, and moreover, this conic belongs to $\mathscr{V}$. In $\mathscr{V}$, the conic C determined by c_1 and c_2 (which is part of the tube $[c_1, c_2]$) also intersects C_3 in a point. As such, we obtain that $[c_1, c_2] \cap C_3 = C \cap C_3$ belongs to W_3 indeed.

Finally, we show that with these choices of W_1, W_2 and W_3, the claim holds. Take any point $x \in X$. If $\rho^*(x) \in W_i \cap X$ for some $i \in \{1,2,3\}$, then of course $\rho^*(x) \in X$. So assume $\rho^*(x) \notin W_1 \cup W_2 \cup W_3$. We consider the $\mathbb{K}$-subplane π^* of $(\rho^*(X), \rho^*(\Xi))$ determined by the points $\rho^*(x_1), \rho^*(x_2), \rho^*(x_3), \rho^*(x)$ which in F^* gives, as above, a copy $\mathscr{V}^*$ of $\mathscr{V}_2(\mathbb{K},\mathbb{K})$. Now, inside $\mathscr{V}^*$, $\rho^*(x)$ lies on some conic C_x^* intersecting $W_1 \cap X$, $W_2 \cap X$ and $W_3 \cap X$ in three distinct points, say c_1, c_2, c_3, respectively. Now, the points c_1, c_2, c_3 belong to X and $[c_1, c_2] \cap C_3 = c_3$ by our choice of W_1 and W_3. As such, $\rho^*(x) \in C_x^* = \langle c_1, c_2, c_3 \rangle \cap X$. The claim follows, ending the proof. $\square$

From now on we assume that ρ has target subspace F such that $F \cap X = \rho(X)$. We now endow Y with the following natural structure and deduce some more properties of it.

Definition 6.4.14 (The point-line geometry $\mathbb{P}_Y$). Let $\mathscr{P}_Y = \{V \mid V$ is the vertex of a tube $C\}$ and $\mathscr{L}_Y = \{\Pi_x^Y \mid x \in X\}$ and let $\mathbb{P}_Y$ denote the point-line geometry $(\mathscr{P}_Y, \mathscr{L}_Y)$ with containment made symmetric as incidence relation. Its dual, $(\mathscr{L}_Y, \mathscr{P}_Y)$ is denoted by $\mathbb{P}_Y^*$.

Lemma 6.4.15. *The point-line geometry $\mathbb{P}_Y$ has the following properties:*

(i) *For each element of $\mathscr{L}_Y$, all members of $\mathscr{P}_Y$ not disjoint with it, are entirely contained in it and they form a regular spread. In particular, $\mathscr{P}_Y$ is a regular spread of Y;*

(ii) *the point-line geometries $\mathbb{P}_Y^*$ and $(\rho(X), \rho(\Xi))$ are isomorphic projective planes;*

(iii) *the projective plane $\mathbb{P}_Y^*$ is desarguesian.*

Moreover,

(iv) *the connection map $\chi : (\rho(X), \rho(\Xi)) \to \mathbb{P}_Y^* : x \mapsto \Pi_x^Y$ is a projectivity;*

(v) *X is the union over $x \in \rho(X)$ of all subspaces $\langle x, \chi(x)\rangle$ and each member $\xi \in \Xi$ with vertex V is such that $\rho_V(X(\xi))$ is a $\mathfrak{R}_d(\mathbb{K})$-quadric of the regular d-scroll $\mathfrak{R}_d(\mathbb{K})$ defined by the regular spread $\mathscr{R}_V$, the quadric $\rho(\mathscr{C}_V)$ and the projectivity χ_V, and vice versa.*

(vi) *all (d,v)-tubes entirely contained in X are induced by the members of Ξ,*

(vii) *(X, Ξ) is projectively unique if it exists.*

Proof. (i) Let Π_z^Y be an arbitrary member of $\mathscr{L}_Y$ and take a vertex V not collinear to z (exists by (V)). Then Π_z^Y is complementary to V in Y by Lemma 6.5.15(ii) and hence we can identify the projection $\tilde{Y}$ of Y from V (cf. Proposition 6.4.11) with Π_z^Y. This proposition then implies that, for each point x collinear to V (and hence not collinear to z), the vertices of the tubes $[x,z]$ (i.e., the v-spaces $\Pi_z^Y \cap \Pi_x^Y$) form a regular spread of Π_z^Y. Since each pair of vertices is disjoint by (V), all other elements of $\mathscr{P}_Y$ are disjoint from Π_z^Y. We conclude that the elements of $\mathscr{P}_Y$ having a non-trivial intersection with Π_z^Y are contained in it and form a regular spread of it indeed. In order for $\mathscr{P}_Y$ to be a regular spread of Y, we need that each two elements V_1 and V_2 of $\mathscr{P}_Y$ induce a regular spread on $\langle V_1, V_2\rangle$, and they do: take two tubes C_1 and C_2 through V_1 and V_2, respectively, and let z the unique intersection point of C_1 and C_2 (which exists by (H2) and is unique by (V)), then V_1 and V_2 span the subspace Π_z^Y and hence the assertion follows from what we deduced just before.

(ii) Let Π_x^Y be an arbitrary element of $\mathscr{L}_Y$. By Corollary 6.2.8, Π_x is the set of points of X collinear to Π_x^Y. Also, Lemma 6.4.7 implies that Π_x is the set of points of X mapped by ρ onto $\rho(x)$. Hence $\psi(\Pi_x^Y) := \rho(x)$ defines a bijective correspondence between $\mathscr{L}_Y$ and $\rho(X)$. Now consider the set of elements of $\mathscr{L}_Y$ incident with a fixed element of $\mathscr{P}_Y$, i.e., all subspaces Π_x^Y through to a certain vertex V, which means all subspaces Π_x^Y with x collinear to V. Then $\{\rho(x) \mid x \perp V\} = \rho(C)$ for any tube C through V by Proposition 6.5.15(i) and Lemma 6.4.7; even stronger: each member Π_x^Y of $\mathscr{L}_Y$ through V corresponds to a unique point of $\rho(C)$ and vice versa. Hence if we set $\psi(V) := \rho(C)$, then $\psi : (\mathscr{L}_Y, \mathscr{P}_Y) \to (\rho(X), \rho(\Xi))$ is a collineation. As $\psi(\mathbb{P}_Y^*)$ is a projective plane by Proposition 6.4.8, so is $\mathbb{P}_Y^*$.

(iii) Since the Desargues theorem is self-dual, it is equivalent to show that $\mathbb{P}_Y$ is desarguesian. Let Δ be a triangle with vertices V_1, V_2 and V_3 and Δ' a triangle with vertices V_1', V_2' and V_3'. Suppose Δ and Δ' are in central perspective from V. We claim that they are in axial perspective too.

Take any point $v \in V$. Then there are unique lines L_i, $i = 1,2,3$ through v such that $L_i \cap V_i$ is a point v_i and $L_i \cap V_i'$ is a point v_i'. Then the triangles $v_1v_2v_3$ and $v_1'v_2'v_3'$ are centrally in perspective from v. Since Y is a subspace of $\mathsf{PG}(N,\mathbb{K})$, it is desarguesian, so there is an axis L, i.e., each intersection point $p_{ij} := v_iv_j \cap v_i'v_j'$, with $i,j \in \{1,2,3\}$, $i \neq j$, lies on this line L. Let V_{ij} be the unique members of the spread containing the points p_{ij}, respectively. The line L is entirely contained in $\langle V_{13}, V_{23}\rangle$, and since V_{12} shares a point with L, item (i) of this lemma implies that $V_{12} \subseteq \langle V_{13}, V_{23}\rangle$. This shows the claim.

In a completely similar fashion, one can show that triangles that are in axial perspective, are also in central perspective. This shows that $\mathbb{P}_Y^*$ is desarguesian.

(iv) Clearly, χ is the inverse image of the above defined collineation ψ, and as such it is a collineation. We now show its linearity. To that end, let Q be a quadric of $\rho(\Xi)$ and let C be a tube with $\rho(C) = Q$. If V is the vertex of C, then the restriction of χ to the points of Q is given by χ_V, with the notation of Proposition 6.4.11. According to this proposition, the map χ_V preserves the cross-ratio and hence so does χ. We conclude that χ is a linear collineation, i.e., a projectivity.

(v) For each point $x \in X$, we have that x belongs to $\Pi_x = \langle \rho(x), \chi(\rho(x))\rangle \setminus \chi(\rho(x))$ (recall $\rho(x) \in X$), showing the first part of the assertion. The second part of the assertion follows immediately from Proposition 6.4.11.

(vi) Suppose C is a (d,v)-tube not contained in a member of Ξ. If its vertex V were not contained in Y, i.e., if C contains a singular affine line L with $\langle L\rangle \cap Y = \emptyset$, then $\rho(L)$ is a line in $\langle \rho(X)\rangle$ containing at least three points of $\rho(X)$ (since $|\mathbb{K}| > 2$), contradicting the properties of ordinary Veronese varieties. Hence $V \subseteq Y$, so $\rho(C)$ is a quadric of $(\rho(X), \rho(\Xi))$. Since for an ordinary Veronese variety with $|\mathbb{K}| > 2$, the elliptic spaces are determined by their point set, we obtain that $\rho(C) = \rho(C')$ for some tube C' with $\langle C'\rangle \in \Xi$. Let V' be the vertex of C'. Let $x, x' \in C$ such that $\check{x}, \check{x}'$ are two distinct points of $\rho(C)$, which are automatically non-collinear. Then x, x' are non-collinear and every point of V is collinear to both x, x'. By Corollary 6.2.8, $V \subseteq V'$ and so $V = V'$ (because they have the same dimension). But then it follows that $\rho_V(C)$ is an $\mathfrak{R}_d(\mathbb{K})$-quadric on the regular d-scroll $\mathfrak{R}_d(\mathbb{K})$ determined by $\mathscr{R}_V$ and $\rho(C')$, and by the previous item, $\langle C\rangle$ belongs to Ξ after all.

(vii) First note that the projective plane $\mathbb{P}_Y^*$ as given above is projectively unique, and so is the Veronese variety $(\rho(X), \rho(\Xi))$. Since all projectivities from $(\rho(X), \rho(\Xi))$ to $\mathbb{P}_Y^*$ are equivalent up to a projectivity of the source geometry $(\rho(X), \rho(\Xi))$, as follows from Main Result 6.1.4 and Proposition 5.1.5, we obtain that (X, Ξ) is projectively unique if it exists. $\square$

The above lemma even allows us to exclude one of the possibilities for d if $\operatorname{char}\mathbb{K} \neq 2$.

Proposition 6.4.16. *The variety* (X,Ξ) *is projectively equivalent to* $\mathcal{V}_2(\mathbb{K},\mathbb{A})$ *where* $\mathbb{A} = \mathsf{CD}(\mathbb{B},0)$ *and* $\mathbb{B}$ *is a quadratic associative division algebra over* $\mathbb{K}$, *and* $\dim_{\mathbb{K}}(\mathbb{B}) = d$. *Hence, if* $\operatorname{char}\mathbb{K} \neq 2$, *then* (X,Ξ) *exists if and only if* $d \in \{1,2,4\}$.

Proof. Assume first that $\operatorname{char}\mathbb{K} \neq 2$. If $d \notin \{1,2,4\}$, the only remaining possibility by Proposition 6.4.8 is $d = 8$. The same proposition, together with Lemma 6.4.15(ii), implies that

$\mathbb{P}^*_Y$ is isomorphic to $\mathsf{PG}(2,\mathbb{A})$, where $\mathbb{A}$ is a strictly alternative division algebra over $\mathbb{K}$ with $\dim_{\mathbb{K}}(\mathbb{A}) = 8$. But then it is impossible that $\mathbb{P}^*_Y$ is desarguesian (cf. Lemma 6.4.15(iii)). Hence $d \neq 8$.

By Proposition 5.2.21, the Veronese representations $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ with $\mathbb{A} = \mathsf{CD}(\mathbb{B},0)$, where $\mathbb{B}$ is a quadratic associative division algebra over $\mathbb{B}$ with $\dim_{\mathbb{K}}(\mathbb{B}) = d$ (d possibly an infinite cardinal) are Hjelmslevean Veronese sets with $(d,d-1)$-tubes. Since we have shown above that these are projectively unique, we conclude that (X,Ξ) is projectively equivalent to $\mathscr{V}_2(\mathbb{K},\mathbb{A})$. $\square$

This finishes the proof of Main Theorem 6.1.2.

Remark 6.4.17. Proposition 6.4.15 shows that one can construct all points and quadrics of $\mathscr{V}_2(\mathbb{K},\mathbb{A})$, with $\mathbb{A} = \mathsf{CD}(\mathbb{B},0)$ where $\mathbb{B}$ is a quadratic associative division algebra over $\mathbb{B}$, by taking a regular $(d-1)$-spread in a $(3d-1)$-dimensional projective space over $\mathbb{K}$ together with an ordinary Veronese variety $\mathscr{V}_2(\mathbb{K},\mathbb{B})$ and a duality χ between these.

6.4.3 Projective Hjelmslev planes of level 2

To conclude, we say some more about (X,Ξ) as an abstract point-line geometry.

Definition 6.4.18. An incidence structure $(\mathscr{P},\mathscr{L},I)$ is called a *projective Hjelmslev plane of level* 2 if for each two points (resp. lines), there is at least one line (resp. point) incident to it, and if there is a canonical epimorphism to a projective plane such that two points (resp. two lines) have the same image if and only if they are not incident with a unique line (resp. point).

Proposition 6.4.19. *The pair $(X,\mathscr{C})$ is a* projective Hjelmslev plane of level 2. *More precisely: the map $\overline{\chi} = \chi \circ \rho : X \to \mathbb{P}^*_Y : x \mapsto \Pi^Y_x$ is an epimorphism satisfying the following properties.*

(Hj1) *Two points x,x' of X are always joined by at least one member of $\mathscr{C}$; this member is unique if and only if $\overline{\chi}(x) \neq \overline{\chi}(x')$;*
(Hj2) *Two members C,C' of $\mathscr{C}$ always intersect in at least one point; this point is unique if and only if $\overline{\chi}(C) \neq \overline{\chi}(C')$;*
(Hj3) *The inverse image under $\overline{\chi}$ of a point of $\mathbb{P}^*_Y$, endowed with all intersections with non-disjoint tubes, is an affine plane;*
(Hj4) *The set of tubes contained in the inverse image under $\overline{\chi}$ of a line of $\mathbb{P}^*_Y$, endowed with all mutual intersections, is an affine plane.*

Proof. Clearly, both χ and ρ are morphisms (they preserve collinearity), hence so is $\overline{\chi}$. Surjectivity follows as each point of $\mathbb{P}^*_Y$ is by definition of the form Π^Y_x.

(Hj1) By (H1), each two points x,x' of X are contained in a tube. This tube is unique if and only if x and x' are non-collinear, which is at its turn equivalent with $\Pi^Y_x \neq \Pi^Y_{x'}$ (cf. Corollary 6.2.8).

(Hj2) By Lemma 6.4.1, two tubes C, C' either intersect each other in precisely one point of X, or they have a generator (and hence also their vertex) in common. As $\overline{\chi}(C)$ and $\overline{\chi}(C')$ are the respective vertices of C and C', the property holds.

(Hj3) The inverse image under $\overline{\chi}$ of a point of $\mathbb{P}^*_Y$, hence of some Π^Y_x, is the affine subspace Π_x. We endow this affine subspace now with the intersections of all tubes having their vertex in Π^Y_x, which yields singular affine $(\nu+1)$-spaces through each element of the spread in Π^Y_x and each point of X of Π_x. Hence we obtain the Brose-Bruck construction of an affine plane.

(Hj4) This follows from Corollary 6.4.5, by dualising.

□

6.5 A test case for $|\mathbb{K}| = 2$

Beware: lines only have three points from now on. Let me first point out why this puts the proof on the line. I would also like to add that I never considered this test case very important, seeing the poor outcome, so the three main reasons to include this section anyway are:

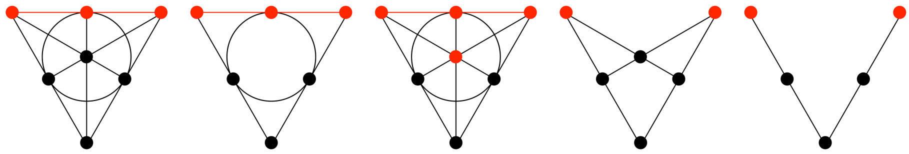
- I want to convince you that this case is tricky. For that: see subsection 6.5.1.
- A representation of a Hjelmslev plane turns up, giving a funny twist to the non-existence of a structure which was "very close to existing". This representation and the "near example" (as far as such a notion makes sense) can be found in subsection 6.5.2.
- I have spent too much time on this not to present it somewhere. The surprisingly long proof for such a limited test case can be found in Subsection 6.5.3.

6.5.1 What is wrong with $\mathbb{F}_2$?

Already quite early in the proof we relied on $|\mathbb{K}| > 2$. Indeed, Lemmas 6.2.2 and 6.2.3 are still fine, and so is Corollary 6.2.4, but then it starts going wrong in Lemma 6.2.5, which says:

Lemma 6.2.5 *Two singular affine subspaces Π and Π' intersecting in at least one point $x \in X$ generate a singular subspace and $\langle \Pi, \Pi' \rangle \cap X$ is a singular affine subspace.*

Non-proof. Consider two singular affine lines L_1 and L_2 which share a point x of X, whose unique respective points in Y we denote by y_1 and y_2. Then both lines only have a unique other point in X left, say x_1 and x_2, respectively. Suppose that x_1 and x_2 are not collinear. By (H1) and (H2*), this implies that x_1 and x_2 determine a unique tube C. One would naively expect to be able to use the fact that tubes are convex, but normally (i.e., when $|\mathbb{K}| > 2$) it is precisely Lemma 6.2.5 that takes care of this fact. As such, there are five options for the Fano plane generated by L_1 and L_2, which we pictured below (the red points are the points in Y, the black those in X):



Such a trivial statement, yet I could find no proof; whereas it really is an important means to discover more structure in the point set $X \cup Y$. It does make sense to at least **require tubes to be convex** by definition, for otherwise we are too far away from a sensible geometry anyway. From convexity it follows that x_1x_2 has to be a singular line, i.e., it contains a third point in Y, say y_3. The three points y_1, y_2 and y_3 then lie on one line of the Fano plane generated by L_1 and L_2. So, using convex tubes, the weaker version of Lemma 6.2.5 reads:

Lemma 6.5.1. *Collinearity is an equivalence relation: if, for three points $x_1, x_2, x \in X$ holds that $x_1 \perp x \perp x_2$, then $x_1 \perp x_2$; moreover, the three points in Y on the lines xx_1, xx_2 and x_1x_2 lie on one line.*

This even allows us to show that Y is a subspace:

Corollary 6.5.2. *The set Y is a subspace.*

Proof. Each two points y_1 and y_2 are contained in vertices of respective tubes C_1 and C_2. If $C_1 = C_2$, clearly y_1 and y_2 are on a line with only points of Y. If $C_1 \neq C_2$, then their intersection contains a point $x \in X$ by (H2*). By Lemma 6.5.1, we obtain that $y_1 y_2 \subseteq Y$. □

We conclude that only the two first options of the above depicted five can occur. Even though this is already an improvement, taking into account that second option remains rather inconvenient. After several unsuccessful attempts (both in trying to prove that the variety (X, Ξ) is still projectively unique and in trying to construct a counterexample), I decided to give it a go with two more additional assumptions, as a way of looking for inspiration:

(H0) v is finite and $d = 1$;

(H3) for each $x \in X$, there are $\xi_1, \xi_2 \in \Xi$ through x such that $\dim(T_x) = \langle T_x(\xi_1), T_x(\xi_2) \rangle$.

There is one good thing about $|\mathbb{K}| = 2$, and that is that d can only take the values 1 or 2. Indeed, there are no anisotropic forms using more than 2 variables over $\mathbb{F}_2$. So by restricting my attention to $d = 1$, I only forget about half of the cases. Yet I believe that my methods could work for $d = 2$ as well, they just got more annoying to write down, so that in the end (it is just a test case anyway) I guess that my proof for $d = 1$ should already give you a feeling of how annoying these things are. Of course, if someone would notice a shortcut or think of an alternative proof, I would be more than happy to hear this.

6.5.2 The near example

In this section I want to give you a quick preview on the nature of the geometric structure that I deduced from the above axioms, so that you can see what I mean by the fact that it is “close to existing” (there is of course no such things as “almost existing”). For further details I refer to the next subsection.

What makes the almost-example very curious is that, the structure that we in fact want to obtain (a Hjelmslev projective plane, cf. Section 6.4.3) turns up as a substructure. I will first present you, seemingly unrelated, the (classical) representation of the Hjelmslev projective plane of level 2 over the dual numbers $\mathsf{CD}(\mathbb{K},0)$ in $\mathsf{PG}(5,\mathbb{K})$ ([1]). For a moment, assume $\mathbb{K}$ is an arbitrary field.

A representation of the Hjelmslev projective plane of level 2 over the dual numbers $\mathsf{CD}(\mathbb{K},0)$ **in** $\mathsf{PG}(5,\mathbb{K})$

Take two disjoint planes π^* and π in $\mathsf{PG}(5,\mathbb{K})$, between which $\alpha : \pi^* \mapsto \pi$ is a collineation. Then for each point $p \in \pi^*$, we let $\mathscr{P}_p$ denote the set of all lines through p in the 3-dimensional subspace $\langle \pi^*, \alpha(p)\rangle$ which are not contained in π^*. Let L be any line in π^* and Π any 3-dimensional subspace in $\langle \pi^*, \alpha(L)\rangle$ not containing π^*; then we define the line $\mathscr{L}_{L,\Pi}$ as the members of $\bigcup_{p\in L} \mathscr{P}_p$ which are contained in Π. We define $\mathscr{P}$ as the set of lines one obtains by taking the union of all $\mathscr{P}_p$, where p varies over the points of π^*; and we define $\mathscr{L}$ as the set which consists of all sets $\mathscr{L}_{L,\Pi}$, where L is any line of π^* and Π any 3-space in $\langle \pi^*, \alpha(L)\rangle$ not containing π^*. Then the geometry $\mathscr{H} = (\mathscr{P}, \mathscr{L})$ is a Hjelmslev projective plane of level 2.

In the special case that $|\mathbb{K}| = 2$, another Hjelsmlev projective plane of level 2 is hidden in here: it has the same point set as $\mathscr{H}$ but its lines are defined differently: we replace each line $\mathscr{L}_{L,\Pi} \in \mathscr{L}$ by its complement (denoted $\mathscr{L}'_{L,\Pi}$), i.e., by the set of lines in $\mathscr{P}$ that also intersect L but which are not contained in $\mathscr{L}_{L,\Pi}$. Since $|\mathscr{P}_p| = 4$ for each point p, there are 12 lines meeting L, and each line $\mathscr{L}_{L,\Pi}$ also contains 6 elements, hence so do their complements. The set of all these complements is denoted by $\mathscr{L}'$. The resulting point-line geometry $\mathscr{H}' = (\mathscr{P}, \mathscr{L}')$ is also a Hjelmslev projective plane of level 2, but it is coordinatised over the ring $\mathbb{Z}/4\mathbb{Z}$ (I state this as a fact).

These two structures are non-isomorphic, and one can for instance distinguish $\mathscr{H}$ and $\mathscr{H}'$ by the fact that the former contains a projective plane over $\mathbb{K}$ as a structure, whereas the latter does not.

Remark 6.5.3. To see that $\mathscr{H}$ contains a projective plane, consider the lines $\mathscr{L}_{L,\Pi_L}$ where L ranges over π^* and $\Pi_L = \langle L, \alpha(L)\rangle$.

A non-existing structure

Consider a Hjelmslevean Veronese set with convex $(1,\nu)$-tubes, additionally satisfying (H0) and (H3). Then firstly, I could show that $\nu = 1$ and that $\dim(Y) = 5$, and that the vertices of the tubes precisely give us the set $\mathscr{P}$ as above.

Secondly, if one considers the projection ρ of X from Y onto a complementary subspace F, then it turns out that this gives a set of 7 points in a 3-dimensional subspace, which are such that each two points are on a unique conic of $\rho(\xi)$ whose inverse image is a set of tubes of Ξ whose vertices all contain a fixed point of π^*. As such, each conic of $\rho(\Xi)$ is linked to a unique point in π^*.

Conversely, each two such conics have a unique point z of $\rho(X)$ in common, and if one considers the three points in π^* related to the three conics through z, then these are on a line L of π^*. It moreover turns out that each point of $\rho^{-1}(z)$ is collinear to six vertices meeting L, which form exactly a line of $\mathscr{L}'$ (these lines vary if we vary the points in $\rho^{-1}(z)$). Choosing the points in $\rho^{-1}(z)$ for $z \in \rho(X)$ wisely, we find a subplane of $\mathscr{H}'$ isomorphic to a projective plane, which is impossible.

Notwithstanding the fact that both the structure in Y exists and that in F too (Lemma 6.5.16), they cannot be joined. This is a similar situation as encountered in Proposition 6.4.16, where it is shown that there are no (d,v)-Hjelsmlevean Veronese sets if $d=8$, $v \geq 0$ and $\mathrm{char}(\mathbb{K}) \neq 2$.

6.5.3 The proof

Now that we have settled the extra assumptions, let us see how far we can take this. A fair warning: this part has not been read by anyone at the moment that I include it in this manuscript. Anyway, here we go.

What will be useful once more is the fact that we may assume that there are two tubes whose vertices are disjoint (when $|\mathbb{K}| > 2$, we could do better though: back then, we could prove that we may assume that two vertices either coincide or are disjoint, cf. Lemma 6.2.9 and Proposition 6.2.10). The proof will resemble that of the latter lemma, but with some extra cases.

Suppose there are tubes C_1 and C_2 with respective vertices V_1 and V_2 such that $\emptyset \subsetneq V_1 \cap V_2 \subsetneq V$. We choose C_1 and C_2 such that $\dim(V_1 \cap V_2)$ is minimal amongst all other non-empty intersections of vertices. Put $V^* = V_1 \cap V_2$. In this section we will show that all tubes go through V^*.

Lemma 6.5.4. *Let C_1 and C_2 be tubes with respective vertices V_1 and V_2 that intersect in a non-trivial subspace V^* of V, minimal with respect to its dimension amongst all pairs of tubes whose vertices intersect non-trivially. Then for each tube C, its vertex V contains V^*; consequently, each point of X is collinear to V^*.*

Proof. By (H2*), the tubes C_1 and C_2 share a point $x \in X$, so $\overline{C_1} \cap \overline{C_2} = \langle x, V^* \rangle$. By the same axiom, an arbitrary tube C intersects C_i in a point $z_i \in X$, $i = 1,2$. First suppose that z_1 is not collinear to z_2. Since both z_1 and z_2 are collinear to V^*, and as tubes are convex, we have $V^* \subseteq V$. Note that, if both z_1 and z_2 are not collinear to x, then all points of $V \cap V_2$ are collinear to both x and z_1 and hence, the same argument implies that $V \cap V_2 \subseteq V_1$; likewise $V \cap V_1 \subseteq V_2$, so we obtain that $V \cap V_i = V^*$, $i = 1,2$.

Next suppose that $z_1 \perp z_2$ (possibly $z_1 = z_2$). Lemma 6.5.1 implies that $z_1 \perp x$ if and only if $z_2 \perp x$. As such, there are two cases:

• *Case 1: $z_1, z_2 \notin x^\perp$ are collinear.*
Since z_1 and x are not collinear, it follows just as above that no point of $V_2 \setminus V_1$ is collinear to z_1; likewise when switching the indices. The minimality property however forces the vertex V of a tube through z_1 and z_2 to have at least a subspace of dimension $\dim(V^*)$ in common with V_1 and V_2, which is only possible if $V \cap V_1 = V^* = V \cap V_2$.

• *Case 2: $z_1, z_2 \in x^\perp$ (and hence z_1 and z_2 are equal or collinear).*
For $i = 1,2$, let z_i' be a point of C_i not collinear to x (and hence non-collinear to z_i either). By the foregoing, it is shown that the vertex V' of a tube C' through z_1' and z_2' (whether they are collinear or not) intersects V_1 and V_2 in exactly V^*. Hence, for each $i \in \{1,2\}$, we can replace the pair (C_1, C_2) by the pair (C_i, C'). As the points $C \cap C_i = z_i$ and $C_i \cap C' = z_i'$ are not collinear, the foregoing again implies that $V \cap V_i$ contains $V' \cap V_i = V^*$.

As C was arbitrary, we conclude that each tube's vertex V contains V^*. It immediately follows that each point x is collinear with V^*. □

For an arbitrary subspace F of $\mathsf{PG}(N,\mathbb{K})$ complimentary to V^*, we now consider the map $\rho : X \mapsto F : x \mapsto \langle x, V^* \rangle \cap F$. The pair $(\rho(X), \rho(\Xi))$ is well defined as we just project from a part of the vertex of each tube; and consists of $(1,v')$-tubes with base Q_1^0, where $v' = v - \dim(V^*) - 1$.

Proposition 6.5.5. *Let C_1 and C_2 be tubes with respective vertices V and V' that intersect in a non-trivial subspace V^* of V, minimal with respect to its dimension amongst all pairs of tubes whose vertices intersect non-trivially. Then $(\rho(X), \rho(\Xi))$ is a Hjelmslevean Veronese set with convex $(1,v')$-tubes for $v' = v - \dim(V^*) - 1 < \infty$ that satisfies (H3).*

Proof. By Lemma 6.5.4, each point of $x \in X$ is collinear with V^* and all elements of Ξ contain V^*. We need to show that $(\rho(X), \rho(\Xi))$ satisfies (H1), (H2*) and (H3).

- *Axiom (H1).* Let x and x' be points of $\rho(X)$. Then Axiom (H1) in (X, Ξ) implies that there is a tube C containing x and x'. By Lemma 6.5.4, the vertex of C contains V^* and hence $\rho(C)$ is a tube through x and x'.
- *Axiom (H2*).* Let ξ and ξ' be members of $\rho(\Xi)$. Then $\langle \xi, V^* \rangle \cap \langle \xi', V^* \rangle$ belongs to $X \cup Y$ and contains at least one point $x \in X$ by Axiom (H2*) in (X, Ξ). It is clear that $\xi \cap \xi'$ belongs to $X \cup Y$ and that it contains $\rho(x) \in \rho(X)$.
- *Axiom (H3).* Take $x \in X$ arbitrary. Let C_1' and C_2' be tubes through x such that $T_x = \langle T_x(C_1'), T_x(C_2') \rangle$ for certain tubes C_1' and C_2' through x. As the vertices of C_1' and C_2' contain V^*, it is clear that $T_{\rho(x)}$ is spanned by $T_{\rho(x)}(\rho(C_1))$ and $T_{\rho(x)}(\rho(C_2))$. □

Proposition 6.5.5 allows us to assume that there is a pair of tubes whose vertices are disjoint.

Notation If a and b are two points on a line, then $a \wedge b$ will denote the unique third point on the line ab.

Lemma 6.5.6. *For each two collinear points x and x' of X, we have $\langle x^\perp \cap Y \rangle = \langle x'^\perp \cap Y \rangle \subseteq T_x \cap T_{x'} \cap Y$. Furthermore, $\dim(T_x \cap Y) \leq 2v + 2$.*

Proof. Take $y \in Y$ on a singular line with x. If $x' = x \wedge y$, then clearly $y \in x'^{\perp} \cap Y$, so suppose $x' \neq x \wedge y$. As collinearity is an equivalence relation, x' and $x \wedge y$ are on a singular line with a unique point, say $y' \in Y$. Then $y \in \langle y', x \wedge x' \rangle \subseteq \langle x'^{\perp} \cap Y \rangle$ indeed. Clearly, $\langle x^{\perp} \cap Y \rangle$ belongs to $T_x \cap Y$. Interchanging the roles of x and x', the first assertion follows.

By (H3), T_x is spanned by $T_x(C_1)$ and $T_x(C_2)$ for two tubes C_1 and C_2 through x and as such, $\dim(T_x) \leq 2(v+2)$. If V_1 and V_2 are the respective vertices of C_1 and C_2, then clearly, $\langle V_1, V_2 \rangle \subseteq T_x \cap Y$. Since Y is a subspace (cf. Corollary 6.5.2) and $X \cap Y = \emptyset$, $T_x \cap Y$ cannot contain $T_x(C_1)$ entirely, so $T_x \cap Y$ has at least codimension 1 in T_x. We obtain $\dim(T_x \cap Y) \leq 2v + 2$. □

Lemma 6.5.7. *Let C_1 and C_2 be tubes such that $C_1 \cap C_2$ is a unique point $x \in X$. If there are points $x_i \in C_i \setminus x^{\perp}$, $i = 1,2$, with $x_1 \perp x_2$, then $v = 0$.*

Proof. Suppose that x_1 and x_2 are collinear points of C_1 and C_2 respectively, with $x_i \notin x^{\perp}$, $i = 1,2$. Denote by V_1 and V_2 the respective vertices of C_1 and C_2 (by assumption, $V_1 \cap V_2 = \emptyset$). First note that $\langle x_1, V_1 \rangle$ and $\langle x_2, V_2 \rangle$ are disjoint subspaces, for no point of $C_1 \setminus x^{\perp}$ can be collinear with V_2. However, since collinearity is a transitive relation (cf. Lemma 6.5.1), x_1 is collinear to each point of $\langle x_2, V_2 \rangle \setminus V_2$. As such, $S := \langle x_1, x_2, V_1, V_2 \rangle$ is a subspace of dimension $2v + 3$ of T_{x_1}, from which we claim that it contains no other points in X than those in $\langle x_1, V_1 \rangle \cup \langle x_2, V_2 \rangle$. Indeed, such a point z would lie on a unique line having points $z_i \in \langle x_i, V_i \rangle$, $i = 1,2$. If z_1, z_2 both belong to X we obtain three points of X on a line, which is not possible; if both points belong to Y then so does z by Corollary 6.5.2, contradicting $z \in X$; and if say $z_1 \in X$ and $z_2 \in Y$ then z_1 would be collinear to a point of V_2 after all, which is not possible either.

Now consider a tube C_3 through x_1 and x_2. Denoting by V_3 its vertex, it is clear that $\langle x_1, V_3 \rangle$ belongs to T_{x_1} as well, and by the previous paragraph, $\langle x_1, V_3 \rangle \cap S = x_1 x_2$, implying that the subspace $\langle V_1, V_2, V_3 \rangle$ has dimension $3v + 2$. We conclude that $\dim(T_{x_1} \cap Y) \geq 3v + 2$.

On the other hand, Lemma 6.5.6 says that $\dim(T_x \cap Y) \leq 2v + 2$. Combined with the above, this results in $3v + 2 \leq 2v + 2$, or equivalently, $v \leq 0$. Since $v \geq 0$, we obtain that $v = 0$. □

In [29] it was shown that, if $v = 0$, then (X, Ξ) is projectively unique (and hence projectively equivalent to $\mathcal{V}_2(\mathbb{K}, \mathsf{CD}(\mathbb{K}, 0))$, as required). **So we may assume that** $v > 0$. In excluding this possibility, the above lemma will be crucial, as it implies that the following projection is well-defined.

Consider the projection ρ of X from the subspace Y onto a subspace F of $\mathsf{PG}(N, \mathbb{K})$ complementary to Y, i.e.,

$$\rho : X \to F : x \mapsto \langle Y, x \rangle \cap F.$$

Lemma 6.5.8. *For points $x, x' \in X$, we have $\rho(x) = \rho(x')$ if and only if x and x' are equal or collinear. Consequently, for each $\xi \in \Xi$, $\rho(\xi) \cap \rho(X) = \rho(X(\xi))$ is an oval in $\rho(\xi) \cong \mathsf{PG}(2, \mathbb{K})$.*

Proof. Let x,x' be distinct points of X. Then $\rho(x)=\rho(x')$ if and only if $\langle x,Y\rangle=\langle x',Y\rangle$, which is equivalent with xx' being singular since it contains a point of Y. This shows the first assertion, and makes the second one clear (note that $\xi\cap Y$ is precisely the vertex of $X(\xi)$, so $\rho(\xi\cap X)=\rho(\xi)\cap\rho(X)$). □

Lemma 6.5.9. *Suppose C_1 and C_2 are tubes with respective vertices V_1 and V_2. Then $\rho(C_1)\neq\rho(C_2)$ if and only if V_1 and V_2 are disjoint if and only if $\rho(C_1)\cap\rho(C_2)=\rho(C_1\cap C_2)$. If $\rho(C_1)=\rho(C_2)$, collinearity induces a 1-1-correspondence between the generators of C_1 and those of C_2. Moreover, the geometry $(\rho(X),\rho(X(\Xi)))$ is a projective plane.*

Proof. Suppose first that C_1 and C_2 are tubes intersecting each other in a unique point $x\in X$ (as mentioned before, such a pair exists). If $\rho(z_1)=\rho(z_2)$ for two points $z_1,z_2\in C_1,C_2$, then z_1 and z_2 are collinear points by Lemma 6.5.8 and hence, $v>0$ and Lemma 6.5.7 imply that $\rho(z_1)=\rho(x)=\rho(z_2)$. We conclude that $\rho(C_1)\cap\rho(C_2)$ is precisely $\rho(x)$. Now take two points z_1,z_2 on C_1,C_2, respectively, with $\rho(z_1)\neq\rho(x)\neq\rho(z_2)$. Then z_1,z_2 are on a unique tube $C_{12}:=[z_1,z_2]$ by (H1) and (H2*); and its vertex V_{12} has to be disjoint from V_1, as z_2 is not collinear to any point of V_1 (otherwise convexity implies that this point belongs to V_2, and $V_1\cap V_2=\emptyset$; likewise V_{12} is disjoint from V_2. The same reasoning as earlier in this paragraph then implies that $\rho(C_{12})\cap\rho(C_i)=\rho(z_i)$, $i=1,2$. Since z_1,z_2 where arbitrary on $C_i\setminus x^{\perp}$, since each two tubes intersect non-trivially by (H2*) and since the roles of C_1,C_2 and C_{12} are interchangeable, we obtain that for each pair of tubes C,C', we have that $\rho(C)\neq\rho(C')$ is equivalent with $\rho(C)\cap\rho(C')=\rho(C\cap C')$ and that $\rho(C)\neq\rho(C')$ implies that their corresponding vertices V and V' need to be distinct (the converse of this statement is contained in Lemma 6.5.7). In particular, $(\rho(X),\rho(X(\xi)))$ is a projective plane.

It follows immediately from Lemma 6.5.8 that, if $\rho(C_1)=\rho(C_2)$ for tubes C_1,C_2, each generator of C_1 is collinear to a unique generator of C_2 and vice versa. □

Lemma 6.5.10. *For each x in X and for each tube C through x, there are 2^{v+1} tubes C' through x with $\rho(C)=\rho(C')$.*

Proof. Let C_0 be a tube with $\rho(x)\notin\rho(C_0)$. Then x is collinear to no point of C_0. As such, each point c of C_0 determines a unique tube $[x,c]$ with x and two tubes $[x,c]$ and $[x,c']$ with $c,c'\in C_0$ have the same image under ρ if and only if c and c' belong to the same generator of C_0. Since each generator contains 2^{v+1} points, the lemma follows. □

Lemma 6.5.11. *For each $x\in X$, there are 2^{2v+2} points of X in $x^{\perp}$. Consequently, $|X|=7\cdot 2^{2v+2}$.*

Proof. Let n be the number of points in $x^{\perp}$. Take a point x' not collinear to x. Lemma 6.5.10 implies that there are 2^{v+1} tubes through x that have the same image under ρ as $[x,x']$. Moreover, it is clear that each tube which is mapped by ρ to $\rho([x,x'])$ has exactly 2^{v+1} points in $x'^{\perp}$ and that for each $x''\in x'^{\perp}$, we have $\rho([x,x'])=\rho([x,x''])$. We hence obtain $n=2^{v+1}\cdot 2^{v+1}$.

As $x^{\perp}$ and $x'^{\perp}$ are disjoint if x and x' are not collinear, i.e., if and only if $\rho(x)\neq\rho(x')$, the rest of the lemma follows as $\rho(X)$ contains 7 points (cf. Lemma 6.5.9). □

Lemma 6.5.12. *Let C_1 and C_2 be two tubes, with respective vertices V_1 and V_2, sharing a unique point x. Then:*

- (i) *all tubes C_1' with $\rho(C_1') = \rho(C_1)$ through some point $c_1 \in C_1 \setminus x^\perp$ are of the form $[c_1, x']$, with $x' \in \langle x, V_2\rangle \setminus V_2$;*
- (ii) *there is a unique $(\nu-1)$-space V_1^* in V_1 such that the vertex V_1' of any tube C_1' with $\rho(C_1') = \rho(C_1)$ either coincides with V_1 or intersects it in V_1^* and both cases occur; if moreover C_1' contains a point $c_1 \in C_1 \setminus x^\perp$ then $V_1' \cap \langle V_1, V_2\rangle \subseteq V_1$;*
- (iii) *the points in $x^\perp \cap \langle x, V_1, V_2\rangle$ are precisely those in the union of the two singular affine subspaces $\langle x, V_1^*, V_2\rangle \setminus \langle V_1^*, V_2\rangle$ and $\langle x, V_2^*, V_1\rangle \setminus \langle V_2^*, V_1\rangle$ (where V_2^* is defined likewise as V_1^*); in particular, $x^\perp$ is not a singular subspace.*

Proof. By Lemma 6.5.9, we have $\rho(C_1) \neq \rho(C_2)$ and $V_1 \cap V_2 = \emptyset$. Let c_1 be a point on $C_1 \setminus \langle x, V_1\rangle$. We divide the proof into smaller claims.

Claim 1. All tubes C_1' through c_1 with $\rho(C_1') = \rho(C)$ are given by $[c_1, x']$ for $x' \in \langle x, V_2\rangle \setminus V_2$.
Clearly, each tube $[c_1, x']$ is a tube through c_1 with $\rho([c_1, x']) = \rho(C)$. For the converse we note that there are $2^{\nu+1}$ points in $\langle x, V_2\rangle \setminus V_2$ and equally many tubes C_1' through c_1 with $\rho(C_1) = \rho(C_1')$ (cf. Lemma 6.5.10). Furthermore, if there would be a tube C' with $\rho(C) = \rho(C')$ containing two points x' and x'' of $\langle x, V_2\rangle \setminus V_2$), then C' also contains $x' \wedge x''$, which belongs to V_2. But then Lemma 6.5.9 implies that $\rho(C') = \rho(C_2)$, whereas we assumed $\rho(C') = \rho(C_1) \neq \rho(C_2)$, a contradiction. This shows the claim and the first assertion.

Claim 2. The point c_1 cannot be collinear to any point of $\langle V_1, V_2\rangle \setminus V_1$.
Suppose for a contradiction that c_1 is collinear to some point $y_0 \in \langle V_1, V_2\rangle \setminus V_1$. Then y_0 is contained in a unique plane $\langle x, y_1, y_2\rangle$ with $y_i \in V_i$, $i = 1, 2$. As such, at least one of x, $x \wedge y_1$ is collinear to y_0, implying that C, which is uniquely determined by either of these points and c_1, has y_0 in its vertex after all, contradicting $y_0 \notin V_1$. This shows the claim.

Now let C_1' be any tube through c_1 with $\rho(C_1) = \rho(C_1')$, i.e., $C_1' = [c_1, x']$ for some $x' \in \langle x, V_2\rangle \setminus (V_2 \cup \{x\})$. Denote its vertex by V_1'.

Claim 3. We have $V_1' \cap \langle V_1, V_2\rangle \subseteq V_1$ and $\dim(V_1 \cap V_1') \geq \nu - 1$.
The vertices V_1, V_1' and V_2 belong to $\langle x^\perp \cap Y\rangle \subseteq T_x$ (since $\langle x^\perp \cap Y\rangle = \langle x'^\perp \cap Y\rangle$ by Lemma 6.5.6). If V_1' would contain a point of $\langle V_1, V_2\rangle$ outside V_1, then this point would be collinear to c_1, contradicting Claim 2. Hence $V_1' \cap \langle V_1, V_2\rangle \subseteq V_1$ indeed. Consequently, noting that $\langle V_1, V_2\rangle$ is a subspace of codimension 1 in $T_x \cap Y$ by Lemma 6.5.6, we obtain that $\dim(V_1 \cap V_1') \geq \nu - 1$.

Claim 4. There always is a tube C_1' through c_1 with $\rho(C_1) = \rho(C_1')$ but $V_1 \neq V_1'$.
Suppose for a contradiction that all tubes C_1' through c_1 with $\rho(C_1) = \rho(C_1')$ are such that $V_1 = V_1'$. Then by Claim 1, each point of $\langle x, V_2\rangle \setminus V_2$ is collinear to V_1 and as such $\langle x, V_1, V_2\rangle \setminus \langle V_1, V_2\rangle$ is a singular affine subspace contained in $x^\perp$, containing $2^{2\nu+2}$ points. By Lemma 6.5.11, it coincides with $x^\perp$. However, if $x^\perp$ is a singular subspace, the first claim implies that each two tubes whose images under ρ coincide and contain $\rho(x)$, have the same vertex.

Now let V_1, V_2 and V_3 be vertices corresponding to three tubes through x having different images under ρ. Since $\nu > 0$, three ν-spaces never cover a $(2\nu+1)$-space, so there is a point

$y \in \langle V_1, V_2\rangle \setminus (V_1 \cup V_2 \cup V_3)$. Yet, since $\langle x, V_1, V_2\rangle$ is singular, xy is a singular line which by (H2*) has to be contained in some tube through x, but whose vertex cannot equal V_1, V_2, or V_3. This contradicts what we derived in the previous paragraph. The claim is proved.

Claim 5. If C_1 and C_1' are two tubes with $\rho(C_1) = \rho(C_1')$ and $\dim(V_1 \cap V_1') = v-1$, then the vertex V_1'' of each tube C_1'' with $\rho(C_1'') = \rho(C_1)$ contains $V_1 \cap V_1'$.
Take a point c_1' on C_1' not collinear with x nor with c_1. Then there is a unique tube $C := [c_1, c_1']$ through these two points (so note that $\rho(C) = \rho(C_1)$), whose vertex V contains $V_1 \cap V_1'$ as the non-collinear points c_1 and c_1' are both collinear to it. Moreover, $V \cap \langle V_1, V_1'\rangle = V_1 \cap V_1'$, for no point of C_1 is collinear to a point in $\langle V_1, V_1'\rangle \setminus V_1'$ (using an argument similar to the one in Claim 2). So now for any tube C_1'' with $\rho(C_1'') = \rho(C)$, we obtain that its vertex V_1'' needs to share a $(v-1)$-space with V_1, V_1' and V (by Lemma 6.5.9 and Claim 3), which is only possible if V_1'' contains $V_1 \cap V_1'$, as required.

Denoting this $(v-1)$-space of V_1 by V_1^*, Claims 2 up to 5 show the second assertion. Let V_2^* be defined likewise as V_1^* but with respect to V_2. We show the last assertion in the next two claims.

Claim 6. $\langle x, V_1^*, V_2\rangle$ is a singular subspace and no point of $\langle x, V_2\rangle \setminus \langle x, V_2^*\rangle$ is collinear to V_1.
Since each tube $[c_1, x']$ with $x' \in \langle x, V_2\rangle \setminus V_2$ has a vertex containing V_1^* by the second assertion, it follows that x' is collinear with V_1^* and hence $\langle x, x', V_1^*\rangle$ is a singular subspace. As this holds for any $x' \in \langle x, V_2\rangle \setminus V_2$ we obtain that $\langle x, V_1^*, V_2\rangle$ is a singular subspace.

If there would be a point $x' \in \langle x, V_2\rangle \setminus \langle x, V_2^*\rangle$ collinear to V_2, then each point x'' of $\langle x, V_1\rangle \setminus V_1$ would be collinear to V_2: since x' is collinear to $x \wedge x''$, also x'' is collinear to $x \wedge x'$, which is a point of $V_2 \setminus V_2^*$. Thus, a tube C_2' through x'' with $\rho(C_2) = \rho(C_2')$ then has a vertex V_2' which contains V_2^* and $x \wedge x'$, i.e., $V_2' = V_2$ and we obtain that x'' is indeed collinear to V_2. But then all points of $\langle x, V_1\rangle$ are collinear to V_2, which we excluded in the first paragraph of Claim 4.

Claim 7. There is no singular line xy with $y \in \langle V_1, V_2\rangle \setminus (\langle V_1, V_2^*\rangle \cup \langle V_1^*, V_2\rangle)$.
Suppose for a contradiction that x is collinear to some point y in $\langle V_1, V_2\rangle \setminus (\langle V_1, V_2^*\rangle \cup \langle V_1^*, V_2\rangle)$. Then the point y is contained in a unique singular line y_1y_2 with $y_i \in V_i \setminus V_i^*$, $i = 1, 2$. But then the point $x \wedge y_2$ is collinear to y_1 and V_1^* and as such to V_1 entirely (since a tube through this point with same image under ρ as C_1 will have a vertex through V_1^* and y_1), contradicting the previous claim. □

Corollary 6.5.13. *Let C_1, C_2 and C_3 be three tubes through a point $x \in X$ with pairwise different images under ρ and denote their respective vertices with V_1, V_2 and V_3. Then $V_3 \cap \langle V_1, V_2\rangle$ has dimension $v-1$.*

Proof. Lemma 6.5.12(iii) implies that V_3 (whose points are of course collinear to x) intersects $\langle V_1, V_2\rangle$ in a subspace of $\langle V_1, V_2^*\rangle$ or of $\langle V_1^*, V_2\rangle$. Recall moreover that $V_i \cap V_3 = \emptyset$ for $i = 1, 2$ by Lemma 6.5.9. It follows that V_3 is not entirely contained in one of $\langle V_1, V_2^*\rangle$, $\langle V_1^*, V_2\rangle$, so V_3 shares just a $(v-1)$-space with one of them (less is not possible since $\langle V_1, V_2\rangle$ is a hyperplane of $T_x \cap Y$). □

We fix a point $x \in X$ and take three tubes C_1, C_2 and C_3, with respective vertices V_1, V_2 and V_3, through x with different images under ρ (so V_1, V_2 and V_3 are pairwise disjoint). By Corollary 6.5.13, there is a point $y_1 \in V_1$ such that no line though y_1 intersects both V_2 and V_3. Denote the point $x \wedge y_1$ by x_1.

Lemma 6.5.14. *Let C_1, C_2 and C_3 be three tubes through a point $x \in X$ with pairwise different images under ρ and denote their respective vertices with V_1, V_2 and V_3. Then*

- (i) *Each tube C_3' through x_1 with $\rho(C_3) = \rho(C_3')$ intersects C_2 in a point $x_2 \in \langle x, V_2\rangle \setminus V_2$ and C_3 in an affine v-space not contained in $x^\perp$, in particular its vertex $V_3' \cap V_3 = V_3^*$;*
- (ii) *there is a unique $(2v-1)$-space S having a $(v-1)$-space in common with each of V_1, V_2, V_3 and $S \cap V_i = \langle V_j, V_k\rangle \cap V_i = V_i^*$, for $\{i,j,k\} = \{1,2,3\}$; in particular, $S = \langle V_1^*, V_2^*\rangle$;*
- (iii) *with $S = \langle V_1^*, V_2^*\rangle$, the points in $x^\perp$ are precisely those in $\bigcup_{i=1}^3 \langle x, S, V_i\rangle \setminus \langle S, V_i\rangle$;*
- (iv) $v = 1$.

Proof. (i) Since $\rho(C_3) = \rho(C_3')$, we obtain that $\rho(C_2 \cap C_3') = \rho(C_2) \cap \rho(C_3) = \rho(x)$, so the intersection of C_2 and C_3' is indeed a point x_2 in $\langle x, V_2\rangle \setminus V_2$. Put $y_2 = x \wedge x_2$. The fact that $\rho(C_3) = \rho(C_3')$ also implies that the vertex V_3' of C_3' intersects V_3 in at least a $(v-1)$-space. If $V_3' = V_3$, then $x_1 \wedge x_2$ belongs to V_3, but then $y_1 \wedge y_2 = x_1 \wedge x_2$, implying that $y_1 y_2$ is a line through y_1 intersecting V_2 and V_3, a contradiction. So $V_3 \cap V_3' = V_3^*$ and alongside we obtained that $x_1 \wedge x_2 \notin V_3$.

If $C_3 \cap C_3'$ would contain a point x_3 in $\langle x, V_3\rangle$, then the plane $\langle x_1, x_2, x_3\rangle$ would intersect V_3' in a line and hence V_3^* in a point. Yet, we already know that $x_1 x_2$ does not intersect V_3, and neither does $x_1 x_3$ for otherwise $x_1 \in \langle x, V_3\rangle$, likewise for $x_2 x_3$. This contradiction implies that $C_3 \cap C_3'$ is an affine v-space through V_3^* inside $\langle c_3, V_3\rangle$, where c_3 is a point on $C_3 \setminus x^\perp$. This shows the first assertion.

(ii) Note that the above means that x_1 is not collinear to V_3 (as otherwise $V_3' = V_3$ after all). By Lemma 6.5.12, this is equivalent with $y_1 \notin V_1^*$ and since y_1 is an arbitrary point in V_1 not contained in $\langle V_2, V_3\rangle$. This implies that $V_1^* \subseteq \langle V_2, V_3\rangle \cap V_1$, or since the dimensions are equal, $V_1^* = \langle V_2, V_3\rangle \cap V_1$. Now $\langle V_1^*, V_2\rangle \cap V_3 = \langle V_1, V_2\rangle \cap V_3$ and the latter equals V_3^* by changing the roles of V_1 and V_3. Likewise, $V_2^* = \langle V_1, V_3\rangle \cap V_2 = \langle V_1^*, V_3\rangle \cap V_2 = \langle V_1^*, V_3^*\rangle \cap V_2$. As such $\langle V_1^*, V_2^*, V_3^*\rangle$ is a $(2v-1)$-space, and this is the unique one in $T_x \cap Y$ intersecting each V_i, $i = 1,2,3$ in a $(v-1)$-space. This shows the second assertion.

(iii) It follows by Lemma 6.5.12(iii) that for each point $y \in \bigcup_{i=1}^3 \langle S, V_i\rangle$, the line xy is collinear. Since there are $2^{2v+2} - 1$ such points y, this yields 2^{2v+2} points in $x^\perp$ (including x), so the maximum amount is reached by Lemma 6.5.11.

(iv) We use the fact that C_3' contains an affine v-space with V_3^* at infinity, contained in one of the two generators of C_3 distinct from $\langle x, V_3\rangle$. In such a generator, there are only two such affine v-spaces with V_3 at infinity, and each of them determines precisely one tube together with x_1. Recalling that there are 2^{v+1} possibilities for C_3' (since there are 2^{v+1} tubes through x_1 which have the same image under ρ as C_3), we obtain $2^{v+1} = 4$, so $v = 1$. This shows the last assertion. □

This now enables us to describe the structure of all vertices. We have seen that, for a given tube C with V, all tubes in $\rho(C)$ have a vertex that goes through a unique point V^* on V. We call V^* a *special point*. Let $\mathscr{V}$ be the set of all vertices, i.e., $\mathscr{V} = \{V \mid V \text{ is the vertex of } C \text{ for } C \in \mathscr{C}\}$.

Lemma 6.5.15. *For each point $x \in X$, x is collinear to precisely* 6 *vertex lines $V_1, V_1', V_2, V_2', V_3, V_3'$ such that $V_i \cap V_i' = V_i^*$ for $i = 1,2,3$ and V_1^*, V_2^*, V_3^* form a line $L_x^* \in \pi^*$, which is moreover the third line through V_i^* in the plane $\langle V_i, V_i' \rangle$, $i = 1,2,3$. We obtain the following structure.*

- (i) *The special points V^* form the points of a projective plane π^* of order* 2.
- (ii) *The vertex lines through some special point V^* span a* 3*-space Π_{V^*} containing π^* and each line in Π_{V^*} through V^* not in π^* occurs as a vertex (hence there are 4 such vertices) and for each line L^* of π^*, the vertices that meet L^* span a* 4*-space Π_{L^*} containing Π_{V^*} for each $V^* \in L^*$.*
- (iii) *The subspace Y has dimension* 5.
- (iv) *The structure formed by the* 3*-spaces Π_{V^*}, where V^* a special point, and Π_{L^*}, where L^* is a line of π^*, is a projective plane of order 2.*

Proof. Taking three tubes C_1, C_2 and C_3 through $x \in X$ with pairwise different images under ρ yields, by Lemma 6.5.14(ii,iii), respective vertex lines V_1, V_2 and V_3 which span a 4-space and their respective special points V_1^*, V_2^* and V_3^* are on a line, say L_x^*.

Lemma 6.5.14(iii) also says that, for $i = 1,2,3$, a point y in $\langle L_x^*, V_i \rangle \setminus (L_x^* \cup V_i)$ is collinear to x. As such, the line xy is contained in some tube C_{xy} with vertex V_{xy}, and the latter has to go through one of V_j^* for some $j \in \{1,2,3\}$ by Lemma 6.5.9. If $j \neq i$ then V_{xy} would intersect V_i in a point distinct from V_i^*, a contradiction to Lemma 6.5.12(ii). Hence V_{xy}, which we will now denote by V_i', is indeed a line through V_i^* in the plane $\langle V_i, L_x^* \rangle$. Again by Lemma 6.5.14$(iii)$, there are no other vertex lines collinear to x, since $\bigcup_{i=1}^3 (V_i \cup V_i') = \bigcup_{i=1}^3 \langle L_x^*, V_i \rangle$.

(i) Let V_1^* and V_2^* be two distinct special points. We take tubes C_1 and C_2 with respective vertices V_1 and V_2 containing V_1^* and V_2^* respectively, and by (H2*) these tubes intersect in a point $x \in X$. By Lemma 6.5.14(ii), the point $V_1^* \wedge V_2^*$ is then also a special point, namely some V_3^* corresponding to a third tube C_3 through x. Note that V_3^* does not depend on x.

Since there are precisely seven special points (one for each oval in $\rho(\Xi)$), the above suffices to deduce that the special points form a projective plane π^* of order 2.

(ii) Take three tubes C_1, C_2 and C_3, with respective vertices V_1, V_2 and V_3, through x as above. We show that their are four vertex lines through V_3^* corresponding to tubes with the same image under ρ as C_3.

Let x_1 be a point on $\langle x, V_1 \rangle$ not on V_1 nor on the line $\langle x, V_1^* \rangle$. By Lemma 6.5.14(i), each of tubes through x_1 with the same image under ρ as C_3 is given as $C_{13} := [x_1, x_3]$ for some point x_3 on $C_3 \setminus x^{\perp}$. Its vertex V_{13} goes through V_3^*. Note that $V_{13} \neq V_3$ (since x_1 is not collinear to V_3 by our choice of x_1) and $V_{13} \neq V_3'$ (since x_3 is not collinear to V_3' for otherwise V_3' would be collinear to both x and x_3 and as such be the vertex of $[x, x_3] = C_3$). By Lemma 6.5.14(i), C_{13} intersects C_2 in a point $x_2 \in \langle x, V_2 \rangle$. Then $x_2 \neq x \wedge V_2^*$, for otherwise

$y_{12} := x_1 \wedge x_2$, which belongs to V_{13}, is a point on V_1', a contradiction. So y_{12} is a point in $\langle V_1, V_2\rangle$ not contained in the planes $\langle V_1, L_x^*\rangle$ and $\langle V_2, L_x^*\rangle$, i.e., contained in the unique other plane α_3 in $\langle V_1, V_2\rangle$ through L_x^*. This means that $V_{13} = \langle V_3^*, y_{13}\rangle$ is a line in α_3 through V_3^*, distinct from L_x^*.

If x_3' is a point on $C_3 \notin \{x^\perp, x_3^\perp\}$, then the vertex V_{13}' of $C_{13}' := [x_1, x_3']$ is not equal to V_{13}, for if these lines coincide, then the non-collinear points x_3 and x_3' of C_3 are both collinear to it, whereas it is distinct from V_3, a contradiction.

Consequently, the two lines in α_3 through V_3^* distinct from L_x^* both occur as a vertex of a tube through x_1 with the same image under ρ as C_3, and conversely, each such tube has one of these two lines as its vertex.

If we vary x_1 among the points on $\langle x, V_1\rangle$ not on V_1 nor on the line $\langle x, V_1^*\rangle$, we obtain the same two lines, since these did not depend on the exact position of x_1 among the points of $\langle x, V_1\rangle \setminus (V_1 \cup \langle x, V_1^*\rangle)$. If $x_1 \in \langle x, V_1^*\rangle$, then x_1 is collinear to both V_3 and V_3' and since a point can only be collinear to two vertex lines through the same special point, each tube through x_1 which is mapped by ρ onto $\rho(C_3)$ has V_3 or V_3' as its vertex.

We conclude that V_3, V_3' and V_{13}, V_{13}' are the four vertex lines through V_3^*. Since $\langle V_3, V_3'\rangle$ and $\langle V_{13}, V_{13}'\rangle$ span planes both containing L_x^*, it is clear that these lines span a 3-dimensional subspace $\Pi_{V_3^*}$. Again letting x_3 be a point of $C_3 \setminus x^\perp$, the corresponding special line $L_{x_3}^*$ (which goes through V_3^*) is the intersection of two planes spanned by two times two lines out of the four vertex lines through V_3^*, likewise for a point $x_3' \in C_3 \setminus (x^\perp \cup x_3^\perp)$. It follows that the lines $L_{x_3}^*$ and $L_{x_3'}^*$ also belong to $\Pi_{V_3^*}$, so $\pi^* \subseteq \Pi_{V^*}$ indeed (more precisely, the plane π^* is the unique plane through L_x^* distinct from $\langle V_3, V_3'\rangle$ and $\langle V_{13}, V_{13}'\rangle$).

It also follows that the vertex lines that meet L_x^* are all contained in the 4-space $\langle V_1, V_2, V_3\rangle$, showing the last part of the second assertion.

(iii) Through each special point there are 4 vertex lines, which gives a set of 9 points of Y. There are seven such sets and all are pairwise disjoint. This amounts to a total of 63 points, which is precisely the number of points in $\mathsf{PG}(5,\mathbb{K})$. Since Y is a subspace, the assertion follows.

(iv) This follows immediately from the second assertion. □

The above shows that the structure of $\mathscr{V}$ is a representation of a projective Hjelmslev plane of level 2. We now get back to the structure of $(\rho(X), \rho(\Xi))$ before reaching a final contradiction.

Lemma 6.5.16. *The set $(\rho(X), \rho(\Xi))$ is the set of seven points in a 3-space complimentary to a non-incident point-plane pair (n, π_n) and the planes of $\rho(X)$ are given by joining the point n with the lines L of the plane π.*

Proof. We obtained (cf. Lemma 6.5.9) that $(\rho(X), \rho(\Xi))$ is a projective plane. We claim that it moreover has the property that for each point p in $\rho(X)$, the three ovals O_1, O_2 and O_3 of $\rho(X)$ going through it, share their tangent line at x, i.e., $T_x(O_1) = T_x(O_2) = T_x(O_3)$. To see this, let x be a point with $\rho(x) = p$ and take three tubes C_1, C_2, C_3 through p with

$\rho(C_i) = O_i$, $i = 1,2,3$. Then $T_x(C_3) \subseteq \langle T_x(C_1), T_x(C_2)\rangle =: T_x$ by (H3). Furthermore, the 6-dimensional subspace T_x obviously contains $\langle x^\perp \cap Y\rangle$, which is a 4-dimensional subspace (cf. Lemma 6.5.14(iii)). Hence $\dim(T_x \cap Y)$ is precisely 4 by Lemma 6.5.6. Applying ρ, i.e., projecting from Y, then maps both $T_x(C_i)$ for each $i \in \{1,2,3\}$ and T_x onto a line, showing the claim.

Let $\pi_1, ..., \pi_7$ be the seven planes in $\rho(\Xi)$ (so each containing three points of $\rho(X)$, as $\rho(\xi) \cap \rho(X) = \rho(X(\xi))$ for each $\xi \in \Xi$). Observe that two such planes intersect each other in a line, since they share a point of $\rho(X)$ and hence also the corresponding tangent line. Take a point $p \in \rho(X)$ and a plane π of $\rho(X)$ with $p \notin \pi$, and put $\rho(x) \cap \pi = \{p_1, p_2, p_3\}$. Then the three planes of $\rho(X)$ through p share the lines T_{p_1}, T_{p_2} and T_{p_3} with π, respectively, and as such they are contained in $\langle p, \pi\rangle$. These three intersection lines have the nucleus n of $\{p_1, p_2, p_3\}$ in common, and hence it is clear that $T_p = np$, being the intersection line of the three planes through p. Now consider a plane $\langle p, T_{p_i}\rangle$, $i = 1,2,3$. It already contains p and p_i of $\rho(X)$, and the third point p'_i cannot lie on the line pp_i, nor on the line $np = T_p$, so it has to lie on the remaining line, but not on T_{p_i}, which leaves a unique possibility for the position p'_i.

We conclude that the seven points of $\rho(X)$ (namely $p, p_1, ..., p'_3$) are contained in the 3-space $\langle p, \pi\rangle$. Moreover, complement of $\{n\} \cup \rho(X)$ is precisely the point set of a plane π_n, as one can verify. It is now clear that all points $\rho(X)$ play the same role, so for each such point p we have that the line np is exactly the tangent line T_p. Since there are only three planes through T_p in $\langle n, \pi_n\rangle$, and these correspond to the three lines in π_n through $n \wedge p$, the assertion follows. □

Corollary 6.5.17. *The point set X spans a 9-dimensional subspace.*

Proof. In Lemma 6.5.15 we deduced that $\dim(Y) = 5$ and since projecting from Y onto a complementary subspace gives a set $\rho(X)$ spanning a 3-dimensional subspace by the previous lemma, we obtain $\dim(\langle X\rangle) = 9$. □

Lemma 6.5.18. *Let x_1, x_2, x_3 and x_4 be four points of X such that their image under ρ yields four points no three of which are on an oval of $\rho(\Xi)$. Then the points x'_i obtained by intersecting $[x_1, x_i]$ with $[x_j, x_k]$, for $\{i, j, k\} = \{2,3,4\}$ are collinear to a common vertex line.*

Proof. It follows by Lemma 6.5.16 that the points x_1, x_2, x_3 and x_4 are mapped by ρ on a plane. So $\langle x_1, x_2, x_3, x_4, Y\rangle$ has dimension 8, whereas $\langle x_1, x_2, x_3, x_4\rangle$ has dimension 3 (no 4 points of X can be contained in the same plane by (H2*)). This means that $\langle x_1, x_2, x_3, x_4\rangle$ intersects Y in a unique point y. We claim that this point is the point of π^* corresponding to the tubes which are mapped by ρ on $\rho(X) \setminus \{x_1, x_2, x_3, x_4\}$.

To that end, we first consider two general tubes C_1 and C_2 with $C_1 \cap C_2$ a unique point $x \in X$. Then it is clear that $\dim(\langle C_1, C_2\rangle) = 8$. Furthermore, $\langle C_1, C_2\rangle$ contains $x^\perp$ entirely since it contains $\langle T_x(C_1), T_x(C_2)\rangle$, which by (H3) spans T_x and the latter contains $x^\perp$. By Lemma 6.5.14(iii) we have that $\dim(\langle x^\perp \cap Y\rangle) = 4$. If x_1 and x_2 are two non-collinear points on $C_1 \setminus x^\perp$ and x_3 and x_4 two non-collinear points on $C_2 \setminus x^\perp$, then $\{x_1, x_2, x_3, x_4\}$ is a set with the above properties, so it contains a unique point $y \in Y$, which necessarily belongs to $\langle T_x \cap Y\rangle$ for otherwise $Y \subseteq \langle C_1, C_2\rangle$, which is impossible as then $\langle \rho(C_1), \rho(C_2)\rangle$ would be a plane instead of a 3-space.

Now taking as C_1 and C_2 the respective tubes we obtain by taking those determined by pairs of $\{x_1, x_2, x_3, x_4\}$, we obtain that y belongs to $\langle T_{x'_i} \cap Y\rangle$ for $i \in \{2,3,4\}$. This intersection equals the 3-space spanned by π^* and the vertex lines through V^*, where the latter is the special point in π^* corresponding to the tubes that are mapped by ρ on $\{\rho(x'_1), \rho(x'_2), \rho(x'_3)\}$.

On the other hand, y cannot be contained in $\langle V_1, V_2\rangle$, where V_1 and V_2 are the respective vertices of C_1 and C_2, since then $\langle C_1, C_2\rangle$ would have dimension less then 8. This excludes the special line L^*_x of π^* corresponding to x, and also the two lines through V^* inside $\langle V_1, V_2\rangle$. According to Lemma 6.5.14, the two other vertex lines through V^* are precisely the ones collinear to x. Hence, using this arguments on the three pairs of tubes with respective intersection points x'_1, x'_2, x'_3 yields that $y \in \bigcap_{i=1}^3 (x_i'^{\perp} \cap Y) \setminus \pi^*$. Since two non-collinear points, like $x_1'^{\perp}$ and $x_2'^{\perp}$, have precisely a vertex line in common, we obtain that there is a vertex line (namely spanned by y and V^*) in the intersection $\bigcap_{i=1}^3 (x_i'^{\perp} \cap Y)$. □

Lemma 6.5.19. *The pair* (X, Ξ) *does not exist when* $v > 0$.

Proof. We take four points x_1, x_2, x_3, x_4 and the corresponding three intersection points x'_1, x'_2, x'_3 as in the statement of the previous lemma. Let $V_1, \ldots, V_6$ be the vertex lines obtained by taking tubes $[x_i, x_j]$ with $i \neq j \in \{1,2,3,4\}$ and let V_7 be the unique vertex line in $\bigcap_{i=1}^3 (x_i'^{\perp} \cap Y)$, whose existence is guaranteed by the previous lemma. Since $\{x_1, x_2, x_2, x_4, x'_1, x'_2, x'_3\}$ bares the structure of a projective plane, the corresponding vertex lines $V_1, \ldots, V_7$ determine a projective plane as well. This plane is embedded in the Hjelmslev plane $\mathscr{V}$ in such a way that each of its induced lines is actually the complement of a projective subline, and as has been explained in Section 6.5.2, this is not possible. This contradiction shows that the pair (X, Ξ) with $v > 0$ does not exist after all. □

6.6 Interesting substructure: scrolls

Recall that a *normal rational curve in* $\mathsf{PG}(m, \mathbb{K})$ is given by $\{(x_0^m, x_0^{m-1}x_1, \ldots, x_0 x_1^{m-1}, x_1^m) \mid (x_0, x_1) \in (\mathbb{K} \times \mathbb{K}) \setminus (0,0)\}$ and recall the definition of a normal rational scroll:

Definition 6.6.1. Let Π_k and Π_ℓ be complementary subspaces of a projective space $\mathsf{PG}(k + \ell + 1, \mathbb{K})$ of respective dimensions k and ℓ. In Π_k and Π_ℓ, respectively, we consider normal rational curves C_k and C_ℓ, between which we have a bijection φ preserving the cross-ratio (i.e., a projectivity). The union of all *transversal lines* $\langle p, \varphi(p)\rangle$ with $p \in C_k$ is called a *normal rational scroll* and is denoted by $\mathfrak{S}_{k,\ell}(\mathbb{K})$.

We are particularly interested in $\mathfrak{S}_{1,2}(\mathbb{K})$, which consists of lines between a normal rational curve in dimension 1 (a line) and one in dimension 2 (a conic) and as such is contained in $\mathsf{PG}(4, \mathbb{K})$. This object is also more specifically called a *normal rational cubic scroll*. Each conic on $\mathfrak{S}_{1,2}(\mathbb{K})$ that intersects all transversals of $\mathfrak{S}_{1,2}(\mathbb{K})$ will be called an $\mathfrak{S}_{1,2}(\mathbb{K})$-conic. The following property is folklore, and will be useful. We give a proof for completeness' sake.

Lemma 6.6.2. *Let $\mathfrak{S} = \mathfrak{S}_{1,2}(\mathbb{K})$ be a normal rational cubic scroll in $\mathsf{PG}(4,\mathbb{K})$, $|\mathbb{K}| > 2$, defined by the line L, the conic C and a projectivity $\varphi : C \to L$. Firstly, given two points p and q on distinct transversals of $\mathfrak{S}$ and with $p, q \notin L$, there is a unique $\mathfrak{S}$-conic through p and q. Secondly, each two $\mathfrak{S}$-conics intersect in a point of $\mathfrak{S}$. Thirdly, if $\mathfrak{S}_c$ is the set of all $\mathfrak{S}$-conics through a point $c \in C$, then all tangent spaces through c to these conics are in the plane spanned by the point $\varphi(c)$ and the tangent line through c at C.*

Proof. We can coordinatise $\mathsf{PG}(4,\mathbb{K})$ such that the points of C are given by $(x_0^2, x_0x_1, x_1^2, 0, 0)$, those of L by $(0,0,0,x_0,x_1)$ and such that φ maps $(x_0^2, x_0x_1, x_1^2, 0, 0)$ to $(0,0,0,x_0,x_1)$. The transversal through the point $(0,0,0,1,x)$ is denoted by T_x, the transversal through $(0,0,0,0,1)$ by T.

(i) Without loss, $p \in T$ and $q \in T_0$. Then p and q have respective coordinates $(0,0,1,0,\lambda)$ and $(1,0,0,\lambda',0)$ for $\lambda', \lambda \in \mathbb{K}$. A general point on $T_x \setminus L$ for $x \in \mathbb{K} \setminus \{0\}$ is then, for some $\lambda_x \in \mathbb{K}$, given by $(1, x, x^2, \lambda_x, x\lambda_x)$. One can verify that the set of points

$$(x_0^2, x_0x_1, x_1^2, x_0^2\lambda' + x_0x_1\lambda, x_0x_1\lambda' + x_1^2\lambda)$$

is contained in the plane $X_3 = X_0 + \lambda X_1$, $X_4 = \lambda' X_1 + \lambda X_2$, satisfies $X_0X_2 = X_1^2$, contains p and q and meets the transversal T_x for $x \in \mathbb{K} \setminus \{0\}$ in the point with $\lambda_x = \lambda' + x\lambda$ and contains no other points of the scroll than these. Hence this set is an $\mathfrak{S}$-conic through p and q.

Next, assume for a contradiction that there are two distinct $\mathfrak{S}$-conics C' and C'' through p and q. Then T_x for $x \in \mathbb{K} \setminus \{0\}$ is intersected by both C' and C'', moreover, $T_x \cap C' \neq T_x \cap C''$ as otherwise $C' = C''$. So $T_x \subseteq \langle C', C'' \rangle$. As this holds for any $x \in \mathbb{K} \setminus \{0\}$, and thus for at least two such values of x, as $|\mathbb{K}| > 2$, we obtain that $\langle C', C'' \rangle$ contains $\langle C, L \rangle$, whereas $3 = \dim(\langle C', C'' \rangle) < \dim(\langle L, C \rangle) = 4$. This contradiction shows uniqueness.

(ii) The (unique) $\mathfrak{S}$-conic through p and q meets C in the point $(\lambda^2, -\lambda'\lambda, \lambda'^2)$. It follows that each two $\mathfrak{S}$-conics meet, as they all play the same role as C.

(iii) Finally, consider the point c on C with coordinates $(1,0,0,0,0)$. Then the $\mathfrak{S}_c$-conics through c are the $\mathfrak{S}$-conics through c and some point $c_\lambda \in T$, i.e., a point c_λ with coordinates $(0,0,1,0,\lambda)$ for $\lambda \in \mathbb{K}$. We already determined the point sets of the $\mathfrak{S}$-quadrics through c and c_λ above and hence we see that its tangent lines through c are all in the plane $X_2 = X_4 = 0$, which indeed contains the point $\varphi(c)$ (which has coordinates $(0,0,0,1,0)$). This shows the assertion. $\square$

We now show that two conics intersecting in a point and between which there is a projectivity (a linear collinearity, i.e., one that preserves the cross-ratio), determine a unique normal rational cubic scroll.

Lemma 6.6.3. *Let C_1 and C_2 be two conics intersecting each other in a point c and spanning $\mathsf{PG}(4,\mathbb{K})$, between which there is a projectivity $\varphi : C_1 \to C_2$ fixing c. Then there is an affine line L intersecting all transversals $\langle x, \varphi(x) \rangle$ for $x \in C_1 \setminus \{c\}$ and all points on L are on such a transversal. The induced mapping $\overline{\varphi}$ between C_1 and $\langle L \rangle$ taking a point $x \in C_1 \setminus \{c\}$ to*

$\langle x,\varphi(x)\rangle\cap L$ and c to $\langle L\rangle\setminus L$ is a projectivity. In particular, C_2 is on the normal rational cubic scroll $\mathfrak{S}_{1,2}$ in $\mathsf{PG}(4,\mathbb{K})$ defined by C_1, $\langle L\rangle$ and $\overline{\varphi}$.

Proof. We coordinatise $\mathsf{PG}(4,\mathbb{K})$. For three distinct points c, p_1, q_1 on C_1, choose coordinates $c(1,0,0,0,0)$, $p_1(1,1,1,0,0)$ and $q_1(0,0,1,0,0)$ and let $z_1(0,1,0,0,0)$ be the intersection point of the tangent lines to C_1 at $(1,0,0,0,0)$ and $(0,0,1,0,0)$. Then C_1 is a conic in the plane $X_3=X_4=0$ with equation $X_0X_2=X_1^2$. Let $p_2=\varphi(p_1)$ and $q_2=\varphi(q_1)$. Then we can attach the following coordinates to them: $p_2(1,0,0,1,1)$ and $q_2(0,0,0,0,1)$. By choosing $z_2(0,0,0,1,0)$ as the intersection point of the tangent lines to C_2 at $(1,0,0,0,0)$ and $(0,0,0,0,1)$, we obtain that C_2 is a conic in the plane $X_1=X_2=0$ with equation $X_3^2=X_0X_4$.

Each point of C_1 can be written as $(x_0^2,x_0x_2,x_2^2,0,0)$ and hence corresponds to a pair $(x_0,x_2)\in\mathbb{K}^2\setminus\{(0,0)\}$; likewise each point of C_2 can be written as $(x_0^2,0,0,x_0x_4,x_4^2)$ and corresponds to a pair $(x_0,x_4)\in\mathbb{K}^2\setminus\{(0,0)\}$. Now writing $\varphi(c)=c$, $\varphi(p_1)=p_2$ and $\varphi(q_1)=q_2$ by means of those pairs, we have $\varphi((1,0))=(1,0)$, $\varphi((1,1))=(1,1)$ and $\varphi((0,1))=(0,1)$. Since φ is a projectivity, it is determined by the image of three points, and by our choice of coordinates $\varphi((x_0,x_2))=(x_0,x_2)$, i.e., $\varphi((x_0^2,x_0x_2,x_2^2,0,0))=(x_0^2,0,0,x_0x_2,x_2^2)$. Taking $x-\varphi(x)$ for all $x\in C_1\setminus\{c\}$ we obtain points with coordinates $(0,x_0x_2,x_2^2,-x_0x_2,-x_2^2)$, which are all on the line $\overline{L}$ given by $X_0=X_1+X_3=X_2+X_4=0$ (if $x=c$ then we obtain $(0,0,0,0,0)$ since $\varphi(c)=c$). The only point on this line that is not intersected by a transversal $\langle x,\varphi(x)\rangle$ is the point $z(0,1,0,-1,0)$, so $L=\overline{L}\setminus\{z\}$ and $\langle L\rangle=\overline{L}$.

The mapping $\overline{\varphi}$ between C_1 and $\overline{L}$ is then given by $(x_0,x_1,x_2,0,0)\mapsto(0,x_0,x_2,-x_0,-x_2)$ for all $(x_0,x_1,x_2)\in K^2\setminus\{(0,0,0)\}$ with $x_0x_2=x_1^2$. Clearly, this is a projectivity, which thus gives a normal rational cubic scroll containing both C_1 and C_2. □

The above lemma has a higher-dimensional analogue, replacing conics by quadrics of Witt index 1 in $\mathsf{PG}(d+1,\mathbb{K})$ for some $d>1$.

Corollary 6.6.4. *Let Q_1 and Q_2 be two quadrics of Witt index 1 in $\mathsf{PG}(d+1,\mathbb{K})$ for $d>1$, intersecting each other in a point c and spanning $\mathsf{PG}(2d+2,\mathbb{K})$ (i.e., $\langle Q_1\rangle\cap\langle Q_2\rangle=\{c\}$ too), between which there is a projectivity $\varphi:Q_1\to Q_2$ fixing c. Then there is an affine d-space α intersecting all transversals $\langle x,\varphi(x)\rangle$ for $x\in Q_1\setminus\{c\}$ and all points of α are on such a transversal. The induced mapping $\overline{\varphi}$ between Q_1 and $\langle\alpha\rangle$ taking a point $x\in Q_1\setminus\{c\}$ to $\langle x,\varphi(x)\rangle\cap L$ and c to $\langle\alpha\rangle\setminus\alpha$ then takes a conic of Q_1 to an affine line of α and preserves the cross-ratio. Moreover, $T_c(Q_2)$ is contained in $\langle T_c(Q_1),\overline{\varphi}(c)\rangle$.*

Proof. Analogously as above, we coordinatise $\mathsf{PG}(2d+2,\mathbb{K})$ such that c has coordinates $(1,0,\ldots,0)$ and such that a point p on $Q_1\setminus\{c\}$ and its image $\varphi(p)$ on $Q_2\setminus\{c\}$, for some anisotropic quadratic form q, have the following respective forms:

$$(q(x_2,\ldots,x_{d+1}),1,x_2,...,x_{d+1},0,\ldots,0)\quad\text{and}\quad(q(x_2,\ldots,x_{d+1}),0,\ldots,0,1,x_2,\cdots,x_{d+1}),$$

for some $x_2,...,x_{d+1}\in\mathbb{K}$. Again taking $x-\varphi(x)$ for all $x\in Q_1\setminus\{c\}$ be obtain points

$$(0,1,x_2,...,x_{d+1},-1,-x_2,...,-x_{d+1}),$$

which are all contained in the projective d-space $\overline{\alpha}$ given by $X_0 = X_1 - X_{d+2} = \cdots = X_{d+1} - X_{2d+2} = 0$. The points on $\overline{\alpha}$ not intersected by a transversal $\langle x, \varphi(x)\rangle$ all belong to the $(d-1)$-space Π given by the intersecting $\overline{\alpha}$ with the hyperplane $X_1 = 0$ and hence $\overline{\alpha} \setminus \Pi$ is indeed an affine plane α as described above. The remaining part of the statement follows from Lemma 6.6.3. □

We now aim for a higher-dimensional analogue for the normal rational cubic scrolls, with a quadric of Witt index 1 in $\mathsf{PG}(d+1,\mathbb{K})$ for $d > 1$ (instead of a conic). It turns out that we need regular $(d-1)$-spreads in $\mathsf{PG}(2d-1,\mathbb{K})$ for this. Indeed, the above implies that the object intersecting all transversals of a pair of projectively equivalent quadrics Q_1 and Q_2 with $\langle Q_1\rangle \cap \langle Q_2\rangle = Q_1 \cap Q_2$ equal to a point c is an affine d-space (instead of an affine line), and the object corresponding to the intersection point c then is its projective $(d-1)$-space at infinity (instead of a point). Hence we should be able to associate to each point of the quadric a $(d-1)$-space intersecting that affine d-space in a unique point. Moreover, two such $(d-1)$-spaces should never intersect, as otherwise there would be quadrics Q_1' and Q_2' on the scroll-to-be with $\langle Q_1'\rangle \cap \langle Q_2'\rangle \neq Q_1' \cap Q_2'$, and this we do not want in view of Lemma 6.6.2. Lastly, we also want a projectivity (i.e., a bijection preserving the cross-ratio) between this set of $(d-1)$-spaces and Q_1, so for this the notion of a line on this set of $(d-1)$-spaces should be well defined, intuitively making it clear that we need a regular spread (at then lines correspond to reguli of it).

To be precise, we define a projectivity between a regular $(d-1)$-spread $\mathscr{R}$ in $\mathsf{PG}(2d-1,\mathbb{K})$ and a quadric Q in $\mathsf{PG}(d+1,\mathbb{K})$ of Witt index 1, as a bijection φ between the elements of $\mathscr{R}$ and the points of Q such that, restricted to a regulus $\mathscr{G}$ of $\mathscr{R}$, $\varphi(\mathscr{G})$ is a conic on Q and $\varphi_{|\mathscr{G}}$ preserves the cross-ratio.

We can now define this generalised type of scroll (which is no longer "normal rational" nor "cubic") as follows.

Definition 6.6.5. Let Q be a quadric of Witt index 1 in $\Pi \cong \mathsf{PG}(d+1,\mathbb{K})$ and $\mathscr{R}$ a regular $(d-1)$-spread in $\Pi' \cong \mathsf{PG}(2d-1,\mathbb{K})$, where Π and Π' are complementary subspaces of a projective space $\mathsf{PG}(3d+1,\mathbb{K})$. Suppose we have a bijection φ between Q and $\mathscr{R}$ preserving the cross-ratio. Then we call the union of all transversal subspaces $\langle p, \varphi(p)\rangle$ with $p \in Q$ a *regular d-scroll* and denote it by $\mathfrak{R}_d(\mathbb{K})$.

Each quadric of Witt index 1 intersecting each transversal subspace $\langle p, \varphi(p)\rangle$ in a point not in $\varphi(p)$ is called a *$\mathfrak{R}_d(\mathbb{K})$-quadric*. We show that this regular d-scroll also exhibits the properties of a normal rational cubic scroll mentioned in Lemma 6.6.2.

Lemma 6.6.6. *Let $\mathfrak{R} := \mathfrak{R}_d(\mathbb{K})$ be a regular d-scroll in $\mathsf{PG}(3d+1,\mathbb{K})$, $|\mathbb{K}| > 2$ defined by a quadric Q of Witt index 1 in a complementary $(d+1)$-space Π of $\mathsf{PG}(3d+1,\mathbb{K})$, a regular $(d-1)$-spread $\mathscr{R}$ in a $(2d-1)$-space Π' of $\mathsf{PG}(3d+1,\mathbb{K})$ and some projectivity φ between Q and $\mathscr{R}$. Then, given two points p and q on distinct transversal subspaces of $\mathfrak{R}$ and with $p, q \notin \Pi'$, there is a unique $\mathfrak{R}$-quadric through p and q. Furthermore, each two $\mathfrak{R}$-quadrics of $\mathfrak{R}$ intersect in a unique point of $\mathfrak{R}$ not on $\mathscr{R}$.*

Proof. First note that for each $\mathfrak{R}$-quadric Q' through a point c of Q, φ induces a projectivity ψ between the quadrics Q and Q' fixing c and hence, by Corollary 6.6.4, ψ extends to a projectivity $Q \to \langle \alpha_c \rangle$ where $\alpha_c = \overline{\psi}(Q)$ is an affine d-space with $\overline{\psi}(c) = \varphi(c)$ its $(d-1)$-space at infinity. Moreover, the corollary also says that $T_c(Q')$ is contained in $\langle T_c(Q), \varphi(c) \rangle$.

Let c and p be distinct points of Q and take an arbitrary point $p' \in \langle p, \varphi(p) \rangle \setminus (p \cup \varphi(p))$. We show that there is a unique $\mathfrak{R}$-quadric through c and p'. Denote by r_p the point of pp' on the spread element $\varphi(p)$. Then $\langle \alpha_c \rangle = \langle r_p, \varphi(c) \rangle$. Now, for each line L in $\langle \alpha_c \rangle$ through r_p, Lemma 6.6.2 yields a unique conic C_L through c and p' whose other points are on the transversals determined by the points of $L \setminus \{r_p, \varphi(c) \cap L\}$ (note that these points are not on L). Varying L amongst all lines in $\langle \alpha_c \rangle$ through r_p, we obtain a set of points containing c and p' and containing a unique point of each transversal subspace $\langle x, \varphi(x) \rangle$ but not on $\varphi(x)$, for each $x \in Q$. This set of points $Q_{c,p'}$ spans a $(d+1)$-space since it is contained in $\langle Q, \alpha_c \rangle$ and since the only subspace of dimension smaller than $d+1$ intersecting all transversals are precisely the d-spaces in Π' through an element of $\mathscr{R}$. Moreover, $Q_{c,p'}$ is projectively equivalent to Q (since both are projectively equivalent to $\mathscr{R}$) and hence $Q_{c,p'}$ is also a quadric of Witt index 1 in a $(d+1)$-dimensional subspace of $\mathsf{PG}(3d+1, \mathbb{K})$. We conclude that $Q_{c,p'}$ is a $\mathfrak{R}$-quadric through c and p' and plays the same role as Q with respect to $\mathscr{R}$. From this it follows that each two points on two distinct transversals (and not contained in Π') determine a unique $\mathfrak{R}$-quadric (uniqueness following from the fact that the conics on it through c and p' were uniquely determined) and that each such quadric plays the same role as Q.

Next, we show that each two $\mathfrak{R}$-quadrics intersect non-trivially. Let R_p denote the transversal subspace $\langle p, \varphi(p) \rangle$. We consider the projection ρ of $\mathfrak{R}$ from R_p onto a complimentary subspace F of $\mathsf{PG}(3d+1, \mathbb{K})$. We claim that this projection is injective. If two points q and q' are mapped onto the same point, then they are on a line intersecting the R_p in a point p', and hence q and q' belong to distinct transversal subspaces. But then the line qq' intersects three distinct transversal subspaces, which is impossible (the three respective lines through p', q and q that meet Q then are contained in a 3-space, yielding a line in $\langle Q \rangle$ intersecting Q in at least three points). This shows the claim.

Clearly, $\mathfrak{R}$-quadrics are projected onto affine d-spaces. By injectivity, it suffices to show that each pair of projections always intersects each other in a point. Note that they never intersect each other in more than a point since by the above. Moreover, the projections of the $\mathfrak{R}$-quadrics through some fixed point $p' \in R_p \setminus \varphi(p)$ share their $(d-1)$-space at infinity, for this is the projection of $\langle T_{p'}(Q), \varphi(p) \rangle$ (which contains each of their tangent spaces at p'). Since two affine d-spaces corresponding to $\mathfrak{R}$-quadrics through different points of R_p intersect in at most one affine point, their affine $(d-1)$-spaces at infinity are disjoint. In F, it follows immediately that these two affine d-spaces have precisely a point in common. This shows the assertion and completes the proof. □

CHAPTER

7

SPLIT VERONESE SETS

In this chapter, we will geometrically characterise the Veronese varieties $P(\mathbb{A})$ associated to the split generalised dual numbers $\mathbb{A}$ (except for $\mathsf{CD}(\mathbb{H}',0)$), as introduced in Chapter 5. We start by introducing the setting in which we will do this and the axioms, afterwards we give a general description of geometries that we will encounter in this chapter (which are slightly more general than the Veronese varieties we are aiming at), and then we can state the main theorem.

Throughout $\mathbb{K}$ denotes an arbitrary (commutative) field.

7.1 Split Veronese sets

We start by defining split Veronese sets formally.

7.1.1 Definition

As in the previous chapter, we will work with a point set in $\mathsf{PG}(N,\mathbb{K})$ and a family of quadrics:

Definition 7.1.1. Let R,V be integers with $V \geq -1$ and $R \geq 1$. An (R,V)-*cone* C is a cone with a V-dimensional vertex and as basis a hyperbolic quadric of rank $R+1$ (i.e., a non-degenerate quadric of maximal Witt index in $\mathsf{PG}(2R+1,\mathbb{K})$); C without its vertex is called an (R,V)-*tube*.

However, as opposed to the situation in the previous chapter, we will now work with two types of point sets (lets say “ordinary points” and “special points”) and likewise, we will also use two families of (R,V)-tubes. Below this is explained in detail, but informally speaking the "special points" are points belonging to vertices of "ordinary tubes" and the "special tubes" are special in the sense that their basis also contains points which are contained in the vertex of "ordinary tubes".

Let r,v,r',v',N be integers which are at least -1 with $r' > r \geq 1$. Suppose that $X \cup Z$ is a spanning point set of $\mathsf{PG}(N,\mathbb{K})$. We define Y as the subspace spanned by the points of Z. Put $d := 2r+v+1$ and $d' := 2r'+v'+1$. Let Ξ be a collection of $(d+1)$-dimensional subspaces of $\mathsf{PG}(N,\mathbb{K})$ with $|\Xi| > 1$ and Θ a possibly empty collection of $(d'+1)$-dimensional subspaces of $\mathsf{PG}(N,\mathbb{K})$ such that:

- For each $\xi \in \Xi$, the intersection $XY(\xi) := (X \cup Y) \cap \xi$ is an (r,v)-cone C_ξ, $X(\xi) := X \cap \xi$ is a (r,v)-tube T_ξ and $Y(\xi) := Y \cap \xi$ is the vertex of C_ξ;
- for each $\theta \in \Theta$, the intersection $XY(\theta) := (X \cup Y) \cap \theta$ is an (r',v')-cone C_θ, $Y(\theta) := Y \cap \theta$ is precisely a generator M of the quadric C_θ (which in particular contains the vertex V_θ of C_θ), and then $Z(\theta) := Z \cap \theta$ is the (disjoint) union of V_θ and some r'-space of M complementary to it; lastly $X(\theta) := X \cap \theta$ is $C_\theta \setminus M$.

Remark 7.1.2. Note that $|\Xi| > 1$ implies $N > d+1$.

A subspace S of $\mathsf{PG}(N,\mathbb{K})$ is called *singular* if all its points are contained in $X \cup Y$; if $S \subseteq X$ then S is called an X-subspace. For each point $x \in X$ we denote by T_x the subspace spanned by all singular lines through x.

A quadruple (X,Z,Ξ,Θ) is called a **split Veronese set** (with parameters (r,v,r',v') if Θ non-empty and parameters (r,v) if Θ empty) if the following axioms are satisfied:

(S1) Each pair of distinct points $p_1,p_2 \in X \cup Z$ is contained in a member of $\Xi \cup \Theta$;

(S2) for each pair of distinct members $\zeta_1,\zeta_2 \in \Xi \cup \Theta$, the intersection $\zeta_1 \cap \zeta_2$ is singular;

(S3) for each point $x \in X$, there exists ξ_1,ξ_2 in Ξ such that $T_x = \langle T_x(\xi_1), T_x(\xi_2)\rangle$;

If (X,Z,Ξ,Θ) satisfies (S1) and (S2), then we call it a **split pre-Veronese set**. A split Veronese set is called **mono-symplectic** if Θ is empty, it is called **duo-symplectic** otherwise.

Our goal is to classify the split Veronese sets, which has already been done for the mono-symplectic ones (see next subsection).

7.1.2 Mono-symplectic split Veronese sets

In Lemma 7.4.16 and Proposition 7.4.17), we will show that a mono-symplectic split Veronese set is projectively equivalent to a cone (with a possibly empty vertex) over a mono-symplectic split Veronese set with parameters $(r,-1)$ in which Z is the empty set. Such sets have been classified by Schillewaert and Van Maldeghem in [39], who call them *Mazzocca-Melone sets of split type (d)* (recall $d = 2r+v+1 = 2r$). There is a slight difference in the axioms used in [39], indeed, they consider the following two variants of (S3):

(S3') for each point $x \in X$, $\dim(T_x) \leq 2d$;
(S3") for each point $x \in X$, $\dim(T_x) = 2d$.

Note that (S3) implies (S3'). It is hence a straightforward verification that the obtained geometries obtained in [39] that additionally satisfy (S3) are precisely the following varieties:

($r=1$) The Segre varieties $\mathsf{S}_{1,2}(\mathbb{K})$ and $\mathsf{S}_{2,2}(\mathbb{K})$;

($r=2$) the line Grassmannians $\mathsf{G}_{4,1}(\mathbb{K})$ and $\mathsf{G}_{5,1}(\mathbb{K})$;

($r=4$) the variety $\mathsf{E}_{6,1}(\mathbb{K})$.

Before stating our main theorem, we introduce some notation necessary for it, whilst giving examples of split (pre-)Veronese sets.

7.2 Examples of duo-symplectic split (pre-)Veronese sets

We first describe some geometries, after which we will explain how they are related to the split Veronese sets.

7.2.1 The (half) dual Segre varieties

Let ℓ and k be natural numbers with $\ell, k \geq 1$. The Segre variety $\mathsf{S}_{\ell,k}(\mathbb{K})$ is the set of points in the image of the following map (called the *Segre map*), where $N := (\ell+1)(k+1)-1$:

$$\sigma : \mathsf{PG}(\ell,\mathbb{K}) \times \mathsf{PG}(k,\mathbb{K}) \to \mathsf{PG}(N,\mathbb{K}) : ((x_0,..,x_\ell),(y_0,...,y_k)) \mapsto (x_i y_j)_{0\leq i\leq \ell, 0\leq j\leq k}.$$

This product can be visualised in $\mathsf{PG}(N,\mathbb{K})$ by taking an ℓ-space Π_ℓ and a k-space Π_k intersecting each other in precisely a point, and then the points in $\mathsf{im}(\sigma)$ are those in the direct product of Π_ℓ and Π_k. We will now use the Segre varieties to define two types of varieties: the half dual and dual Segre varieties, respectively. We will encounter these varieties when studying duo-symplectic pre-split Veronese sets. Figure 7.1 contains a graphic representation of the dual Segre variety for $k = \ell = 2$.

Half dual Segre varieties

Inside $\mathsf{PG}(N+\ell+1,\mathbb{K})$, we consider a Segre variety $\mathsf{S} := \mathsf{S}_{\ell,k}(\mathbb{K})$ and a subspace Y of dimension ℓ complementary to the subspace spanned by S. Let Π be any ℓ-space of S. We consider a linear duality χ_Π between Π and Y, which hence takes a point of Π to a hyperplane of Y. Then we can extend χ_Π to a map χ from all points of S to Y, by defining, for a point x of S not in Π, its image $\chi(x)$ as $\chi_\Pi(x_\Pi)$, where x_Π is the unique point in Π collinear to x. The union of all points in $\{\langle x, \chi(x)\rangle \setminus \chi(x) \mid x \in \mathsf{S}\}$ is the point set of what we call a *half dual Segre variety*, which we will denote by $\mathsf{HDS}_{\ell,k}(\mathbb{K})$.

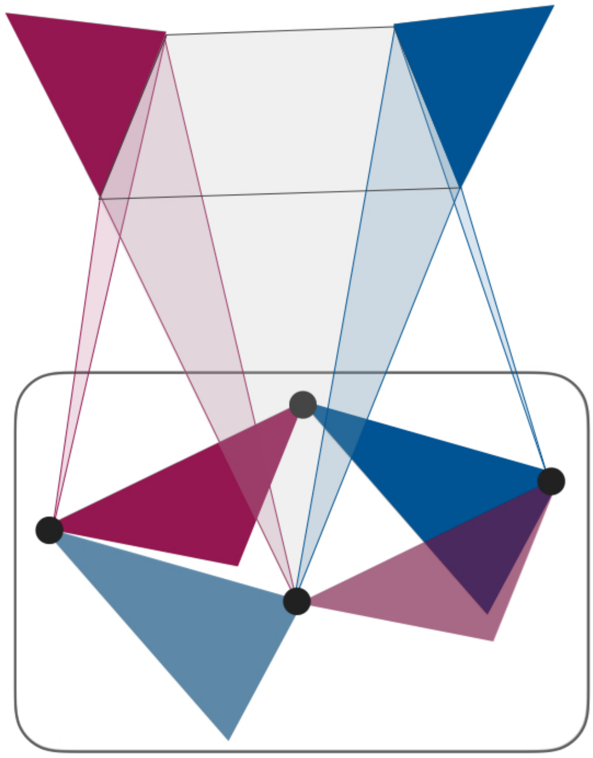


Figure 7.1: A schematic representation of the variety $\mathsf{DS}_{2,2}(\mathbb{K})$.

Dual Segre varieties

We consider, inside $\mathsf{PG}(N+\ell+k+2,\mathbb{K})$, a Segre variety $\mathsf{S} := \mathsf{S}_{\ell,k}(\mathbb{K})$, and in an $(\ell+k+1)$-dimensional subspace Y complementary to it, two disjoint subspaces Z_1 and Z_2 of respective dimensions ℓ and k. As above, let Π_1 be any ℓ-space of S, and now also take any k-space Π_2 of S which intersects Π_1 in a point (and hence is of the other type). For $i=1,2$, let χ_{Π_i} be a linear duality between Π_i and Z_i taking a point of Π_i to a hyperplane of Z_i. Then we can extend the maps χ_{Π_1} and χ_{Π_2} to a map χ from all points of S to $\langle Z_1, Z_2\rangle$, by defining, for a point x of S, its image $\chi(x)$ as $\langle \chi_{\Pi_1}(x_{\Pi_1}), \chi_{\Pi_2}(x_{\Pi_2})\rangle$, where x_{Π_i} is either equal to x if $x \in \Pi_i$, or, if not, it is the unique point in Π_i collinear to x. The union of all points in $\{\langle x, \chi(x)\rangle \setminus \chi(x) \mid x \in \mathsf{S}\}$ is the point set of a *dual Segre variety*, which we will denote by $\mathsf{DS}_{\ell,k}(\mathbb{K})$.

Remark 7.2.1. The half dual Segre variety $\mathsf{S}_{\ell,k}(\mathbb{K})$ is the projection of the dual Segre variety $\mathsf{S}_{\ell,k}(\mathbb{K})$ from the subspace Z_2.

7.2.2 Dual line Grassmannians

Let n be a natural number with $n \geq 2$. The line Grassmannian $\mathsf{G}_{n,1}(\mathbb{K})$ of $\mathsf{PG}(n,\mathbb{K})$ is the set of points in $\mathsf{PG}(\frac{1}{2}(n^2+n)-1,\mathbb{K})$ obtained by taking the images of all lines of $\mathsf{PG}(m,\mathbb{K})$

under the Plücker map

$$\mathsf{pl} : (\langle x_0, x_1, \ldots, x_n), (y_0, y_1, \ldots, y_n)\rangle \mapsto \left(\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \right)_{0 \le i < j \le n}.$$

We consider, inside $\mathsf{PG}(\frac{1}{2}(n^2+3n), \mathbb{K})$, a subspace Y of dimension n and a complementary subspace F of dimension $\frac{1}{2}(n^2+n)-1$. In F, we take a line Grassmannian $\mathsf{G} := \mathsf{G}_{n,1}(\mathbb{K})$, which is the image under pl of a certain n-dimensional projective space P. Let χ' be a linear duality between the projective spaces P and Y, and note that each line of P corresponds to a $(n-2)$-space of Y. As such, we can define a map χ between G and Y which is defined by, for each point $x \in \mathsf{G}$, taking x to $\chi'(\mathsf{pl}^{-1}(x))$. The union of all points in $\{\langle x, \chi(x)\rangle \setminus \chi(x) \mid x \in \mathsf{G}\}$ is the point set of a *dual line Grassmannian*, which we will denote by $\mathsf{DG}_{n,1}(\mathbb{K})$.

Each of these three classes of geometries contains a duo-symplectic split Veronese set (where the convex closures of two points at distance 2 gives the members of $\Xi \cup \Theta$, and such a convex closure belongs to Ξ if and only if it only shares its vertex with Y):

Proposition 7.2.2. *The varieties $\mathsf{HDS}_{2,2}(\mathbb{K})$, $\mathsf{DS}_{2,2}(\mathbb{K})$ and $\mathsf{DG}_{5,1}(\mathbb{K})$ are isomorphic to $P(\mathbb{T}')$, $P(\mathsf{CD}(\mathbb{L}',0))$ and $P(\mathbb{S}')$, respectively, and hence they are split Veronese sets, with respective parameters $(1,0,2,-1)$, $(1,1,2,1)$, $(2,1,4,-1)$.*

Proof. The first assertion follows from the description of the varieties $P(\mathbb{T}')$, $P(\mathsf{CD}(\mathbb{L}',0))$ and $P(\mathbb{S}')$ in Section 5.2: the decomposition into a subspace Y and a "base variety" (which is the Segre variety $\mathsf{S}_{2,2}(\mathbb{K})$ in the smallest two cases and the line Grassmannian $\mathsf{G}_{5,1}(\mathbb{K})$ in the largest case) is given, together with the structure of $p^\perp \cap Y$ for each point p in the base variety. The second assertion then follows from Proposition 5.2.21. $\square$

Remark 7.2.3. The (half) dual Segre varieties and dual line Grassmannians other than these in Proposition 7.2.2 do not satisfy Axiom (S3).

7.2.3 Mutants

We will encounter more general versions of the above defined geometries. This is caused by the fact that the varieties $\mathsf{S}_{\ell,k}(\mathbb{K})$ and $\mathsf{G}_{n,1}(\mathbb{K})$ are embeddings of the abstract geometries $\mathsf{A}_{\ell,1}(\mathbb{K}) \times \mathsf{A}_{k,1}(\mathbb{K})$ and $\mathsf{A}_{n,2}(\mathbb{K})$ (cf. Section 2.2), respectively, but the latter geometries could admit other embeddings. The varieties $\mathsf{S}_{\ell,k}(\mathbb{K})$ and $\mathsf{G}_{n,1}(\mathbb{K})$ are however their (absolutely) universal embeddings, as follows from the main results in Wells ([54]) and Zanella ([57]):

Fact 7.2.4. *Let $(\mathscr{P}, \mathscr{L})$ be a point-line geometry isomorphic to either $\mathsf{A}_{\ell,1}(\mathbb{K}) \times \mathsf{A}_{k,1}(\mathbb{K})$, for $\ell, k \ge 1$, or $\mathsf{A}_{n,2}(\mathbb{K})$, for $n \ge 2$, such that $\mathscr{P}$ be a spanning subset of a projective space $\mathsf{PG}(m,\mathbb{K})$, and each member of $\mathscr{L}$ is the set of all points on a certain line of $\mathsf{PG}(m,\mathbb{K})$. Then $\mathscr{P}$ arises as an injective projection of $\mathsf{S}_{\ell,k}(\mathbb{K})$, or $\mathsf{G}_{n,1}(\mathbb{K})$, respectively, from a suitable subspace.*

We could now, in the above descriptions of the (half) dual Segre varieties and dual line Grassmannians, replace the Segre varieties and line Grassmannians by injective projections of them, if any, and apply the same construction. However, if we want the resulting geometry to satisfy (S2), we cannot use *any* injective projection, as we will now explain.

Let Ω be either a Segre variety $\mathsf{S}_{\ell,k}(\mathbb{K})$ or a line Grassmannian $\mathsf{G}_{n,1}(\mathbb{K})$. Let X be its point set and Ξ be its set of symps (viewing Ω as a parapolar space). As can be verified from their respective algebraic definitions, (X,Ξ) is a mono-symplectic pre-split Veronese set with $Z=\emptyset$. We will only consider projections of them which preserve the fact that they are pre-split Veronese sets. In general, let (X,Ξ) is a mono-symplectic pre-split Veronese set with $Z=\emptyset$ and $\langle X\rangle = \mathsf{PG}(m,\mathbb{K})$ for some m.

Definition 7.2.5. We say that a subspace S of $\mathsf{PG}(m,\mathbb{K})$ is *legal with respect to* (X,Ξ) if S is disjoint from $\langle \xi_1,\xi_2\rangle$ for each pair of symps $\xi_1,\xi_2\in\Xi$. The projection of (X,Ξ) from S onto a subspace of $\mathsf{PG}(m,\mathbb{K})$ complementary to S is called a *legal projection* of (X,Ξ).

Note that a legal projection is automatically injective. By definition, any legal projection of (X,Ξ) is also a mono-symplectic pre-split Veronese set.

Definition 7.2.6. Consider the (half) dual variety $\mathsf{(H)D}\Omega$ associated to Ω as defined above. Then we may replace Ω by a legal projection Π of Ω, and we may re-position Y in such a way that, after applying the same construction and obtaining a point set X, the projection of $\langle\Pi\rangle\cap X$ from $\langle\Pi\rangle\cap Y$ (onto a subspace of $\langle\Pi\rangle$ complementary to $\langle\Pi\rangle\cap Y$) is injective; the ambient projective space is then restricted to $\langle\Pi,Y\rangle$. The resulting structure is called a *mutant* of $\mathsf{(H)D}\Omega$.

Luckily, for the split Veronese sets, (S3) forces the occurring Segre varieties and line Grassmannians to be rather small, in which case they do not admit proper legal projections:

Proposition 7.2.7. (i) *No Segre variety* $\mathsf{S}_{\ell,k}(\mathbb{K})$ *with* $\ell\le 3$ *and* $k\ge 1$ *admits proper legal projections;*

(ii) *also the line Grassmannians* $\mathsf{G}_{n,1}(\mathbb{K})$ *for* $n\in\{4,5\}$ *do not admit proper legal projections.*

Proof. Since $\mathsf{S}_{\ell,k}(\mathbb{K})$ with $\ell<3$ is contained in $\mathsf{S}_{3,k}(\mathbb{K})$ it suffices to show that the latter does not admit proper legal projections. The Segre variety $\mathsf{S}_{3,k}(\mathbb{K})$ is defined by the rank 1 $4\times(k+1)$ matrices over $\mathbb{K}$ in the projective space defined by the vector space of all $4\times(k+1)$ matrices over $\mathbb{K}$. If M is such a matrix of rank 4, then M is the sum of four rank 1 matrices M_1, M_2, M_3 and M_4 which are pairwise not collinear. Then either the symp determined by M_1 and M_2 and the one determined by M_3 and M_4 are distinct or they coincide, but either way, it follows that M is not a legal point w.r.t. $\mathsf{S}_{3,k}(\mathbb{K})$. If M has rank 2 or 3, M is already the sum of 2 or 3 rank 1 matrices, respectively, and the same conclusion holds.

The second assertion is a direct consequence of the main result in [39]. □

Moreover, in these small cases we will later on (in Lemmas 7.5.30, 7.6.12 and 7.7.17) be able to show that $\langle\Pi\rangle\cap Y$ needs to be empty, implying that no mutants occur for the split Veronese sets. The following two lemmas will help us with that:

Lemma 7.2.8. *Let Σ be a 4-space contained in the 8-dimensional projective space generated by the Segre variety $S := \mathsf{S}_{2,2}(\mathbb{K})$ and suppose that Σ intersects S in exactly a grid G. Then there exists a grid G' of S such that $\langle G' \rangle$ intersects $\Sigma \setminus G$ non-trivially.*

Proof. Let π_1 and π_2 be two planes of S of different type (i.e., not disjoint) which are disjoint from G. Let p be the unique intersection point of π_1 and π_2. Then the 4-space $\langle \pi_1, \pi_2 \rangle$ has a point q in common with Σ, which does not belong to G as $\langle \pi_1, \pi_2 \rangle \cap S = \pi_1 \cup \pi_2$. Then q is contained in a unique plane $\langle L_1, L_2 \rangle$ with L_i a line of π_i through p, for $i = 1, 2$. Hence q belongs to the subspaces spanned by the grid of S determined by L_1 and L_2. □

Lemma 7.2.9. *Let Σ be a 10-space contained in the 14-dimensional projective space generated by the line Grassmannian $A := \mathsf{A}_{5,2}(\mathbb{K})$. Then, if $\Sigma \cap A$ contains a line Grassmannian $A' := \mathsf{A}_{4,2}(\mathbb{K})$, then $A' \subsetneq \Sigma \cap A$.*

Proof. Consider a singular 4-space V of A disjoint from A' (which exists). Then $V \cap \Sigma \setminus A'$ contains a point. □

7.3 Main results

We are ready to state our main results.

7.3.1 The results

Theorem 7.3.1. *Let (X,Z,Ξ,Θ) be a split Veronese set with parameters (r,v,r',v') where $\langle X,Z\rangle = \mathsf{PG}(N,\mathbb{K})$ for some arbitrary field $\mathbb{K}$ with $|\mathbb{K}|>2$. If mono-symplectic, then X is projectively equivalent to a cone with a vertex of dimension v^* (possibly, $v^*=-1$), whose points are those of Z, over one of the following geometries:*

- (i) *A Segre variety $\mathsf{S}_{1,2}(\mathbb{K})$ or $\mathsf{S}_{2,2}(\mathbb{K})$, a line Grassmannian $\mathsf{G}_{4,1}(\mathbb{K})$ or $\mathsf{G}_{5,1}(\mathbb{K})$, or the variety $\mathsf{E}_{6,1}(\mathbb{K})$; in this case $v=v'=v^*$.*

If duo-symplectic, then X is either projectively equivalent to a cone with a vertex of dimension v^* (possibly, $v^*=-1$) over one of the following geometries:

- (ii) *A half dual Segre variety $\mathsf{HDS}_{2,k}(\mathbb{K})$, where $k\in\{1,2\}$, which is a split Veronese set with parameters $(1,0,2,-1)$;*
- (iii) *A dual line Grassmannian variety $\mathsf{DG}_{5,1}(\mathbb{K})$, which is a split Veronese set with parameters $(2,1,4,-1)$,*

or projectively equivalent to the following geometry:

- (iv) *A dual Segre variety $\mathsf{DS}_{2,2}(\mathbb{K})$, with parameters $(1,1,2,1)$.*

In particular, the varieties in (i) up to (iv) are subvarieties of the Veronese variety $P(\mathbb{O}')$ over the split octonions $\mathbb{O}'$, and apart from $\mathsf{S}_{1,2}(\mathbb{K})$ and $\mathsf{G}_{4,1}(\mathbb{K})$, all of them are a Veronese variety $P(\mathbb{A})$ for some split quadratic alternative algebra $\mathbb{A}$ whose radical is either empty or generated by a single element t.

Remark 7.3.2. The reason that we exclude the field of 2 elements in the above theorem is that already in one of the very preliminary lemmas (Lemma 7.4.2), things can go wrong if $|\mathbb{K}|=2$. An alternative approach is required, and seeing the high cost and low benefits, we did not pursue this. No counterexamples are known.

7.3.2 Structure of the proof

In **Section 7.4**, we deduce some general properties and show that, if a split (pre-)Veronese set is mono-symplectic, then all quadrics in the members of Ξ have the same vertex, from which we will be able to show Case (i) of the Main theorem (see Proposition 7.4.16). From that point onwards, we will assume that the split (pre-)Veronese set is **duo-symplectic**, and we will deduce that we may then assume that for each $x\in X$, there is at least one member of Θ containing x.

Our proof uses **induction on** r, so we will first deal with the cases where $r = 1$. For a split (pre-)Veronese set with parameters (r, v, r', v') with $r > 1$, we can consider its point-residues for any $x \in X$, which are split pre-Veronese sets with parameters $(r-1, v, r'-1, v')$ (as we will show later on). Even if the Veronese set we started from satisfies (S3), this cannot be guaranteed for its point-residue. Therefore we will have to study split **pre**-Veronese sets as well.

In **Section 7.5** we will deal with split (pre-)Veronese set with parameters $(1, v, r', v')$, with the additional assumption that for each point $x \in X$, there are at least two members of Θ containing x. This case will lead to the **dual Segre varieties** (Case (iv) of the Main theorem).

In **Section 7.6**, we deal with split (pre-)Veronese set in which there is a point in X through which there is a unique member of Θ. In this case, there will be a unique member $\theta_x \in \Theta$ through each point x of X. This case will lead us to **half dual Segre varieties** (in particular, it follows that $r = 1$) (Case (ii) of the Main theorem).

In **Section 7.7**, finally, we turn our attention to the split (pre-)Veronese set with parameters (r, v, r', v') where $r \geq 2$. We will show that $r = 2$ and we will obtain **dual line Grassmannians** (Case (iii) of the Main theorem).

7.4 Basic properties

Let (X, Z, Ξ, Θ) be a split pre-Veronese set with parameters (r, v, r', v') with $\langle X, Z \rangle = \mathsf{PG}(N, \mathbb{K})$ for an arbitrary field $|\mathbb{K}| > 2$.

Recall that a subspace S of $\mathsf{PG}(N, \mathbb{K})$ is called *singular* if its points are contained in $X \cup Y$. If moreover $S \subseteq X$, then we call S an X-space. Two subspaces are called *collinear* if there is a singular subspace containing them.

Lemma 7.4.1. *If a line L of $\mathsf{PG}(N, \mathbb{K})$ contains at least three points of $X \cup Y$, it is singular. Moreover, a singular line contains at most one point in Y.*

Proof. If L contains two points of Y, then L belongs to Y since the latter is a subspace by definition. So, if $|L \cap (X \cup Y)| \geq 3$, then we may assume that L contains at least two points x_1, x_2 of X. By (S1), these are contained in a member of $\Xi \cup \Theta$, in which it is clear that L is a singular line. □

Lemma 7.4.2. *If p_1 and p_2 are non-collinear points of $X \cup Z$, then there is a unique member of $\Xi \cup \Theta$ containing them, which is denoted by $[p_1, p_2]$.*

Proof. By (S1), there exists a member of $\Xi \cup \Theta$ containing p_1 and p_2. If there were at least two of them, then by (S2) their intersection is a singular subspace containing $p_1 p_2$. But then the line $p_1 p_2$ is singular, a contradiction. □

Lemma 7.4.3. *Let L_1 and L_2 be two singular lines not entirely contained in Y, intersecting each other in a point s. Then either $\langle L_1, L_2\rangle$ is a singular plane or L_1 and L_2 are contained in a unique member of $\Xi \cup \Theta$.*

Proof. Take X-points $x_i \in L_i \setminus \{s\}$, $i = 1,2$. Suppose first that x_1 is not collinear to x_2. Then by Lemma 7.4.1, x_1 and x_2 are the only points of $X \cup Y$ on x_1x_2. Since $|\mathbb{K}| > 2$, we can take X-points $x_i' \in L_i \setminus \{s\}$ distinct from x_i, $i = 1,2$. Since $\langle L_1, L_2\rangle$ is a plane of $\mathsf{PG}(N,\mathbb{K})$, the line $x_1'x_2'$ intersects x_1x_2 in a point p, distinct from x_1, x_2 and hence, by the foregoing, $p \notin X \cup Y$. Therefore, the line $x_1'x_2'$ is not singular. If $[x_1, x_2]$ and $[x_1', x_2']$ are distinct, then by (S2) we get $p \in X \cup Y$ after all, a contradiction. Hence $[x_1, x_2]$ and $[x_1', x_2']$ coincide and contain both L_1 and L_2.

So we may assume that $x_1 \perp x_2$ for each pair of X-points $x_i \in L_i \setminus \{s\}$, $i = 1,2$. Let $p \in \langle L_1, L_2\rangle \setminus (L_1 \cup L_2)$ be arbitrary. As $|\mathbb{K}| > 2$, there is a line through p meeting L_1 and L_2 in distinct X-points. By our assumption, these X-points are collinear and hence $p \in X \cup Y$. We conclude that the plane $\langle L_1, L_2\rangle$ is singular indeed. □

Definition 7.4.4. For each point $p \in X \cup Y$, we denote by $p^\perp$ the union of all singular lines through p with at most one point in Y (i.e., not entirely contained in Y).

Lemma 7.4.5. *Let $\zeta \in \Xi \cup \Theta$ and $p \in (X \cup Y) \setminus \zeta$ arbitrary. Then, if $p \in X$, the set $p^\perp \cap XY(\zeta)$ is a singular subspace of $XY(\zeta)$; if $p \in Y$, then $p^\perp \cap X(\zeta)$ contains no two non-collinear points. Consequently, $XY(\zeta)$ is a convex subspace of $X \cup Y$ with respect to singular lines not entirely contained in Y, whose vertex is the subspace of Y collinear to any two non-collinear X-points of ζ.*

Proof. Suppose $p \in X$ (resp. $p \in Y$) and suppose p_1, p_2 are non-collinear points in $p^\perp \cap XY(\zeta)$ (resp. $p^\perp \cap X(\zeta)$). Put $L_i := pp_i$. Then L_1 and L_2 are singular lines not entirely contained in Y and hence, Lemma 7.4.3 implies that $p \in [p_1, p_2] = \zeta$, a contradiction. For the second assertion, it now suffices to note that the unique line between two collinear points of $XY(\zeta)$ is contained in $XY(\zeta)$. □

We can now extend Lemma 7.4.3 to higher-dimensional subspaces.

Lemma 7.4.6. *Let S_1 and S_2 be two singular subspaces of dimension k, with $k > 1$, not entirely contained in Y, intersecting each other in a $(k-1)$-space S. Then either $\langle S_1, S_2\rangle$ is a singular $(k+1)$-space, or S_1 and S_2 are contained in a unique member of $\Xi \cup \Theta$.*

Proof. Let x_1, x_2 be X-points of $S_1 \setminus S$ and $S_2 \setminus S$, respectively. If x_1 and x_2 are not collinear, then by Lemma 7.4.5, $[x_1, x_2]$ contains S and, as such, $[x_1, x_2]$ contains $\langle S_1, S_2\rangle$.

Hence we may suppose that each pair of X-points x_1, x_2 with $x_1 \in S_1 \setminus S$ and $x_2 \in S_2 \setminus S$ is collinear. Let p be any point in $\langle S_1, S_2\rangle \setminus (S_1 \cup S_2)$. For any X-point $x_1 \in S_1 \setminus S$, the line x_1p intersects $S_2 \setminus S$ in a point p_2. If $p_2 \in X$, then p is on the singular line x_1p_2 and as such belongs to $X \cup Y$. If $p_2 \in Y$, then we take an X-point x_1' in $S_1 \setminus (S \cup \{x_1\})$ such that the line x_1x_1' intersects S in an X-point x (note that $S \nsubseteq Y$ since $p_2 \in Y$ and $S_2 \nsubseteq Y$). Let p_2' be the point in $S_2 \setminus S$ on $x_1'p$. Then p_2' belongs to the line xp_2 and as $x \in X$, also $p_2' \in X$. So $p \in X \cup Y$ as before. We conclude that $\langle S_1, S_2\rangle$ is singular indeed. □

Definition 7.4.7. For each X-space S, we will denote by Y_S the set of points of Y collinear to (all points p of) S, i.e., $Y_S := \bigcap_{p\in S}(p^\perp \cap Y)$.

Corollary 7.4.8. *For each X-space S of dimension $k-1 \geq 0$, Y_S is a subspace of Y.*

Proof. Let S_1 and S_2 be singular k-spaces through S such that $S \setminus S_i$ contains a point $y_i \in Y$, $i = 1,2$. By Lemma 7.4.3 or Lemma 7.4.6, $\langle S_1, S_2 \rangle$ is either singular or contained in a member ζ of $\Xi \cup \Theta$. In the latter case however, it is also singular, since $y_1 \perp y_2$ in the polar space ζ. □

Lemma 7.4.9. *Let S be a singular subspace of dimension k and let ζ be a member of $\Xi \cup \Theta$. If $S \cap \zeta$ is a hyperplane of S not entirely contained in Y and $S \cap \zeta$ is not a maximal singular subspace of ζ, then there is a member ζ' of $\Xi \cup \Theta$ containing S such that $\zeta' \cap \zeta$ is not collinear to S.*

Proof. By the assumption on $S \cap \zeta$, there exists a pair of non-collinear singular subspaces S_1 and S_2 of ζ through $S \cap \zeta$. Since the latter is not entirely contained in Y, neither are S_1 or S_2. Lemma 7.4.5 implies that not both $\langle S, S_1 \rangle$ and $\langle S, S_2 \rangle$ can be singular subspaces, for then an X-point of $S \setminus \zeta$ (which exists because $S \nsubseteq Y$) would be collinear to a pair of non-collinear points of $X(\zeta)$. So we may assume that S is not collinear to S_1 and, as such, $\langle S, S_1 \rangle$ is contained in a member of $\Xi \cap \Theta$ by Lemma 7.4.6. This shows the lemma. □

We record a special case of the previous lemma. Observe that the crucial difference between members of Ξ and members of Θ is that for each member ξ of Ξ, we have that $Y(\xi)$ is collinear to each point of $X(\xi)$, whereas for a member θ of Θ holds that for each point $x \in X(\theta)$ there is a point $y \in Y(\theta)$ not collinear to it.

Corollary 7.4.10. *Let x be a point of $X(\zeta)$ for some $\zeta \in \Xi \cap \Theta$ and let L be a singular line through x with a unique point $y \in Y$. Then, if the line L is not contained in ζ, it is contained in a member of Θ together with an X-line of ζ through x.*

Proof. Lemma 7.4.9 implies that L is contained in a member ζ' of $\Xi \cup \Theta$, $i = 1,2$, with $\zeta \cap \zeta'$ a singular subspace through x not collinear to L. Since all lines through x containing a point of Y are collinear to L by Corollary 7.4.8, this implies that $\zeta \cap \zeta'$ contains an X-line L'. Since y is not collinear to the points of L', it then follows that ζ' belongs to Θ. □

7.4.1 Projections of (X, Z, Ξ, Θ)

The following lemma will allow us to assume that there is no non-empty subspace in Y collinear to all points of X (it explains why, in the main theorem, we speak of "a cone with vertex V^* over...").

Lemma 7.4.11. *If $V^* \subseteq Y$ is a subspace of Y of dimension v^* such that all points of X are collinear to V^*, then the projection of a split (pre-)Veronese set (X, Z, Ξ, Θ) with parameters (r, v, r', v') from V^* onto a complementary subspace of $\mathsf{PG}(N, \mathbb{K})$ is a split (pre-)Veronese set with parameters $(r, v - v^* - 1, r', v' - v^* - 1)$ inside $\mathsf{PG}(N - v^* - 1, \mathbb{K})$.*

Proof. If all points of X are collinear to V^*, then obviously all members of Ξ and Θ have V^* in their vertex. It is trivial that the projection of (X,Z,Ξ,Θ) from V^* satisfies (Si) if (X,Z,Ξ,Θ) does, for each $i \in \{1,2,3\}$. $\square$

Throughout we will always use the above projection whenever there is a non-trivial subspace V^* in Y collinear to all points of X. There is another projection that we will frequently make use of. Let F be a subspace of $\mathsf{PG}(N,\mathbb{K})$ complementary to Y.

Definition 7.4.12. The projection of (X,Z,Ξ,Θ) onto F is induced by the following map.

$$\rho : X \to F : x \mapsto \langle Y,x\rangle \cap F.$$

Definition 7.4.13. The *connection* map between $\rho(X)$ and Y (recalling $Y_x = x^\perp \cap Y$) is defined as follows:

$$\chi : \rho(X) \mapsto Y : \rho(x) \mapsto Y_x.$$

We show some general properties about ρ and χ (in particular, we show that the latter is well-defined).

Lemma 7.4.14. *(i) For each $x \in X$, $\rho^{-1}(\rho(x)) = \langle x, Y_x\rangle \cap X$ and hence χ is well defined;*
(ii) for each $\xi \in \Xi$, $\rho(X(\xi))$ is a non-degenerate hyperbolic quadric of rank $r+1$;
(iii) for each $\theta \in \Theta$, $\rho(X(\theta))$ is a singular subspace of dimension r'.

Proof. (i) If $\rho(x) = \rho(x')$ for points $x,x' \in X$ with $x \neq x'$, then $\langle x,x',Y\rangle$ contains Y as a hyperplane. Therefore, the line xx' meets Y in a point $y \in Y$, which by Lemma 7.4.1 means that xx' is a singular line. In particular, $y \in Y_x$, and so $x' \in \langle x, Y_x\rangle \cap X$ indeed. Conversely it is clear that all points of the latter set are mapped onto the same point by ρ. The fact that χ is well-defined then follows immediately.

(ii) and (iii) are obvious noting that, for each $\xi \in \Xi$, $Y \cap \xi$ is the vertex of ξ and for each $\theta \in \Theta$, $Y \cap \theta$ is a maximal singular subspace of θ. $\square$

Remark 7.4.15. Notwithstanding the fact that Y is a subspace, we cannot just immediately use the projection ρ and study the pair $(\rho(X),\rho(\Xi))$. Indeed, if p_1,p_2 are two non-collinear points of $\rho(X)$, and $x_i,x_i' \in \rho^{-1}(p_i)$ for $i=1,2$, then x_1 and x_2 and also x_1' and x_2' are non-collinear points of X for sure, but we do not even know whether $\rho([x_1,x_2]) = \rho([x_1',x_2'])$. Moreover, to establish the inverse image of a line of $\rho(X)$, we need to know more on the structure of Y, et cetera.

7.4.2 Mono-symplectic split Veronese sets

In the following lemma, we express the fact that (X,Z,Ξ,Θ) is mono-symplectic (i.e., $|\Theta| = 0$) in terms of the vertices of members of Ξ.

Lemma 7.4.16. *The following are equivalent:*

(i) Θ is empty;

(ii) *each member of* Ξ *has* Y *as its vertex;*
(iii) *there is a member of* Ξ *having* Y *as its vertex.*

Proof. Since $(ii) \Rightarrow (iii)$ is trivial, it suffices to show that $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$.

$\boxed{(i) \Rightarrow (ii)}$ Suppose that Θ is empty. Take any member $\xi \in \Xi$, say with vertex V, and let x be one of its X-points. If x would be collinear to some point $y \in Y \setminus V$ then, by Corollary 7.4.10, xy is contained in a member of Θ together with an X-line of ξ through x, contradicting our assumption. Hence $Y_x = V$. Now, take $x' \in X \setminus \{x\}$ arbitrary. Then (S1) and our assumption imply that x and x' are contained in a member of Ξ, whose vertex V' is contained in Y_x and hence coincides with it since $\dim(V) = \dim(V') = v'$. As a consequence, all points of X are collinear to V, and therefore also all members of Ξ have vertex V. We now claim that this implies that $Y = V$. Indeed, if not, then since $\langle Z \rangle = Y$, there is a point $z \in Z \setminus V$. But then, for any point $x \in X$ we have by (S1) that there is a member $\zeta \in \Xi \cup \Theta$ containing x and z. Since x is not collinear to z, this member should be an element of Θ, a contradiction. Hence $Y = V$ indeed, showing (ii).

$\boxed{(iii) \Rightarrow (i)}$ Suppose that there is some $\xi \in \Xi$ having Y as its vertex. Let x be a point of $X \setminus X(\xi)$. We show that $x \perp Y$ as well. By Lemma 7.4.5, $X(\xi)$ contains a point x' non-collinear to x and hence $[x,x'] \in \Xi \cup \Theta$. If $[x,x'] \in \Xi$, then it has vertex Y (since $v = \dim(Y)$); if $[x,x'] \in \Theta$ then $Y([x,x'])$ should contain a point non-collinear to x', a contradiction. Hence each point $x \in X$ is collinear to Y and consequently, Θ is empty. ☐

Since we are dealing with mono-symplectic sets, Θ is empty. Moreover, by the previous lemma, all members of Ξ have $Y = \langle Z \rangle$ is their vertex. Hence, when considering the projection ρ of (X,Z,Ξ,Θ) from Y, only $\rho(X)$ and $\rho(\Xi)$ carry information:

Proposition 7.4.17. *Let* (X,Z,Ξ,Θ) *be a mono-symplectic split Veronese set. Then* $(\rho(X),\rho(\Xi))$ *is isomorphic to one of the following point-quadric varieties: a Segre variety* $\mathsf{S}_{1,2}(\mathbb{K})$ *or* $\mathsf{S}_{2,2}(\mathbb{K})$ $(r=1)$*, a line Grassmannian* $\mathsf{G}_{4,1}(\mathbb{K})$ *or* $\mathsf{G}_{5,1}(\mathbb{K})$ $(r=2)$ *or the variety* $\mathsf{E}_{6,1}(\mathbb{K})$ $(r=3)$*.*

Proof. As Θ is empty by assumption, Lemma 7.4.16 implies that each member of Ξ has Y as its vertex. In particular, for each point $x \in X$ holds that $x \perp Y$. By Lemma 7.4.11, the projection $(\rho(X),\rho(\Xi))$ satisfies axioms (S1), (S2) and (S3) as well and as such, it is a *Mazzocca-Melone set of split type* $2r$. The result follows from [39]. ☐

We have shown Theorem 7.3.1(i). Henceforth we assume that (X,Z,Ξ,Θ) is a duo-symplectic split (pre-)Veronese set.

7.4.3 The members of Θ through X-spaces

Since now $|\Theta| \geq 1$ by assumption, one can show that each point of X is contained in at least one member of Θ.

Lemma 7.4.18. *Let x be a point of X. Then there is always a point $z \in Z$ not collinear to x and as such, x is contained in at least one member of Θ.*

Proof. Suppose for a contradiction that x is collinear to each point of Z. Then $x \perp \langle Z \rangle = Y$ and x is not contained in any member of Θ. By (S1), this means that x is contained in some $\xi \in \Xi$, say with vertex V. By Lemma 7.4.16 and $|\Theta| \geq 1$, $V \subsetneq Y$ and hence there is a point $y \in Y \setminus V$, which is collinear to x by assumption. Corollary 7.4.10 implies that xy is contained in a member of Θ together with an X-line of ξ through x. But then $Y(\Theta)$ contains a point non-collinear to x after all. We conclude that there is a point $z \in Z$ non-collinear to x and $[x,z]$ is a member of Θ containing x. □

Lemma 7.4.19. *Let ζ be a member of $\Xi \cup \Theta$ and θ a member of Θ with $\zeta \neq \theta$ such that $\zeta \cap \theta$ contains an X-line L. Then $\zeta \cap \theta$ contains $L^\perp \cap Y(\zeta)$. Moreover, if $\zeta \in \Xi$, then $Y(\zeta) \subseteq \theta$; if $\zeta \in \Theta$, then $\zeta \cap \theta = \langle L, L^\perp \cap Y(\zeta) \rangle = \langle L, L^\perp \cap Y(\theta) \rangle$.*

Proof. Let y be a point of $Y(\zeta) \cap L^\perp$ and take any point $z \in Z(\theta)$ collinear to a unique point x of L. Let x' on $L \setminus \{x\}$ be arbitrary. By Corollary 7.4.8, the lines xy and xz are contained in a singular plane. As such, convexity (cf. Lemma 7.4.5) implies that each X-point on the line xy belongs to $[x',z] = \theta$, and hence $xy \subseteq \theta$. So $L^\perp \cap Y(\zeta)$ belongs to θ indeed.

So, if $\zeta \in \Xi$, then its vertex $Y(\zeta)$ belongs to θ; if $\zeta \in \Theta$, then $\langle L, L^\perp \cap Y(\zeta) \rangle$ is a maximal singular subspace of both ζ and θ, and hence coincides with both $\zeta \cap \theta$ and $\langle L, L^\perp \cap Y(\theta) \rangle$. □

Corollary 7.4.20. *If $\theta_1, \theta_2 \in \Theta$ share an X-plane, they coincide.*

Proof. If $\theta_1 \cap \theta_2$ share an X-plane, they in particular share an X-line L. So if $\theta_1 \neq \theta_2$, then Lemma 7.4.19 implies that $\theta_1 \cap \theta_2 = \langle L, L^\perp \cap Y(\theta_1) \rangle$, which contains no X-plane, a contradiction. □

Recall that for each X-space S, Y_S denotes the subspace $S^\perp \cap Y$ is denoted by Y_S (cf. Definition 7.4.7 and Corollary 7.4.8). We divide the X-lines in three categories:

Definition 7.4.21. An X-line contained in 0, 1 or at least 2 members of Θ is called a 0-line, a 1-line or a 2-line, respectively.

Lemma 7.4.22. *Let L be an X-line and x any of its points. Then, if $\theta \in \Theta$ contains L, then $\theta \cap Y_x$ is a subspace of Y_x which contains Y_L as a hyperplane. Conversely, for each subspace H of Y_x containing Y_L as a hyperplane, there is a unique member of Θ containing H and L. Consequently:*

- (i) *L is a 0-line if and only if $Y_x = Y_L = Y_{x'}$ for each $x' \in L$;*
- (ii) *L is a 1-line if and only if Y_L is a hyperplane of Y_x;*
- (iii) *L is a 2-line if and only if Y_L is strictly contained in a hyperplane of Y_x.*

Proof. Suppose first that $Y_L \subsetneq Y_x$. Take a point $y \in Y_x \setminus Y_L$. By Corollary 7.4.8 and $y \notin Y_L$, the lines xy and L are contained in a member $\theta_{L,y}$ of Θ. By Lemma 7.4.5, Y_L is contained in $\theta_{L,z}$. In fact we even have $\theta_{L,y} \cap Y_x = \langle Y_L, y\rangle$, since in the polar space $\theta_{L,y}$, the X-line L is collinear to precisely a hyperplane of $x^\perp \cap Y(\theta_{L,y})$. Moreover, each $\theta \in \Theta$ containing L arises as $\theta_{L,y}$ for some $y \in Y_x \setminus Y_L$, since in the polar space θ, there is always a point $y \in Y$ which is collinear to x and not to L.

(i) It follows from the previous sentence that there is no member of Θ containing L if and only if $Y_L = Y_x$. Since $x \in L$ was arbitrary, we obtain that also $Y_{x'} = Y_L = Y_x$ for each other point $x' \in L$.

(ii) Now, if Y_L is a hyperplane of Y_x, then for each $y \in Y_x \setminus Y_L$, we have that $\theta_{L,y}$ contains $\langle Y_L, y\rangle = Y_x$. Therefore, $\theta_{L,y}$ is the unique member of Θ containing L.

(iii) If Y_L is strictly contained in a hyperplane of Y_x, then there are points $y_1, y_2 \in Y_x \setminus Y_L$ such that $\langle Y_L, y_1\rangle \neq \langle Y_L, y_2\rangle$ and hence $\theta_{L,y_1} \neq \theta_{L,y_2}$. Clearly, each of these members of Θ contains Y_L. □

Lemma 7.4.23. *Let ζ_1 and ζ_2 be two members of $\Xi \cup \Theta$, and put $S = \zeta_1 \cap \zeta_2$. If two points x_1, x_2 in $X(\zeta_1)\setminus S$ and $X(\zeta_2)\setminus S$, respectively, are such that $H_1 := x_1^\perp \cap S \neq x_2^\perp \cap S =: H_2$, then they are not collinear.*

Proof. Suppose for a contradiction that $x_1 \perp x_2$ and $H_1 \neq H_2$. Let h_1 be a point of $H_1 \setminus H_2$. Then $x_1 \in [x_2, h_1] = \zeta_2$ by Lemma 7.4.5, a contradiction. □

Lemma 7.4.24. *Suppose that there exist θ_1 and θ_2 in Θ, with respective vertices V_1 and V_2, such that $\theta_1 \cap \theta_2$ is a maximal singular subspace of both θ_1 and θ_2 of the form $\langle x, H\rangle$, with $x \in X$ and $H \subseteq Y$. For $i \in \{1,2\}$, let $L_i \in \theta_i$ be an X-line through x and put $H_i := L_i^\perp \cap H$, $i = 1,2$. Then:*

- (i) *if $H_1 \neq H_2$, then L_1 and L_2 are contained in a member of Ξ (and there are always L_1 and L_2 such that this occurs);*
- (ii) *if $H_1 = H_2$, then L_1 and L_2 are collinear (and there are L_1 and L_2 such that this occurs if and only if $V_1 = V_2$).*

Furthermore, $v = v' + r' - 2$.

Proof. (i) Suppose that $H_1 \neq H_2$. It follows from Lemma 7.4.23 that the L_1 and L_2 are not collinear. Suppose for a contradiction that L_1 and L_2 are contained in some $\theta \in \Theta$. By Lemma 7.4.22(iii), H_1 and H_2 belong to θ. But then, since L_1 and H_2 are not collinear, $\theta = \theta_1$, contradicting $L_2 \subsetneq \theta_1$. We conclude that L_1 and L_2 determine a member of Ξ, whose vertex is precisely $H_1 \cap H_2$ by Lemmas 7.4.5 and 7.4.19. As such, provided that such a pair of X-lines L_1 and L_2 exists, we have $v = \dim(H) - 2 = ((v' + r' + 1) - 1) - 2$. We claim such a pair exists. Indeed, if all pairs L_1, L_2 are collinear to the same hyperplane of H, then this hyperplane coincides with V_1 and V_2, whereas these vertices are however strictly contained in hyperplanes of H, for $\dim(H) = v' + r' \geq v' + 2$. So $v = v' + r' - 2$ indeed.

(ii) Suppose that $H_1 = H_2$ and denote this hyperplane of H by T. Let θ be any member of Θ containing $\langle T, x\rangle$. Then θ contains precisely two maximal singular subspaces through the submaximal singular subspace $\langle T,x\rangle$, only one of which contains X-lines through X. Suppose for a contradiction that L_1 and L_2 are not collinear. Then they are contained in a member ζ of $\Xi\cup\Theta$, which contains T (and hence $\langle T,x\rangle$) by Lemma 7.4.19. If $\zeta\in\Xi$, then T is its vertex, which is not possible since $\dim(T)=\dim(H)-1$, contradicting the earlier obtained value of v. So $\zeta\in\Theta$, but then $\langle L_1,T\rangle$ and $\langle L_2,T\rangle$ are two maximal singular subspaces of ζ through $\langle T,x\rangle$ containing X-lines through x, contradicting what we have just deduced. Therefore, L_1 and L_2 are collinear.

Finally, suppose that the respective vertices V_1 and V_2 of θ_1 and θ_2 do not coincide. By definition, $Z(\theta_i)$ is the disjoint union of V_i and some r'-dimensional subspace, say R_i ($i=1,2$). So if $V_1\neq V_2$, then $V_1\subseteq R_2$ and $V_2\subseteq R_1$; in particular, $V_1\cap V_2=\emptyset$. Then for each X-line L_1 in θ_1 through x, L_1 is not collinear to $V_2\subseteq R_1$, whereas for each X-line L_2 in θ_2 through x, L_2 is collinear to V_2. So if $V_1\neq V_2$, we cannot find a pair L_1,L_2 with $H_1=H_2$. On the other hand, if $V_1=V_2$, then for each X-line L_1 in θ_1 through x, there is a unique maximal singular subspace N_2 in θ_2 through $\langle x,H_1\rangle$ containing an X-line L_2 through x, for which clearly $H_2=H_1$. □

7.5 The dual Segre varieties

Throughout this section, we suppose that (X,Z,Ξ,Θ) is a duo-symplectic split pre-Veronese set with $r=1$ such that, through each X-point, there are at least two members of Θ.

Our first goal is to show that there is at least one 1-line (which takes quite long).

7.5.1 The subcase where there is no 1-line

During this subsection, we assume that (X,Z,Ξ,Θ) possesses no 1-lines, and show that this eventually leads to a contradiction.

Claim: All members of Θ have the same vertex.

Lemma 7.5.1. *There is at least one 2-line through each X-point. Moreover, $r'=2$ and any two members of Θ sharing a 2-line have the same vertex.*

Proof. By assumption, there is at least one member $\theta\in\Theta$ containing x. As all X-lines in θ are 2-lines, this already shows the first assertion.

Let L be any 2-line and take two members $\theta_1,\theta_2\in\Theta$ containing L. Recall that $\theta_1\cap\theta_2=\langle L,Y_L\rangle$ by Lemma 7.4.22. Take X-planes π_i through L in θ_i, for $i=1,2$. Then π_1 and π_2 are necessarily collinear, for they can neither be contained in a member of Ξ since $r=1$, nor in a member of Θ for this would violate Corollary 7.4.20. Since $\pi_1^\perp\cap\theta_2$ is a singular subspace by Lemma 7.4.5, this implies that there cannot be two non-collinear X-planes in θ_2 through L. Now if $r'>2$, such X-planes would exist and hence $r'=2$ indeed.

The fact that π_1 and π_2 are collinear also implies that $\pi_1^\perp \cap (\theta_1 \cap \theta_2) = \pi_2^\perp \cap (\theta_1 \cap \theta_2)$ (cf. Lemma 7.4.23). Note that, by definition, the points of Z in $Y(\theta_i)$, $i = 1,2$, are precisely the points of V_i and of some r'-dimensional subspace, say R_i, complementary to V_i in $Y(\theta_i)$. Since $r' = 2$, R_i contains a unique point r_i collinear to L. So $Y_L = \langle V_i, r_i \rangle$. If $V_1 \neq V_2$, then necessarily $V_1 = r_2$ and $V_2 = r_1$. In particular, Y_L is the line $\langle V_1, V_2 \rangle$. So, an X-plane through L in θ_i is collinear to precisely the point V_i of Y_L (since $\langle L, V_1, V_2 \rangle = \langle L, Y_L \rangle$ is a maximal singular subspace of θ_i), contradicting the first assertion of this paragraph. We conclude that $V_1 = V_2$. □

Remark 7.5.2. We already mentioned that the reason that we postpone using Axiom (S3) is that, a residue of a split Veronese set with $r > 1$ not necessarily satisfies (S3). Later on, we will show that a residue of a split pre-Veronese set can never have 2-lines (essentially this is because two members of Θ can never share an X-plane (cf. Corollary 7.4.20)). We will come back to this later, but say this in advance to justify our use of Axiom (S3) later on in this subcase, as we have just shown that 2-lines occur.

The above lemma says that all members of Θ sharing an X-line have the same vertex, and we want to extend this to all members of Θ. This requires some work.

Lemma 7.5.3. *Suppose that there are two members of Θ with distinct vertices. Then, for each $x \in X$, $Y_x \cap Z$ is the disjoint union of two v'-spaces V_1 and V_2 with $Y_x = \langle V_1, V_2 \rangle$ such that each $\theta \in \Theta$ through x either has V_1 or V_2 as its vertex (and both occur). Moreover, $v = 1$ and $v' \geq 2$ in this case.*

Proof. Take $x \in X$ arbitrary and take any $\theta_1 \in \Theta$ through x, say with vertex V_1. Clearly, each 0-line L through x is collinear to V since, by Lemma 7.4.22(i), $Y_x = Y_L$, and hence $V \subseteq Y_L$. Suppose for a contradiction that each 2-line through x is collinear to V too. Then all points of X are collinear to V: if $x' \in X$ is a point not collinear to x, then $[x, x'] \in \Xi \cup \Theta$ has V in its vertex (as two non-collinear X-lines through x are collinear to V), and if x and x' are on a singular line with a unique point in Y then $V \subseteq Y_x = Y_{x'}$. As such, all members of Θ would have the same vertex, contradicting our assumption.

Hence there is a 2-line M_2 through x with M_2 not collinear to V_1. Take any $\theta_2 \in \Theta$ containing M_2 and denote its vertex by V_2. Clearly, $V_1 \neq V_2$. By Lemma 7.5.1, $\theta_1 \cap \theta_2$ does not contain an X-line, nor is there a $\theta \in \Theta$ meeting both θ_1 and θ_2 in respective X-lines, for in both cases, we would obtain $V_1 = V_2$, a contradiction. As such, $\theta_1 \cap \theta_2$ equals $\langle x, H \rangle$ for a possibly empty subspace $H \subseteq Y_x$.

Claim 1: For each pair of X-lines L_1 and L_2 through x in θ_1 and θ_2 respectively, L_1 and L_2 are contained in a member $\xi(L_1, L_2)$ of Ξ with vertex $V(L_1, L_2) = Y_{L_1} \cap Y_{L_2}$.
The previous paragraph implies that no member of Θ contains $L_1 \cup L_2$. So suppose for a contradiction that L_1 and L_2 are collinear. Then firstly, the plane $\langle L_1, L_2 \rangle$ is an X-plane, since all points of Y_{L_1} are contained in θ_1 (cf. Lemma 7.4.22(iii)) and $L_2 \nsubseteq \theta_1$; secondly, for at least one plane π in θ_1 through L_1, the planes $\langle L_1, L_2 \rangle$ and π are not collinear and as such, they are contained in some member of Θ, contradicting the beginning of this paragraph. Hence L_1 and L_2' are contained in a member $\xi(L_1, L_2) \in \Xi$ indeed, say with vertex $V(L_1, L_2)$.

Clearly, $V(L_1,L_2)=Y_{L_1}\cap Y_{L_2}$. Lemma 7.4.19 implies that $V(L_1,L_2)$ belongs to H. This shows the claim. Observe that this in particular means that H is non-empty, as $v\geq 0$.

By definition, $Y(\theta_1)\cap Z$ is the disjoint union of V_1 and some plane R_1 (recall $r'=2$) and $\langle V_1,R_1\rangle = Y(\theta_1)$; likewise, $Y(\theta_2)\cap Z$ is the disjoint union of V_2 and a plane R_2 and $\langle V_2,R_2\rangle = Y(\theta_2)$. Put $H_V:=V_1\cap H$ and $H_R:=R_1\cap H$. As $H=Y(\theta_1)\cap Y(\theta_2)$, it also contains two disjoint subspaces in Z, and so it follows that $H\cap Z=H_V\cup H_R$ and $\langle H_R,H_Z\rangle\subseteq H$. There are two options: either $(H_V,H_R)=(V_2\cap H,R_2\cap H)$ or $(H_V,H_R)=(R_2\cap H,V_2\cap H)$. Note moreover that each r'-space in $Y(\theta_i)$ complementary to V_i, whether it belongs to Z or not, plays the same role as R_i, $i=1,2$.

Case 1: $(H_V,H_R)=(V_2\cap H,R_2\cap H)$.
Suppose first that $M_2\perp H$. As by assumption, M_2 is not collinear to V_1, there is a point $z\in V_1$ not collinear to M_2 (necessarily, $z\notin H$ then). The lines zx and M_2 are contained in a member θ_2' of Θ, which also contains $\langle x,H\rangle$ by Lemma 7.4.5. As θ_2' contains M_2, it also has vertex V_2 (cf. Lemma 7.5.1), and hence the fact that $z\in Z(\theta_2')\setminus V_2$ means that $z\in R_2'$ (with notation as above). However, $\langle z,H_V\rangle\subseteq V_1\subseteq Z$, whereas inside θ_2', lines joining points of R_2' and V_2 are not entirely contained in Z. We conclude that H_V is empty.

The above implies that $H_V\subsetneq H$. Indeed, if $H=H_V$, then $M_2\perp H$ as $H_V\subseteq V_2$, from which we concluded above that H_V is empty, and hence H is also empty, contradicting the observation at the end of Claim 1. Take a subspace R in H complementary to H_V; which is non-empty by the foregoing. As $H\subseteq x^\perp$ and $r'=2$, R is either a point or a line. Either way, let $p\in R$ be a point. Then there is an X-line L_1 through x in θ_1 with $L_1^\perp\cap H=\langle H_V,p\rangle$, and there are X-lines L_2 and L_2' through x in θ_2 such that $L_2^\perp\cap H=\langle H_V,p\rangle$ and $p\notin L_2'^\perp$. Using Claim 1, we obtain that $V(L_1,L_2)=\langle H_V,p\rangle$ whereas $V(L_1,L_2')=H_V$. As the dimensions of the latter vertices should be equal (to v), this is a contradiction. We ruled out this case.

Case 2: $(H_V,H_R)=(R_2\cap H,V_2\cap H)$.
Since $r'\leq 2$ and $R_i\subseteq x^\perp$, we get that $\dim(H_V)$ and $\dim(H_R)$ are at most 1. Suppose first that both H_R and H_V are empty (so $H\cap Z$ is empty). As H is non-empty and complimentary to V_i, we have $0\leq\dim(H)\leq 1$, and then we take any X-line L_1 in θ_1 through x such that $L_1^\perp\cap H$ is precisely a point $p\in H$, and (as in the previous case) there are X-lines L_2 and L_2' in θ_2 through x such that $p\in L_2^\perp\cap H$ and $p\notin L_2^\perp\cap H$, which again yields $\dim(V(L_1,L_2))\neq\dim(V(L_1,L_2'))$, a contradiction.

So we may suppose that H_R contains at least a point z_R. Suppose for a contradiction that $R_2\cap x^\perp$ contains a point z outside H_V (i.e., outside θ_1). Let L_1 be an X-line in θ_1 through x such that $L_1\perp z_R$. As $L_1^\perp\cap Y\subseteq\theta_1$, it follows that L_1 and z are not collinear. Consequently, the lines zx and L_1 are contained in a member $\theta_1'\in\Theta$, which also has vertex V_1 by Lemma 7.5.1. Since $z\in Z(\theta_1')\setminus V_1$, we have $z\in R_1'$ (with notation as before). Since $L_1\perp xz_R\perp xz$, we obtain $xz_R\in\theta_1'$ and hence, as $z_R\notin V_1$, we get that $z_R\in R_1'$ too. As such, the line $zz_R\subseteq R_1'\subseteq Z$, whereas we see in θ_2 that the line zz_R intersects Z in $\{z,z_R\}$ only, a contradiction. Hence $H_V=R_2\cap x^\perp$. In particular, H_V contains at least a point and hence, reversing the roles of θ_1 and θ_2 in the foregoing, we obtain that $H_R=R_1\cap x^\perp$.

Conclusion. First of all, we obtained that $V_1\cap V_2=\emptyset$. Secondly, we have that $H=\langle x^\perp\cap R_1,x^\perp\cap R_2\rangle$ (since $H\subseteq x^\perp$, H cannot be bigger than this). For X-lines L_i in θ_i through x, let z_i be the unique point of R_i collinear to L_i. As $L_i^\perp \subseteq Y(\theta_i)$, we have $z_1 = L_1^\perp \cap V_2$ and $z_2 = L_2^\perp \cap V_1$, so $Y_{L_i} = \langle V_i, z_i \rangle$. Moreover, the vertex of $\xi(L_1, L_2)$ is $Y_{L_1} \cap Y_{L_2} = z_1 z_2$ (cf. Claim 1) and contains z_1 and z_2 as its only two points in Z. In particular we have $v = 1$.

Claim 2: Y_x is generated by V_1 and V_2; and $Y_x \cap Z = V_1 \cup V_2$. Moreover, each $\theta \in \Theta$ containing x has either V_1 or V_2 as its vertex.
Let V' be any $(v'+1)$-space in Y_x through V_1. Let L_1 be any X-line through x in θ_1. Suppose that $V' \neq Y_{L_1}$. Take any point $y \in V' \setminus V_1$. Then L_1 and xy are not collinear and hence determine a member $\theta_1' \in \Theta_1$, having V_1 as vertex. Inside θ_1', there is an X-line L_1' through x collinear to $\langle V_1, y \rangle = V'$, and hence $Y_{L_1'} = V'$. As θ_1 and θ_1' play the same role with respect to L_2 (they both contain x and have V_1 as their vertex), the above conclusion implies that $Y_{L_1'}$ has a (unique) point in common with V_2. Since $y \in Y_x \setminus Y_{L_1}$ was arbitrary, we obtain $Y_x = \langle V_1, V_2 \rangle$. Likewise, each $(v'+1)$-space in Y_x through V_2 occurs as Y_{L_2} for some X-line L_2 through x (we can indeed switch the roles of θ_1 and θ_2 as θ_1 contains an X-line through x not collinear to V_2). So, let $z_1 \in V_2$ and $z_2 \in V_1$ be arbitrary and take X-lines L_1 and L_2 through x such that $Y_{L_i} = \langle V_i, z_i \rangle$, $i = 1, 2$. By Claim 1, $z_1 z_2$ is the vertex of ξ_{L_1, L_2} and hence $z_1 z_2 \cap Z = \{z_1, z_2\}$. As $z_1 \in V_2$ and $z_2 \in V_1$ were arbitrary, we obtain $Y_x \cap Z = V_1 \cup V_2$. Lastly, take any θ in Θ_x and denote its vertex by V. As $V \subseteq Y_x \cap Z$ and $v' \geq 1$, either $V = V_1$ or $V = V_2$. This shows the claim. Note that, since Y_L is less than a hyperplane of Y_x, we get $v' \geq 2$. □

Lemma 7.5.4. *All members of Θ have the same vertex and hence there is a v'-space V collinear to all points of X.*

Proof. Let $x \in X$ be arbitrary. If there are two members of Θ with distinct vertices, then Lemma 7.5.3 implies that there are members $\theta_1, \theta_2 \in \Theta$ through x such that their respective vertices V_1 and V_2 are disjoint and generate Y_x. Let $R_i \in Y(\theta_i)$ be complementary to V_i. As moreover $Y_x \cap Z = V_1 \cup V_2$, we see that $x^\perp \cap R_i \subseteq V_j$ for $\{i, j\} = \{1, 2\}$. Hence $\theta_1 \cap \theta_2 = \langle x, x^\perp \cap R_1, x^\perp \cap R_2 \rangle$ and therefore has dimension 4 (recall $r' = 2$).

By Remark 7.5.2 and Lemma 7.5.1, we may use (S3), which implies that there are $\xi_1, \xi_2 \in \Xi$ through x with $T_x = \langle T_x(\xi_1), T_x(\xi_2) \rangle$. Since $r = 1$ and $v = 1$ (cf. Lemma 7.5.3), we get $\dim(T_x) \leq 8$. On the other hand, we have $\langle T_x(\theta_1), T_x(\theta_2) \rangle \subseteq T_x$, and by the above and $v' \geq 2$ (cf. Lemma 7.5.3), we get $\dim(T_x) \geq 10$, a contradiction. We conclude that all members of Θ have the same vertex, say V, and as each X-point is contained in a member of Θ by Lemma 7.4.18, all X-points are collinear to V. □

By Lemmas 7.5.4 and Lemma 7.4.11, we can project (X, Z, Ξ, Θ) from V and obtain a split (pre-)Veronese set with parameters $(1, v - v' - 1, 2, -1)$; with which we will continue to work without changing our notation, i.e., we just assume that (X, Z, Ξ, Θ) has $v' = -1$. An obvious consequence of this is the following.

Corollary 7.5.5. *Each 2-line L is collinear to a unique point $y_L \in Z$.*

Proof. If $\theta_L \in \Theta$ contains L, then we know that $L^\perp \cap Y = L^\perp \cap Y(\theta_L)$, and the latter is precisely a point, which belongs to Z by definition. □

The connection between 2-lines L and the points y_L.

We deepen the connections between the 2-lines L and the points y_L and show that this leads to non-compatible structures.

Lemma 7.5.6. *Let L and M be 2-lines having a point $x \in X$ in common. Then L and M are contained in a member of Θ; moreover, $y_L = y_M$ if and only if $\langle L, M\rangle$ is a singular plane containing a unique point of Y.*

Proof. Suppose first that $\langle L, M\rangle$ is a singular plane π. We claim that there is a member of Θ containing π (and hence both L and M). If π contains a point of Y (which necessarily coincides with both y_L and y_M), then each member of Θ containing L also contains y_L and therefore it contains π, showing the claim in this case. So let π be an X-plane. Take a member $\theta_L \in \Theta$ containing $\langle L, y_L\rangle$. If π were collinear to y_L, then π is not collinear to the unique X-plane π' in θ_L through L, implying that π and π' determine an element of $\Xi \cup \Theta$, which however is excluded by $r = 1$ and by Corollary 7.4.20, respectively. So π is not collinear to y_L (in particular, $y_L \neq y_M$) and hence π and y_L determine a member of Θ containing π. This shows the claim.

Next, suppose for a contradiction that L and M are contained in some $\xi \in \Xi$. Then the vertex of ξ is contained in $y_L \cap y_M$ and since $v \geq 0$, we have $y_L = y_M =: y$. Since L and M are 2-lines, we know by Lemma 7.4.22 that Y_x contains a point $y' \notin Y_L$. Let θ_L and θ_M be the members of Θ determined by L and xy' and by M and xy', respectively. Then $\theta_L \cap \theta_M = \langle x, y, y'\rangle$ and since $r' = 2$, we can apply Lemma 7.4.24, which says that L and M need to be collinear as $H_L = y = H_M$, a contradiction.

The above contradiction shows that L and M are indeed contained in some $\theta \in \Theta$. In θ, it is clear that $y_L = y_M$ if and only if $\langle L, M\rangle$ is a singular plane containing $y_L = y_M$. The lemma follows. □

Every degenerate hyperbolic quadric contains two natural systems of generators (maximal singular subspaces). Each such system is called a *regulus*.

Lemma 7.5.7. *For each $\xi \in \Xi$, the set of 2-lines contained in ξ are all X-lines contained in some member of a fixed regulus. Moreover, $v = 0$.*

Proof. Take $x \in X(\xi)$ arbitrary and let L_1 and L_2 be X-lines through x belonging to different reguli of $X(\xi)$. By Lemma 7.5.6, L_1 and L_2 cannot be both 2-lines. Suppose for a contradiction that they are both 0-lines. Then $Y(\xi) = L_1^\perp \cap L_2^\perp \cap Y = Y_x$ by Lemma 7.4.22(i). By Lemma 7.5.1, there is a 2-line L through x. Then L is collinear to at most one of L_1 and L_2, say L and L_1 are not collinear. Since 0-lines are contained in no member of Θ, we have that L and L_1 belong to a member $\xi_1 \in \Xi$, which then has y_L as its vertex. Since $0 = \dim(y_L) < \dim(Y_x)$ as L is a 2-line, this is a contradiction. Hence two X-lines of $X(\xi)$ belonging to members of different reguli, have different types, from which the first assertion follows. If L is one of the 2-lines of $X(\xi)$, we get that y_L is the vertex of ξ and as such $v = 0$. □

Corollary 7.5.8. *The set Z coincides with Y.*

Proof. Let z_1 and z_2 be any two points of Z. By (S1), there is a $\zeta \in \Xi \cup \Theta$ through them, and since $v = 0$, $\zeta \in \Theta$. Now, in ζ, the line $z_1 z_2$ belongs to Z as $v' = -1$. It follows that $Z = \langle Z \rangle = Y$. □

We will now make use of the maps ρ and χ, as defined in Definitions 7.4.12 and 7.4.13.

Lemma 7.5.9. *Let L be a 2-line and suppose that M is an X-line with $\rho(L) = \rho(M)$. Then M is a 2-line too and $y_L = y_M$.*

Proof. For each point x of L there is, as $\rho(L) = \rho(M)$, a unique point $\overline{x}$ of M with $\rho(x) = \rho(\overline{x})$. If $x = \overline{x}$ for some $x \in L$, then either $L = M$ or $\langle M, L \rangle$ coincides with the plane $\langle L, y_L \rangle$. In both cases, the assertion follows immediately.

So suppose $x \neq \overline{x}$ for all $x \in L$. It then follows form Lemma 7.4.14 that $x\overline{x}$ is a singular line with a unique point y_x in Y. Put $K := \{y_x \mid x \in L\}$. If L and M would be collinear, then for each $x \in L$, we get $y_x \perp L$, i.e., $y_x = y_L$. This however implies that $\langle M, L \rangle = \langle L, y_L \rangle$ again, contradicting the fact that M and L are disjoint. So L and M are not collinear. Let x_1, x_2 be two points of L. Then x_1 and $\overline{x}_2$ are not collinear and therefore they uniquely determine a $\zeta \in \Xi \cup \Theta$. Since $x_2, \overline{x}_1 \in x_1^\perp \cap x_2'^\perp$, ζ contains $K \cup L \cup M$. Hence, as $y_{x_1} \neq y_{x_2}$ (otherwise $x_1 \perp \overline{x}_2$) and $v = 0$, we get that $\zeta \in \Theta$. This means that M is a 2-line too. In θ, it follows that K is a line (inside $Y(\theta)$) and that the point y_L, being collinear to both L and K, is also collinear to M, so $y_L = y_M$ indeed. This shows the assertion. □

By the previous lemma, it makes sense to keep speaking about 0-lines and 2-lines in $\rho(X)$ (note though that we do not claim that each line in $\rho(X)$ is the image of an X-line, and we will not need this), and of the unique point y_L of Y collinear to such a 2-line L in $\rho(X)$.

Lemma 7.5.10. *Let L and M be 2-lines such that $\rho(L)$ and $\rho(M)$ are distinct lines through a point $p \in \rho(X)$. Then $\langle \rho(L), \rho(M) \rangle$ is a singular plane π of $\rho(X)$ each line of which is a 2-line. Furthermore, $y_L \neq y_M$, and the map taking a 2-line K of $\rho(X)$ through p inside π to the point y_K gives a bijection between this line set and the set of points on $y_L y_M$.*

Proof. Since $p \in \rho(L) \cap \rho(M)$, there are points p_L and p_M on L and M, respectively, with $\rho(p_L) = \rho(p_M) = p$. We claim that we may assume that $p_L = p_M$. So suppose that they are distinct. Then the line $p_L p_M$ contains a point $y \in Y$ by Lemma 7.5.9. Let p_L' be a point on $L \setminus \{p_L\}$. Now either $y = y_L$ or y is not collinear to L, yet in both cases there is a member $\theta \in \Theta$ containing the lines L and $\langle p_L, p_M \rangle$. In θ, it is clear that there is a line L' through p_M with $\rho(L) = \rho(L')$. Replacing L by L', this shows the claim.

Let $x \in X$ denote the intersection point of L and M. By Lemma 7.5.6, L and M are contained in some $\theta \in \Theta$. As such, the lines $\rho(L)$ and $\rho(M)$ span the singular plane $\rho(\theta)$ and each line in this plane is reached by some X-line in θ. This shows the first assertion. By the same lemma, $y_M \neq y_L$, as otherwise $\rho(L) = \rho(M)$. In the polar space $XY(\theta)$, collinearity gives a bijective correspondence between the X-lines K through x and the points on y_K on y_Ly_M, and since all X-lines in $\rho^{-1}(\rho(K))$ correspond to the same point y_K by Lemma 7.5.9, this gives us the required bijective correspondence. $\square$

Take a connected component Π of $\rho(X)$ with respect to 2-lines intersecting each other in points and let Y_Π be the subset of Y consisting of all points of $\{y_L \mid \rho(L)$ a line of $\Pi\}$.

Lemma 7.5.11. *A connected component Π of $\rho(X)$ with respect to 2-lines intersecting each other in points is a singular subspace of $\rho(X)$, whose lines are in bijective correspondence to the points of Y_Π.*

Proof. Let p be any point of Π. We claim that all other points of Π are on a 2-line with p. If not, then there are points $p', p'' \in \Pi$ such that pp' and $p'p''$ are 2-lines, and pp'' is not. But then, looking in p', it follows from Lemma 7.5.10 that $\langle p, p', p''\rangle$ is a singular plane all of whose lines are 2-lines and hence p and p' are on a 2-line after all. So Π is indeed a singular subspace of $\rho(X)$.

Suppose that there are two distinct 2-lines $\rho(L)$ and $\rho(M)$ in Π with $y_L = y_M$. It follows from Lemma 7.5.10 that $\rho(L)$ and $\rho(M)$ do not share a point. Let $\rho(K)$ be a 2-line in Π joining a point of $\rho(L)$ and a point of $\rho(M)$. Then $y_L = y_M$ is collinear to two distinct points of K (those corresponding to the intersection points $\rho(K)\cap\rho(L)$ and $\rho(K)\cap\rho(M)$) and hence $y_K = y_L = y_M$, contradicting Lemma 7.5.10. As by definition, each line of Π corresponds to a unique point of Y_Π, this shows the lemma. $\square$

Lemma 7.5.12. *For each two points y_L and y_M of Y_Π, the lines $\rho(L)$ and $\rho(M)$ of Π have a point in common.*

Proof. We claim that there is a point $x \in X$ with $\rho(x) \in \Pi$ such that x is collinear to y_Ly_M. Since $Y = Z$ (cf. Lemma 7.5.8) and $v = 0$, there is a $\theta \in \Theta$ through y_L and y_M. In θ, there is a point $x \in X$ collinear to the line y_Ly_M. If $\rho(x) \in \Pi$ we are good, so suppose it is not. Let $x' \in X$ be any point with $\rho(x') \in \Pi$. Suppose first that x and x' are collinear. As $\rho(x) \neq \rho(x')$, the line xx' is an X-line (cf. Lemma 7.4.14). From $\rho(x) \notin \Pi$, it follows that xx' is a 0-line, and hence $x'^{\perp} \cap Y = x^{\perp} \cap Y$ (cf. Lemma 7.4.22(i)). In particular, x' is also collinear to y_Ly_M and hence is a valid choice. Secondly, suppose x' and x are not collinear. Then they are contained in a member ζ of Ξ or of Θ. In the first case, Lemma 7.5.7 implies that there is a 2-line through x' in ζ meeting a 0-line through x, say in a point x''. Then x'' is a good choice: it is collinear to y_Ly_M as it is on a 0-line with x, and it is contained in Π since it is on a 2-line with x'. If $\zeta \in \Theta$, then since x and x' are joined by two intersecting 2-lines in θ, we obtain $x \in \Pi$, a contradiction. The claim follows.

So, let $x \in X$ be a point collinear to y_Ly_M with $p := \rho(x) \in \Pi$. By Lemma 7.5.11, $\rho(L)$ and $\rho(M)$ are the unique respective lines in Π collinear to y_L and y_M. Let p' be any point of $\rho(L)$, distinct from p if p were on it. Then p and p' are on a 2-line $\rho(K)$ of Π by Lemma 7.5.11. Since both p and p' are collinear to y_L, so is K. Consequently, $\rho(K) = \rho(L)$ since this was the unique line in Π collinear to y_L, so $p \in \rho(L)$. Likewise, $\rho(M)$ contains p and hence $\rho(L)$ and $\rho(M)$ intersect in p. $\square$

Conclusion

Finally, we reach a contradiction.

Proposition 7.5.13. *There is at least one* 1*-line.*

Proof. Let L be any 2-line and let θ_1, θ_2 be two elements of Θ containing L. Let π_1 and π_2 be the unique X-planes through L in θ_1 and θ_2, respectively. Then, as noted before, π_1 and π_2 span a singular 3-space S (as they cannot be contained in a member of Ξ nor of Θ). If $S \cap Y$ were non-empty, then $S \cap Y$ is a point y (since the planes π_1 and π_2 are X-planes). But then $y \perp L$ and hence $y = y_L$. Consequently $\pi_2 \subseteq \langle \pi_1, y_L \rangle \subseteq \theta_1$, a contradiction. We conclude that $\rho(S)$ is 3-dimensional. Moreover, since each line K in S is contained in a plane together with two 2-lines M_1 and M_2 of $\pi_1 \cup \pi_2$, Lemma 7.5.6 implies that K belongs to a member of Θ containing M_1 and M_2 and hence is a 2-line as well. This however implies that the connected component Π of $\rho(X)$ containing $\rho(S)$ contains a pair of disjoint 2-lines, contradicting Lemma 7.5.12. We conclude that our assumption that there are no 1-lines must be false. □

7.5.2 The subcase where there is a 1-line

By Proposition 7.5.13, we know that (X, Z, Ξ, Θ) has at least one 1-line.

Proposition 7.5.14. *All X-lines are* 1*-lines.*

We need a series of lemmas before we can show this.

Lemma 7.5.15. *For each point $x \in X$ on a* 1*-line L, we have that $Y_x = x^\perp \cap Y(\theta)$ for each $\theta \in \Theta$ containing x.*

Proof. Let θ_L be the unique member of Θ containing L. By Lemma 7.4.22(ii), $Y_x = x^\perp \cap Y(\theta_L)$. In particular, $\dim(Y_x) = r' + v'$ and hence, for each $\theta \in \Theta$ containing x we obtain that $x^\perp \cap Y(\theta)$, which also has dimension $r' + v'$, coincides with Y_x. □

The above lemma allows us to make use of Lemma 7.4.24, which already tells us that $v = v' + r' - 2$.

Lemma 7.5.16. *Let x be a point contained in a* 1*-line. Then there are exactly two members θ_1^x and θ_2^x of Θ containing x. Their respective vertices V_1^x and V_2^x are disjoint and generate Y_x. Moreover, $v' = r' - 1$.*

Proof. Recall that we assume that there are at least two members of Θ through x, say θ_1^x and θ_2^x. By Lemma 7.5.15, both contain the maximal singular subspace $\langle x, Y_x \rangle$ and hence $\theta_1^x \cap \theta_2^x = \langle x, Y_x \rangle$. Take X-lines L_1 and L_2 through x in θ_1^x and θ_2^x, respectively. By Lemma 7.4.24, L_1 and L_2 are collinear if and only if $H_1 := L_1^\perp \cap Y_x = L_2^\perp \cap Y_x =: H_2$. We claim that this is not possible. Suppose for a contradiction that L_1 and L_2 span a singular plane π. Then L_2 is collinear to the maximal singular subspace $\langle L_1, H_1 \rangle$ of θ_1^x, and hence for any X-plane π_1 through L_1 in θ_1^x, L_2 is not collinear to π_1. As such, π and π_1 are contained in some $\zeta \in \Xi \cup \Theta$. However, if $\zeta \in \Xi$ this contradicts $r = 1$ and if $\zeta \in \Theta$ this contradicts Corollary 7.4.20, showing the claim. We conclude from Lemma 7.4.24 that the respective vertices V_1^x and V_2^x of θ_1^x and θ_2^x do not coincide. Since the points of Z in θ_i^x, $i = 1,2$, are precisely those of V_i^x and of some disjoint r'-dimensional subspace R_i^x (such that $\langle R_i^x, V_i^x\rangle = Y(\theta_i^x)$), we have that $V_1^x = R_2^x \cap x^\perp$ and $V_2^x = R_1^x \cap x^\perp$. So in particular, $v' = r' - 1$ and V_1^x and V_2^x are disjoint subspaces spanning Y_x.

So, Y_x has two v'-dimensional subspaces in Z and by the above, each of them is the vertex of precisely one member of Θ through x. This implies that θ_1^x and θ_2^x are the only members of Θ containing x. □

Notation For each X-point x on a 1-line, we keep denoting the unique two members of Θ through x by θ_1^x and θ_2^x, and their respective vertices by V_1^x and V_2^x.

Corollary 7.5.17. *Let x be an X-point contained in a 1-line, and let L_1 and L_2 be X-lines through x in θ_1^x and θ_2^x, respectively. Then L_1 and L_2 are non-collinear and determine a unique member of Ξ. Furthermore, $v = 2v' - 1 = 2r' - 3$.*

Proof. By Lemma 7.5.16, the respective vertices V_1^x and V_2^x of θ_1^x and θ_2^x do not coincide. So according to Lemma 7.4.24, the lines L_1 and L_2 are not collinear and hence contained in a unique member of Ξ. Secondly, we have $v = v' + r' - 2$ and $v' = r' - 1$. □

Lemma 7.5.18. *Let x be an X-point contained in a 1-line. Then each X-line through x is contained in $\theta_1^x \cup \theta_2^x$.*

Proof. Suppose for a contradiction that K is an X-line through x not contained in $\theta_1^x \cup \theta_2^x$. Then K is a 0-line, for otherwise an element of Θ through K would coincide with one of θ_1^x, θ_2^x. So by Lemma 7.4.22(i), K is collinear to Y_x. Now take any X-line L through x in θ_1^x. Then L and K are not collinear, because $K^\perp \cap \theta_1^x$ is the maximal singular subspace $\langle x, Y_x\rangle \not\supseteq L$. As such, L and K are contained in some $\xi \in \Xi$, which then has Y_L as its vertex. But then $v = r' + v' - 1$, whereas we deduced before that $v = r' + v' - 2$ (cf. Lemma 7.5.16). This contradiction shows the lemma. □

Proof of Proposition 7.5.18. Take any point $x \in X$ contained on a 1-line L. Firstly, let M be an X-line through x. Then M is contained in θ_1^x or θ_2^x and, in there, it is clear that Y_M is a hyperplane of Y_x, so it follows form Lemma 7.4.22 that M is a 1-line indeed. Since x was just any point on a 1-line, we obtain by connectivity (via X-lines) that all X-lines are 1-lines. □

<u>The structure of Y and of Y_x for $x \in X$</u>

We now have, for any $x \in X$, that there are two members of Θ containing x, again denoted by θ_1^x and θ_2^x, and their respective vertices by V_1^x and V_2^x. For $i = 1,2$, we denote by R_i^x the unique r'-space in $Z(\theta_i^x)$ disjoint from the vertex V_i^x. We show that $R_1^x \cup R_2^x$ does not depend on x.

Lemma 7.5.19. *For any point $x \in X$, we have $Z = R_1^x \cup R_2^x$ and hence $Y = \langle Y(\theta_1^x), Y(\theta_2^x)\rangle$; moreover, we can choose the numbering in such a way that $R_1^x = R_1^{x'}$ and $R_2^x = R_2^{x'}$ for each other point $x' \in X$. In particular,* $\dim Y = 2r' + 1$.

Proof. Each point $z \in Z$ which is not contained in Y_x determines a unique member $\theta \in \Theta$ together with x. So, by Lemma 7.5.16, $\theta = \theta_1^x$ or $\theta = \theta_2^x$, and hence $z \in Y(\theta_1^x)$ or $z \in Y(\theta_2^x)$, or more precisely, $z \in R_1^x \cup R_2^x$. So indeed, $Z = R_1^x \cup R_2^x$ and $Y = \langle Z\rangle = \langle Y(\theta_1^x), Y(\theta_2^x)\rangle$. Since a union of two r'-spaces only contains two r'-spaces, we have for each pair of points $x, x' \in X$ that $\{R_1^x, R_2^x\} = \{R_1^{x'}, R_2^{x'}\}$. So far, our numbering was arbitrary but of course this gives a canonical numbering: for each $x' \in X \setminus \{x\}$, we choose it such that $R_1^x = R_1^{x'}$ and $R_2^x = R_2^{x'}$. □

Put $R_i = R_i^x$, $i = 1,2$, for any $x \in X$. This hence divides the set Θ in two: for $i = 1,2$, we define Θ_i as the set $\{\theta \in \Theta \mid R_i \subseteq Z(\theta)\}$. It also divides each member of Ξ in two natural reguli, as we show in Lemma 7.5.21 below..

Corollary 7.5.20. *For each $x \in X$, we have $V_1^x = x^\perp \cap R_2$ and $V_2^x = x^\perp \cap R_1$.*

Proof. Considering θ_1^x and θ_2^x, this follows immediately from $R_i^x = R_i$ and the fact that $x^\perp \cap R_i^x$ and V_i^x are the two subspaces of $Z(\theta_i^x) \cap x^\perp$, $i = 1,2$. □

Lemma 7.5.21. *Let $\xi \in \Xi$ be arbitrary and denote its vertex by T. Then, for each $x \in X(\xi)$:*

(i) *ξ shares a generator $G_i = \langle T, L_i\rangle$ with θ_i^x for some X-line L_i, $i = 1,2$, and $T = \langle L_1^\perp \cap V_2^x, L_2^\perp \cap V_1^x\rangle$;*

(ii) *for each $x' \in X(\xi)$: if $x' \in G_1$, put $(x_1, x_2) = (x', x)$; if $x' \in G_2$, put $(x_1, x_2) = (x, x')$; and if $x' \notin G_1 \cup G_2$, put $(x_1, x_2) = (x'^\perp \cap L_1, x'^\perp \cap L_2)$. Then $(V_1^{x'}, V_2^{x'}) = (V_1^{x_2}, V_2^{x_1})$ and hence $Y_{x'} = \langle R_1 \cap x_1^\perp, R_2 \cap x_2^\perp\rangle$.*

Proof. (i) Since all X-lines through x are contained in $\theta_1^x \cup \theta_2^x$ by Lemma 7.5.18 and since T belongs to $Y_x = \langle V_1^x, V_2^x\rangle$ and is determined by the Y-points collinear to both L_1 and L_2, the first statement follows immediately.

(ii) Recall from Corollary 7.5.20 that $V_1^{x'} = x'^\perp \cap R_2$ and $V_2^{x'} = x'^\perp \cap R_1$ for each $x' \in X$. Firstly, take $x' \in G_1$. Then $V_1^{x'} = V_1^{x_2}$ since $x_2 = x$ and $x, x' \in \theta_1^x$ so both points are collinear to the latter's vertex; and $V_2^{x'} = V_2^{x_1}$ is trivial since $x_1 = x'$. Likewise, the statement is true if $x' \in G_2$, so suppose $x' \notin G_1 \cup G_2$. Then $V_2^{x'} = V_2^{x_1}$ since $x_1 x'$ is contained in $\theta_2^{x_1}$ by Lemma 7.5.18, so again both points are collinear to the latter's vertex. Likewise, we obtain $V_1^{x'} = V_1^{x_2}$.

By Lemma 7.5.16, it then follows that $Y_{x'} = \langle V_1^{x'}, V_2^{x'}\rangle = \langle V_1^{x_2}, V_2^{x_1}\rangle = \langle x_2^\perp \cap R_2, x_1^\perp \cap R_2\rangle$, from which the statement follows. □

Lemma 7.5.22. *Let V_i by any hyperplane of R_i, $i = 1,2$. Then the points of X collinear to $\langle V_1, V_2\rangle$ are precisely the points of $\langle x, V_1, V_2\rangle \cap X$ for some $x \in X$.*

Proof. Let $x \in X$ be arbitrary. Suppose first that x is collinear to $\langle V_1, V_2\rangle$. We claim that $\langle x, V_1, V_2\rangle \cap X$ is precisely the set of X-points collinear to $\langle V_1, V_2\rangle$. Clearly, all points of $\langle x, V_1, V_2\rangle \cap X$ are collinear to $\langle V_1, V_2\rangle$. Suppose for a contradiction that $x' \notin \langle x, V_1, V_2\rangle$ is an X-point collinear to $\langle V_1, V_2\rangle$. If x and x' determine a unique member of Ξ, then $\langle V_1, V_2\rangle$ would be contained in its vertex, contradicting the fact that $\dim(\langle V_1, V_2\rangle) = 2v' + 1 > v$ by Corollary 7.5.17. In all other cases, Lemma 7.5.15, Lemma 7.5.18 or (S1) implies that x and x' belong to a member of Θ, in which $\langle x, V_1, V_2\rangle$ is the unique maximal singular subspace containing $\langle V_1, V_2\rangle$ and not contained in Y; hence $\langle x', V_1, V_2\rangle = \langle x, V_1, V_2\rangle$. From this contradiction the claim follows.

Next, suppose that x is not collinear to $\langle V_1, V_2\rangle$. In this case, $(V_1^x, V_2^x) \neq (V_2, V_1)$. Without loss of generality, $V_2^x \neq V_1$. Then there is an X-point x_1 in θ_1^x on an X-line with x that is collinear to V_1. If $V_1^x = V_2$, then x_1 is collinear to $\langle V_1, V_1^x\rangle$ (as x and x_1 both belong to θ_1^x, we have $V_1^x = V_1^{x_1}$). If $V_1^x \neq V_2$, then we can likewise find a point $x_2 \in X(\theta_2^x)$ on an X-line with x that is collinear to V_2. Putting $L_i := xx_i$ for $i = 1,2$, we obtain from Corollary 7.5.17 that L_1 and L_2 are contained in a member ξ of Ξ. Let x' be a point on $X(\xi)$ collinear to x_1 and x_2, but not equal or collinear to x. Then Lemma 7.5.21 says that $Y_{x'} = \langle x_1^\perp \cap R_1, x_2^\perp \cap R_2\rangle = \langle V_1, V_2\rangle$. The lemma follows. $\square$

There are some interesting consequences.

Lemma 7.5.23. *Let θ and θ' be two members of Θ, with respective vertices V and V'. Then $\theta \cap \theta'$ contains an X-point if and only if they belong to the different classes Θ_1 and Θ_2 of Θ. If $\theta, \theta' \in \Theta_i$ for some $i \in \{1,2\}$, then $\theta \cap \theta'$ is $\langle R_i, V \cap V'\rangle$ and $V \cap V'$ is a hyperplane of V and V'.*

Proof. If $\theta \cap \theta'$ contains an X-point x, then without loss, $\theta = \theta_1^x \in \Theta_1$ and $\theta' = \theta_2^x \in \Theta_2$, so they belong to different classes indeed. Now let $\theta_1 \in \Theta_1$ and $\theta_2 \in \Theta_2$ be arbitrary and denote their respective vertices by V_1 and V_2. By definition, θ_i contains R_i for $i = 1,2$, and therefore $V_1 \subseteq R_2 \subseteq \theta_2$ and $V_2 \subseteq R_1 \subseteq \theta_1$. So $\langle V_1, V_2\rangle \subseteq \theta_1 \cap \theta_2$. In θ_1 and θ_2, there are (unique) maximal singular subspace through $\langle V_1, V_2\rangle$ containing a point of X. By Lemma 7.5.22, these two subspaces coincide, implying that $\theta_1 \cap \theta_2$ contains an X-point.

Secondly, take two arbitrary members $\theta, \theta' \in \Theta_1$. Again, R_1 is contained in $\theta \cap \theta'$ by definition. Their respective vertices V and V' are both hyperplanes of R_2. Let $x \in X(\theta)$ and $x' \in X(\theta')$ be points with $x^\perp \cap R_1 = x'^\perp \cap R_1$. Then, if $V = V'$, $Y_x = Y_{x'}$, which by Lemma 7.5.22 implies that $x' \in \langle x, Y_x\rangle \subseteq \theta$. However, as θ and θ' belong to the same class, they cannot share a point of X. So $V \neq V'$ and hence they intersect each other in a hyperplane. $\square$

Corollary 7.5.24. *For each hyperplane V of R_i, there is a unique member of Θ_j having V as its vertex; $\{i, j\} = \{1,2\}$.*

Proof. Without loss, $V \subseteq R_1$. Let V' by a hyperplane of R_2. Then by Lemma 7.5.22, there is a point $x \in X$ with $Y_x = \langle V, V'\rangle$. Let $z \in R_2 \setminus V'$ be arbitrary. Then $[x, z]$ is a member of Θ

containing R_2, i.e., $[x,z] \in \Theta_2$. Moreover, $[x,z]$ contains Y_x by Lemma 7.5.15 and therefore, $[x,z]$ has V as its vertex. By Lemma 7.5.23, there is no other member of Θ_2 having V as its vertex. □

The relation between two X-points can be expressed in terms of the subspaces of Y they are collinear to.

Lemma 7.5.25. *Take two distinct points $x_1, x_2 \in X$. Then*

(i) *x_1x_2 is a singular line with a unique point in Y $\Leftrightarrow V_i^{x_1} = V_i^{x_2}$ for all $i \in \{1,2\}$;*
(ii) *x_1 and x_2 belong to a member of Θ $\Leftrightarrow V_i^{x_1} = V_i^{x_2}$ for precisely one $i \in \{1,2\}$;*
(iii) *x_1 and x_2 are non-collinear points of a member of Ξ $\Leftrightarrow V_i^{x_1} \neq V_i^{x_2}$ for all $i \in \{1,2\}$.*

Proof. Assertion (i) follows immediately from Lemma 7.5.22. The "⇒"s of (ii) and (iii) are also easily verified. So, next, suppose that for x_1, x_2 holds that $V_i^{x_1} = V_i^{x_2}$ for precisely one $i \in \{1,2\}$. Then x_1 and x_2 cannot be contained in a member of Ξ for this would violate our deduced value of v; nor can they be on a singular line with a unique point of Y by assertion (i), and hence, as each X-line is also contained in a member of Θ by Lemma 7.5.18, "⇐" of assertion (ii) follows. By lack of other options, also "⇐" of assertion (iii) now follows. □

We again consider the maps ρ and χ (cf. Definitions 7.4.12 and 7.4.13).

The projection $\rho(X)$ and its connection to Y

Let $\xi \in \Xi$ be arbitrary. We already noted in Lemma 7.4.14 that $\rho(X(\xi))$ is a quadric Q in $\rho(X)$. Let p_1 and p_2 be arbitrary points in Q. We show that, for each pair of points $x_1, x_2 \in X$ with $\rho(x_1) = p_1$ and $\rho(x_2) = p_2$, the points x_1 and x_2 determine a unique member of Ξ, and $\rho(X([x_1, x_2])) = Q$. We also determine the inverse image of Q.

Lemma 7.5.26. *Let x_1 and x_2 be non-collinear X-points of some $\xi \in \Xi$, and denote the vertex of the latter by T. Then:*

(i) *If $x_i' \in \rho^{-1}(\rho(x_i))$ for $i = 1,2$, then also x_1' and x_2' determine a unique member $\xi' \in \Xi$, which also has vertex T and with $\rho(X(\xi)) = \rho(X(\xi'))$.*
(ii) *$\rho^{-1}(\rho(X(\xi)))$ is precisely the set of X-points collinear to T, which coincides with the set of X-points on members of Ξ having T as their vertex.*

Proof. By Lemma 7.4.14, $x_i' \in \langle x_i, Y_{x_i}\rangle \cap X$, for $i = 1,2$. Since $\xi = [x_1, x_2]$, Lemma 7.5.25 implies that also x_1' and x_2' determine a unique member $[x_1', x_2']$ of Ξ, which moreover has the same vertex as ξ for both are given by $Y_{x_1} \cap Y_{x_2}$. Now let ξ' be any member of Ξ with vertex T. We show that $\rho(X(\xi)) = \rho(X(\xi'))$.

Let $x \in X(\xi)$ be a point in $x_1^\perp \cap x_2^\perp$ and consider θ_1^x and θ_2^x. Let $i = 1,2$. Lemma 7.5.18 implies (possibly up to switching x_1 and x_2) that $xx_i \subseteq \theta_i^x$ (note that $\langle x_i, Y_{x_i}\rangle \subseteq \theta_i^x$ as well). Put $T_i := R_i \cap (xx_i)^\perp$ and note that T is spanned by T_1 and T_2. Let $\overline{x} \in X(\xi)$ be arbitrary and consider $Y_{\overline{x}}$. By Lemma 7.5.21(ii) and with the same notation concerning the points $\overline{x}_1$ and $\overline{x}_2$, the subspace $Y_{\overline{x}}$ is spanned by the respective hyperplanes $\overline{x}_1^{\perp} \cap R_1$ and $\overline{x}_2^{\perp} \cap R_2$ of R_1 and R_2 (which contain T_1 and T_2, respectively). Conversely, each pair H_1, H_2 with $T_i \subseteq H_i \subseteq R_i$ ($i = 1,2$) occurs as the perp of the X-points of a *unique* generator $\langle \overline{x}, T\rangle$ of $X(\xi)$ (unique indeed, for no two generators of $X(\xi)$ could be collinear to exactly the same subspace of Y by Lemma 7.5.25). That means that also $X(\xi')$ contains a unique generator, say $\langle \overline{x}', T\rangle$, such that $Y_{\overline{x}'} = Y_{\overline{x}}$. For such a pair of points, Lemma 7.4.14 implies that $\rho(\overline{x}) = \rho(\overline{x}')$. Reversing the roles of ξ and ξ', we conclude that $\rho(X(\xi)) = \rho(X(\xi'))$.

Lastly, note that each X-point x collinear to T is contained in a member of Ξ having T as its vertex: by the above $X(\xi)$ contains a point $\tilde{x}$ such that $Y_{\tilde{x}} \cap Y_x = T$, and for such a pair of points, Lemma 7.5.25 implies that x and $\tilde{x}$ determine a member of Ξ, which clearly has vertex T. This shows the lemma. □

Lemma 7.5.27. *Let $\rho(x_1)$ and $\rho(x_2)$ be distinct points on a line of $\rho(X)$, for $x_1, x_2 \in X$. Then:*

(i) *there is a unique $\theta \in \Theta$ containing $\rho^{-1}(\rho(x_1)) \cup \rho^{-1}(\rho(x_2))$;*
(ii) *for each $x_1' \in \rho^{-1}(\rho(X))$, we can choose $x_2'' \in \rho^{-1}(\rho(x_2))$ such that $x_1'x_2''$ is an X-line.*

Proof. Again, $\rho^{-1}(\rho(x_i)) \in \langle x_i, Y_{x_i}\rangle \cap X$ for $i = 1,2$ by Lemma 7.4.14. By Lemma 7.5.25, it suffices to show that $\dim(Y_{x_1} \cap Y_{x_2}) = 2r' - 2$, as this implies that there is a member $\theta \in \Theta$ containing x_1 and x_2 and hence also $\langle x_1, Y_{x_1}\rangle$ and $\langle x_2, Y_{x_2}\rangle$ (cf. Lemma 7.5.15). So suppose for a contradiction that $\dim(Y_{x_1} \cap Y_{x_2}) \neq 2r' - 2$. Then $\dim(Y_{x_1} \cap Y_{x_2}) = 2r' - 3$ (and hence $\langle Y_{x_1}, Y_{x_2}\rangle = Y$) for if $Y_{x_1} = Y_{x_2}$, then also $\rho(x_1) = \rho(x_2)$ by Lemma 7.4.14, contradicting our assumption.

Consider the $(2r'+3)$-space $\langle x_1, x_2, Y\rangle$, which, as noted above, equals $\langle x_1, x_2, Y_{x_1}, Y_{x_2}\rangle$. Recall that $\dim(\langle x_i, Y_{x_i}\rangle) = 2r'$. The fact that $\rho(x_1)$ and $\rho(x_2)$ are on a line of $\rho(X)$, implies that there is a point $x_3 \in X$ with $\rho(x_3)$ on $\rho(x_1)\rho(x_2) \setminus \{\rho(x_1), \rho(x_2)\}$, and hence x_3 is a point of $\langle x_1, x_2, Y\rangle$. In here, we see that $\langle x_3, x_1, Y_{x_1}\rangle$ intersects $\langle x_2, Y_{x_2}\rangle$ in a $(2r'-2)$-space containing $Y_{x_1} \cap Y_{x_2}$ and some other point, say x_2'. Then the line $x_2'x_3$ intersects $\langle x_1, Y_{x_1}\rangle$ in a point x_1'. Since x_3 does not belong to Y, neither does $x_1'x_2'$. As this line does contain three points of $X \cup Y$, it is singular (cf. Lemma 7.4.1), implying that it is an X-line (since $x_3 \notin \langle x_1, Y_{x_1}\rangle$). Then $x_1'x_2'$ is a 1-line by Lemma **??**, and as such it belongs to a member of Θ. By Lemma 7.5.25, this contradicts our assumption on $Y_{x_1} \cap Y_{x_2}$. We conclude that there is a unique $\theta \in \Theta$ containing $\langle x_1, Y_{x_1}\rangle \cup \langle x_2, Y_{x_2}\rangle$, showing ($i$).

For assertion (ii), let $x_1' \in \langle x_1, Y_{x_1}\rangle \cap X$ be arbitrary. In θ, x_1' is collinear to a hyperplane of $\langle x_2, Y_{x_2}\rangle$, which does not coincide with Y_{x_2} (since $Y_{x_1} \neq Y_{x_2}$) and hence contains an X-point x_2'. The lemma follows. □

Let L be the set of X-lines. The previous lemma showed that, for each $\rho(X)$-line L', there is an X-line L with $\rho(L) = L'$.

Proposition 7.5.28. *The pair $(\rho(X), \rho(\mathsf{L}))$ is isomorphic to an injective projection of the Segre geometry $\mathsf{S}_{r',r'}(\mathbb{K})$, each of whose maximal singular subspaces corresponds to a unique* *member of Θ and each of whose symps (i.e., a direct product of two lines intersecting in a point) corresponds to a v-space V of Y in the sense that the members of Ξ with vertex V are precisely those whose image under ρ is this symp.*

Proof. We first determine the maximal singular subspaces of $(\rho(X),\rho(\mathsf{L}))$. Let $\theta \in \Theta$ be arbitrary. Recall from Lemma 7.4.14 that $\rho(X(\theta))$ is a singular r'-space of $\rho(X)$, which we will denote by S_θ. Moreover, if $\theta \neq \theta' \in \Theta$, then $S_\theta \neq S_{\theta'}$ as $S_\theta \cap S_{\theta'}$ cannot contain a $\rho(X)$-line by Lemma 7.5.27. We first show the following general claim.

Claim 1: each point $p \in \rho(X) \setminus S_\theta$ is collinear to at most one point of S_θ.
Suppose for a contradiction that there is a point $p \in \rho(X) \setminus S_\theta$ collinear to two points s_1 and s_2 of S_θ. Let $x \in X$ be a point with $\rho(x) = p$. Lemma 7.5.27 implies that, for $i = 1,2$, we can choose $x_i \in \rho^{-1}(s_i)$ with xx_i an X-line (and by the same lemma, $x_1, x_2 \in \theta$). Since $x \notin \theta$, x_1 and x_2 are on an X-line of θ. Now, $\langle x, x_1, x_2 \rangle$ is a singular plane and as before contained in a member of Θ together with some plane of θ through x_1x_2. This contradicts x_1x_2 being a 1-line and shows the claim.

Now let S be an arbitrary singular subspace of $\rho(X)$ containing at least a line L. By Lemma 7.5.27, L is contained in S_θ for a unique $\theta \in \Theta$. If some point $p \in S \setminus L$ would not be contained in S_θ, then this would violate the above claim. Hence $S \subseteq S_\theta$. We conclude that $\{S_\theta \mid \theta \in \Theta\}$ is precisely the set of maximal singular subspaces of $\rho(X)$.

For $i = 1,2$, put $\mathsf{S}_i := \{S_\theta \mid \theta \in \Theta_i\}$. Let $p \in \rho(X)$ be arbitrary and take $x \in \rho^{-1}(p)$. By the previous paragraph, all lines of $\rho(X)$ through p are contained in precisely one of $S_{\theta_1^x}$, $S_{\theta_2^x}$ (cf. Lemma 7.5.18). In particular, $S_{\theta_1^x} \cap S_{\theta_2^x} = \{p\}$. Moreover, by Lemma 7.5.23, each pair $(S_1, S_2) \in \mathsf{S}_1 \times \mathsf{S}_2$ is such that $S_1 \cap S_2$ contains at least one point, which is again unique by the previous paragraph. We can now determine the structure of $\rho(X)$. To that end, let $(S_1, S_2) \in \mathsf{S}_1 \times \mathsf{S}_2$ be arbitrary and denote their unique intersection point by p.

Claim 2: $\rho(X)$ is the direct product of S_1 and S_2.
Let $q \in \rho(X)$ be arbitrary. If $q \in S_1 \cup S_2$, then $q = (p, s_2) \in \{p\} \times S_2$ or $q = (s_1, p) \in S_1 \times \{p\}$ for some $s_i \in S_i$, $i = 1,2$. So suppose $q \notin S_1 \cup S_2$. As noted above, all singular lines of $\rho(X)$ through q are contained in S_1^q and S_2^q, and also as noted above, $S_1^q \cap S_2$ is a unique point, say s_2^q; likewise, $S_2^q \cap S_1$ is a unique point, say s_1^q. Moreover, the points s_1^q and s_2^q determine q uniquely: by the above, there is a unique member $S_2' \in \mathsf{S}_2$ through s_1^q and a unique member $S_1' \in \mathsf{S}_1$ through s_2^q, and $S_1' \cap S_2' = \{q\}$. This shows the claim. Observe that Claim 1 implies that s_i^q is the unique point of S_i collinear to q, $i = 1,2$ and this also implies that there are precisely two pairs of $\rho(X)$-lines through s_1^q and s_2^q, respectively, which intersect each other non-trivially.

Take $\xi \in \Xi$ such that $p \in \rho(X(\xi))$ and note that the two lines L_1 and L_2 of $\rho(X(\xi))$ through p are contained in S_1 and S_2, respectively. Then $\rho(X(\xi)) = L_1 \times L_2$. Conversely, let ps_1 and ps_2 be arbitrary lines through p in S_1 and S_2, respectively, i.e., $s_i \in S_i \setminus \{p\}$, and let G be the grid of $\rho(X)$ determined by $ps_1 \times ps_2$. Put $T = \chi(s_1) \cap \chi(s_2)$.

Claim 3: $\rho^{-1}(G)$ is the set of X-points collinear to T.
Let x_1 and x_2 be X-points such that $\rho(x_i) = s_i$. Since s_1 and s_2 are distinct points not on a line of $\rho(X)$, x_1 and x_2 are non-collinear points of some $\xi \in \Xi$ with vertex $Y_{x_1} \cap Y_{x_2} =$

$\chi(s_1)\cap\chi(s_2)=T$. Since $\rho(X(\xi))$ is a full grid of $\rho(X)$-lines containing the points s_1 and s_2, the observation made at the end of Claim 2 implies that $\rho(X(\xi))=G$. It then follows from Lemma 7.5.26(ii) that the inverse image of G, i.e., of $\rho(X(\xi))$, under ρ is indeed precisely the set of X-points on members of Ξ with T as vertex, or equivalently, the set of X-points collinear to T. This shows the claim.

Lastly, $\rho(\mathsf{L})=\{\{s_1\}\times L_2\,|\,s_1\in S_1, L_2 \text{ a line of } S_2\}\cup\{L_1\times\{s_2\}\,|\,L_1 \text{ a line of } S_1, s_2\in S_2\}$ follows from the fact that each line L of $\rho(\mathsf{L})$ is contained in a unique member of $\mathsf{S}_1\cup\mathsf{S}_2$ (indeed, if $L\subseteq S_1'$ for some $S_1'\subseteq\mathsf{S}_1$, then $s_2:=S_2\cap S_1'$ and $L_1=\{s_1^q\,|\,q\in L\}$, and the latter set is indeed a *line* of S_1 for it belongs to the grid $L\times qs_1^q$ for $q\in L$ arbitrary).

Since $\dim(S)=r'$ for each maximal singular subspace of $\rho(X)$, the above implies that $(\rho(X),\rho(\mathsf{L}))$ is an injective projection of $\mathsf{S}_{r',r'}(\mathbb{K})$. This concludes the proposition. □

Corollary 7.5.29. *We have $N\le r'^2+4r'+2$.*

Proof. By Lemma 7.5.19, we know $\dim(Y)=2r'+1$ and by Proposition 7.5.28, $\dim(F)\le(r'+1)^2-1$. Since F and Y are complementary subspaces of $\mathsf{PG}(N,\mathbb{K})$, we obtain $N\le r'^2+4r'+2$. □

Lemma 7.5.30. *There exists a subspace F^* such that $F^*\cap X$ contains a legal projection Π of $\mathsf{S}_{r',r'}(\mathbb{K})$ with $\langle\Pi\rangle=F^*$ which is such that, for each $x\in X$, there is a point $\overline{x}\in\Pi$ such that $x\in\langle\overline{x},Y_{\overline{x}}\rangle$. The projection of $F^*\cap X$ from $F^*\cap Y$ is injective. If $r'=2$, then $F^*\cap Y=\emptyset$.*

Proof. By Proposition 7.5.28, $\rho(X)$ is the point set of an injective projection of a Segre geometry $\mathsf{S}_{r',r'}(\mathbb{K})$, and the elements $\theta\in\Theta$ are in $1-1$-correspondence to the set of r'-spaces S_θ. We now show that we can construct a legal projection of $\mathsf{S}_{r',r'}(\mathbb{K})$ in X using well-chosen r'-dimensional X-spaces in certain members of Θ.

To that end, take a basis of hyperplanes $V_1^0,\dots,V_1^{r'}$ in R_2 and a basis of hyperplanes $V_2^0,\dots,V_2^{r'}$ in R_1. By Corollary 7.5.24, there is, for each $0\le t\le r'$, $i=1,2$, a unique member $\theta_i^t\in\Theta_i$ having V_i^t as its vertex. For any pair t,u with $0\le t,u\le r'$, we claim that we can select X-points $x_{t,u}$ in $\theta_1^t\cap\theta_2^u$ such that $x_{t,u}\perp x_{t,u'}$ for all $0\le u'<u$ and $x_{t,u}\perp x_{t',u}$ for all $0\le t'<t$. We proceed inductively (taking lexicographic order on the pairs $\{(t,u)\,|\,1\le u,t\le r'\}$).

First note that, for each pair t,u, the subspace $\theta_1^t\cap\theta_2^u$ has dimension $2r'$ and coincides with $\Pi_{t,u}:=\langle x'_{t,u},V_1^t,V_2^u\rangle$, where $x'_{t,u}$ is any X-point in $\theta_1^t\cap\theta_2^u$, i.e., any X-point collinear to $\langle V_1^t,V_2^u\rangle$. The first point $x_{0,0}$ can be chosen as any X-point in $\theta_1^0\cap\theta_2^0$. Assume that we now have to choose the point $x_{t,u}$ and that all previous points are fine. Observe that the points $x_{t,u'}$ with $u'<u$ belong to θ_1^t, and the points $x_{t',u}$ with $t'<t$ belong to θ_2^u, whereas $\Pi_{t,u}\subseteq\theta_1^t\cap\theta_2^u$. This observation, together with our choice of the vertices, implies that:

$$\dim\left(\Pi_{t,u}\cap\bigcap_{0\le u'<u}x_{t,u'}^{\perp}\cap\bigcap_{0\le t'<t}x_{t',u}^{\perp}\right)=2r'-(t+u).$$

We also have

$$\dim\left(\Pi_{t,u}\cap Y\cap\bigcap_{0\le u'<u}x_{t,u'}^{\perp}\cap\bigcap_{0\le t'<t}x_{t',u}^{\perp}\right)=(r'-u-1)+(r'-t-1)+1=2r'-(t+u)-1,$$

and so, as $2r'-(t+u)\geq 0$, it follows that $\Pi_{t,u}\cap\bigcap_{0\leq u'<u}x_{t,u'}^{\perp}\cap\bigcap_{0\leq t'<t}x_{t',u}^{\perp}$ contains at least one X-point $x_{t,u}$, which is as required. This shows the claim.

For each $t,u\in\{0,\dots,r'\}$, put $S_1^t:=\langle x_{t,0},\dots,x_{t,r'}\rangle$ and $S_2^u:=\langle x_{0,u},\dots,x_{r',u}\rangle$, which are singular subspaces by construction. Also by construction, $S_1^t\cap Y\subseteq\bigcap_{u'=0}^{r'}(x_{t,u'}^{\perp}\cap Y)=V_1^t$, and as $S_1^t\subseteq\theta_1^t$, this implies that S_1^t is an r'-dimensional X-space in θ_1^t (if not, there would be points in R_1 collinear to S_1^t). Likewise, S_2^u is an r'-dimensional X-space in θ_u^2.

We claim that each point x_0 on S_1^0 is contained in a unique r'-dimensional X-space intersecting each of the X-spaces S_1^t with $t\in\{0,\dots,r'\}$. If $x_0=x_{0,u}$ for some $0\leq u\leq r'$, then $x_0\in S_2^u$ and the assertion is clear. Next, suppose that x_0 is on a line joining two of the r' chosen X-points, say, $x_0\in\langle x_{0,0},x_{0,1}\rangle$. Let $t\in\{1,\dots,r'\}$ be arbitrary. Then $x_{0,0}$ and $x_{t,1}$ determine a unique member $\xi\in\Xi$, as follows from Lemma 7.5.25. The points $x_{0,1}$ and $x_{t,0}$ also belong to ξ and as such, the point x_0 is collinear to a unique point x_t on the line $\langle x_{t,0},x_{t,1}\rangle$. Note that the line x_0x_t belongs to $\theta_2^{x_0}$ by Lemma 7.5.18 and the fact that $x_t\notin\theta_1^{x_0}=\theta_1^0$. This implies that x_0x_t is the unique singular line through x_0 meeting S_1^t: a singular line through x_0 with a unique point in Y does not leave θ_1^0, and an X-line through x_0 meeting S_1^t in a point different from x_t would imply that $\theta_2^{x_0}\cap\theta_1^t$ have an X-line in common, contradicting Lemma 7.5.23.

Now take $t'\in\{1,\dots,r'\}\setminus\{t\}$ arbitrary. On the one hand x_t and $x_{t'}$ belong to the member of Ξ determined by $x_{t,0}$ and $x_{t',1}$; on the other hand they belong to $\theta_2^{x_0}$; from which we obtain $x_t\perp x_{t'}$. We conclude that $\langle x_0,\dots,x_{r'}\rangle$ is a singular subspace through x_0 intersecting S_1^t in a point for each $t'\in\{1,\dots,r'\}$. This subspace belongs to $\theta_2^{x_0}$ and no point of R_2 is collinear to it (as $x_u^{\perp}\cap R_2=V_1^u$ for $u\in\{0,\dots,r'\}$), from which we conclude that it is an r'-dimensional X-space. It follows from the last sentence of the previous paragraph that this X-space is the unique one through x_0 with the property that it intersects each S_1^t for $1\leq t\leq r'$. We can now repeat the above argument for points on lines $\langle x,x'\rangle$ with x and x' on lines joining two points of $\{x_{0,0},\dots,x_{0,r'}\}$, et cetera. This shows the claim. In hindsight, we can also describe this r'-space through x_0 as generated by the points $x_t\in S_1^t$ for $t\in\{0,\dots,r'\}$ that are collinear to $x_0^{\perp}\cap R_1$ (which is the vertex of $\theta_2^{x_0}$); and varying $x_0\in S_1^0$, all hyperplanes of R_1 are reached precisely once (collinearity between the points of S_1^0 and the hyperplanes of R_1 is a duality, as is easily seen in θ_1^0).

Finally, we put $F^*=\langle S_1^0,\dots,S_1^{r'}\rangle$ and Π is the union of all r'-spaces intersecting S_1^t for each $t\in\{0,\dots,r'\}$. By (S2), Π is a legal projection of $\mathsf{S}_{r',r'}(\mathbb{K})$. We now show that $\langle F^*,Y\rangle=X$.

Let $x\in X$ be arbitrary and recall that $Y_x=\langle V_1^x,V_2^x\rangle$. As S_1^0 belongs to θ_1^0, it contains a unique point x_1 with $V_2^{x_1}=V_2^x$. It follows from Corollary 7.5.24 that $\theta_2^{x_1}=\theta_2^x$. We deduced above that there is a unique r'-dimensional X-space, say $\pi_2^{x_1}$, through x_1 intersecting each S_1^t with $t\in\{0,\dots,r'\}$. Obviously, $\pi_2^{x_1}\subseteq F$. Moreover, $\pi_2^{x_1}$ belongs to $\theta_2^{x_1}$ by construction, and the latter coincides with θ_2^x. So θ_2^x contains the X-space $\pi_2^{x_1}$, has V_1^x as its vertex and contains R_2, which means that $\pi_2^{x_1}$ contains a unique point x' collinear to $\langle V_1^x,V_2^x\rangle$. By Lemma 7.5.22, we obtain $x\in\langle x',V_1^x,V_2^x\rangle=\langle x,Y_x\rangle$.

The fact that the projection of $F^*\cap X$ from $F^*\cap Y$ is injective follows from Proposition 7.5.28. Finally, suppose that $r'=2$ (in which case Π is isomorphic to $\mathsf{S}_{2,2}(\mathbb{K})$ by Proposition 7.2.7) and suppose for a contradiction that $F^* \cap Y$ contains a point y. This point is contained in at least one line joining the planes R_1 and R_2, which clearly occurs as the vertex of some member $\xi \in \Xi$. In Π, ξ corresponds to a grid G. The 4-space $\langle G, y\rangle$ is hence contained in the 8-dimensional subspace generated by the Segre variety Π and intersects Π in precisely G. However, using Lemma 7.2.8, we then obtain a contradiction to (S2).

The lemma is proved. □

Conclusion

Henceforth we assume that F^* is as described in the statement of the previous lemma. Be careful, we will use both F and F^*. We now focus on the connection between $\rho(X)$ and Y. Consider the partial connection map $\chi_i : \rho(X) \to R_i : \rho(x) \mapsto x^\perp \cap R_i$. Clearly, $\chi(\rho(x)) = \langle \chi_1(\rho(x)), \chi_2(\rho(x))\rangle$ for each $x \in X$.

Proposition 7.5.31. *Let (X, Z, Ξ, Θ) be a duo-symplectic split pre-Veronese set with $r = 1$ for which there are at least two members of Θ through each X-point. Then*

(i) *X is the point set of a mutant of the dual Segre variety $\mathsf{DS}_{r',r'}(\mathbb{K})$ and*
(ii) *if additionally, (X, Z, Ξ, Θ) satisfies (S3), then $r' = 2$ and X is projectively unique.*

Proof. (i) By Proposition 7.5.30, we know that there is a subspace F^* in $\mathsf{PG}(N, \mathbb{K})$ containing a legal projection Π of $\mathsf{S}_{r',r'}(\mathbb{K})$, and that the projection of $X \cap F^*$ from $Y \cap F^*$ is injective. We also know that, if S_1 and S_2 denote the two sets of maximal singular subspaces of Π, then each of their members is contained in a unique member of Θ_1 or Θ_2, respectively. We also showed that $X = \{\langle x, Y_x\rangle \setminus Y_x \mid x \in \Pi\}$.

We now show that the correspondence $\Pi \to Y : x \mapsto Y_x$ (which we will also denote by χ) is as described in Section 7.2.1. For $i = 1, 2$, take $S_i \in \mathsf{S}_i$ arbitrary and let θ_i be the member of Θ_i containing S_i. Denote by χ_i the restriction of χ to S_i. Inside the polar space $XY(\theta_i)$, it is clear that χ_i coincides with the collinearity relation between the opposite subspaces S_i and R_i, and as such we get that it is a linear duality between S_i and R_i.

Let x_0 denote the intersection of S_1 and S_2. Take $x \in X$ arbitrary. If $x \notin S_i$, then there is a unique point $x_i \in S_i$ collinear to x. If $x \in S_i$, we put $x_i = x$. Note that all points x' of each member $S_1' \in \mathsf{S}_1$ have the property that $x'^\perp \cap R_2$ is independent of x', as for each such x', it coincides with the vertex of the unique member $\theta_1' \in \Theta_1$ containing S_1'; likewise all points x' of each member $S_2' \in \mathsf{S}_2$ have the property that $x'^\perp \cap R_1$ is independent of x'. As such, $\chi_i(x_i) = x^\perp \cap R_i$ for $i = 1, 2$ and since $Y_x = \langle x^\perp \cap R_1, x^\perp \cap R_2\rangle$ by definition, we obtain that $Y_x = \langle \chi_1(x_1), \chi_2(x_2)\rangle$. This shows (i).

(ii) By Proposition 7.5.28, we know that, for any subspace F in $\mathsf{PG}(N, \mathbb{K})$ complementary to Y, $\rho(X)$ is an injective projection of a Segre variety $\mathsf{S}_{r',r'}(\mathbb{K})$ in F. Let $T^F_{\rho(x)}$ be the set of $\rho(X)$-lines in F through $\rho(x)$ and $T^F_{\rho(x)}(\rho(X(\xi)))$ be the tangent hyperplane to $\rho(X(\xi))$ at $\rho(x)$ for some $\xi \in \Xi$ with $\rho(x) \in \rho(\xi)$. By (S3), there are members $\xi_1, \xi_2 \in \Xi$ through x such that T_x is generated by $T_x(\xi_1)$ and $T_x(\xi_2)$. For $i = 1, 2$, we have that $T_x(\xi_i) =$

$\langle Y(\xi_i), T^F_{\rho(x)}(X(\xi_i))\rangle$. So $T_x = \langle T_x(\xi_1), T_x(\xi_2)\rangle$ is equivalent with $Y_x = \langle Y(\xi_1), Y(\xi_2)\rangle$ and $T^F_{\rho(x)} = \langle T^F_{\rho(x)}(X(\xi_1)), T^F_{\rho(x)}(X(\xi_2))\rangle$.

On the other hand, $\dim(T^F_{\rho(x)}) = 2r'$ as the tangent space at $\rho(x)$ is generated by the unique r'–dimensional subspaces $S_{\theta_1^x}$ and $S_{\theta_2^x}$ of $\rho(X)$ through $\rho(x)$. As $r=1$, we deduce that $\langle S_{\theta_1^x}, S_{\theta_2^x}\rangle$ can only be generated by $T^F_{\rho(x)}(X(\xi_1))$ and $T^F_{\rho(x)}(X(\xi_2))$ if $2r' \le 2+2=4$, hence if $r'=2$ (because $r'>r\ge 1$ by assumption). Recalling that $v = 2r'-1 = 1$ and $\dim(Y_x) = 2r'+1 = 3$, the first requirement only implies that ξ_1 and ξ_2 have disjoint vertices.

In case $r'=2$, then the variety $\mathsf{S}_{2,2}(\mathbb{K})$ does not admit legal projections (cf. Proposition 7.2.7), and hence R_1, R_2 and $F^*\cap X$ are projectively unique, and, for $i=1,2$, the projectivity χ_{S_i} between S_i and the dual of R_i is unique up to a projectivity of S_i. As such, X is projectively unique. $\square$

7.6 The half dual Segre varieties

Throughout this section, we suppose that (X,Z,Ξ,Θ) is a duo-symplectic split pre-Veronese set containing at least one X-point through which there is exactly one member of Θ.

The X-lines and members of Θ through a point of X

Lemma 7.6.1. *Let $x\in X$ be such that there is a unique member $\theta^x \in \Theta$ containing $x\in X$. Then $Y(\theta^x) = Y$ and for each $x'\in X$, there is a unique member $\theta^{x'}\in\Theta$ containing x'. In particular,* $\dim(Y) = r'+v'+1$ *and Θ induces a partition of X.*

Proof. Suppose for a contradiction that $Y(\theta^x) \subsetneq Y$. Then there is a point $z\in Z$ with $z\notin\theta^x$. By Lemma 7.4.22(ii) (applied to any X-line L in θ^x through x), θ^x contains Y_x and hence x is not collinear to z. As such, (S1) implies that $[x,z]$ is a second member of Θ through x, a contradiction. We conclude that $Y = Y(\theta^x)$ indeed. In particular, $\dim(Y) = r'+v'+1$.

Now let $x'\in X$ be arbitrary. Then each θ of Θ through x' (note that there is at least one such member by Lemma 7.4.18) contains Y, for it needs to contain an $(r'+v'+1)$-dimensional subspace of Y. Hence $Y_{x'}$ is a hyperplane of Y and therefore, taking a point $z\in Y\setminus Y_{x'}$, $\theta = [x',z]$ and as such this is the unique member of Θ through x'. This shows the lemma. $\square$

Lemma 7.6.2. *Let $\theta\in\Theta$ be arbitrary and suppose L is an X-line intersecting θ in a unique point $x\in X$. Then $Y_L = Y_x$. Consequently, each point $x'\in X\setminus X(\theta)$ is collinear to the vertex V of θ.*

Proof. Note that no member of Θ contains L, for this would yield a second member of Θ through x, contradicting Lemma 7.6.1. It then follows from Lemma 7.4.22(i) that $Y_L = Y_x$, and $Y_{x'} = Y_x$ for each $x'\in L$.

Now take any point $x' \in X \setminus X(\theta)$. Suppose x' is collinear to some point $x \in X(\theta)$. By Lemma 7.6.1 and $x' \notin \theta$, xx' is an X-line. By the above paragraph, $Y_{x'} = Y_x$, so in particular, $x' \perp V$. Next, suppose that x' is not collinear to any point of $X(\theta)$. Taking $x \in X(\theta)$ arbitrary, (S1) and Lemma 7.6.1 imply that x and x' are contained in a member ξ of Ξ. Let L_1 and L_2 be two non-collinear X-lines of ξ through x. For $i = 1, 2$, we have that L_i is collinear to V because, if L_i belongs to θ then $L_i \perp V$ by definition and if L_i does not belong to θ then $L_i \perp V$ by the first paragraph. Consequently, V is contained in the vertex of ξ and, in particular, $x' \perp V$. □

By Lemma 7.4.11, we can project (X, Z, Ξ, Θ) from V and obtain a split pre-Veronese set with parameters $(r, v - v' - 1, r', -1)$; with which we will continue to work without changing our notation, i.e., we just assume that (X, Z, Ξ, Θ) has $v' = -1$.

Corollary 7.6.3. *We have $Y = Z$ and $\dim Y = r'$.*

Proof. Take $\theta \in \Theta$ arbitrary. By Lemma 7.6.1 and $v' = -1$, we have $Y = Y(\theta) \subseteq Z$, so $Y = Z$ as $\langle Z \rangle = Y$. The same lemma yields $\dim Y = r' + v' + 1 = r' - 1 + 1 = r'$. □

Lemma 7.6.4. *Let $\theta \in \Theta$ be arbitrary and take any point $x \in X(\theta)$. Then each pair of X-lines L_1, L_2 through x not contained in θ is collinear and $v = r' - 2$.*

Proof. By Lemma 7.6.2, the lines L_1 and L_2 are collinear to Y_x. Take any X-line L through x inside θ which is not collinear to L_1. Then by (S1) and Lemma 7.6.1, L and L_1 determine a member ξ' of Ξ, whose vertex is Y_L and therefore has dimension $r' - 2$. In particular, $v = r' - 2$.

Now, if L_1 and L_2 would not be collinear, they are also contained in some member ξ of Ξ, whose vertex is Y_x and hence has dimension $r' - 1$, contradicting $v = r' - 2$. Hence L_1 and L_2 are collinear indeed. □

Lemma 7.6.5. *Let $\theta \in \Theta$ be arbitrary and suppose $\xi \in \Xi$ shares an X-point with θ. Then $\xi \cap \theta$ is a maximal singular subspace of both θ and ξ of the form $\langle L, Y_L \rangle$, with L an X-line. In particular, $r = 1$. Moreover, each point $x' \in X \setminus X(\theta)$ is collinear to a maximal singular subspace of θ of the form $\langle x, Y_x \rangle = \langle x, Y_{x'} \rangle$, where $x \in X(\theta)$.*

Proof. Let x be a point of $X(\theta) \cap X(\xi)$. In ξ, take any two non-collinear X-lines L_1 and L_2 through x. By Lemma 7.6.4, precisely one of them, say L_1, is contained in θ. This implies that $r = 1$ and that $\theta \cap \xi = \langle L_1, V \rangle$, where V is the vertex of ξ. As $V = Y_{L_1} \cap Y_{L_2} = Y_{L_1} \cap Y_x = Y_{L_1}$ by Lemma 7.6.2, the subspace $\langle L_1, V \rangle$ is a maximal singular subspace of both ξ and θ.

Next, take any point $x' \in X \setminus X(\theta)$ and any point $x \in X(\theta)$. If x' is on an X-line with x, then $x'^{\perp} \cap \theta = \langle x, Y_x \rangle = \langle x, Y_{x'} \rangle$, as $Y_x = Y_{x'}$ by Lemma 7.5.4. So suppose that x' and x are not on an X-line. Then by Lemma 7.6.1 and (S1), x and x' are contained in a member $\xi \in \Xi$. The above implies that $\xi \cap \theta = \langle L, Y_L \rangle$ for an X-line L of ξ. As such, x' is collinear to a point of L, which we now take as x. The lemma follows. □

Lemma 7.6.6. *Let H be a hyperplane of Y. Then the set of all points of X collinear to H, forms, together with H, a (maximal) singular subspace $\pi(H)$, which moreover intersects each $\theta \in \Theta$ in a maximal singular subspace of θ of the form $\langle x, H\rangle$ with $x \in X(\theta)$.*

Proof. Take two points x_1, x_2 collinear to H. Let θ_i be the unique member of Θ containing x_i (cf. Lemma 7.6.1), for $i = 1,2$. If $\theta_1 = \theta_2$, then $x_2 \in \langle x_1, H\rangle$ and hence x_1x_2 is a singular line (with a unique point in Y). So suppose $\theta_1 \neq \theta_2$. If x_1 and x_2 are not collinear, then they determine a member of Ξ, which has H as its vertex, contradicting $v = r' - 2$. Hence π_H is a singular subspace indeed, maximal by definition. Since each $\theta \in \Theta$ contains an X-point x collinear to H and $\langle x, H\rangle$ is a maximal singular subspace of θ, the lemma follows. □

For $x \in X$, we keep denoting by θ^x the unique member of Θ containing x and, with the notation of the previous lemma, we denote by π^x the subspace $\pi(H)$ where $H = Y_x$. Let Π be the set $\{\pi^x \mid x \in X\}$.

Corollary 7.6.7. *For each point $x \in X$, all X-lines through it are contained in $\theta^x \cup \pi^x$.*

Proof. This follows immediately from Lemmas 7.6.2 and 7.6.6. □

The projection $\rho(X)$ and its connection to Y

What now follows contains many similarities with the situation in Section 7.5.2. This is caused by the fact that we will, in the end, obtain a half dual Segre variety $\mathsf{HSD}_{r',k}(\mathbb{K})$ for some natural number $k \geq 1$, and in case $k = r'$ this is the projection of the dual Segre variety $\mathsf{HSD}_{r',r'}(\mathbb{K})$ (as encountered in the previous section) from one of its two r'-spaces in Z. Despite the similarities between both cases there is no upshot in treating them simultaneously as it would boil down to a similar amount of work and obscure some of our arguments.

We again consider the maps ρ and χ (cf. Definitions 7.4.12 and 7.4.13).

Lemma 7.6.8. *Let x_1 and x_2 be non-collinear X-points of some $\xi \in \Xi$, and denote the vertex of the latter by T. Then, if $x_i' \in \rho^{-1}(\rho(x_i))$ for $i = 1,2$, then also x_1' and x_2' determine a unique member $\xi' \in \Xi$, which also has vertex T and with $\rho(X(\xi)) = \rho(X(\xi'))$.*

Proof. By Lemma 7.4.14, $x_i' \in \langle x_i, Y_{x_i}\rangle \cap X$ for $i = 1,2$. Since $Y_{x_1} \cap Y_{x_2}$ is the vertex T of ξ and $v = r' - 2$, we have that $Y_{x_1} \neq Y_{x_2}$. Clearly, also x_1' and x_2' are collinear to T. Firstly, suppose x_1' and x_2' belong to a member $\theta \in \Theta$. Then θ also contains x_1 and x_2, which implies $\xi = \theta$, a contradiction. Secondly, suppose $x_1'x_2'$ is a singular line. Then, since $x_2' \notin \langle x_1, Y_{x_1}\rangle$ by assumption, the line $x_1'x_2'$ is an X-line, which is not contained in a member of Θ by the foregoing. By Lemma 7.5.4 however, this implies $Y_{x_1} = Y_{x_2}$, a contradiction. So x_1' and x_2' are also non-collinear points of some member ξ' of Ξ, with vertex T. We show that $\rho(X(\xi)) = \rho(X(\xi'))$.

Let $x \in X(\xi)$ be a point contained in $x_1^\perp \cap x_2^\perp$. Then either $x \in \theta^{x_1} \cap \pi^{x_2}$ or $x \in \theta^{x_2} \cap \pi^{x_1}$. The two situations are the same, and one of them will suffice, so assume the former. Note that $\langle x, T\rangle$ is the unique generator of $X(\xi)$ in $\theta^{x_1} \cap \pi^{x_2}$ and that the X-points in $\theta^{x_1} \cap \pi^{x_2}$ can equivalently be described as the points in $X(\theta^{x_1})$ that are collinear to Y_{x_2}. Likewise, also $X(\xi')$ has a unique generator, say $\langle x', T\rangle$, contained in $\theta^{x'_1} \cap \pi^{x'_2}$, i.e., a unique generator whose X-points belong to $\theta^{x'_1} = \theta^{x_1}$ and are collinear to $Y_{x'_2} = Y_{x_2}$. As such, it is clear that $x' \in \theta^{x_1} \cap \pi^{x_2} = \langle x, Y_{x_2}\rangle = \langle x, Y_x\rangle$, and hence $\rho(x') = \rho(x)$.

Next, we show that also for each point $\overline{x}_2$ on $xx_2 \setminus \{x, x_2\}$ holds that $X(\xi')$ has a point $\overline{x}'_2$ with $\rho(\overline{x}'_2) = \rho(\overline{x}_2)$. Indeed, the line $x'x'_2$ is contained in the singular subspace $\langle x, x_2, Y_{x_2}\rangle$ (the points x, x', x_2 and x'_2 are all collinear to Y_{x_2}) and hence the unique point $\overline{x}'_2$ in $\langle \overline{x}_2, Y_{x_2}\rangle \cap x'x'_2$ is such that $\rho(\overline{x}_2) = \rho(\overline{x}'_2)$. Now consider a point $\overline{x}_1$ on $xx_1 \setminus \{x, x_1\}$. Inside $\theta^{x_1} = \theta^{x'_1}$, which also contains the line $x'x'_1$, we see that $\overline{x}_1$ is collinear to a unique point $\overline{x}'_1$ of $x'x'_1$ and that this line has a unique point of Y (as also $x_1x'_1$ and xx' do). Hence $\rho(\overline{x}_1) = \rho(\overline{x}'_1)$.

For a general point $\overline{x}$ on $X(\xi)$ not on $\langle xx_2, T\rangle \cup \langle xx_1, T\rangle$, we have that $\overline{x}$ is collinear to unique points $\overline{x}_2$ on xx_2 and $\overline{x}_1$ on xx_1. Then $\langle \overline{x}, T\rangle$ is the unique generator of $X(\xi)$ such that $\overline{x}$ belongs to $\theta^{\overline{x}_2}$ and such that $Y_{\overline{x}} = Y_{\overline{x}_1}$. Similarly as above, and also relying on the above obtained points of $X(\xi')$, we have that $X(\xi')$ contains a unique generator $\langle \overline{x}', T\rangle$ such that $\overline{x}'$ belongs to $\theta^{\overline{x}'_2} = \theta^{\overline{x}_2}$ and such that $Y_{\overline{x}'} = Y_{\overline{x}'_1} = Y_{\overline{x}_1}$, and for such a pair of points we have $\rho(\overline{x}') = \rho(\overline{x})$. We conclude that $\rho(X(\xi)) = \rho(X(\xi'))$. This shows the lemma. □

Lemma 7.6.9. *Let $\rho(x_1)$ and $\rho(x_2)$ be distinct points on a line of $\rho(X)$, for $x_1, x_2 \in X$. Then*

- (i) *there is a unique $\zeta \in \Theta \cup \Pi$ containing $\rho^{-1}(\rho(x_1)) \cup \rho^{-1}(\rho(x_2))$ and $\zeta \in \Theta$ if and only if $Y_{x_1} \neq Y_{x_2}$.*
- (ii) *for each $x'_1 \in \rho^{-1}(\rho(X))$, we can choose $x''_2 \in \rho^{-1}(\rho(x_2))$ such that $x'_1x''_2$ is an X-line.*

Proof. Again, $\rho^{-1}(\rho(x_i)) = \langle x_i, Y_{x_i}\rangle \cap X$ for $i = 1,2$ by Lemma 7.4.14. By Lemma 7.6.6, x_1 and x_2 belong to the same (unique) member of Π if and only if $Y_{x_1} = Y_{x_2}$; if so, this member also contains $\langle x_i, Y_{x_i}\rangle$ for $i = 1,2$, and each X-point of $\langle x_1, Y_{x_1}\rangle$ is on an X-line with each X-point of $\langle x_2, Y_{x_2}\rangle$, so in this case, (ii) follows.

So suppose $Y_{x_1} \neq Y_{x_2}$. We first claim that there are points $x'_1 \in \rho^{-1}(\rho(x_1))$ and $x'_2 \in \rho^{-1}(\rho(x_2))$ such that $x'_1x'_2$ is an X-line. To that end, consider the $(r'+2)$-space $\langle x_1, x_2, Y\rangle$ in which Y is a subspace of dimension r', and note that our assumption implies that $\langle x_1, x_2, Y\rangle = \langle x_1, x_2, Y_{x_1}, Y_{x_2}\rangle$. The fact that $\rho(x_1)$ and $\rho(x_2)$ are on a line of $\rho(X)$ implies that there is a point $x_3 \in X$ with $\rho(x_3)$ on $\rho(x_1)\rho(x_2) \setminus \{\rho(x_1), \rho(x_2)\}$. As such, $x_3 \in \langle x_1, x_2, Y\rangle \setminus (\langle x_1, Y_{x_1}\rangle \cup \langle x_2, Y_{x_2}\rangle)$. Inside $\langle x_1, x_2, Y\rangle$, we then get that $\langle x_3, x_1, Y_{x_1}\rangle$ intersects $\langle x_2, Y_{x_2}\rangle$ in an $(r'-1)$-space containing $Y_{x_1} \cap Y_{x_2}$ and some other point, say x'_2. Then the line x'_2x_3 intersects $\langle x_1, Y_{x_1}\rangle$ in a point x'_1. Since x_3 does not belong to Y, neither does the line $x'_1x'_2$. As this line does contain three points of $X \cup Y$, it is singular, implying that it is an X-line (since $x_3 \notin \langle x_1, Y_{x_1}\rangle$). The claim follows.

By Corollary 7.6.7, we conclude that the X-line $x'_1x'_2$ belongs to a member ζ of $\Theta \cup \Pi$, and since $Y_{x_1} \neq Y_{x_2}$, $\zeta \in \Theta$. Moreover, ζ contains $\langle x'_1, Y_{x_1}\rangle \cup \langle x'_2, Y_{x_2}\rangle$. Let $x'_1 \in \langle x_1, Y_{x_1}\rangle \cap X$

be arbitrary. Now, in ζ is it clear that x_1' is collinear to a hyperplane of $\langle x_2, Y_{x_2}\rangle$, which is definitely not Y_{x_2} since $Y_{x_1} \neq Y_{x_2}$, and hence x_1' it is collinear to some X-point x_2' of $\langle x_2, Y_{x_2}\rangle$. This shows (ii) also in this case. □

Let L be the set of X-lines. The previous lemma showed that, for each $\rho(X)$-line L', there is an $L \in \mathsf{L}$ with $\rho(L) = L'$.

Proposition 7.6.10. *The pair $(\rho(X), \rho(\mathsf{L}))$ is isomorphic to an injective projection of the Segre variety $\mathsf{S}_{r',k}(\mathbb{K})$, where $k = \dim(\pi) - r'$ for any $\pi \in \Pi$. The two natural sets of maximal singular subspaces of $\mathsf{S}_{r',k}(\mathbb{K})$ are such that one of them is in $1-1$-correspondence to Θ, and the other to Π; each grid $L_1 \times L_2$ for intersecting lines $L_1, L_2 \in \rho(\mathsf{L})$ is the image under ρ of some subset of Ξ which all have as vertex $\chi(x_1) \cap \chi(x_2)$, where x_1, x_2 is an arbitrary pair of non-collinear points on this grid.*

Proof. We first determine the maximal singular subspaces of $(\rho(X), \rho(\mathsf{L}))$. Let $\zeta \in \Theta \cup \Pi$ be arbitrary. It is clear that $\rho(X(\zeta))$ is a singular subspace of $\rho(X)$, which we will denote by S_ζ. Moreover, if $\zeta \neq \zeta' \in \Theta \cup \Pi$, then $S_\zeta \neq S_{\zeta'}$ as $S_\zeta \cap S_{\zeta'}$ cannot contain a $\rho(X)$-line by Lemma 7.6.9. We first show the following general claim.

Claim 1: each point $p \in \rho(X) \setminus S_\zeta$ is collinear to a unique point of S_ζ.
Suppose for a contradiction that there is a point $p \in \rho(X) \setminus S_\zeta$ collinear to two points s_1 and s_2 of S_ζ. Let $x \in X$ be a point with $\rho(x) = p$. Lemma 7.6.9 implies that, for $i = 1,2$, we can choose $x_i \in \rho^{-1}(s_i)$ such that xx_i an X-line (and by the same lemma, $x_1, x_2 \in \zeta$). If $\zeta \in \Pi$, then x_1x_2 is an X-line and by Corollary 7.6.7 together with $x \notin \zeta$, the X-lines xx_i, $i = 1,2$ are contained in $\theta^{x_i} = \theta^x$. However, this means that θ^x contains an X-line collinear to a hyperplane of Y, a contradiction. If $\theta \in \Theta$, then again x_1x_2 is an X-line since $x \notin \zeta$ and, as x is collinear to x_1x_2, we immediately obtain a contradiction against Lemma 7.6.5. This shows the claim.

Now let S be an arbitrary singular subspace of $\rho(X)$ containing at least a line L. By Lemma 7.6.9, L is contained in S_ζ for a unique $\zeta \in \Theta \cup \Pi$. If some point $p \in S \setminus L$ would not be contained in S_ζ, then this would violate the above claim. Hence $S \subseteq S_\zeta$. We conclude that $\{S_\zeta \mid \zeta \in \Theta \cup \Pi\}$ is precisely the set of maximal singular subspaces of $\rho(X)$.

Put $\mathsf{S}_\Theta := \{S_\theta \mid \theta \in \Theta\}$ and $\mathsf{S}_\Pi := \{S_\pi \mid \pi \in \Pi\}$. Let $p \in \rho(X)$ be arbitrary and take $x \in \rho^{-1}(p)$. By the previous paragraph, all lines of $\rho(X)$ through p are contained in precisely one of S_{θ^x}, S_{π^x} (cf. Corollary 7.6.7). In particular, $S_{\theta^x} \cap S_{\pi^x} = \{p\}$. Moreover, by Lemma 7.6.6, each pair $(S_\theta, S_\pi) \in \mathsf{S}_\Theta \times \mathsf{S}_\Pi$ is such that $S_\Theta \cap S_\Pi$ contains at least one point, which is again unique by the previous paragraph. We can now determine the structure of $\rho(X)$. To that end, let $(S_\theta, S_\pi) \in \mathsf{S}_\Theta \times \mathsf{S}_\Pi$ be arbitrary and denote their unique intersection point by p.

Claim 2: $\rho(X)$ is the direct product of S_θ and S_π.
Let q be an arbitrary point of $\rho(X)$. If $q \in S_\theta \cup S_\pi$, then $q = (p,s) \in \{p\} \times S_\pi$ for some $s \in S_\pi$ or $q = (s,p) \in S_\theta \times \{p\}$ for some $s \in S_\theta$. So suppose $q \notin S_1 \cup S_2$. By Claim 1, q is collinear to unique points s_θ^q and s_π^q of S_Θ and S_Π respectively. Moreover, the points s_θ^q and s_π^q determine q uniquely: Indeed, by the above, there is a unique member $S'_\pi \in \mathsf{S}_\Pi$ through

s_θ^q and a unique member $S'_\theta \in \mathsf{S}_\Theta$ through s_π^q, and then we have $S'_\theta \cap S'_\pi = \{q\}$. This shows the claim. Observe that this also implies that there are unique $\rho(X)$-lines through s_θ^q and s_π^q not in S_θ, S_π respectively, which intersect each other. Another important consequence is that $\dim(S_\pi) = \dim(S_{\pi'})$ for each two $\pi, \pi' \in \Pi$, as such the value k is well-defined.

Take $\xi \in \Xi$ such that $p \in \rho(X(\xi))$ and note that the two lines L_1 and L_2 of $\rho(X(\xi))$ through p are contained in S_θ and S_π, respectively. Then $\rho(X(\xi)) = L_1 \times L_2$. Conversely, let ps_θ and ps_π be arbitrary lines through p in S_θ and S_π, respectively, i.e., $s_\theta \in S_\theta \setminus \{p\}$ and $s_\pi \in S_\pi \setminus \{p\}$ and let G be the grid of $\rho(X)$ determined by $ps_\pi \times ps_\theta$.

Claim 3: there is a $\xi \in \Xi$ such that $\rho(X(\xi)) = G$, and $Y(\xi) = \chi(s_\theta) \cap \chi(s_\pi)$.
Let x_θ and x_π be X-points such that $\rho(x_\theta) = s_\theta$ and $\rho(x_\pi) = s_\pi$. Since s_θ and s_π are distinct points not on a line of $\rho(X)$, $[x_1, x_2]$ is a member ξ of Ξ with vertex $T := Y_{x_1} \cap Y_{x_2} = \chi(s_\theta) \cap \chi(s_\pi)$. By Lemma 7.6.8, $\rho(X(\xi))$ does not depend on our choice of X-points in the inverse images of s_θ and s_π. Since $\rho(X(\xi))$ is a full grid of $\rho(X)$-lines containing the points s_θ and s_π, the observation at the end of Claim 2 implies $\rho(X(\xi))$ has to coincide with $ps_1 \times ps_2$. This shows the claim.

Concerning the lines: $\rho(\mathsf{L})$ is the union of the sets $\{\{s_\theta\} \times L_\pi \mid s_\theta \in S_\theta, L_\pi \text{ a line of } S_\pi\}$ and $\{L_\theta \times \{s_\pi\} \mid L_\theta \text{ a line of } S_\theta, s_\pi \in S_\pi\}$, as follows from the fact that each line $L \in \rho(\mathsf{L})$ is contained in a unique member of $\mathsf{S}_\Theta \cup \mathsf{S}_\Pi$ (indeed, if $L \subseteq S'_\theta$ for some $S'_\theta \subseteq \mathsf{S}_\Theta$, then $s_\pi := S_\pi \cap S'_\theta = s_\pi^q$ for each $q \in L$, and $L_\theta = \{s_\theta^q \mid q \in L\}$, and the latter set is indeed a *line* of S_θ for it belongs to the grid $L \times qs_\theta^q$ for $q \in L$ arbitrary).

Since $\dim(S_\theta) = r'$ for each $S_\theta \in \mathsf{S}_\Theta$ and $\dim(S_\pi) = \dim(\pi) - (\dim(Y) - 1) - 1 = k$, this concludes the proof of the proposition. □

Corollary 7.6.11. *We have $N \leq (r'+1)(k+2) - 1$.*

Proof. By Lemma 7.6.1, we know $\dim(Y) = r'$ and by Proposition 7.6.10, $\dim(F) \leq (r'+1)(k+1) - 1$. Since F and Y are complementary subspaces of $\mathsf{PG}(N, \mathbb{K})$, we obtain $N \leq (r'+1)(k+2) - 1$. □

Also the next lemma is highly similar to Lemma 7.6.12. We give the proof up to the part where there are differences, and do not repeat the parts which are exactly the same.

Lemma 7.6.12. *There exists a subspace F^* such that $F^* \cap X$ contains a legal projection Ω of $\mathsf{S}_{r',k}(\mathbb{K})$ with $\langle \Omega \rangle = F^*$, which is such that, for each $x \in X$, there is a point $\overline{x} \in \Omega$ such that $x \in \langle x, Y_x \rangle$. The projection of $F^* \cap X$ from $F^* \cap Y$ is injective, and $F^* \cap Y$ is empty if $r' = 2$ and $k \leq 2$.*

Proof. By Proposition 7.6.10, $\rho(X)$ is the point set of an injective projection of the Segre geometry $\mathsf{S}_{r',k}(\mathbb{K})$, and the elements $\zeta \in \Theta \cup \Pi$ are in $1-1$-correspondence to the set of maximal singular subspaces S_ζ. We now show that we can construct a legal projection of a Segre variety using r'-dimensional X-spaces well-chosen in certain members of Θ.

To that end, take a basis of hyperplanes $V^0, \ldots, V^{r'}$ in Y. For $t \in \{0, \ldots, r'\}$, denote by π^t the maximal singular subspace $\pi(V^t)$ generated by X-points collinear to V^t (cf. Lemma 7.6.6).

Take any X-space of dimension k in π^0 complementary to V^0 and let $(x_{0,0},\dots,x_{0,k})$ be a basis of π^0. Consider the unique respective members $\theta^0,\dots,\theta^k$ of Θ containing the points $x_{0,0},\dots,x_{0,k}$. Just like in the proof of Lemma 7.5.30, we can consecutively select X-points $x_{t,u} \in \pi^t \cap \theta^u$ with $t \in \{1,\dots,r'\}$ and $u \in \{0,\dots,k\}$ such that $x_{t,u} \perp x_{t',u}$ with $0 \le t' < t$ (the condition $x_{t,u} \perp x_{t,u'}$ with $u' < u$ being trivially fulfilled now as π^t is a singular subspace).

Put $S^u := \langle x_{t,u} \mid 0 \le t \le r' \rangle$ for each $u \in \{0,\dots,k\}$. For each point of S^0, there is a unique singular k-space through it intersecting S^u for each $u \in \{0,\dots,k\}$, and this k-space belongs to X, as can be proven just like in the proof of Lemma 7.5.30. We define Ω as the union of all these k-spaces; and by (S2), Ω is a legal projection of a Segre variety $\mathsf{S}_{r',k}(\mathbb{K})$. Put $F^* = \langle S^u \mid 0 \le u \le k \rangle$. The fact that each X-point is contained in $\langle x, Y_x \rangle$ for some $x \in X$ also follows as in the proof of Lemma 7.5.30; as does the fact that the projection of $F^* \cap X$ from $F^* \cap Y$ is injective.

We now show that $F^* \cap Y$ is empty if $r' = 2$ and $k \le 2$. If $k = 2$, then this is the same argument as in the proof of Lemma 7.5.30; so suppose $k = 1$. If y is a point of Y contained in F^*, then y is on a unique line intersecting two planes of Ω. This line contains three points of $X \cup Y$ and is hence singular, and by the above, this line is an X-line, a contradiction.

This concludes the proof. $\square$

Henceforth we assume that F^* is as in the previous lemma.We now focus on the connection between $\rho(X)$ and Y.

Proposition 7.6.13. *Let (X, Z, Ξ, Θ) be a duo-symplectic split pre-Veronese set with an X-point through which there is precisely one member of Θ. Then*

- (i) *Projecting X from a v'-space $V \subseteq Y$ collinear to all points of X, we obtain the point set of a mutant of a half dual Segre variety $\mathsf{HDS}_{r',k}(\mathbb{K})$;*
- (ii) *if additionally, (X, Z, Ξ, Θ) satisfies (S3), then $(r', k) \in \{(2,1),(2,2)\}$ and X is projectively unique.*

Proof. (i) Lemma 7.6.2 yields the v'-space V collinear to all points of X. Lemma 7.4.11 again allows us to project from V, so that we only need to deal with the case where $v' = -1$.

By Proposition 7.6.12, we know that there is a subspace F^* in $\mathsf{PG}(N, \mathbb{K})$ containing a legal projection Ω of $\mathsf{S}_{r',k}(\mathbb{K})$, and that the projection of $X \cap F^*$ from $Y \cap F^*$ is injective. We also know that, if S_Θ and S_Π denote the two sets of maximal singular subspaces of Ω, then each of their members is contained in a unique member of Θ or Π, respectively. We also showed that $X = \{\langle x, Y_x \rangle \setminus Y_x \mid x \in \Omega\}$. Hence $X = \{\langle x, Y_x \rangle \setminus \chi(x) \mid x \in \Omega\}$.

We now show that the correspondence $\Omega \to Y : x \mapsto Y_x$ (which we will also denote by χ) is as described in Sections 7.2.1 or 7.2.3. Take $S_\theta \in \mathsf{S}_\Theta$ arbitrary. Denote by χ_S the restriction of χ to S. Inside the polar space $XY(\theta)$ (in which S_θ is an X-space of dimension r') it is clear that χ_S coincides with the collinearity relation between the opposite subspaces S_θ and Y, and as such we get that it is a linear duality between S_θ and Y. Let $x \in X$ be arbitrary. If $x \notin S_\theta$, then there is a unique point $s_\theta^x \in S_\theta$ collinear to x. For such a point, we know that $Y_{s_\theta^x} = Y_x$ (cf. Lemma 7.6.2), and hence $\chi(s_\theta^x) = \chi(x)$. We conclude that χ is indeed as described in Section 7.2.1.

(ii) By Proposition 7.6.10, we know that, for any subspace F in $\mathsf{PG}(N,\mathbb{K})$ complementary to Y, $\rho(X)$ is an injective projection of a Segre variety $\mathsf{S}_{r',k}(\mathbb{K})$ in F. Let $T^F_{\rho(x)}$ be the set of $\rho(X)$-lines in F through $\rho(x)$ and $T^F_{\rho(x)}(\rho(X(\xi)))$ be the tangent hyperplane to $\rho(X(\xi))$ at $\rho(x)$ for some $\xi\in\Xi$ with $\rho(x)\in\rho(\Xi)$. By (S3), there are members $\xi_1,\xi_2\in\Xi$ through x such that T_x is generated by $T_x(\xi_1)$ and $T_x(\xi_2)$. For $i=1,2$, we have that $T_x(\xi_i)=\langle Y(\xi_i),T^F_{\rho(x)}(X(\xi_i))\rangle$. So $T_x=\langle T_x(\xi_1),T_x(\xi_2)\rangle$ is equivalent with $Y_x=\langle Y(\xi_1),Y(\xi_2)\rangle$ and $T^F_{\rho(x)}=\langle T^F_{\rho(x)}(X(\xi_1)),T^F_{\rho(x)}(X(\xi_2))\rangle$.

In $\rho(X)$, we see that $T^F_{\rho(x)}$ is generated by the unique r'- and k-dimensional subspaces S_{θ^x} and S_{π^x} of this variety through $\rho(x)$, and hence $\dim(T^F_{\rho(x)})=r'+k$. As $r=1$, we deduce that $\langle S_{\theta^x},S_{\pi^x}\rangle$ can only be generated by $T^F_{\rho(x)}(X(\xi_1))$ and $T^F_{\rho(x)}(X(\xi_2))$ if $r'+k\leq 2+2=4$, which implies $r'=2$ and $k\in\{1,2\}$ (because $r'>r\geq 1$ by assumption). Recalling that $v=r'-2=0$ and $\dim(Y_x)=r'-1=1$, the first requirement only implies that ξ_1 and ξ_2 will have distinct vertices.

Now, if $(r',k)\in\{(2,1),(2,2)\}$, then we know that $F^*\cap Y=\emptyset$ and that $F^*\cap X=\Omega$ is projectively unique (cf. Proposition 7.2.7). Since Ω and Y are projectively unique, and, for $i=1,2$, the projectivity χ_{S_θ} between S_θ and the dual of Y is unique up to a projectivity of S_θ, also X is projectively unique. This shows (ii). □

7.7 The dual line Grassmannians

7.7.1 Point residues

Throughout this section, (X,Z,Ξ,Θ) is a duo-symplectic split pre-Veronese set with parameters (r,v,r',v') with $r\geq 2$.

To benefit from our earlier work on $r=1$, we will consider residues:

Definition 7.7.1. For each $x\in X$, we define the *point residue* $\mathrm{Res}(x):=(X_x,Z_x,\Xi_x,\Theta_x)$ as follows: X_x consists of all X-lines through x; Z_x consists of all lines through x having a point in Z; Ξ_x is defined as $\{\mathrm{Res}_\xi(x)\mid x\in\xi\in\Xi\}$ and Θ_x as $\{\mathrm{Res}_\theta(x)\mid x\in\theta\in\Theta\}$.

In order to show that this residue is a pre-Veronese set as well, we first need that $\langle Z_x\rangle=\langle x,Y_x\rangle$ (with $Y_x=x^\perp\cap Y$ as before).

Lemma 7.7.2. *For each $x\in X$, the set $Y_x\cap Z$ generates Y_x.*

Proof. Suppose for a contradiction that $\langle Y_x\cap Z\rangle$ is a strict subspace H of Y_x. Let $\theta\in\Theta$ through x be arbitrary (cf. Lemma 7.4.18). Take an X-line L through x in θ. By Lemma 7.4.22(i), Y_L is strictly contained in Y_x. As such, there is a point $y\in Y_x\setminus(Y_L\cup H)$. By Lemma 7.4.3, xy and L are contained in some $\theta'\in\Theta$. By definition, $\langle Z(\theta')\rangle=Y(\theta')$. However, $Y_x\cap Y(\theta')$ is a hyperplane of $Y(\theta')$ whose Z-points are contained in $H\cap\theta'$ (which is at most a hyperplane of $Y_x\cap Y(\theta')$), implying that $\langle Z(\theta')\rangle<Y(\theta')$. This contradiction shows that $\langle Y_x\cap Z\rangle=Y_x$. □

Lemma 7.7.3. *For each $x \in X$, the point-residue $(X_x, Z_x, \Xi_x, \Theta_x)$ is a duo-symplectic pre-Veronese set in a projective space $\mathsf{PG}(N_x, \mathbb{K})$ with $N_x = \dim(T_x) - 1$, with parameters $(r - 1, v, r' - 1, v')$.*

Proof. Put $\overline{Y}_x = \langle Z_x \rangle$ (this notation should avoid confusion with the closely related set Y_x). By Lemma 7.7.2, $\overline{Y}_x$ consists of all singular lines through x having a unique point in Y (in Y_x, to be precise). Now T_x is generated by the lines in $X_x \cup Y_x$ by definition, and hence $T_x = \langle X_x, Z_x \rangle$. Let $\mathsf{PG}(N_x, \mathbb{K})$ be the projective space $\mathrm{Res}_{T_x}(x)$ (so $N_x = \dim(T_x) - 1$), in which $X_x \cup Z_x$ is a spanning point set. It is then a straightforward verification that each $\zeta \in \Xi \cup \Theta$ intersects the sets X_x, Y_x and Z_x as described in Definition A.1.8, and that the associated parameters are $(r - 1, v, r' - 1, v')$.

The fact that (S2) holds in the residue is clear, so we show that this is also the case for (S1). Let L and M be two X-lines through x, and let x_L and x_M be points on $L \setminus \{x\}$ and $M \setminus \{x\}$, respectively. By Lemmas 7.4.3 and 7.4.5, we either have that x_L and x_M determine a member of $\Xi \cup \Theta$ containing x, in which case (S1) follows, or that $\langle L, M \rangle$ is a singular plane. In the latter case, (S1) implies the existence of a member ζ of $\Xi \cup \Theta$ containing L. If ζ also contains M, we are good, so suppose it does not. Since $r, r' \geq 2$, there is a singular plane π in ζ through L not collinear to M (cf. Lemma 7.4.5). For any point $x' \in \pi \setminus L$, we then have that $[x', x_M] \in \Xi \cup \Theta$ contains L and M and hence also in this case, (S1) follows. □

Remark 7.7.4. Let H be a hyperplane of T_x containing Y_x and not containing x. Then we can identify the lines in X_x, Z_x and $\overline{Y}_x$ with their intersection points with H. Doing so, we can identify $\overline{Y}_x$ and Y_x, making things easier to picture.

Lemma 7.7.5. *Let $x \in X$ be arbitrary. If there is a subspace $V^* \subseteq Y_x$ such that all X-points collinear to x are collinear to V^*, then all X-points are collinear to V^*.*

Proof. Take any point $x' \in X$ not collinear to x. Then $[x, x'] \in \Xi \cup \Theta$. Take a pair of non-collinear lines L_1 and L_2 through x in $[x, x']$. Then $L_1 \perp V^* \perp L_2$, and hence $[x, x']$ has V^* in its vertex, so $x' \perp V^*$. We conclude that all X-points are collinear to V^* indeed. □

By Lemma 7.4.11, we may hence assume that no residue $\mathrm{Res}(x)$ contains points in Y_x collinear to all points of X_x.

The following property will help us to distinguish cases.

Lemma 7.7.6. *If there is an X-line which is contained in a unique member of Θ, then so are the others.*

Proof. For any point x on an X-line L which is contained in a unique member of Θ, we have that, in $\mathrm{Res}(x)$, the point corresponding to the line L is contained in a unique member of Θ_x. Hence, by Lemma 7.6.1, this holds for each point of $\mathrm{Res}(x)$, i.e., each X-line through x is contained in a unique member of Θ. Since x on L was arbitrary and since X is connected via X-lines by (S1), the lemma follows. □

7.7.2 Case distinction

Let (X, Z, Ξ, Θ) be a duo-symplectic split pre-Veronese set with parameters (r, v, r', v') with $r \geq 2$, containing no Y-points collinear to all points of X.

By Lemma 7.7.5, also each residue of (X, Z, Ξ, Θ) contains no Y-points collinear to all X-points of the residue.

We consider subsequent point-residues until we obtain one in which $r = 1$ (or equivalently, we take an S-residue where S is an X-space of dimension $r-2$). By Sections 7.5 and 7.6, such a residue is either a dual Segre variety $\mathsf{DS}_{R,R}(\mathbb{K})$ for some natural number $R \geq 2$, or a half dual Segre variety $\mathsf{HDS}_{R,K}(\mathbb{K})$ for some natural numbers R, K with $R \geq 2$ and $K \geq 1$, respectively. There are three options:

1. *There is an X-line contained in a unique member of Θ.* This means that, for $x \in L$, there is a point of $\text{Res}(x)$ contained in a unique member of Θ_x. In this case Proposition 7.6.13 implies that $\text{Res}(x)$ is a mutant of a half dual Segre variety. In particular, $r = 2$.

2. *Each X-line is contained in at least two members of Θ_x.* Here, there are two subcases:

   (a) If $r = 2$, then Proposition 7.5.31 implies that $\text{Res}(x)$ is a mutant of a dual Segre variety, for every $x \in X$. We will show that this situation conflicts with (S3).

   (b) If $r > 2$, then by Corollary 7.4.20 and Proposition 7.6.13, the residue $\text{Res}(L)$ is a mutant of a half dual Segre variety $\mathsf{HDS}_{r'-2,k}(\mathbb{K})$, for every X-line L. In particular, $r = 3$. We will show that this situation also conflicts with (S3), unless $k = 1$, in which case a further analysis is required to rule out these.

Only Case 1 will lead to an existing case. We can immediately rule out Case 2(a).

Lemma 7.7.7. *If (X, Z, Ξ, Θ) is a duo-symplectic split Veronese set with $r = 2$, then for each $x \in X$,* $\text{Res}(x)$ *is a mutant of a half dual Segre variety.*

Proof. Suppose that there is a point $x \in X$ for which $\text{Res}(x)$ is not isomorphic to a mutant of a half dual Segre variety. Since $r - 1 = 1$, the only alternative is that $\text{Res}(x)$ is isomorphic to a mutant of the dual Segre variety $\mathsf{DS}_{r'-1,r'-1}(\mathbb{K})$. By Lemma 7.5.19, Y_x is generated by $(r'-1)$-spaces $R_x'^1$ and $R_x'^2$, so we have $\dim(Y_x) = 2r' - 1$. Now (S3) yields members ξ_1, ξ_2 of Ξ through x such that T_x is generated by $T_x(\xi_1)$ and $T_x(\xi_2)$. Let V_1 and V_2 be the respective vertices of ξ_1 and ξ_2. Then $\langle V_1, V_2 \rangle \subseteq Y_x$, so by the above $\dim(\langle V_1, V_2 \rangle) \leq 2r' - 1$. Since $r = 2$, it follows that $\dim\langle T_x(\xi_1), T_x(\xi_2) \rangle \leq 2r' + 8$.

So $\text{Res}(x)$ is contained in a projective space of dimension $2r' + 7$, implying that a legal projection of $\mathsf{S}_{r'-1,r'-1}(\mathbb{K})$ (cf. Lemma 7.5.30) is contained in a projective space F_x^* in $\text{Res}(x)$. Since $r' > r = 2$, Π contains a copy Π' of $\mathsf{S}_{2,2}(\mathbb{K})$, which lives in 8 dimensions and hence $\langle \Pi' \rangle \cap Y$ contains a point y. Then y is contained in the vertex of a quadric determined by a grid G of Π' (note that the set of vertices induced by the grids of Π' is the set of $(2r'-3)$-spaces of Y_x intersecting $R_x'^i$ in an $(r'-3)$-space through a fixed $(r'-4)$-space H_x^i, $i=1,2$). The 4-space $\langle G,y\rangle$ is hence contained in the 8-dimensional subspace generated by the Segre variety Π' and intersects Π' in precisely G. Just like in the proof of Lemma 7.5.30, Lemma 7.2.8 then leads to a contradiction to (S2). This shows the lemma. $\square$

To rule out Case 2(b), we need a little more work. But some of it will be very useful to classify Case 1, which does lead to examples.

We conclude this subsection with the following result concerning the case $r=2$.

Lemma 7.7.8. *Let (X,Z,Ξ,Θ) be a duo-symplectic split Veronese set with $r=2$. Suppose some $x\in X$ is such that* $\mathrm{Res}(x)$ *is a mutant of the half dual Segre variety* $\mathsf{HDS}_{r'-1,k}(\mathbb{K})$ *for some natural number k with $k\geq 1$. Then* $\dim(Y)=r'+k$.

Proof. Take a hyperplane H in T_x complementary to x and containing Y_x. By Remark 7.7.4, $H\cap(X_x\cup Y_x)$ is isomorphic to $\mathrm{Res}(x)$, i.e., a mutant of a half dual Segre variety $\mathsf{HDS}_{r'-1,k}(\mathbb{K})$ for some natural number $k\geq 1$. By abuse of notation, we denote the point sets $H\cap X_x$ and $H\cap Z_x$ by X_x and Z_x, respectively; the point set $H\cap\overline{Y}_x$ is, by choice of H, precisely Y_x.

By Proposition 7.6.10 and Lemma 7.6.12, there is a subspace F_x^* with $\langle F_x^*,Y_x\rangle=H$ such that $F_x^*\cap X_x$ contains the point set of the Segre variety $\mathsf{S}_{r'-1,k}(\mathbb{K})$. The latter's $(r'-1)$-spaces are in $1-1$-correspondence to the members of Θ_x, which are of course also in $1-1$-correspondence to the members of Θ containing x. Note moreover that the members of Θ containing x all contain Y_x by Lemma 7.6.1, and by Lemma 7.6.2 we also know that $v'=-1$, so $Y_x=Z_x$ and $Y(\theta)\subseteq Z$ for each $\theta\in\Theta$.

Let I be an index set such that $\{\theta_i\in\Theta\mid i\in\mathsf{I}\}$ ranges over all members of Θ containing x. Let $i\in\mathsf{I}$ be arbitrary. Denote by S_i the unique $(r'-1)$-space in F_x corresponding to θ_i (i.e., $S_i\subseteq\theta_i$), and let z_i be the unique point in $Y(\theta_i)$ collinear to S_i ($z_i\in Z$ as $v'=-1$). Since both $\langle x,S_i\rangle$ and $\langle z,S_i\rangle$ are maximal singular subspaces of θ_i, the points x and z_i are not collinear, i.e., $\theta_i=[x,z_i]$ and $Y(\theta_i)=\langle Y_x,z_i\rangle$. Since, for $i,i'\in\mathsf{I}$, we by definition have that $\theta_i=\theta_{i'}$ if and only if $i=i'$, the foregoing implies that also $z_{i'}\in\theta_i$ if and only if $i=i'$. We claim that the points $\{z_i\mid i\in\mathsf{I}\}$ are the points of a k-dimensional subspace K_x of Y complementary to Y_x. To that end, take two arbitrary members $\theta_1,\theta_2\in\{\theta_i\mid i\in\mathsf{I}\}$ and let $\mathsf{J}\subseteq\mathsf{I}$ be such that $\{S_j\mid j\in\mathsf{J}\}$ is the unique regulus of $F_x\cap X_x$ determined by S_1 and S_2. We show that the points $\{z_j\mid j\in\mathsf{J}\}$ are the points of a line of Y.

Take any point $x_1\in S_1$. Then there is a unique line L through x_1 meeting each S_j for $j\in\mathsf{J}$ in a point, which we will denote by x_j. Note that, for each $j\in\mathsf{J}$, θ_j is the unique member of Θ containing the X-line xx_j; so Lemma 7.4.22(ii) then implies that $Y_{x_j}\subseteq\theta_j$. In particular we obtain that x_1 is not collinear to z_2 (as otherwise $z_2\in Y(x_1)\subseteq\theta_1$, a contradiction); and hence $[x_1,z_2]\in\Theta$; likewise $[x_2,z_1]\in\Theta$. By Lemma 7.7.6, there is only one member of Θ containing x_1x_2, so $[x_1,z_1]=[x_2,z_1]$. As such, the line z_1z_2 contains a unique point z_j' collinear to x_j for each $j\in\mathsf{J}$ (clearly, $z_1'=z_1$ and $z_2'=z_2$). Since $z_j'\in Y_{x_j}\subseteq\theta_j$, we obtain $\langle z_1,z_2\rangle\cap\theta_j=\{z_j'\}$ for each $j\in\mathsf{J}$. If we vary the point $x_1\in S_1$, then this means that the corresponding point on $\langle z_1,z_2\rangle$ collinear to it is always given by $\langle z_1,z_2\rangle\cap\theta_j$, i.e., the point

z'_j is collinear to all points of S_j, so $z'_j = z_j$. This shows the claim: the point set $\{z_i \mid i \in \mathsf{I}\}$ carries the same structure as a k-space of the legal projection of the Segre variety in F^*_x intersecting all subspaces $\{S_i \mid i \in \mathsf{I}\}$.

As each point z of $Y \setminus Y_x$ determines a unique member of Θ through x we obtain that $Y = \langle Y_x, K_x \rangle$, so $\dim(Y) = r' + k$ indeed. □

7.7.3 Eliminating split Veronese sets with $r = 3$

Our first result restricts the possibilities for the line residues.

Lemma 7.7.9. *Let (X, Z, Ξ, Θ) be a duo-symplectic split Veronese set with $r = 3$, and let L be any X-line. Then* $\mathrm{Res}(L)$ *is a mutant of a half dual Segre variety* $\mathsf{HDS}_{r'-2,1}(\mathbb{K})$.

Proof. Suppose for a contradiction that for some line L, $\mathrm{Res}(L)$ is a mutant of a half dual Segre variety $\mathsf{HDS}_{r'-2,k}(\mathbb{K})$, with $k > 1$ (cf. Section 7.7.2). Let $x \in L$ be arbitrary. According to (S3), there are members ξ_1, ξ_2 of Ξ through x such that T_x is generated by $T_x(\xi_1)$ and $T_x(\xi_2)$. By Lemma 7.6.4, we know $v = (r'-2)-2$, and hence $\dim T_x(\xi_i) = r' + 3$, $i = 1, 2$. It follows that $\dim T_x \leq 2r' + 6$.

Clearly, T_x contains T_L and Y_x, and $T_L \cap Y_x = Y_L$, so $\dim(T_x) \geq \dim(T_L) + \dim(Y_x) - \dim(Y_L)$. Since T_L is generated by L and a mutant of $\mathsf{HDS}_{r'-2,k}(\mathbb{K})$, we may set $\dim T_L = A + (r'-1) + 1 + 1$, where A is the dimension of an injective projection, say Ω, of $\mathsf{S}_{r'-2,k}(\mathbb{K})$ (obtained from projecting $\mathsf{HDS}_{r'-2,k}$ from Y_L, cf. Lemma 7.6.10). Since $\dim Y_x = (r'-1) + k$ (cf. Lemma 7.7.8) and $\dim Y_L = r' - 2$, we obtain $\dim(T_x) \geq A + r' + k + 2$. Combined with the previous paragraph, this yields $A \leq r' - k + 4$. Trivially, $A \geq \max\{r' - 2 + k, 2r' - 3\}$, which leads to $2 \leq k \leq 3$ and $r' + k \leq 7$, i.e., $r' \in \{4, 5\}$ (recall $r' > r = 3$).

The above implies that $A \leq 7$. We also know that $\mathrm{Res}(L) = \langle A, Y_L \rangle$ contains a legal projection Π of $\mathsf{S}_{r'-2,k}(\mathbb{K})$ (cf. Lemma 7.6.12). Since $r' \in \{4, 5\}$ and $k \geq 2$, Π contains a copy Π' of $\mathsf{S}_{2,2}(\mathbb{K})$, which lives in 8 dimensions and hence $\langle \Pi' \rangle \cap Y$ contains a point y. Just as in the proof of Lemma 7.7.7, y is contained in the vertex of a quadric determined by a grid G of Π' (this time, the set of vertices induced by the grids of Π' is the set of $(r'-4)$-spaces of Y_L through a fixed $(r'-5)$-space). The 4-space $\langle G, y \rangle$ is hence contained in the 8-dimensional subspace generated by the Segre variety Π' and intersects Π' in precisely G. As before, Lemma 7.2.8 then leads to a contradiction to (S2). This shows the lemma. □

So, considering $\mathrm{Res}(x)$ for any point $x \in X$, our next task is to analyse split pre-Veronese sets with $r = 2$ such that point-residues are isomorphic to mutants of dual half Segre varieties $\mathsf{DHS}_{r'-1,1}(\mathbb{K})$. In fact we wil classify these. As already mentioned, this classification will not only lead to the elimination of Case 2(b), but will also prepare for classifying the examples of Case 1.

For the remainder of this subsection, (X, Z, Ξ, Θ) is a duo-symplectic split pre-Veronese set with parameters $(2, v, r', v')$, containing no Y-points collinear to all points of X, and such that for each point $x \in X$, $\mathrm{Res}(x)$ is isomorphic to a mutant of a half dual Segre variety $\mathsf{HDS}_{r'-1,1}(\mathbb{K})$.

Note that these assumptions imply $v = r' - 3$ and $v' = -1$.

We now proceed similarly as in the previous sections and nail down the entire structure of (X, Z, Ξ, θ), using the projection ρ and the connection map χ (cf. Definitions 7.4.12 and 7.4.13).

Lemma 7.7.10. *Take two distinct points $x_1, x_2 \in X$. Then*

(i) *x_1x_2 is a singular line with a unique point in $Y \iff Y_{x_1} = Y_{x_2}$;*
(ii) *x_1 and x_2 belong to a member of $\Theta \iff \dim(Y_{x_1} \cap Y_{x_2}) = r' - 2$;*
(iii) *x_1 and x_2 are non-collinear points of a member of $\Xi \iff \dim(Y_{x_1} \cap Y_{x_2}) = r' - 3$.*

Proof. Recall that $\dim(Y_{x_i}) = r' - 1$ (cf. Lemma 7.6.1), that $\dim(Y) = r' + 1$ (by Lemma 7.7.8 and $k = 1$) and that $v = r' - 3$.

Firstly, if x_1x_2 is a singular line with a unique point in Y, then $x_2 \in \langle x_1, Y_{x_1} \rangle$ and as such it is clear that $Y_{x_1} = Y_{x_2}$. Secondly, suppose that x_1x_2 is an X-line. Since x_1x_2 is contained in a unique member of Θ (cf. Lemma 7.7.6), Lemma 7.4.22(i) implies that $Y_{x_1x_2}$ is a hyperplane of Y_{x_1}, from which we obtain that $\dim(Y_{x_1} \cap Y_{x_2}) = r' - 2$. Thirdly, suppose that x_1 and x_2 are not collinear. If $[x_1, x_2] \in \Xi$, then $\dim(Y_{x_1} \cap Y_{x_2}) = v = r' - 3$. If $[x_1, x_2] \in \Theta$, then since $\dim(Y([x_1, x_2])) = r'$, we obtain that $\dim(Y_{x_1} \cap Y_{x_2}) = r' - 2$ as x_1 and x_2 are non-collinear X-points. Since each X-line is contained in a member of Θ, the lemma follows. $\square$

We record an obvious but important consequence.

Corollary 7.7.11. *If $x_1, x_2 \in X$ are such that $Y_{x_1} = Y_{x_2}$, then $\rho(x_1) = \rho(x_2)$.*

Proof. By the previous lemma, $Y_{x_1} = Y_{x_2}$ implies that x_1x_2 is a singular line with a unique point in Y. As such, $x_2 \in \langle x_1, Y_{x_1} \rangle$ and thus, by Lemma 7.4.14, we get $\rho(x_1) = \rho(x_2)$. $\square$

Lemma 7.7.12. *Let $\rho(x_1)$ and $\rho(x_2)$ be distinct points on a line of $\rho(X)$, for $x_1, x_2 \in X$. Then:*

(i) *there is a unique $\theta \in \Theta$ containing $\rho^{-1}(\rho(x_1)) \cup \rho^{-1}(\rho(x_2))$;*
(ii) *for each $x_1' \in \rho^{-1}(\rho(X))$, we can choose $x_2'' \in \rho^{-1}(\rho(x_2))$ such that $x_1'x_2''$ is an X-line.*
(iii) *$\{Y_x \mid \rho(x) \in \langle \rho(x_1), \rho(x_2) \rangle\}$ is the set of all $(r'-1)$-spaces through the $(r'-2)$-space $Y_{x_1} \cap Y_{x_2}$ inside the r'-space $Y(\theta)$.*

Proof. Lemma 7.4.14 implies that $\rho^{-1}(\rho(x_i)) = \langle x_i, Y_{x_i} \rangle \cap X$ for $i = 1, 2$.

(i) By Lemma 7.7.10, there is a member of θ containing x_1 and x_2 if and only if $\dim(Y_{x_1} \cap Y_{x_2}) = r' - 2$, and such a member always contains $\langle x_i, Y_{x_i} \rangle$ for $i = 1, 2$ (cf. Lemma 7.4.22(ii)). Suppose for a contradiction that no such member exists. Since $\rho(x_1) \neq \rho(x_2)$ by assumption, Corollary 7.7.11 implies that $Y_{x_1} \neq Y_{x_2}$, so we obtain from Lemma 7.7.10 that $\dim(Y_{x_1} \cap Y_{x_2}) = r' - 3$, i.e., $Y = \langle Y_{x_1}, Y_{x_2} \rangle$. Completely similarly as in Lemma 7.5.27, we can then show that there are points x_1' and x_2' in $\langle x_1, Y_{x_1} \rangle$ and $\langle x_2, Y_{x_2} \rangle$, respectively, which are on an X-line. As such a line is contained in a member of Θ after all, we obtain a contradiction. We conclude that there is a unique $\theta \in \Theta$ containing x_1 and x_2, which then also contains $\langle x_1, Y_{x_1} \rangle \cup \langle x_2, Y_{x_2} \rangle$.

(ii) Let $x_1' \in \langle x_1, Y_{x_1}\rangle \cap X$ be arbitrary. In θ, x_1' is collinear to a hyperplane of $\langle x_2, Y_{x_2}\rangle$ distinct from Y_{x_2}, yielding an X-point $x_2' \in \langle x_2, Y_{x_2}\rangle$ collinear to x_1'.

(iii) Clearly, ρ gives a bijective correspondence between the points of the X-line $x_1'x_2'$ and the points of the $\rho(X)$-line $\langle \rho(x_1), \rho(x_2)\rangle$. Furthermore, looking in θ, it is also clear that the collinearity relation $x \mapsto Y_x$ is a bijection between the points on $x_1'x_2'$ and the $(r'-1)$-space of $Y(\theta)$ containing $Y_{x_1} \cap Y_{x_2}$. □

Lemma 7.7.13. *Let x_1 and x_2 be non-collinear X-points of some $\xi \in \Xi$, and denote the vertex of the latter by T. Then:*

- (i) *If $x_i' \in \rho^{-1}(\rho(x_i))$ for $i = 1,2$, then also x_1' and x_2' determine a unique member $\xi' \in \Xi$, which also has vertex T and with $\rho(X(\xi)) = \rho(X(\xi'))$.*
- (ii) *$\rho^{-1}(\rho(X(\xi)))$ is precisely the set of X-points collinear to T, which coincides with the set of X-points on members of Ξ having T as their vertex.*
- (iii) *$\{Y_x \mid \rho(x) \in \rho(X(\xi))\}$ is the set of all $(r'-1)$-spaces through T in Y.*

Proof. By Lemma 7.4.14, $x_i' \in \langle x_i, Y_{x_i}\rangle \cap X$ for $i = 1,2$. It follows from Lemma 7.7.10 that also x_1' and x_2' are non-collinear points in some $\xi' \in \Xi$. Moreover, the respective vertices T and T' of ξ and ξ' coincide as they are both given by $Y_{x_1} \cap Y_{x_2}$. Now let ξ' be any member of Ξ with vertex T. We show that $\rho(X(\xi)) = \rho(X(\xi'))$.

To that end, we study the set $\{Y_x \mid x \in X(\xi)\}$. We show that this gives, in $\mathrm{Res}_T(Y)$, the set of all lines of a 3-space. Since $\dim(T) = r'-3$ and $\dim(Y) = r'+1$, we indeed have that $\dim(\mathrm{Res}_T(Y))$ is a 3-space, say Π_T. It is also clear that for each $x \in X(\xi)$, the subspace Y_x is an $(r'-1)$-space through T which hence corresponds to a line, say $L(x)$, in Π_T. Let x, x' be two points of $X(\xi)$. By Lemma 7.7.10, $L(x) = L(x')$ if and only if x and x' belong to the same generator of $X(\xi)$, $L(x)$ and $L(x')$ intersect in a point if and only if xx' is an X-line in $X(\xi)$ and $L(x)$ and $L(x')$ are disjoint if and only if x and x' are non-collinear. Moreover, Lemma 7.7.12 implies that each X-line of $X(\xi)$ corresponds to a full planar point pencil in Π_T.

On the other hand, the Klein correspondence yields that the point-line geometry whose point set is the set of all lines of Π_T and whose lines are the planar line pencils of Π_T is isomorphic to the point-line geometry associated to a Klein quadric Q in $\mathsf{PG}(5,\mathbb{K})$. As such, the foregoing implies that $\rho(X(\xi))$ (which is also a Klein quadric in $\mathsf{PG}(5,\mathbb{K})$ as $r = 2$) embeds isometrically into Q, and as such coincides with Q. We conclude that $\{L(x) \mid x \in X(\xi)\}$ is indeed exactly the set of all lines in Π_T. In particular, for each $(r'-1)$-space H in Y through T, we have that both ξ and ξ' contain unique respective generators $\langle x, T\rangle$ and $\langle x', T\rangle$ such that $Y_x = H = Y_{x'}$, and hence $\rho(x) = \rho(x')$. This shows that $\rho(X(\xi)) = \rho(X(\xi'))$ indeed. We have shown all three assertions. □

Lemma 7.7.14. *For each $(r'-1)$-space H in Y, there is a point $x \in X$ such that $Y_x = H$, and all X-points collinear to H are contained in $\langle x, H\rangle$.*

Proof. Take $x \in X$ arbitrary and suppose that $H' := Y_x \neq H$. Since $\dim(Y) = r'+1$, we know $\dim(H \cap H') \geq r-3$. Suppose first that $\dim(H \cap H') = r'-2$. Let $z \in H \setminus H'$ be arbitrary. Then

$[z,x]$ is a member of Θ containing $\langle H',z\rangle = H$ and hence it is clear that $[x,z]$ contains an X-point x' collinear to H, i.e., $Y_{x'} = H$. Next, suppose that $\dim(H \cap H') = r'-3$. Considering an $(r'-1)$-space $H'' \subseteq Y$ with $\dim(H \cap H'') = \dim(H' \cap H'') = r'-2$ and applying the first paragraph twice, we obtain an X-point collinear to H.

The second assertion follows from Lemma 7.7.10. □

Corollary 7.7.15. *For each r'-space Y' in Y, there is a unique $\theta \in \Theta$ with $Y(\theta) = Y'$. Moreover, if $x \in X$ has $Y_x \subseteq Y'$, then $x \in \theta$.*

Proof. Take any $(r'-1)$-space H in Y'. By Lemma 7.7.14, we know that $H = Y_x$ for some $x \in X$. Take any point $z \in Y' \setminus H$. Then $\theta := [x,z]$ is a member of Θ with $Y(\theta) = Y'$. Now let $x' \in X$ be such that $Y_{x'} \subseteq Y'$. Then $X(\theta)$ contains a point x'' with $Y_{x''} = Y_{x'}$, and hence $x' \in \langle x'', Y_{x''}\rangle \subseteq \theta$. This also shows that θ is the unique member of Θ containing Y'. □

Let L be the set of X-lines. Lemma 7.7.12 showed that, for each $\rho(X)$-line L', there is an X-line L with $\rho(L) = L'$.

Proposition 7.7.16. *The pair $(\rho(X), \rho(\mathsf{L}))$ is an injective projection of the line Grassmannian $\mathsf{G}_{r'+1,2}(\mathbb{K})$; its set of singular r'-spaces is in $1-1$-correspondence to Θ; its set of symps (viewing $\mathsf{G}_{r'+1,2}(\mathbb{K})$ as a parapolar space) are in $1-1$-correspondence to the $(r'-3)$-spaces of Y.*

Proof. Recall that $\dim(Y) = r'+1$. Let $\Pi_i(Y)$ denote the set of i-dimensional subspaces of Y. Put $\mathsf{P} = \Pi_{r'-1}(Y)$. For $S_1 \in \Pi_{r'-2}(Y)$ and $S_2 \in \Pi_{r'}(Y)$, we define the pencil $P(S_1,S_2)$ as the set $\{P \in \mathsf{P} : S_1 \subseteq P \subseteq S_2\}$. Then we denote by B the set $\{P(S_1,S_2) \mid S_1 \in \Pi_{r'-2}(Y), S_2 \in \Pi_{r'}(Y), S_1 \subseteq S_2\}$. Since a projective space is self-dual, the point-line geometry (P,B) (with natural incidence) is isomorphic to the line Grassmannian $\mathsf{G}_{r'+1,2}(\mathbb{K})$.

We now claim that χ induces an isomorphism between $(\rho(X), \rho(\mathsf{L}))$ and (P,B). Indeed, the fact that $\chi : \rho(X) \to \mathsf{P} : x \mapsto \chi(x) = Y_x$ is a bijection between $\rho(X)$ and P follows immediately from Corollary 7.7.11 (injectivity) and Lemma 7.7.14 (surjectivity). The fact that a line of $\rho(\mathsf{L})$ is mapped by χ to a line of B follows from Lemma 7.7.12(iii). This shows the claim. It then follows that $(\rho(X), \rho(\mathsf{L}))$ is indeed isomorphic to the point-line truncation of the line Grassmannian $\mathsf{G}_{r'+1,2}(\mathbb{K})$.

Let S be a maximal singular subspace of $\rho(X)$ of dimension r' and take a line L in S. By Lemma 7.7.12(i), there is a unique $\theta \in \Theta$ containing L. Since $\rho(X(\theta))$ is an r'-space containing L, we see that $\rho(X(\theta)) = S$ (as in $\mathsf{G}_{r'+1,2}(\mathbb{K})$, there is a unique r'-space through each line).

Lastly, let Q be any symp of $\rho(X)$, i.e., Q is isomorphic to a Klein quadric. Let p_1 and p_2 be non-collinear points of Q and take points $x_1, x_2 \in X$ with $\rho(x_i) = p_i$. Then, since p_1 and p_2 are distinct and non-collinear, the points x_1 and x_2 are non-collinear points of a member $\xi \in \Xi$ (remembering members of Θ are mapped onto singular spaces), with vertex $T := Y_{x_1} \cap Y_{x_2}$. Now, $\rho(X(\xi))$ is a Klein quadric in $\rho(X)$ containing the points p_1 and p_2, and since two non-collinear points determine a unique quadric in $\rho(X)$, we get that $\rho(X(\xi)) = Q$. By Lemma 7.7.13(ii), $\rho^{-1}(Q)$ is precisely the subset of Ξ whose members have vertex T. □

Lemma 7.7.17. *There exists a subspace F^* such that $F^* \cap X$ contains a legal projection Π of $\mathsf{G}_{r'+1,2}(\mathbb{K})$ with $\langle \Pi \rangle = F^*$, which is such that, for each $x \in X$, there is a point $\overline{x} \in \Pi$ such that $x \in \langle x, Y_x \rangle$. The projection of $F^* \cap X$ from $F^* \cap Y$ is injective, and if $r' = 4$, then $F^* \cap Y = \emptyset$.*

Proof. By Proposition 7.7.16, the projection $\rho(X)$ is the point set of an injective projection of the line Grassmannian $\mathsf{G}_{r'+1,2}(\mathbb{K})$, and its set of singular r'-spaces is in $1-1$-correspondence to the members of Θ.

Let $B := \{p_0, ..., p_{r'+1}\}$ be the set of points of a basis of Y. Put $P = \{(i,j) \mid 0 \leq i,j \leq r'+1 \text{ and } i \neq j\}$. For each pair $(i,j) \in P$, let H_{ij} be the $(r'-1)$-space generated by the points of $B \setminus \{p_i, p_j\}$. By Lemma 7.7.14 and Corollary 7.7.11, each $H_{i,j} \in \mathscr{H}$ corresponds to a unique r'-space $\overline{H}_{i,j} := \langle x'_{i,j}, H \rangle$ whose X-points are collinear to $H_{i,j}$. Note that, for two pairs $(i,j),(i',j') \in P$, $|\{i,j,i',j'\}| = 3$ implies that $\dim(H_{i,j} \cap H_{i',j'}) = r'-2$ and hence, by Lemma 7.7.10, the points $x'_{i,j}$ and $x'_{i',j'}$ determine a member of Θ, which then also contains the subspaces $\overline{H}_{i,j}$ and $\overline{H}_{i',j'}$. In fact, for each $i \in \{0, ..., r'+1\}$, there is a unique $\theta_i \in \Theta$ containing all members $\overline{H}_{i,j}$ where $j \in \{0, ..., r'+1\}$ (with $Y(\theta_i) = \langle B \setminus \{p_i\} \rangle$). Just like in the proof of Lemma 7.5.30, we now select r'-spaces in each of these θ_i, $0 \leq i \leq r'+1$ to create a legal projection of $\mathsf{G}_{r'+1,2}(\mathbb{K})$ in X: following the lexicographic ordering on the pairs of P, we select points $x_{i,j} \in \overline{H}_{i,j} \cap X$ such that $x_{i,j}$ is collinear to $x_{i,j'}$ for each $j' \in \{0, ..., j-1\}$ and to $x_{i',j}$ with $i' \in \{0, ..., i-1\}$. By construction, $R'_i := \langle x_{i,j} \mid 0 \leq j \leq r' \rangle$ is an X-space of dimension r' in θ_i (it is singular and no point of Y is collinear to each of these points).

Next, we note that subsets $\{a,b,c\}$ of size 3 of $\{0, ..., r'\}$ gives rise to singular planes $\pi_{a,b,c} := \langle x_{a,b}, x_{b,c}, x_{a,c} \rangle$. Moreover, for a subset $\{a,b,c,d\}$ of size 4 of $\{0, ..., r'\}$, the planes $\pi_{a,b,c}$ and $\pi_{a,b,d}$ belong to a member of Ξ (for instance determined by the points $x_{a,c}$ and $x_{b,d}$), with vertex $H_{a,c} \cap H_{b,d}$. Since the planes $\langle x_{a,b}, x_{a,c}, x_{a,d} \rangle$ and $\langle x_{a,b}, x_{b,c}, x_{b,d} \rangle$ are X-planes (being contained in R'_a and R'_b, respectively), so are the planes $\pi_{a,b,c}$ and $\pi_{a,b,d}$.

Put $F^* = \langle R'_0, ..., R'_{r'} \rangle$. We now claim that, for each r'-space Y' in Y, the unique member of Θ through it (cf. Corollary 7.7.15), contains an r'-space in F^*. We start by the r'-spaces through $H_{0,1}$. So let H be any r'-space through $H_{0,1}$ meeting the line $\langle p_0, p_1 \rangle$ in a point $q \notin \{p_0, p_1\}$, and let θ_H be the unique member of Θ with $Y(\theta) = H$. Now, for each $j \in \{2, ..., r'+1\}$, the line $\langle x_{0,j}, x_{1,j} \rangle$ contains a unique point q_j which is collinear to the $(r'-1)$-space generated by p and the points in $\{p_i \mid 2 \leq i \leq r'+1, i \neq j\}$ (since $\langle x_{0,j}, x_{1,j} \rangle \subseteq \theta_j$); and $q_j \in \theta_H$ by Corollary 7.7.15.

We claim that $\langle x_{0,1}, q_2, ..., q_{r'+1} \rangle$ is a singular r'-space in X (clearly, it is contained in F^*). The fact that it is singular follows from the fact that the lines $\langle x_{0,1}, q_j \rangle$ and $\langle x_{0,1}, x_{j'} \rangle$ for $j \neq j' \in \{2, ..., r'+1\}$ are not only contained in θ_H but also in a member of Ξ determined by the quadruple $\{0,1,j,j'\}$ (see above); so by (S2) they are collinear. The fact that it is an r'-space in X then follows as it is a set of pairwise collinear points in θ_H which are, by construction, collinear to no point of Y. Hence, for each subspace through one of the $(r'-1)$-spaces $H_{i,j}$ with $(i,j) \in P$, we can do this; and we can repeat this for each $(r'-1)$-space that arises as the intersection of any two such r'-spaces. Continuing like this, the claim follows. We define Π as the union of all the thus obtained r'-spaces.

Let $x \in X$ be arbitrary and take any $\theta \in \Theta$ through x. Then θ contains an X-space R_θ of dimension r' in F^* and clearly, R_θ contains a point x' with $Y_x = Y_{x'}$. As such, $x \in \langle x', Y_x \rangle$ indeed. The correspondence $x \mapsto Y_x$ gives an isomorphism between Π and the $(r'-1)$-Grassmannian of Y, and hence Π is the point set of a projection of $\mathsf{G}_{r'+1,2}(\mathbb{K})$, which is moreover legal by (S2).

The fact that the projection of $F^* \cap X$ from $F^* \cap Y$ is injective follows from Lemma 7.7.16. Finally, we show that, if $r' = 4$, then $F^* \cap Y = \emptyset$. So suppose for a contradiction that $r' = 4$ and that y is a point of Y contained in $F^* \cap Y$. Since $r' = 4$, we have that Π is actually isomorphic to $\mathsf{G}_{5,1}(\mathbb{K})$, for the latter admits no proper legal projection by Proposition 7.2.7. We see that y is collinear to the points of a sub-variety Π' of Π isomorphic to $\mathsf{G}_{4,1}(\mathbb{K})$. By Lemma 7.2.9, the subspace $\langle y, \Pi' \rangle$ contains a point p of $\Pi \setminus \Pi'$. Let p' be the unique point of $\langle y, p \rangle \cap \langle \Pi' \rangle$. Note that $p' \notin \Pi'$, for otherwise $y \in \Pi$, a contradiction as $\Pi \subseteq X$. But then, by the properties of $\mathsf{G}_{4,1}(\mathbb{K})$, we get that p' is on a secant of Π', i.e., there are non-collinear points $x_1, x_2 \in \Pi'$ such that $p' \in \langle x_1, x_2 \rangle$. But then $p \in [x_1, x_2]$ is a contradiction to the prescribed structure of $X([x_1, x_2])$.

This concludes the proof. □

Putting everything together, we obtain the following rather general classification result.

Proposition 7.7.18. *Let (X, Z, Ξ, Θ) be a duo-symplectic split pre-Veronese set with $r = 2$ and containing an X-point through which there is precisely one member of Θ. Then, projecting X from a v'-space $V \subseteq Y$ collinear to all points of X, we obtain the point set of a mutant of the dual line Grassmannian $\mathsf{DG}_{r'+1,1}(\mathbb{K})$.*

Proof. As before, Lemmas 7.6.2 and 7.4.11 imply that there is a v'-space V (the common vertex of the members of Θ) collinear to all points of X. So Lemma 7.7.5 allows us to project from V. By Proposition 7.7.16 and Lemma 7.7.17, there is a subspace F^* in $\mathsf{PG}(N, \mathbb{K})$ containing a legal projection Π of $\mathsf{DG}_{r'+1,1}(\mathbb{K})$ such that X is the union of all affine subspaces $\langle x, Y_x \rangle \setminus Y_x$ where x ranges over the points of Π, and such that the projection of $F^* \cap X$ from $F^* \cap Y$ is injective. The fact that the correspondence $\Pi \to Y : x \mapsto Y_x$ is as described in Section 7.2.2 is shown in Proposition 7.7.16. □

We can now eliminate Case 2(b) completely.

Proposition 7.7.19. *) There does not exist any split Veronese set with parameters (r, v, r', v') with $r = 3$.*

Proof. Suppose for a contradiction that such a split Veronese set exists. Let $x \in X$ be arbitrary. By Lemma 7.7.3, $\mathrm{Res}(x)$ is a split pre-Veronese set with parameters $(2, v, r'-1, v')$ for which there is precisely one member of Θ_x through each point of X_x (cf. Lemma 7.7.9). Hence, by Proposition 7.7.18 and Lemma 7.7.5, projecting X from a v'-space in Y collinear to all points of X gives that $\mathrm{Res}(x)$ is a mutant of the dual line Grassmannian $\mathsf{DG}_{r',1}(\mathbb{K})$. Finally, by Axiom (S3), $\mathrm{Res}(x)$ is generated by two members ξ_1, ξ_2 of Ξ_x. Noting that $v = r' - 4$, we get $\dim \xi_i = r' + 2$, $i = 1, 2$, and hence $\dim\langle \mathrm{Res}(x) \rangle \le 2r' + 5$. Moreover,

$\dim(Y_x) = r'$ and projected from Y_x, we get an injective projection Π of $\mathsf{G}_{r',1}(\mathbb{K})$ (cf. Proposition 7.7.16), and we put $B := \dim\langle\Pi\rangle$. So $\dim\langle\mathrm{Res}(x)\rangle = B + r' + 1$. Combined, this yields $B \le r' + 4$. Since $B \ge 2r'$ (an injective projection of $\mathsf{G}_{r',1}(\mathbb{K})$ contains two r'-spaces intersecting each other in precisely a point), we obtain $r' \le 4$ and together with $r' > r = 3$, we get $r' = 4$. Now, since $B \le \dim\langle\mathsf{G}_{4,1}(\mathbb{K})\rangle = 9$ and $\langle\xi_1, \xi_2\rangle = \langle\mathrm{Res}(x)\rangle$, we get that $\dim(\langle\Pi\rangle \cap Y_x) \ge 2$ (the vertices of ξ_1 and ξ_2 generate at most a line of the 4-space Y_x), implying $B \le 6$, contradicting $B \ge 8$. This shows the proposition. $\square$

7.7.4 The only surviving examples for $r = 2$

> Let (X, Z, Ξ, Θ) be a duo-symplectic split Veronese set with parameters $(2, v, r', v')$, containing no Y-points collinear to all points of X, and such that each point residue is isomorphic to a mutant of a half dual Segre variety.

Using (S3), we narrow down the possibilities for r' and k.

Lemma 7.7.20. *We have $r' = 4$ and $k = 1$.*

Proof. Take $x \in X$ arbitrary. Axiom (S3) implies that there are $\xi_1, \xi_2 \in \Xi$ through x such that T_x is generated by $T_x(\xi_1)$ and $T_x(\xi_2)$. Note that $\dim(T_x(\xi_i)) = r' + 2$ for $i = 1, 2$ and that the respective vertices V_1 and V_2 of ξ_1 and ξ_2 at most generate Y_x, which by Lemma 7.6.1, has dimension $r' - 1$. As ξ_1 and ξ_2 share x, we obtain that $\dim(T_x) \le \min\{r' + 9, 2r' + 4\}$.

On the other hand, we know, by the structure of $\mathrm{Res}(x)$, that $\dim(T_x) = A + r'$, with A the dimension of a legal projection of a Segre variety $\mathsf{S}_{r'-1,k}(\mathbb{K})$. Combining these, we get $A \le 9$. Since $r' > r = 2$, we see that $k \ge 2$ leads to $A \ge 11$, a contradiction. If $k = 1$, then $A = 2r' - 1$, and hence $r' \le 4$.

Now let $(r', k) = (3, 1)$. Then T_x is isomorphic to $\mathsf{HDS}_{2,1}(\mathbb{K})$, where, with obvious notation, it is easily seen that two members of Ξ_x never generate the whole space, as their vertices are 0-dimensional, and the whole vertex space Y_x is 2-dimensional. $\square$

This now implies:

Proposition 7.7.21. *Let (X, Z, Ξ, Θ) be a duo-symplectic split Veronese set with $r = 2$. Then, projecting X from a v'-space $V \subseteq Y$ collinear to all points of X, we obtain the point set of a dual line Grassmannian $\mathsf{DG}_{5,1}(\mathbb{K})$. In particular, such a set is projectively unique.*

Proof. The first part follows from Proposition 7.7.18 and Lemma 7.7.20 and the fact that $\mathsf{G}_{5,1}(\mathbb{K})$ does not admit proper legal projections by Proposition 7.2.7. By Lemma 7.7.17, F^* is disjoint from Y. It is then clear that $F^* \cap X$ is isomorphic to $\mathsf{G}_{5,1}(\mathbb{K})$, and as such is projectively unique, and so are Y and the projectivity $x \mapsto Y_x$ between the line Grassmannian $F^* \cap X$ and the dual of the line Grassmannian in Y. As such, X is projectively unique. This shows the proposition. $\square$

This finishes the proof of Theorem 7.3.1.

Part II

Lacunary parapolar spaces

This part of the thesis is based on the paper "On exceptional Lie geometries" ([18]) of myself, J. Schillewaert, H. Van Maldeghem and M. Victoor. In this paper, we classified k-lacunary parapolar spaces, i.e., parapolar spaces in which the intersection of two symplecta never has dimension k, with some minor additional assumptions. It is the only of my collaborations that I incorporate in this thesis. One of my reasons to do so, is because it fits in: not only are some of the obtained geometries (see below), geometries that we already encountered in Chapter 7 of Part 1, also the flavour of the arguments feels "of the same kind" – and I particularly like these ways of reasoning geometrically/synthetically.

CHAPTER

8

INTRODUCTION

A lot of work in the past went into characterising, using additional properties, certain classes of parapolar spaces, preferably containing as many of exceptional type as possible. Especially worth mentioning in that respect is a relatively recent characterisation by Shult ([43]) with basically only one additional axiom, which he called the“Haircut Axiom”. This axiom expresses a gap in the spectrum of dimensions arising from intersecting symplecta with the perp of a point: the set of points collinear to a given point and belonging to a given symplecton can never be a submaximal singular subspace.

We now start from the observation that almost all of the "popular" exceptional Lie incidence geometries have certain gaps in the spectrum of the dimensions of the singular subspaces that occur as intersections of two symplecta. In this thesis, the main interest are the parapolar spaces in which two symplecta never have an empty intersection. Yet we also consider some parapolar spaces in which certain natural numbers $k \geq 0$ cannot occur as the dimension of the intersection of two symplecta, for it shows the exceptionality of this behaviour and provides an elegant characterisation of an interesting class of parapolar spaces encompassing a Grassmannian for each of the exceptional Lie incidence geometries.

8.1 Context

As mentioned in the previous part, the geometries on the second row of the Freudenthal-Tits magic square are geometries consisting of points and quadrics where the quadrics are either of minimal or maximal Witt index with the property that

- Each pair of points is contained in a quadric;
- the intersection of each pair of quadrics is at least a point.

In fact, these properties were already observed by Freudenthal ([22]) and Springer and Veldkamp ([44]), who considered these geometries as "projective planes over rings", and by Tits ([45]), who viewed the buildings of type E_6 as "projective planes over split octonions". The corresponding geometry is also called *"the Hjelmslev-Moufang plane"* in the literature. Its structure resembles that of an ordinary projective plane, except that each line has the structure of a polar space and two lines can meet in more than one point.

We would now like to know which other geometries also behave like projective planes but have (convex) quadrics as lines, i.e., we want to know which point-quadric geometries satisfy the two above properties if we allow the quadrics to be of any Witt index greater than 1. In the language of parapolar spaces this is equivalent to the question: *"Which strong parapolar spaces of diameter 2 have no pair of disjoint symplecta?"*

We can even answer a more general question by not bounding the diameter and, when there are no symps of rank 2, not requiring the parapolar space to be strong.

8.2 Main result

A parapolar space with at least one gap in that spectrum will be called *lacunary*. More precisely, we have the following definition.

Let k be an integer with $k \geq -1$. We say that a parapolar space is k-**lacunary** if

- k never occurs as the dimension of the intersection of two symplecta, and
- the symplectic rank is at least $k+1$.

Concerning (-1)-lacunary parapolar spaces, we have the following theorem.

Main Theorem 8.2.1. *Let $\Omega = (X, \mathscr{L})$ be a (-1)-lacunary parapolar space of minimum symplectic rank d with $d \geq 2$, which is strong in case there are symplecta of rank 2. Denote the set of symplecta by Ξ as usual. Then, Ω is strong if and only it contains at least one sympthick line. Moreover, if strong, Ω arises from one of the following Lie incidence geometries:*

$(d=2)$ $\mathsf{A}_{1,1}(*) \times \mathsf{A}_{2,1}(*)$ *or* $\mathsf{A}_{2,1}(*) \times \mathsf{A}_{2,1}(*)$ *;*
$(d=3)$ $\mathsf{A}_{4,2}(\mathbb{L})$ *or* $\mathsf{A}_{5,2}(\mathbb{L})$*;*
$(d=5)$ $\mathsf{E}_{6,1}(\mathbb{K})$ *(where $\mathbb{K}$ is a field).*

If Ω is non-strong, then (X, Ξ) is a non-trivial partial linear space such that:
$(*)$ *If $p_0 \in X$ belongs to two distinct members ξ_1, ξ_2 of Ξ, and $p_i \in \xi_i$, $i = 1, 2$, and $p_3 \in X$ is contained in a common member ξ_{i3} of Ξ together with p_i, $i = 1, 2$, where $p_0 \notin \{p_1, p_2, p_3\}$, then*

$$\delta_{\xi_1}(p_0, p_1) + \delta_{\xi_{13}}(p_1, p_3) + \delta_{\xi_{23}}(p_2, p_3) + \delta_{\xi_2}(p_0, p_2) \geq 5,$$

where δ_ξ is the distance in $\xi \in \Xi$, i.e., 0 if the arguments are equal, 1 if they are collinear in ξ, and 2 otherwise.

The above theorem will allow us to classify the k-lacunary parapolar spaces of symplectic rank at least d with $d \geq k+3$, as we will now explain. The following lemma hints at an inductive approach.

Lemma 8.2.2. *A parapolar space Ω is locally connected and k-lacunary with $k \geq 0$ of minimum symplectic rank $d \geq 3$ if and only if, for each point p of Ω, the point-residual Ω_p is a strong $(k-1)$-lacunary parapolar space of minimum symplectic rank $d-1$.*

Proof. By Proposition 2.1.43 and $d \geq 3$, it follows that Ω is a locally connected parapolar space of minimum symplectic rank $d \geq 3$ if and only if, for each point p of Ω, the point-residual Ω_p is a strong parapolar space of minimum symplectic rank $d-1$.

Now assume that Ω is k-lacunary. Suppose that, for some point p, there are two symps ξ_1 and ξ_2 in Ω_p intersecting each other in precisely a $(k-1)$-space K. Then the corresponding symps ξ_1' and ξ_2' in Ω intersect each other in precisely the k-space corresponding to K, contradicting the fact that Ω is k-lacunary. The converse statement is similar. □

So suppose that Ω is a locally connected k-lacunary parapolar space of minimum symplectic rank d with $k \geq 0$ and $d \geq k+3$. Since Ω is locally connected, we have that for each $p \in X$, the point-residue Ω_p is a strong $(k-1)$-lacunary parapolar space of minimum symplectic rank $d-1$ with $d-1 \geq (k-1)+3$, and we will be able to deduce the structure of Ω from Ω_p. Inductively we can go back until we reach a strong (-1)-lacunary parapolar space of symplectic rank at least 2 (since we assume $d \geq k+3$), so Theorem 8.2.1 will serve as induction basis. The following table depicts what these chains of lacunary parapolar spaces look like ($\mathbb{K}$ denotes a field and $\mathbb{L}$ a skew field):

$k=-1$		$k=0$		$k=1$		$k=2$		$k=3$		$k=4$
$\mathsf{A}_{1,1}(*) \times \mathsf{A}_{2,1}(*)$	$\leftarrow$	$\mathsf{A}_{4,2}(\mathbb{L})$	$\leftarrow$	$\mathsf{D}_{5,5}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{6,1}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{7,7}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{8,8}(\mathbb{K})$
$\mathsf{A}_{2,1}(*) \times \mathsf{A}_{2,1}(*)$	$\leftarrow$	$\mathsf{A}_{5,3}(\mathbb{L})$	$\leftarrow$	$\mathsf{E}_{6,2}(\mathbb{K})$						
$\mathsf{A}_{4,2}(\mathbb{L})$	$\leftarrow$	$\mathsf{D}_{5,5}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{6,1}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{7,7}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{8,8}(\mathbb{K})$		
$\mathsf{A}_{5,2}(\mathbb{L})$	$\leftarrow$	$\mathsf{D}_{6,6}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{7,1}(\mathbb{K})$						
$\mathsf{E}_{6,1}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{7,7}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{8,8}(\mathbb{K})$						

Table 8.1: The locally connected k-lacunary parapolar spaces of minimum rank at least $k+3$.

Remark 8.2.3. One could wonder why the above lists end. Well, they end because the rightmost parapolar space in each sequence is a *non-strong* parapolar space, which cannot occur as a point-residual, as these are always strong (cf. Proposition 2.1.43).

Note that, if we would not require that the minimum rank d is at least $k+3$, then the induction process could also end if a residue possesses rank 2 symplecta. Yet, there are not too many possibilities for $d < k+3$, since the definition of k-lacunary implies $d \geq k+1$. So, when $d = k+2$, the induction process ends with a strong 0-lacunary parapolar space of minimum symplectic rank 2; when $d = k+1$, it stops at a strong 1-lacunary parapolar space of minimum symplectic rank 2. The latter is however easily excluded:

Lemma 8.2.4. *There are no strong 1-lacunary parapolar spaces of minimum symplectic rank 2.*

Proof. Suppose for a contradiction that Ω is a strong 1-lacunary parapolar space containing a symp ξ of rank 2. By connectivity, there is a symp ξ' intersecting ξ non-trivially. Then $\xi \cap \xi'$ is a point p, as otherwise 1-lacunarity forces $\xi \cap \xi'$ to contain a plane, which is impossible as ξ only has rank 2. Let L and L' be lines through p in ξ and ξ', respectively. Since Ω is strong, there is a symp through L and L', which intersects ξ in L, contradicting what we have just deduced. $\square$

In this thesis, will not deal with the k-lacunary parapolar spaces of minimum symplectic rank $k+2$ (in particular, we disregard the 0-lacunary parapolar spaces of minimum symplectic rank 2). The result for the other k-lacunary parapolar spaces goes as follows:

Main Theorem 8.2.5. *Let $\Omega = (X, \mathscr{L})$ be a k-lacunary parapolar space of minimum symplectic rank d with $d \geq k+3$, which is strong in case there are symplecta of rank 2. Then, if locally connected, Ω is one of the Lie incidence geometries occurring in Table 8.1. If Ω is not locally connected, then Ω arises from a collection of locally connected k-lacunary parapolar spaces and polar spaces of rank at least $k+1$ as described in* Construction 2 *(called a k-*buttoned parapolar space*).*

8.3 Structure of the proof

The methods we will use are elementary and can be understood without more knowledge on parapolar spaces than is given in the preliminary chapter. We divide the proof into the following parts.

In **Chapter 9**, we will determine the geometric structure of the (-1)-lacunary parapolar spaces of minimum symplectic rank d with $d \geq 2$:

- In Section 9.1, we suppose that $d = 2$, i.e., that there are symps of rank 2 (in which case our assumption in that Ω is strong). We can then show that Ω has *uniform* symplectic rank, after which we are able to deduce the geometric structure of Ω (Subsection 9.1.2).

- In Section 9.2, we deal with the case where $d \geq 3$. The essential case is the one in which Ω has at least one sympthick line, i.e., a line which is contained in more than one symp. We shall prove that in this case, Ω is locally connected and that $d \in \{3, 5\}$; after which we deal with the cases $d = 3$ and $d = 5$ separately. The (almost trivial) case where each line of Ω is contained in a unique symp is dealt with in Chapter 11.

After having deduced a certain amount of properties, the classification in fact follows from Shult's Haircut theorem (Theorem 8.4.6 in the next subsection), and this is what is done in ([18]). However, in this thesis, I present an elementary approach (i.e., not relying on other classifications or theorems), showing that this classification is in fact independent.

In **Chapter 10**, we determine the geometric structure of the locally connected k-lacunary parapolar spaces spaces Ω of minimum symplectic rank d with $d \geq k+3$ and $k \geq 0$. The results on (-1)-lacunary parapolar spaces allow to determine the structure of Ω; even without much effort if one now makes use of other classification results (which are gathered in the next subsection). It would also be possible to do it by hand, but this would take us too far now.

Finally, in **Chapter 11**, we deal with the locally disconnected k-lacunary parapolar spaces. We show that $k \neq 0$ and that Ω arises by gluing together a collection of locally connected k-lacunary parapolar spaces "in a good way" (if $k = -1$, then in fact one only glues polar spaces together).

8.4 Useful theorems

For easy reference, we give a brief overview of the work which has been done before on classifications of certain classes of parapolar spaces relevant for the proof of our main results.

Combining Theorem 17.1.2 of [42] by Shult and Proposition 11.5.21 of [6] by Buekenhout and Cohen results in the following theorem.

Theorem 8.4.1. *Suppose $\Omega = (X, \mathscr{L})$ is a strong parapolar space with these three properties:*

— *For every point-symplecton pair (p, ξ), we have $p^\perp \cap \xi \neq \emptyset$.*
— *For each point p, the set of points at distance at most 2 from p is a proper subspace of Ω.*
— *If the symplectic rank is at least 3, every maximal singular subspace has finite dimension.*

Then Ω is one of the following:

(i) $\mathsf{D}_{6,6}(\mathbb{K}), \mathsf{A}_{5,3}(\mathbb{L})$ *or* $\mathsf{E}_{7,7}(\mathbb{K})$[1]*, for $\mathbb{K}$ any commutative field and $\mathbb{L}$ any skew field.*
(ii) *A dual polar space of rank 3.*
(iii) *A Cartesian product geometry $L \times \Delta$, where L is a thick line, and Δ is a polar space of rank at least 2.*

We now recall Theorem 15.4.5 of [42], which is an updated version of a result of Cohen & Cooperstein [10]. Consider the following property in a parapolar space Ω.

$(\mathsf{CC})_{d-2}$ If, for any point p and any symp ξ with $p \notin \xi$, the intersection $p^\perp \cap \xi$ has dimension at least $d-2$, then it has dimension $d-1$.

Theorem 8.4.2. *Let Ω be a locally connected parapolar space of uniform symplectic rank $d \geq 3$. If $d \geq 4$, it is also assumed that Ω is a strong parapolar space of which all singular subspaces have finite dimension. Then, if Ω satisfies property $(\mathsf{CC})_{d-2}$, then Ω is one of the following (where $\mathbb{K}$ denotes a commutative field and $\mathbb{L}$ a skew field):*

(i) *If $d = 3$, then Ω is either:*

[1]The book [42] contains the misprint $\mathsf{E}_{7,1}$ instead of $\mathsf{E}_{7,7}$

(a) *The Grassmannian of ℓ-spaces distinct from the hyperplanes of a vector space V over $\mathbb{L}$ of dimension m (possibly infinite), with* $\dim V \geq 4$ *and* $\ell \in \mathbb{N}_{\geq 2}$ *with, if m is finite,* $\ell \leq \lceil \frac{m-1}{2} \rceil$*, or*

(b) *The quotient* $\mathsf{A}_{2n-1,n}(\mathbb{L})/\langle \sigma \rangle$*, where σ is a polarity of V of Witt index at most* $n-5$*,* $n \geq 5$.

(ii) *If $d = 4$, Ω is a homomorphic (isomorphic if $n \leq 9$) image of* $\mathsf{D}_{n,n}(\mathbb{K})$*,* $n \geq 5$.

(iii) *If $d = 5$, then Ω is the Lie incidence geometry* $\mathsf{E}_{6,1}(\mathbb{K})$.

(iv) *If $d = 6$, then Ω is the Lie incidence geometry* $\mathsf{E}_{7,7}(\mathbb{K})$.

The following is Lemma 3.2 in [43].

Lemma 8.4.3. *Suppose* $\Omega = \mathsf{A}_{2n-1,n}(\mathbb{L})/\langle \sigma \rangle$*, where σ is a polarity of the associated vector space V of Witt index at most $n-5$ and $\mathbb{L}$ a skew field. Then* $\mathrm{Diam}\,\Omega \geq 5$.

We also use the following Theorem from [41], which itself strengthened [27].

Theorem 8.4.4. *Suppose Ω is a parapolar space of symplectic rank d with $d \geq 3$ satisfying these axioms:*

(NP) *Given a point p not incident with a symp ξ, the intersection $p^\perp \cap \xi$ is never just a point.*

(F) *If $d \geq 4$, every maximal singular subspace has finite singular rank.*

Then Ω is one of the following[2] *(where $\mathbb{K}$ is an arbitrary commutative field):*

(i) $\mathsf{E}_{6,2}(\mathbb{K})$*,* $\mathsf{E}_{7,1}(\mathbb{K})$ *or* $\mathsf{E}_{8,8}(\mathbb{K})$*;*

(ii) *a metasymplectic space;*

(iii) $\mathsf{B}_{n,2}(*)$ *or* $\mathsf{D}_{n,2}(\mathbb{K}), n \geq 4$*;*

(iv) *a strong parapolar space of diameter* 2.

Finally we use Shult's Haircut Theorem, see [43]. This uses the following property (called the "Haircut Axiom") in a parapolar space of minimum symplectic rank $d \geq 3$ (if the rank is uniform then this coincides with property $(\mathsf{CC})_{d-2}$).

(H) For any point p and any symp ξ with $p \notin \xi$, the intersection $p^\perp \cap \xi$ is never a submaximal singular subspace of ξ.

The above is a residual property (cf. Lemma 2.4 of [43]):

Lemma 8.4.5. *Suppose Ω is a parapolar space of symplectic rank at least* 3*. Then Ω satisfies property* (H) *if and only if Ω_p also has the property* (H) *for each point p.*

Shult's Haircut Theorem then goes as follows.

Theorem 8.4.6. *Suppose Ω is a locally connected parapolar space of symplectic rank at least* 3*, satisfying the following:*

[2] [41] uses a different convention for the numbering of the nodes, and allows infinite symplectic rank in (iii)

— *each singular space possesses a finite projective dimension; moreover, there exists an upper bound to the rank of a symplecton.*
— *the Haircut Axiom* (H).

Then Ω has a uniform symplectic rank $d \geq 3$ and one of the following occurs ($\mathbb{K}$ is any commutative field, $\mathbb{L}$ is any skew field):

(i) $d = 3$ *and Ω is either the d-Grassmannian* $\mathsf{A}_{n,d}(\mathbb{L})$ *or a homomorphic image* $\mathsf{A}_{2n-1,n}(\mathbb{L})/\langle\sigma\rangle$ *where σ is a polarity of* $\mathsf{A}_{2n-1}(\mathbb{L})$ *of Witt index at most $n-5$, $n \geq 5$;*
(ii) $d = 4$ *and Ω is a* Y_1 *geometry or a twisted version thereof (these include* $\mathsf{E}_{6,2}(\mathbb{K})$*,* $\mathsf{D}_{n,n}(\mathbb{K}), n \geq 5$*);*
(iii) $d = 5$ *and Ω is a homomorphic image of a building geometry* $\mathsf{E}_{m+4,1}(*)$ *with $m \geq 2$ (this includes* $\mathsf{E}_{6,1}(\mathbb{K})$*,* $\mathsf{E}_{7,1}(\mathbb{K})$*,* $\mathsf{E}_{8,1}(\mathbb{K})$*);*
(iv) $d = 6$ *and Ω is* $\mathsf{E}_{7,7}(\mathbb{K})$*;*
(v) $d = 7$ *and Ω is* $\mathsf{E}_{8,8}(\mathbb{K})$.

Conversely, all of the listed geometries satisfy the hypotheses.

The object mentioned in (ii) and named "a Y_1 geometry or a twisted version thereof" will be of no importance to us (and hence we will not define or discuss them). The reason is that such parapolar space has hyperbolic symplecta of rank 4, and the point-residuals are parapolar spaces of type $\mathsf{A}_{n,\ell}$, with $n \geq 4$ and $\ell \in \{2, \ldots, n-1\}$. Only the cases $\mathsf{A}_{5,3}$ and $\mathsf{A}_{n,2}$ will be of interest to us, but then the corresponding Y_1 geometry has type $\mathsf{E}_{6,2}$ and $\mathsf{D}_{n+1,n+1}$, respectively.

CHAPTER

9

(−1)-LACUNARY PARAPOLAR SPACES WITH AT LEAST ONE SYMPTHICK LINE

In this chapter, $\Omega = (X, \mathscr{L})$ is a (-1)-lacunary parapolar space of minimum symplectic rank d, with $d \geq 2$; i.e., in Ω, each two symplecta intersect non-trivially. We will classify this class of parapolar spaces. As mentioned before, we start with the case where $d = 2$.

9.1 Case 1: minimum symplectic rank 2

In this section, $\Omega = (X, \mathscr{L})$ is a (-1)-lacunary strong parapolar space of minimum symplectic rank 2. As usual we denote the set of symps with Ξ. For convenience, we will call a symp of rank 2 a *quad* (from "quadrangle").

Lemma 9.1.1. *Let L_1, L_2 be disjoint lines of an arbitrary quad ξ of Ω. Then at least one of L_1, L_2 is properly contained in a singular subspace, or some line of ξ intersecting both L_1, L_2 is properly contained in a singular subspace.*

Proof. We claim that there exist lines M_1, M_2 not contained in ξ and meeting L_1, L_2 in points q_1, q_2, respectively. Let $i \in \{1,2\}$. By Axiom (PPS1), there is a point $p \in X \setminus \xi$. Connectivity of $(X, \mathscr{L})$ yields a shortest path $(p, p_1, \ldots, p_n, q_i)$ from p to L_i (so $q_i \in L_i$). Now, if $p_n q_i$ does not belong to ξ, then we can put $M_i = p_n q_i$. If $p_n \in \xi$, then $p_{n-1} \notin \xi$ (as otherwise we could shorten the path) and so, by strongness, p_{n-1} and q_i determine a symp ξ_i and then there is a line M_i in ξ_i through q_i not contained in ξ. The claim is proved.

We may assume that L_i ($i \in \{1,2\}$ still) is not properly contained in a singular subspace. Consequently, since $(X, \mathscr{L})$ is strong, L_i and M_i are contained in a unique symp ξ_i and the singular subspace $\xi \cap \xi_i$ equals L_i. Hence $\xi_1 \cap \xi_2$, nonempty by (-1)-lacunarity, is not contained in ξ. For any point $q \in \xi_1 \cap \xi_2$, q is collinear to a point $r_1 \in L_1$ and to a point $r_2 \in L_2$. Necessarily, $r_1 \perp r_2$ since $q \notin \xi$. So r_1, r_2, q are contained in a singular subspace properly containing the line $r_1 r_2$. □

Lemma 9.1.2. *No singular subspace of Ω contains two disjoint lines which are both contained in a quad. Moreover, the singular subspaces of Ω are projective.*

Proof. Assume for a contradiction that there is a singular subspace S containing two disjoint lines L_1 and L_2 that are both contained in a quad; say $L_i \subseteq \xi_i \in \Xi$, for $i \in \{1,2\}$. Then $\xi_i \cap S = L_i$ and $\xi_1 \cap \xi_2$ contains a point $q \notin S$. Now q is collinear to unique points p_1 and p_2 on L_1 and L_2, respectively. Let $r \in L_1 \setminus \{p_1\}$. Then ξ_1 is determined by r and q, but since $p_2 \in \{r,q\}^\perp$, we obtain $p_2 \in \xi_1$, a contradiction.

By definition, the points and lines of Ω are projective, so let S be a singular subspace properly containing a line. To obtain that S is projective, it suffices to show Veblen's axiom ([52]). Let L_1 and L_2 be lines of S, each meeting two intersecting lines K_1, K_2 of S in two distinct points. Denote by p the intersection of K_1 and K_2. Suppose for a contradiction that L_1 and L_2 are disjoint. Then the previous paragraph yields a symp ζ of rank at least 3 containing, without loss, L_1. Since p is collinear to all points of L_1, Lemma 2.1.37 implies that p and L_1 are contained in a projective plane, which then also contains K_1, K_2 and L_2. Consequently, L_1 and L_2 intersect after all. We conclude that S is projective. □

9.1.1 Reduction to uniform symplectic rank 2

The aim of this subsection is to show that all symps of Ω have rank 2.

Lemma 9.1.3. *Let ξ be a quad of Ω and let $L \subseteq \xi$ be a line contained in a singular plane π. Then a symp ζ of Ω disjoint from L has rank 2.*

Proof. We divide the proof into two parts, based on whether or not there is a point in ζ collinear to a point in $\pi \setminus L$. Before heading off, we observe that $\zeta \cap \pi$ is empty. Indeed, suppose $\zeta \cap \pi$ is a point p (not on L, by assumption). By (-1)-lacunarity, $\zeta \cap \xi$ contains a point p' (also off L). Then p' is not collinear to p, as otherwise $p \in \xi(r,p') = \xi$ for some point $r \in L$ not collinear to p', a contradiction. However, if p and p' are not collinear, $\zeta = \xi(p,p')$ contains a point on L after all, violating our assumption.

Case I: *There is a point q of ζ collinear to some point x of $\pi \setminus L$.*

Note that $q \notin \xi$, for otherwise $x \in \xi$ since x is collinear to both q and a point of L not collinear to q.

Claim. Each point of ζ is collinear to at least one point of π.
Denote by Z be the subset of points of ζ collinear to at least one point of π. We (subsequently) show that Z is a subspace containing $q^\perp \cap \zeta$ and at least one point of ζ not belonging to $q^\perp \cap \zeta$, as then Fact 2.1.31 implies that $Z = \zeta$, proving the claim.

- *Z is a subspace of ζ:* Let q_1, q_2 be collinear points of Z. If they are collinear to a common point of π, then every point of q_1q_2 is collinear to that point. If not, then they are collinear to distinct points x_1, x_2, respectively, with $\delta(q_1, x_2) = \delta(q_2, x_1) = 2$. But then the symp $\xi(q_1, x_2)$ contains the lines q_1q_2 and x_1x_2 and hence every point of q_1q_2 is collinear to a unique point of the line $x_1x_2 \subseteq \pi$ (note though that the lines q_1q_2 and x_1x_2 are not necessarily opposite).

- *Z contains $q^\perp \cap \zeta$:* Let $r \in \zeta$ be a point collinear to q. We show that $r \in Z$. If $r \perp x$, then there is nothing to prove, so suppose $x \notin r^\perp$. Then $\xi(r, x) \cap \xi$ contains at least some point p^*. If $p^* \in L$, then $\xi(r, q)$ contains the line xp^* of π; and if $p^* \notin L$, then $\delta(p^*, x) = 2$ (otherwise $x \in \xi$), so, by convexity, also in this case we obtain that π contains a line of $\xi(r, x)$. In both cases we obtain that r is collinear to a point of the line $\pi \cap \xi(r, x)$, so $r \in Z$ indeed.

- *At least one point r of ζ not belonging to $q^\perp$ belongs to Z:* Firstly, if some point p of $\xi \cap \zeta$ is not collinear to q, then we can take $r = p$. Hence we may assume that q is collinear to $\xi \cap \zeta$. Secondly, note that it suffices to find a point $r \in \zeta \cap q^\perp \setminus \{q\}$ which is collinear to a point of $\pi \setminus L$ (because the previous item then implies that $r^\perp \subseteq Z$, and $r^\perp \nsubseteq q^\perp$). Assume for a contradiction that for each such point r, we have $r^\perp \cap \pi \subseteq L$. Take such a point r and let $p^* \in L$ be a point collinear to it. Then it follows that each point of the line rq, in particular q, has to be collinear to p^*. If some point p of $\xi \cap \zeta$ were not collinear to p^*, then $\xi = \xi(p, p^*)$ contains q (recall $p \perp q$), a contradiction. So $\xi \cap \zeta$ is just a point, say p, which is collinear to p^*. It also follows that $r \perp p$ (since $p^* \notin \zeta$). Since it is impossible that $q^\perp$ belongs to $p^\perp$, we obtain a contradiction. This shows that r is collinear to some point of $\pi \setminus L$.

As mentioned above, this shows the claim. We now show that ζ is a quad indeed. Suppose for a contradiction that ζ has rank at least 3. Let p be a point of $\xi \cap \zeta$ and let p' be the unique point on L collinear to p. Then we consider a plane α in ζ intersecting both $\xi \cap \zeta$ and $p'^\perp$ in exactly the point p. If there is a point $z \in \alpha \setminus \{p\}$ collinear to a point p^* of L (note that our choice of α implies $p' \neq p^*$), then $z \in p^\perp \cap p^{*\perp} \subseteq \xi(p, p^*) = \xi$, again contradiction our choice of α. The above claim implies that each point of $\alpha \setminus \{p\}$ is collinear to a unique point of $\pi \setminus L$. Just like above, it then follows that each line through p in α corresponds bijectively (via collinearity) to a line of π through p' (by assumption distinct from L). If two lines of M_1 and M_2 through p in α are mapped to the same line M of π through p', then $M \setminus \{p'\}$ contains distinct points $z, \overline{z}$ collinear to points $m_1, \overline{m}_1, m_2, \overline{m}_2$ on $M_1 \setminus \{p\}$ and $M_2 \setminus \{p\}$, respectively. But then the point $m = m_1m_1 \cap \overline{m}_1\overline{m}_2$ (which is on $\alpha \setminus \{p\}$) is collinear to M and in particular to p', a contradiction. Hence, collinearity defines a collineation between α and π indeed, but then some points of α different from p are collinear to points of L after all, a contradiction. This proves the lemma in Case I.

Case II: *No point of ζ is collinear to a point of $\pi \setminus L$.*

Claim. No line of π is contained in a symp of rank at least 3.
Suppose for a contradiction that some line of π were contained in a symp of rank at least 3. Lemma 2.1.37 then yields a symp ξ^* containing π. By (-1)-lacunarity, the symps ξ^* and ζ

share a point z and our current assumption implies that $z^\perp \cap \pi = L$. Let $p \in \xi \cap \zeta$ be arbitrary and set $p' = p^\perp \cap L$. Then $p' \perp z$ and, consequently, $z \perp p$ (as otherwise $p' \in \zeta = \xi(p,z)$, which is not the case). However, this means that ξ, which is defined by L and p, also contains z, contradicting the fact that $z \perp L$. The claim follows.

We now show that ζ is a quad, distinguishing between the following two cases.

- *Case IIa: $\zeta \cap \xi$ is a single point p.*
  Let p' be the unique point on L collinear with p. Pick an arbitrary point $y \in \pi \setminus L$ and an arbitrary point $z \in \xi \setminus L$ such that z is collinear to a point $z' \in L \setminus \{p'\}$. Then y and z are not collinear as otherwise $\xi = \xi(p',z)$ contains y. Set $\xi^* = \xi(y,z)$. Then ξ^* contains a line M of π, namely $M = yz'$. By the above claim, ξ^* is a quad and hence $\xi^* \cap \pi = M$. Noting that $p \in \zeta$ is collinear to $p' \in \pi \setminus M$, we can replace (ξ, L) by (ξ^*, M) and then Case I applies again, showing that ζ is a quad.

- *Case IIb: $\zeta \cap \xi$ is a line K.*
  Select $p \in K$ arbitrarily and set $p' = p^\perp \cap L$. Select a line $M \neq K$ of ζ through p not contained in $p'^\perp$ and consider the symp ξ_1 defined by p' and M. If ξ_1 has a line in common with π, then ξ_1 is a quad by the above claim and hence the points of $M \setminus \{p\}$ are collinear to points of $\pi \setminus L$, contradicting our hypothesis. Hence there is a line $N \neq pp'$ of ξ_1 through p' not contained in π.

  If N and L are contained in a singular plane π', then we replace π by π' and observe that the points of $M \setminus \{p\}$ are collinear to points of $\pi' \setminus L$ (namely to points of N). If N and L are not contained in a singular plane, then there is a line $L' \neq L$ in π through p' which is not collinear to N and hence they determine a symp ξ', which is a quad by the above claim. We then replace ξ by ξ' and observe that the points of $K \setminus \{p\}$ are collinear to points of $\pi \setminus L'$ (namely to points on L). In both cases, these replacements imply that Case I applies again, yielding that ζ has rank 2.

This completes the proof of the lemma. □

Lemma 9.1.4. *Every symp of Ω that intersects a quad in a line has rank 2.*

Proof. Suppose for a contradiction that a quad ξ and a symp ζ of rank at least 3 intersect in a line L. Let M be a line in ξ opposite L. As in the proof of Lemma 9.1.1, there is a line M' intersecting M not contained in ξ. Then Lemma 9.1.3 implies that M and M' are not contained in a plane. Hence there is a symp ξ' containing M and M'. Since L is contained in a plane of ζ, Lemma 9.1.3 again implies that ξ' is a quad.

Claim 1: The intersection $\zeta \cap \xi'$ is a point q.
By (-1)-lacunarity, $\zeta \cap \xi' \neq \emptyset$. Assume for a contradiction that $\zeta \cap \xi'$ is a line K. Since $\xi \cap \xi' = M$, the lines K and L are disjoint. For every point $q \in K$, the unique point in $q^\perp \cap M$ and every point in $q^\perp \cap L$ are collinear as $q \notin \xi$ (implying that also $q^\perp \cap L$ is unique). It follows that each point $q \in K$ is contained in a unique plane α_q intersecting M and L in collinear points. Since α_q contains a line of ζ, and ζ has rank at least 3, Lemma 2.1.37 implies the existence of a symp ξ_q of rank at least 3 containing the plane α_q. Then $\xi_q \cap \xi$

equals the line $\alpha_q \cap \xi'$. Now, for $q' \in K$ with $q' \neq q$, the line $\alpha_{q'} \cap \xi$ is disjoint from ξ_q, once again contradicting Lemma 9.1.3. The claim is proved.

Denote by q' the point $q^\perp \cap M$. As deduced in the previous paragraph, $q^\perp \cap L$ is a point p (since it has to be collinear to q'). Let π be any plane of ζ containing L. Then there is a point $x \in \pi \setminus L$ collinear to q and a point $r \in \xi' \cap q^\perp \setminus \{q\}$ such that the line rq does not intersect M.

Claim 2: r is collinear to some point of $\pi \setminus L$.
If $r \perp x$, then this is trivial. If not, there is a symp $\xi(r,x)$, which intersects ξ and hence, by convexity (as in the previous proof), it has a line R in common with π. Let $x' \in R \cap r^\perp$ and suppose for a contradiction that $x' \in L$. Then the unique point x'' on M collinear with x' is collinear to r too (since $x' \notin \xi'$) and hence $x'' \neq q'$ (since $q' \perp q$). This also implies that $p \neq x'$ and hence $x' \notin q^\perp$. But then $r \in \xi(q,x') = \zeta$, a contradiction. This shows the claim.

Now we replace π by another plane π^* of ζ containing L and not collinear to π. Then r is also collinear to a point x^* of $\pi^* \setminus L$. Since x and x^* are not collinear, we obtain $r \in \zeta$, a contradiction. The lemma is proved. □

Proposition 9.1.5. *The strong* (−1)*-lacunary parapolar spaces of symplectic rank at least 2 have uniform symplectic rank 2.*

Proof. Assume for a contradiction that Ω has a symp ζ of rank at least 3. By assumption, Ω also has a quad ξ and by (−1)-lacunarity, $\xi \cap \zeta \neq \emptyset$. Moreover, by Lemma 9.1.4, $\xi \cap \zeta$ is a point p. Pick lines $L \subseteq \xi$ and $M \subseteq \zeta$ both through p. If L and M are contained in a plane, then by Lemma 2.1.37, this plane is contained in a symp of rank at least 3 intersecting ξ in the line L, contradicting Lemma 9.1.4. Hence, by strongness, L and M define a symp, which has a line in common with both ξ and ζ and hence, again by Lemma 9.1.4, it can neither have rank at least 3 nor rank 2. This impossibility shows the proof. □

If a strong (−1)-lacunary parapolar space contains a rank 2 symp, then all its symps have rank 2.

9.1.2 Uniform symplectic rank 2

In this subsection, we suppose that Ω has uniform symplectic rank 2 (and is still a strong (−1)-lacunary parapolar space of course).

In a general parapolar space, it is the existence of symps of rank at least 3 that guarantees that the singular subspaces are projective. However, by (−1)-lacunarity and strongness, we still enjoy this property even when there are no symps or rank at least 3 around, as a consequence of Lemma 9.1.2. We can even say more.

Lemma 9.1.6. *Every singular subspace of Ω properly containing a line is a projective plane.*

Proof. By Lemma 9.1.2 and the lack of symps of rank at least 3, a singular subspace S of Ω does not contain disjoint lines. So as soon as S properly contains a line, it is a projective plane. □

Recall that Lemma 9.1.1 implies the existence of many singular planes (for each symp, at least one of its lines is contained in a singular plane). The previous lemma implies that two singular planes do not intersect in a line:

Lemma 9.1.7. *If two singular planes of Ω have a line in common, then they coincide.*

Proof. Suppose for a contradiction that π and π' are distinct singular planes of Ω having a line L in common. Take points p,p' in $\pi \setminus L$ and $\pi' \setminus L$, respectively. Since Ω has symplectic rank 2, p and p' need to be collinear. But then $\langle \pi, \pi' \rangle$ is a singular subspace properly containing a projective plane, contradicting Lemma 9.1.6. □

Lemma 9.1.8. *Every symp and every singular plane in Ω that share a point, share a line.*

Proof. Let ξ be a symp and π a singular plane and suppose for a contradiction that $\xi \cap \pi$ is a point $p \in X$. Let L be a line in π not containing p (and hence disjoint from ξ) and let ξ_L be a symp containing L. Since ξ_L does not contain planes, $p \notin \xi \cap \xi_L$. By (-1)-lacunarity, $\xi \cap \xi_L$ contains a point q. Denote by r the unique point of L collinear to q; then $p \perp r \perp q$. If p and q are not collinear, then $r \in \xi$, contradicting $L \cap \xi = \emptyset$. If p and q are collinear, then p,q,r generate a singular plane π' which intersects π in the line pr, contradicting Lemma 9.1.7. These contradictions show the lemma. □

Lemma 9.1.9. *For every point p and every singular plane π of Ω with $p \notin \pi$, there is a unique point in π collinear to p.*

Proof. First note that p cannot be collinear with more than one point of π by Lemma 9.1.7. We claim that no point can be at a finite distance greater than 1 to π. Indeed, if there were such a point, then there also is a point q at distance 2 from π. By strongness and Lemma 9.1.8, this yields a symp through q meeting π in a line L. But then L contains a point collinear to q, a contradiction. Since by (PPS1), the distance $\delta(p,\pi)$ of p to π is finite, it follows that $\delta(p,\pi)=1$. □

In case there is a singular plane intersecting every symp non-trivially, we can show that Ω is a Segre geometry of type $(1,2)$. We first show, under this assumption, that each symp is non-thick.

Lemma 9.1.10. *If Ω has a singular plane π intersecting every symp non-trivially, then each symp is non-thick and all lines through a point $q \notin \pi$ disjoint from π are contained in a singular plane disjoint from π.*

Proof. By Lemma 9.1.8, our assumption implies that π intersects each symp in a line. Let $q \notin \pi$ be a point. By Lemma 9.1.9, π has a unique point p collinear to q. Let L be any line through q disjoint from π and take a symp ξ_L through L. By our assumption and

Lemma 9.1.8, ξ_L has a line L' in common with π. Since $q^\perp \cap \pi = \{p\}$, the line L' contains p. Note that this implies that ξ_L is the unique symp through L.

Now take a line K' in π through p distinct from L'. By Lemma 9.1.9, q is not collinear to K' and hence, by strongness, there is a symp ξ_K through K' and q. Our choice of K' implies that $\xi_K \cap \xi_L = pq$. Let K be a line in ξ_K through q distinct from pq. Clearly, $K \neq L$. If K and L were not collinear, then by strongness, they are contained in a symp; but as deduced above ξ_L is the unique symp through L and $K \nsubseteq \xi_L$. Hence K and L are contained in a singular plane π'. Note that π' is disjoint from π, since $q^\perp \cap \pi = \{p\}$ and p is not collinear to L.

Since L was an arbitrary line through q disjoint from π, we obtain that each such line needs to be contained in π' as this is the unique singular plane through K (cf. Lemma 9.1.7). In particular, the symp ξ_L has only one line through q distinct from pq, i.e., ξ_L is non-thick. As q was an arbitrary point outside π, we conclude that all symps are non-thick. □

Proposition 9.1.11. *If Ω has a singular plane π intersecting every symp non-trivially, then Ω is isomorphic to a Segre geometry of type* $(1,2)$*.*

Proof. Again, Lemma 9.1.8 and our assumption imply that π intersects each symp in a line, and by Lemma 9.1.10, each symp is non-thick.

Claim. For each point $x \in \pi$, there is a unique line L_x through it not contained in π.
Since there is a symp through x, we know there is at least one such line L_x. Suppose for a contradiction that there is a second such line, say L'_x. If $L_x \perp L'_x$, then taking some point $q \in L_x \setminus \{x\}$, we see that Lemma 9.1.10 leads to a contradiction since $\langle L_x, L'_x \rangle$ intersects π. If L_x and L'_x are non-collinear, strongness implies that they are contained in a symp ξ_x which by assumption intersects π in a line through x, implying that ξ_x has three lines through t, contradicting that it is non-thick. The claim is proved.

We now show that $(X, \mathscr{L})$ is isomorphic to the direct product space $\pi \times L$, for any line L meeting π in a point t. Let $x \in X$ be arbitrary. If $x \in \pi \cup L$, then x can be uniquely written in $L \times \{t\} \cup \{t\} \times \pi$. So suppose $x \notin L \cup \pi$. By Lemma 9.1.9, x is collinear to a unique point x_π of π, which does not coincide with t by the above claim. Hence, by strongness, there is a unique symp ξ through x and t and, again by the above claim, ξ contains L as one of its two lines through t. So there is a unique point $x_L \in L$ collinear to x, and $x_L \neq t$. Just like L was the unique line through t not in π, the line xx_π is the unique line through x_π not contained in π. Therefore, the points x_L and x_π determine x uniquely as the unique point other than t contained in their common perp. As such, the point set of $\pi \times L$ coincides with X.

Lastly, it follows from Lemma 9.1.10 that the lines distinct from L through any point $x \in L \setminus \{t\}$ belong to a singular plane π_x and all these singular planes are disjoint from each other and from π. Let x and x' be two points of $X \setminus (\pi \cup L)$. If $x_L = x'_L$, then we just deduced that x_L and x'_L are contained in the plane π_{x_L} and hence they are collinear. By the above claim it is clear that if $x_\pi = x'_\pi$ for points $x, x' \in X$, then x' belongs to the line $\langle x, x_\pi \rangle$ so x and x' are collinear. Again by the above claim and because collinearity gives a one-one correspondence between each pair of distinct planes in $\pi \cup \{\pi_x \mid x \in L \setminus \{t\}\}$, it follows that two points x, x' of distinct planes (i.e., $x_L \neq x'_L$) can only be collinear if $\pi_x = \pi_{x'}$. Hence also the line set of $\pi \times L$ corresponds with $\mathscr{L}$. The proposition is proved. $\square$

We now arrive at the crux of the proof. Since we already know what happens if Ω has a plane meeting every symp non-trivially, we now assume that there is a symp disjoint from some plane.

Lemma 9.1.12. *Suppose Ω contains a disjoint plane-symp pair (π, ξ). Then ξ is non-thick and there is a unique line in π for which collinearity is a bijection from the points of this line to the lines of one system of generators of ξ.*

Proof. Each point q of ξ is collinear with a unique point $\tilde{q}$ of π by Lemma 9.1.9. Let q_1 and q_2 be collinear points of ξ. If $\tilde{q}_1 = \tilde{q}_2$ then $\langle q_1, q_2, \tilde{q}_1 \rangle$ is a singular plane; if $\tilde{q}_1 \neq \tilde{q}_2$, then $\xi(q_1, \tilde{q}_2)$ contains q_2 and $\tilde{q}_1$ as well, and collinearity is a bijection between the points of $q_1 q_2$ and those of $\tilde{q}_1 \tilde{q}_2$. In the first case we say that L is *π-triangular (with center $\tilde{q}_1 = \tilde{q}_2$)*, in the second case *$\pi$-quadrangular (with axis $\tilde{q}_1 \tilde{q}_2$)*. We show three properties on these lines:

(1) *Each pencil of lines in ξ contains at most one π-triangular line.*
Let L_1, L_2 be two intersecting lines of ξ. If both are π-triangular, the corresponding planes meet in a line, contradiction Lemma 9.1.7 and showing the claim.

It follows from (1) that ξ has a pair (M_1, M_2) of disjoint π-quadrangular lines: if there is no π-triangular line this is trivial, otherwise, take M_1 and M_2 any two lines through distinct points of a π-triangular line.

(2) *One or all lines meeting both M_1 and M_2 are π-triangular, according to whether the axes of M_1, M_2 are distinct or not.*
Let A_1 and A_2 denote the respective axes of M_1 and M_2. Since A_1 and A_2 are contained in the projective plane π, the intersection $A_1 \cap A_2$ is either a unique point or $A_1 = A_2$. Let $r \in A_1 \cap A_2$. Then r is collinear to unique points m_1, m_2 on M_1, M_2, respectively. Since $r \notin \xi$, we have $m_1 \perp m_2$ and the line $m_1 m_2$ is π-triangular with center r. Conversely, if a line $m_1 m_2$, with $m_i \in M_i$ for $i \in \{1, 2\}$, is a π-triangular line with center r, then $r \in A_1 \cap A_2$. The claim follows.

It now follows from (2) that ξ has at least two π-triangular lines T_1 and T_2: if all lines meeting M_1 and M_2 are π-triangular this is clear; if there is only one such line, say T_1, then we take two π-quadrangular lines M'_1 and M'_2 each meeting both M_1 and M_2 in a point, as (2) then guarantees the existence of a π-triangular line T_2 meeting both M'_1 and M'_2. Clearly, T_1 and T_2 are distinct and hence, by (1), they are disjoint. Denote their respective centers by t_1 and t_2 (note that $t_1 \neq t_2$, since T_1 and T_2 contain a pair of non-collinear points) and let U_1, U_2, U_3 be three lines each intersecting both T_1 and T_2 in a point.

(3) *The lines T_1 and T_2 define a (full) grid G in ξ, one of which reguli consisting of π-triangular lines and the other of π-quadrangular lines.*
For each $j \in \{1,2,3\}$, the axis of U_j is the line t_1t_2. Let t be an arbitrary point on t_1t_2. Then the points on U_1, U_2, U_3 collinear to t are pairwise collinear. This implies that, varying $t \in t_1t_2$, each line intersecting U_1 and U_2 non-trivially also intersects U_3 non-trivially, and, on top, is π-triangular. This shows the claim.

By (3), it suffices to show that ξ is non-thick to finish the proof. Put $u_i = U_1 \cap T_i$, $i \in \{1,2\}$. Then $\xi_1 := \xi(u_1, t_2) = \xi(u_2, t_1)$ is a symp containing U_1 and t_1t_2. We first show that U_1 and t_1t_2 determine a full grid G^* in ξ_1. To that end, let M be any line intersecting both u_1t_1 and u_2t_2, and let N be any line intersecting both U_1 and t_1t_2. Clearly, N belongs to ξ_1. We claim that M and N intersect in a point. If $M = U_1$ this is trivially true, so suppose $M \neq U_1$. Set $m_i = M \cap u_it_i$, $u = N \cap U_1$ and $t = N \cap t_1t_2$. Let ξ_2 be the symp determined by U_2 and t_1t_2 and note that $\xi_1 \cap \xi_2 = M$. In ξ_1, we see that m_i is collinear to a unique point w_i of N, for $i \in \{1,2\}$. Let T be the π-triangular line of G incident with u and $u' = T \cap U_2$. Then w_i is collinear to both u' and m_i, for $i \in \{1,2\}$, which are non-collinear points of ξ_2. Hence w_1 and w_2 belong to $\xi_1 \cap \xi_2 = M$, which implies $w_1 = w_2 \in M$. The claim follows. As such, U_1 and t_1t_2 are contained in a (full) grid $G^* \subseteq \xi_1$.

Finally, suppose for a contradiction that ξ is thick. Then thickness yields a line L_i through u_i distinct from T_i and U_1, which is π-quadrangular by (1). By (2), there is a unique π-quadrangular line T^* meeting L_1 and L_2 in a point. Since T^* is disjoint from U_1, it is not a line of the grid G and hence, by (1), it is disjoint from the point set of G. Denote by $t^* \in \pi$ the center of T^* and note that $t^* \notin t_1t_2$. Let π^* be the plane containing T^* and t^*. If ξ_1 would intersect π^*, then the intersection would be a line K by Lemma 9.1.8 and since $t_1 \cap \pi^* = \{t^*\}$, we get $t^* \in K$, but then $\pi \subseteq \xi$, a contradiction. Hence π^* and ξ_1 are disjoint and as such, (1) and (2) hold for them. Since U_1 is π^*-quadrangular with axis T^* and t_1t_2 is π^*-triangular with center t^*, (2) implies that all lines of G^* disjoint from t_1t_2 are π^*-quadrangular. But then (2) implies that at least one line of G^* not disjoint from t_1t_2 is π^*-triangular, contradicting (1).

This contradiction shows that ξ is non-thick and the proof of the lemma is complete. □

We henceforth call the singular planes of Ω *blocks* and denote the set of blocks by $\mathscr{B}$. Using the previous proposition, we show that the point-block geometry $\mathscr{G} = (\mathscr{P}, \mathscr{B})$ is a generalized quadrangle (at least in case that for every singular plane, there is a symp disjoint from it). We prove the essential axiom separately.

Lemma 9.1.13. *Suppose that for each singular plane of Ω, there is a symp disjoint from it. Then*

- (i) *for each non-incident point-block pair (p, π) of Ω, there is a unique block π' through p intersecting π in a unique point;*
- (ii) *each line is contained in a unique block.*

Proof. By assumption, there is a symp ξ disjoint from π. It follows from Proposition 9.1.12 that ξ is non-thick and that each line L_i of one system of generators of ξ is contained in a singular plane π_i intersecting π in a point p_i, for i in some index set I. If p belongs to π_i for some $i \in I$, then (i) follows immediately. So suppose $p \notin \bigcup_{i \in I} \pi_i$. Then, by Lemma 9.1.7, p is collinear to a unique point q_i of π_i for each $i \in I$. We may assume that $p_1 \neq q_1$ (if $p_i = q_i$ for each $i \in I$ then p is collinear with the line $p_1 p_2$ of π, a contradiction). Let M be a line through q_1 not through p_1. Since p is not collinear to M, strongness implies a symp ξ' through p and M. Then $\xi' \cap \pi = \emptyset$, as no point of π is collinear to exactly one point of M. Applying Proposition 9.1.12 again on the pair (π, ξ'), we obtain that p is contained in a plane intersecting π in a unique point.

From the above we deduce that each point is contained in at least one plane. Then (ii) follows as well: let L be a line spanned by two points p and q and take a plane π_q through q. If π_q contains p we are done. If not, it follows from (i) that there is a plane π_p through p intersecting π_q in a point r. By Lemma 9.1.7, $r = q$ and hence $L \subseteq \pi_p$. The uniqueness follows from Lemma 9.1.7. □

Recall that an *ideal* subquadrangle Δ' of a generalized quadrangle Δ is a subquadrangle with the property that every line of Δ through a point of Δ' is a line of Δ'.

Lemma 9.1.14. *Suppose that for each singular plane of Ω, there is a symp disjoint from it. Then $\mathscr{G} = (\mathscr{P}, \mathscr{B})$, endowed with natural incidence, is a generalized quadrangle with thick blocks and every symp of Ω is an ideal subquadrangle of $\mathscr{G}$.*

Proof. We verify the axioms of a generalized 4-gon. By Lemma 9.1.7, each pair of points of $\mathscr{G}$ is contained in at most one block (i.e., $\mathscr{G}$ is a partial linear space). Since every block contains a line, all lines of $(\mathscr{P}, \mathscr{B})$ are thick; moreover, each point is contained in at least two lines and hence, by Lemma 9.1.13(ii), in at least two blocks. By considering the blocks containing the lines of a symp, we see that $(\mathscr{P}, \mathscr{B})$ contains a quadrangle. The remaining axioms follow from Lemma 9.1.13.

Let ξ be a symp of Ω. Since each of its points belongs to $\mathscr{P}$ and each of its lines is contained in $\mathscr{B}$ by Lemma 9.1.13(ii), ξ is a subquadrangle of $\mathscr{G}$ indeed. Now let p be a point of ξ and let π be a block through p. Then by Lemma 9.1.8, ξ shares a line with π, showing that ξ is an ideal subquadrangle indeed. □

Proposition 9.1.15. *Suppose that for each singular plane of Ω, there is a symp disjoint from it. Then Ω is isomorphic to a Segre geometry of type* $(2,2)$.

Proof. Our assumption implies that at least one symp is non-thick (cf. Lemma 9.1.12). Since the symps are ideal subquadrangles of the generalized quadrangle $\mathscr{G}$, the latter is non-thick. This means that $\mathscr{G}$, and hence also Ω, is a direct product of two linear spaces Y and Z. As the maximal singular subspaces of Ω are projective planes (cf. Lemmas 9.1.6 and 9.1.13(ii)), also Y and Z are projective planes. The assertion follows. □

A strong (−1)-lacunary parapolar space of minimum symplectic rank 2 is isomorphic to a Segre geometry of type $(1,2)$ or $(2,2)$, i.e., it is the direct product of a thick line or plane with a thick plane (which are not necessarily of the same order).

9.2 Case 2: symplectic rank at least 3 and at least one line is sympthick

In this section, Ω is a parapolar space of minimum symplectic rank d with $d \geq 3$ with lacunary index -1, containing at least one sympthick line. Recall that a singular subspace is called sympthick if it is contained in more than one symp. A symp not containing a sympthick line will be called *isolated*. We start with some general properties and deduce, amongst others, that $d \in \{3,5\}$, after which we split up the prove into those two cases.

9.2.1 General properties and reduction to uniform symplectic rank $d \in \{3,5\}$

Lemma 9.2.1. *Let ξ, ξ' be symps of Ω intersecting each other in exactly a point p. Then there is a singular plane through p intersecting both symps in a line.*

Proof. Take a $(d-2)$-space S in ξ' which is not contained in $p^\perp$. By Lemma 2.1.40, there is a symp $\xi'' \neq \xi'$ through S such that $\dim(\xi' \cap \xi'') \geq d-1$. By (-1)-lacunarity, $\xi \cap \xi''$ contains a point q; and $q \neq p$ because $\xi' \cap \xi''$ is a singular subspace, whereas $p \notin S^\perp$. Since $d-1 \geq 2$, the intersection $\xi' \cap \xi''$ contains at least a point r collinear to both p and q. The point r does not belong to ξ but is collinear to the distinct points $p, q \in \xi$, implying that p and q are collinear. We conclude that $\langle p, q, r\rangle$ is a singular plane intersecting both ξ and ξ' in a line. □

Lemma 9.2.2. *No symp of Ω is isolated.*

Proof. Suppose for a contradiction that some symp ξ of Ω is isolated, i.e., none of its lines is sympthick. Since Ω contains at least one sympthick line, there is a non-isolated symp ξ'. By Lemma 2.1.40, each line of ξ' is sympthick. Consequently, ξ and ξ' cannot share a line and hence, by (-1)-lacunarity, $\xi \cap \xi'$ is exactly a point p. By Lemma 9.2.1, there is a singular plane π intersecting ξ and ξ' in respective lines L and L'. Lemma 2.1.37 says that π is contained in a symp, so in particular, there is a second symp containing L after all, a contradiction. □

As a consequence, Lemma 2.1.40 holds for all symps of Ω. As we will use it often, we record its consequences on Ω:

(i) Each singular subspace of Ω of dimension at most $d-2$ is sympthick.

(ii) If a $(d-1)$-space M is not sympthick, then there is a unique symp ξ_M containing it, and each symp $\xi^* \neq \xi_M$ with $\dim(M \cap \xi^*) \geq d-2$ is non-thick, has rank d and intersects ξ_M in a $(d-1)$-space distinct from M.

Lemma 9.2.3. *Let ξ be any symp of Ω of rank d. Then*

(i) *for each symp ξ' with* $\dim(\xi \cap \xi') \geq d-2$*, the rank of ξ' is d and* $\dim(\xi \cap \xi') = d-1$*;*

(ii) *ξ is non-thick and two generators of ξ of different natural type are never both sympthick;*

(iii) *d is odd.*

Proof. (i) Consider opposite subspaces S_1 and S_2 of ξ of dimension $d-2$. By Lemmas 2.1.40 and 9.2.2, there are symps ξ_1^* and ξ_2^* intersecting ξ in maximal singular subspaces M_1 and M_2 of ξ through S_1 and S_2, respectively. Suppose that $M_1 \cap M_2 = \emptyset$ (so M_1 and M_2 are opposite). Then $\xi_1^* \cap \xi_2^*$, which contains at least a point p by (−1)-lacunarity, is disjoint from ξ. But then the $(d-2)$-spaces $p^\perp \cap M_1$ and $p^\perp \cap M_2$ of ξ contain a pair of non-collinear points since M_1 and M_2 are opposite and $d-2 \geq 1$. This contradiction implies that $M_1 \cap M_2$ intersect in a point.

Observe that this implies that for each $(d-2)$-space of ξ, there is a $(d-1)$-spaces of ξ through it which is not sympthick. Consequently, by Lemma 2.1.40(ii), any symp ξ^* with $\dim(\xi \cap \xi^*) \geq d-2$ is non-thick, has rank d and $\dim(\xi \cap \xi^*) = d-1$. This shows the first assertion, so we continue with the second one.

(ii) Firstly, suppose for a contradiction that ξ is thick. Let M_2^* be a $(d-1)$-space in ξ_2^* through S_2 distinct from M_2. Then M_2^* is collinear to at most one of the maximal singular subspaces of ξ through S_2 and, as there are at least three such subspaces, M_2^* is contained in a symp with a maximal singular subspace M_2' of ξ through S_2 which is disjoint from M_1, contradicting the first paragraph. We conclude that ξ is non-thick indeed.

Secondly, suppose that N_1 and N_2 are two generators of ξ of distinct natural type and assume both are sympthick. By Lemma 2.1.30, there exists a subspace S_3 of ξ of dimension $d-2$ disjoint from N_1 and N_2. Again by Lemma 2.1.40, there is a symp $\xi_3^* \neq \xi$ with $S_3 \subseteq \xi \cap \xi_3^*$. By the above observation, $\xi \cap \xi_3^*$ is a maximal singular subspace N_3 of ξ through S_3. Moreover, the first paragraph implies that both $N_1 \cap N_3$ and $N_2 \cap N_3$ are exactly a point, but then the type of N_3 should either equal the types of N_1 and N_2 or be distinct from both of them, a contradiction.

(iii) It now follows that the sympthick subspaces M_1 and M_2, which intersect each other in exactly a point, need to be of the same natural type. We conclude that d is odd. □

Let ξ be a symp of Ω of rank d. Then we denote by $\Phi^{\xi}_{>1}$ the set of generators of ξ that are sympthick (contained in more than one symp) and by Φ^{ξ}_{d} the set of generators of ξ that are contained in a singular d-space. We show that these two sets are precisely the two natural families of generators of ξ.

Lemma 9.2.4. *Let ξ be a symp of Ω rank d.*

(i) *The set $\Phi^{\xi}_{>1}$ of sympthick generators of ξ coincides with one natural family of ξ;*

(ii) *the set Φ^{ξ}_{d} of generators of ξ contained in a singular d-space coincides with the other natural family of ξ.*

In particular, sympthick generators of ξ are maximal singular subspaces of Ω.

Proof. (i) By Lemma 9.2.3, no two generators of ξ of distinct natural type are both sympthick. Then for each submaximal subspace S of ξ, Lemma 2.1.40 implies that there is a unique maximal singular subspace M through S which is sympthick. As such it is clear that $\Phi^{\xi}_{>1}$ contains all generators of one of the two natural types.

(ii) Consider a generator M of ξ which is not sympthick. We first show that M is contained in a singular d-space. To that end, let M' be any generator of ξ intersecting M in a $(d-2)$-space S. Then $M' = \xi \cap \xi'$, for some $\xi' \in \Xi$ by Lemma 9.2.4. By Lemma 9.2.3(i), ξ' has rank d and hence, by (ii) of the same lemma, it is non-thick. In ξ', we consider the generator M'' containing S and distinct from M', and some point $p \in M'' \setminus M'$. If p were not collinear to all points of M, then for every $q \in M \setminus M''$ the symp $\xi(p,q)$ contains M and is different from ξ, contradicting our assumption on M. Hence p and M generate a singular subspace of dimension d.

Now let N be a sympthick generator of ξ and suppose for a contradiction that N is contained in a singular d-space N'. Like above, let M be a generator of ξ intersecting N in a $(d-2)$-space. Then M is not sympthick by Lemma 9.2.4 and hence we have just shown above that M is contained in a singular d-space M'. Let p_M and p_N be points in $M \setminus S$ and $N \setminus S$, respectively. Clearly, those points are not collinear, so $\xi = \xi(p_M, p_N)$. This implies $N' \cap M' = S$, for if a point $x \in N' \cap M' \setminus S$, then $x \in \xi(p_M, p_N) = \xi$, and hence $\langle S, x\rangle$ would be a third $(d-1)$-space through S in ξ, a contradiction. Let q_N be a point in $N' \setminus N$ not collinear to p_M (if all points of $N' \setminus N$ were collinear to p_M, then also all points of N are, and they are not). But then q_N and p_M are symplectic, since they are both collinear to $M \cap N$ and the latter is at least a line since $d \geq 3$. As such, M belongs to the symp $\xi(p_M, q_N)$ distinct from ξ (as $q_N \notin \xi$), contradicting that M is not sympthick. We conclude that sympthick generators of ξ are not contained in singular d-spaces. $\square$

Lemma 9.2.5. *Each symp of Ω has rank d and is non-thick.*

Proof. Let ξ be any symp of rank d. By Lemma 9.2.3(i), any symp ξ' with $\dim(\xi \cap \xi') \geq d-2$ has rank d as well. Now let ξ^* be an arbitrary symp. We claim that we can find a finite sequence of symps between ξ^* and ξ such that successive symps in the sequence intersect each other in a subspace of dimension at least $d-2$, from which it then follows that each symp in this sequence has rank d.

By (-1)-lacunarity, $\xi \cap \xi^*$ is non-empty. If $\xi \cap \xi^*$ is a point, Lemma 9.2.1 implies the existence of a plane π intersecting both ξ and ξ^* in a line, and since $d \geq 3$, Fact 2.1.38 yields a symp through π which then shares at least a line with both ξ and ξ^*. Hence, if $d = 3$, we are done. If $d > 3$, we may already assume that $1 \leq \dim(\xi \cap \xi^*) \leq d-3$. Under this assumption we can take points p and p^* in ξ and ξ^*, respectively, collinear to $\xi \cap \xi^*$ and not collinear to each other. The symp determined by p and p^* intersects both ξ and ξ^* in a subspace strictly bigger than $\xi \cap \xi^*$. Recursively, the claim follows. We conclude that each symp ξ has rank d and by Lemma 9.2.3(ii), ξ is non-thick. $\square$

Local connectivity now follows as a consequence of Lemma 9.2.1.

Lemma 9.2.6. *Ω is locally connected.*

Proof. Consider two lines L_1 and L_2 through p. Let ξ_1 and ξ_2 be symps through L_1 and L_2, respectively. If $\xi_1 = \xi_2$, then L_1 and L_2 are, if not contained in a plane, connected by a sequence of two planes intersecting each other in a line through p. So suppose $\xi_1 \neq \xi_2$. If $\xi_1 \cap \xi_2$ contains a line L through p then there is a sequence between L_1 and L_2 via L. If $\xi_1 \cap \xi_2 = \{p\}$, then there is a symp ξ intersecting both ξ_1 and ξ_2 in a line by Lemma 9.2.1 and Fact 2.1.38, and hence we can walk via these lines. □

We have the following important corollary.

Corollary 9.2.7. *A* (−1)*-lacunary parapolar space* Ω *of symplectic rank at least 3 is locally connected if and only if it contains a sympthick line if and only if all its lines are sympthick.*

Proof. It follows from Lemma 2.1.40 that, if Ω contains at least one sympthick line, then all its lines are sympthick. From Lemma 9.2.6 it follows that Ω is locally connected. Conversely, if Ω is locally connected, then each of its lines is sympthick: through each point of any point-residual (which is a strong parapolar space), there are at least two symps. □

We proceed by showing boundedness of the singular rank.

Lemma 9.2.8. *A singular subspace of* Ω *has dimension at most* $2(d-1)$.

Proof. Suppose for a contradiction that there is a singular $(2d-1)$-space W in Ω. Let M_1 and M_2 be two disjoint $(d-1)$-subspaces in W. By Fact 2.1.38, there are symps ξ_1 and ξ_2 containing M_1 and M_2, respectively, and (−1)-lacunarity yields a point $p \in \xi_1 \cap \xi_2$. Since M_i is a maximal singular subspace in ξ_i, $i = 1,2$, we know $p \notin W$. In particular, $p \notin M_1 \cup M_2$ and so we can find points $q_1 \in M_1$ and $q_2 \in M_2$ with $q_1 \notin p^\perp$ and $q_2 \in p^\perp$. Then $q_2 \in p^\perp \cap q_1^\perp \subseteq \xi_1$, a contradiction. □

We now show that Ω is also k-lacunary for each natural number k with $1 \leq k \leq d-2$.

Lemma 9.2.9. *Let* ξ_1 *and* ξ_2 *be two symps of* Ω*. Then* $\dim(\xi \cap \xi') \in \{0, d-1\}$.

Proof. Recall that we know from Lemma 9.2.5 that each symp has rank d. Put $U = \xi_1 \cap \xi_2$ and $u = \dim(U)$. By (−1)-lacunarity, $u \geq 0$. We first claim that if $u \geq 2$, then $u = d-1$. If $d = 3$, this is trivial, so suppose $d \geq 4$. Select a sympthick generator M in ξ_1 disjoint from U (which is possible by Lemma 9.2.4) and let ξ be a symp such that $M = \xi_1 \cap \xi$. By (−1)-lacunarity, there is a point $p \in \xi_2 \cap \xi$. Then $p \notin \xi_1$ since $\xi_1 \neq \xi$. However, p is collinear to all points of a $(d-2)$-space in M (since $p \in \xi$) and $\dim(p^\perp \cap U) \geq u-1$ (since $p \in \xi_2$). Since $p^\perp \cap \xi_1$ is a singular subspace, its dimension is at most $d-1$, i.e., $(d-2)+(u-1)+1 \leq d-1$, implying $u \leq 1$. The claim is proved.

Now suppose for a contradiction that $u = 1$. Take a point $p \in \xi_1 \cap U^\perp \setminus U$ and a point $p_2 \in x_2 \cap U^\perp \setminus p_1^\perp$. Then $\xi := \xi(p_1, p_2)$ shares a plane with both ξ and ξ' and hence the above implies that $\xi \cap \xi_i$ is a generator M_i for $i \in \{1,2\}$. The generators M_1 and M_2, being contained in ξ_1 and ξ_2, respectively, intersect each other in precisely the line U. As such, they are sympthick generators in ξ of different natural types, contradicting Lemma 9.2.4. We conclude that $u \in \{0, d-1\}$. $\square$

Lemma 9.2.10. *The symplectic rank d of Ω is either* 3 *or* 5.

Proof. Since d is odd by Lemma 9.2.3(iii), it suffices to show that $d \geq 5$ implies $d = 5$. So suppose $d \geq 5$. Let ξ be a symp and choose two non-sympthick generators M, M' of ξ intersecting each other in a plane π. By Lemma 9.2.4, there are d-spaces W and W'through M, M', respectively. If all points of $W \setminus M$ were collinear to all points of $W' \setminus M'$, then all points of M would be collinear to all points of M', a contradiction. So there are points $p \in W \setminus M$ and $p' \in W' \setminus M'$ which are not collinear. Since π belongs to $p^\perp \cap p'^\perp$, the pair p and p' determine a unique symp ξ^*. Since $\xi \cap \xi^*$ contains π, Lemma 9.2.9 implies that $\xi \cap \xi^*$ is generator M^* of ξ. Since $p^\perp \cap \xi = M$, we have $p^\perp \cap M^* \subseteq M$; likewise $p'^\perp \cap M^* \subseteq M'$. Both subspaces have dimension $d-2$ and are contained in M^*, and hence intersect in a $d-3$-space. On the other hand, since $M \cap M' = \pi$, they intersect in π only, so $d - 3 \leq 2$, implying $d \leq 5$. $\square$

A (-1)-lacunary parapolar space of minimum symplectic rank $d \geq 3$ containing at least one sympthick line is locally connected and has $d \in \{3, 5\}$.

We now consider the cases $d = 3$ and $d = 5$ separately.

9.2.2 Case 2a: uniform symplectic rank $d = 3$

In this subsection we assume $d = 3$. This case should lead to two examples, $\mathsf{A}_{4,2}$ and $\mathsf{A}_{5,2}$, depending on the singular rank of Ω. Recall that Lemma 9.2.8 implies that the singular rank is at most 4. By Lemma 9.2.4, we know that the sympthick planes are maximal singular subspaces. Denote the set of sympthick singular planes by $\mathscr{V}$ and denote by $\mathscr{M}$ the set of maximal singular subspaces of Ω which are not planes (so these have dimension 3 or 4).

Lemma 9.2.11. *Two subspaces M_1 and M_2 of $\mathscr{M}$ never have a line in common.*

Proof. Suppose first that $M_1 \cap M_2$ contains a plane π. Take a point $p_1 \in M_1 \setminus M_2$. By maximality of M_2, the subspace $p_1^\perp \cap M_2$ is at most a hyperplane of M_2. So there is a point $p_2 \in M_2 \setminus p_1^\perp$. But then the symp $\xi(p_1, p_2)$ contains the 3-space $\langle p_1, \pi \rangle$, violating $d = 3$.

Hence we may assume that $M_1 \cap M_2$ is a line L. Note that this implies that for each point $p_1 \in M_1 \setminus L$ holds that $p_1^\perp \cap M_2 = L$, for otherwise the element of $\mathscr{M}$ containing p and $p^\perp \cap M_2$ would intersect M_2 in a plane, contradicting what we obtained in the previous paragraph. So take any point $p_2 \in M_2 \setminus L$, any point $q_1 \in M_1 \setminus \langle p_1, L \rangle$ and any point $q_2 \in M_2 \setminus \langle p_2, L \rangle$. So the pairs (p_1, p_2), (p_1, q_2) and (p_2, q_1) are symplectic (they all have L in their common perp). Then the plane $\langle p_1, L \rangle$ is contained in two symps, namely $\xi(p_1, q_1)$ and $\xi(p_1, q_2)$ (these are distinct as otherwise they contain the 3-space $\langle L, q_1, q_2 \rangle$). Likewise, $\langle p_2, L \rangle$ is sympthick. Yet, the planes $\langle p_1, L \rangle$ and $\langle p_2, L \rangle$ are generators of distinct natural type in the symp $\xi(p_1, p_2)$, contradicting Lemma 9.2.4 and proving the assertion. $\square$

This immediately implies the following.

Corollary 9.2.12. *Two singular subspaces of Ω of dimension at least 3 intersect in at most a point.*

Lemma 9.2.13. *For each line L of Ω, there are unique elements V_L and M_L of $\mathscr{V}$ and $\mathscr{M}$, respectively, containing L. Moreover, for each symp ξ through L, we have $V_L \subseteq \xi$ and $\dim(M_L \cap \xi) = 2$.*

Proof. Let ξ be a symp through L and let π_1 and π_2 be the two planes of ξ through L. Assume π_1 is the one which is sympthick (and hence $\pi_1 \in \mathscr{V}$) and π_2 the one which is strictly contained in some maximal singular subspace $M \in \mathscr{M}$ (cf. Lemma 9.2.4). We put $V_L := \pi_1$ and $M_L := M$. The uniqueness of M_L follows from Lemma 9.2.11. Suppose for a contradiction that L is contained in two planes of $\mathscr{V}$. Then these planes are contained in a common symp, in which it is clear that they are of a different natural type. As they are both not contained in a singular d-space, this contradicts Lemma 9.2.4. Since ξ was an arbitrary symp through L, it follows that V_L belongs to each symp through L and that $M_L \cap \xi$ is a plane through L. □

The previous lemma says that, if a symp and subspace of $\mathscr{M}$ share a line, then they share a plane. We now show a slightly stronger statement.

Lemma 9.2.14. *Let ξ be a symp of Ω. If ξ shares a point with some $M \in \mathscr{M}$, then $\xi \cap M$ is a plane.*

Proof. Suppose by way of contradiction $\dim(\xi \cap M) < 2$. If $\xi \cap M$ were a line L, then $M = M_L$ intersects ξ in a plane by Lemma 9.2.13. Hence $\xi \cap M$ is a point p. Since $\dim(M) \geq 3$, there is a plane π in M disjoint from p and in particular from ξ. Let ξ' be a symp containing π. By (-1)-lacunarity, there is a point $p' \in \xi \cap \xi'$. Let L be the line in π collinear to p' and note that $L \subseteq p^\perp \cap p'^\perp$. Because L is disjoint from ξ, this implies $p \perp p'$. But then $\langle p, p', L\rangle$ is a singular subspace of dimension 3, distinct from M but intersecting it in $\langle p, L\rangle$, contradicting Corollary 9.2.12. □

Lemma 9.2.15. *Ω is strong.*

Proof. Suppose for a contradiction that p_1 and p_2 is a special pair. Let p be the unique point in $p_1^\perp \cap p_2^\perp$. Take a symp ξ containing p_1p and consider the unique element M_{pp_2} of $\mathscr{M}$ through the line pp_2. Then Lemma 9.2.14 yields a plane π in $\xi \cap M$. In ξ, the intersection $p_1^\perp \cap \pi$ is a line L through p. Clearly, $L \subseteq p_1^\perp \cap p_2^\perp$ and hence $\{p_1, p_2\}$ is symplectic after all, a contradiction. □

Lemma 9.2.16. *Let p be a point and M a member of $\mathscr{M}$. If $p \notin M$, then $p^\perp \cap M$ is a line L and $\langle p, L\rangle \in \mathscr{V}$.*

Proof. By connectivity of Ω, there is a shortest path $(p_0, p_1, \dots, p_\ell)$ with $p_{i-1} \perp p_i$, for all $i \in \{1, 2 \dots, \ell\}$, and with $p_0 \in M$ and $p_\ell = p$. Suppose for a contradiction that $\ell \geq 2$. Lemma 9.2.15 yields a symp $\xi \supseteq \{p_0, p_1, p_2\}$. Lemma 9.2.14 yields a plane $\pi \subseteq \xi \cap W$ and hence there are points of π collinear to p_2, contradicting the minimality of ℓ. Consequently, $\ell = 1$. Considering a symp through $p_0 p_1$ and using Lemma 9.2.14 again, we deduce that p is collinear to a line of W. By Corollary 9.2.12, $\langle p, L \rangle \in \mathscr{V}$ and in particular, $p^\perp \cap M = L$. □

Lemma 9.2.17. *Two distinct elements M_1 and M_2 of $\mathscr{M}$ intersect in a unique point.*

Proof. By Lemma 9.2.11, it suffices to show that M_1 and M_2 are not disjoint. So suppose they are. Pick $p_1 \in M_1$ and let L_2 be the unique line in M_2 collinear to p_1 (cf. Lemma 9.2.16). Take a point $p_2 \in M_2 \setminus L_2$. Then p_1 and p_2 determine a symp ξ. Since $\xi \cap M_1$ contains p_1, Lemma 9.2.14 implies that $\xi \cap M_1$ is a plane π_1. By construction, $\xi \cap M_2$ is the plane $\pi_2 := \langle p_2, L_2 \rangle$. The two planes π_1 and π_2 in ξ are both contained in a member of $\mathscr{M}$, whereas they have distinct natural type, contradicting Lemma 9.2.4. □

Lemma 9.2.18. *The geometry $(\mathscr{M}, \mathscr{P})$, with inclusion made symmetric as incidence, is the point-line truncation of a projective space $\mathbb{P}$ of dimension 4 or 5. Moreover, the lines incident with a point of $\mathbb{P}$ correspond to the points of a member $M \in \mathscr{M}$ in Ω and the lines incident with a plane of $\mathbb{P}$ correspond to the points of a member $V \in \mathscr{V}$ in Ω.*

Proof. We verify that $(\mathscr{M}, \mathscr{P})$ satisfies the Veblen-Young axioms of a projective space.

• *Each line contains at least three points.* Let p be a point of Ω. We have to show that there are at least 3 members of $\mathscr{M}$ through p. To that end, consider a symp ξ through p. Then there are at least three non-sympthick planes in ξ through p and these planes are all contained in members of $\mathscr{M}$ by Lemma 9.2.4, and clearly, they are all distinct.

• *Each pair of distinct point is contained in a unique line.* Two elements of $\mathscr{M}$ share a unique point by Lemma 9.2.17.

• *There are three points not in one common line.* Take three points p_1, p_2, p_3 spanning an element of $\mathscr{V}$. Then the unique member $M_{p_1 p_2}$ of $\mathscr{M}$ through $p_1 p_2$ does not contain p_3.

• *Axiom of Pasch.* Take two intersecting lines L_1 and L_2 of $(\mathscr{M}, \mathscr{P})$, i.e., a pair of points p_1 and p_2 of Ω which are collinear (and hence contained in $M_{p_1 p_2} \in \mathscr{M}$). Let L_3 and L_4 be lines of $(\mathscr{M}, \mathscr{P})$ intersecting L_1 and L_2 in distinct points; and let p_3 and p_4 be the points in Ω corresponding to L_3 and L_4, respectively. We show that L_3 and L_4 intersect, i.e., that p_3 and p_4 are collinear (as then they are contained in a member of $\mathscr{M}$, namely, in $M_{p_3 p_4}$). Note that, for $i = 3, 4$, the point p_i is collinear with p_1 and p_2 but is not contained $M_{p_1 p_2}$. Then Lemma 9.2.11 implies that $\langle p_1, p_2, p_3 \rangle$ is the unique maximal singular plane $\pi_{p_1 p_2}$ through $p_1 p_2$. It follows that p_3 and p_4, being contained in $\pi_{p_1 p_2}$, are indeed collinear.

Hence $(\mathscr{M}, \mathscr{P})$ is the point-line truncation of a projective space $\mathbb{P}$. Clearly, the lines through a point of $\mathbb{P}$ correspond to the points in a member $M \in \mathscr{M}$ of Ω. This implies that the dimension of $\mathbb{P}$ is precisely $\dim(M) + 1$, so either 4 or 5. In particular, all members of $\mathscr{M}$ have equal dimension. Note that we deduced in the previous paragraph that the lines incident with a plane of $\mathbb{P}$ correspond to the points of a maximal singular plane of Ω, i.e., a member of $\mathscr{V}$. □

Proposition 9.2.19. *Ω is the line-Grassmannian of $\mathbb{P}$.*

Proof. Let $\mathscr{G} := \mathscr{G}_1(\mathbb{P})$ denote the line-Grassmannian of $\mathbb{P}$. The points of $\mathscr{G}$ are the lines of $\mathscr{P}$, which are exactly the points of Ω. We now verify that also the line sets coincide. To that end, consider a line of $\mathscr{G}$, which is a planar line pencil in $\mathbb{P}$ (i.e., the set of lines through a point p in a plane π). As already mentioned above, the lines through a point p correspond to the points in Ω of a member $M \in \mathscr{M}$ and the lines of $\mathbb{P}$ in the plane π correspond to the points in Ω of a plane $V \in \mathscr{V}$. Now each line of $\mathbb{P}$ incident with both p and π then corresponds to a point of Ω contained in $M \cap V$ and vice versa. Since there are at least two lines through p in π, there are at least two points in $M \cap V$ and hence this intersection is precisely a line (by maximality of V, $V \nsubseteq M$). As such, a line of $\mathscr{G}$ corresponds with a line of Ω. The proposition follows. □

A (−1)-lacunary parapolar space Ω of minimum symplectic rank 3 containing at least one sympthick line is the line Grassmannian of a projective space of dimension 4 or 5, i.e., Ω is a Lie incidence geometry of type $\mathsf{A}_{4,2}(\mathbb{L})$ or $\mathsf{A}_{5,2}(\mathbb{L})$, where $\mathbb{L}$ is a skew field.

9.2.3 Case 2b: uniform symplectic rank $d = 5$

In this subsection we assume that $d = 5$. This should lead to the $\mathsf{E}_{6,1}$ example. We first determine the singular rank.

Lemma 9.2.20. *Let $\Omega = (X, \mathscr{L})$ be a parapolar space of symplectic rank $d = 5$ with lacunary index -1 and containing at least one sympthick line. Then the singular rank is equal to 5.*

Proof. By Lemma 9.2.4, singular subspaces of projective dimension 5 occur. Suppose for a contradiction that there is a singular subspace W of dimension 6. Select two subspaces $U_1, U_2 \subseteq W$ of dimension 4 which intersect in a plane π. Then Lemma 2.1.38 yields symps ξ_1, ξ_2 containing U_1, U_2, respectively. Since $\xi_1 \cap \xi_2$ contains π, Lemma 9.2.9 implies that $\xi_1 \cap \xi_2$ is a 4-space U. In ξ_i, for $i = 1,2$, the 4-spaces U_i and U need to be of distinct type by Lemma 9.2.4, and hence, since $\pi \subseteq U_i \cap U$, it follows that $U \cap U_1$ and $U \cap U_2$ are 3-dimensional. But then the sympthick subspace U is contained in S (as it is spanned by $U_1 \cap U$ and $U_2 \cap U$), contradicting Lemma 9.2.4. □

As in the previous case, denote by $\mathscr{M}$ the set of maximal singular subspaces of Ω which are not contained in a symp (i.e., of dimension 5 by the above lemma); and by $\mathscr{V}$ the set of maximal singular subspaces of Ω which are contained in a symp (i.e., the sympthick 4-spaces). Furthermore, let Π denote the set of singular planes of Ω.

Define the following geometry $\mathscr{G} = (\mathscr{P}, \mathscr{M}, \mathscr{L}, \Pi, \mathscr{V}, \Xi)$ of rank 6 , where the incidence is inclusion made symmetric, except for pairs $(W, U) \in \mathscr{M} \times \mathscr{V}$, which are incident if $\dim(U \cap W) = 3$ and pairs $(W, \xi) \in \mathscr{M} \times \Xi$ are incident if $\dim(W \cap \xi) = 4$. Putting $\mathscr{G} = (X_1, ..., X_6)$, we say that the elements of X_i are the elements of $\mathscr{G}$ of type i, for $i \in \{1,\dots,6\}$. We will show that this geometry has the E_6 diagram as depicted below. Once we have shown that $\mathscr{G}$ is residually connected, it follows that $\mathscr{G}$ is a building of type E_6.

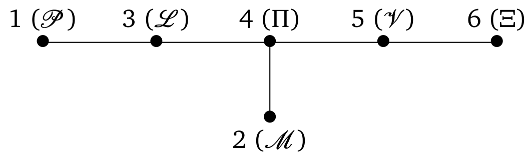


Figure 9.1: The Dynkin diagram of type E_6 with Bourbaki labeling

Lemma 9.2.21. *The geometry $\mathscr{G} = (\mathscr{P}, \mathscr{M}, \mathscr{L}, \Pi, \mathscr{V}, \Xi)$ is of type* E_6.

Proof. We have to verify the diagram of all rank 2 residues $\mathsf{Res}(R_i, R_j, R_k, R_\ell)$, where i, j, k, ℓ is a subset of size 4 of $\{1,\dots,6\}$ and R_i an element of type i, etc.

Now, each member ξ of Ξ is a hyperbolic quadric of rank 5 by Lemma 9.2.4. This implies that the diagram restricted to the types 1 up to 5 is as it should be (all rank 2 residues of the form $\mathsf{Res}(\xi,\dots)$, being part of $\mathsf{Res}(\xi)$, are okay). Next, consider any incident point-line pair $(p, L) \in \mathscr{P} \times \mathscr{L}$. Then the residue $\mathsf{Res}(p, L)$ is a parapolar space with sympthick lines (as it has sympthick maximal singular subspaces) which is (−1)-lacunary by Lemma 9.2.9 and which contains 3-dimensional maximal singular subspaces. Hence Proposition 9.2.19 implies that $\mathsf{Res}(L)$ is of type $\mathsf{A}_{4,2}$, so the diagram restricted to the types $2, 4, 5, 6$ is as it should be. Lastly, we need to verify that there are no edges in the diagram between types 6 and both 1 and 2. Consider $\mathsf{Res}(\pi)$ for a plane $\pi \in \Pi$. Since for each point $p \in \mathscr{P}$ and each line $L \in \mathscr{P}$ incident with π, we have that each symp ξ through π is incident with both p and L, the diagrams of the rank 2 residues of types $(1, 6)$ and $(2, 6)$ are digons indeed. The lemma follows. □

Proposition 9.2.22. *The geometry $\mathscr{G} = (\mathscr{P}, \mathscr{M}, \mathscr{L}, \Pi, \mathscr{V}, \Xi)$ is residually connected and hence Ω is a Lie incidence geometry of type* $\mathsf{E}_{6,1}$.

Proof. We have to verify that each residue of $\mathscr{G}$ is connected. Note that, if a residue has a disconnected diagram, then it is trivially connected, so we only have to check residues conforming to a connected subdiagram of the diagram of type E_6. Let $R := \mathsf{Res}(S)$—where S is a set of pairwise incident elements of distinct types—be a residue with a connected diagram.

- If S contains a member ξ of Ξ, then R is indeed connected since $\mathsf{Res}(\xi)$ is a Lie incidence geometry of type $\mathsf{D}_{5,1}$, which is both connected and residually connected.
- If S contains a point p and a line L (with $p \in L$), then R is connected since $\mathsf{Res}(p, L)$ is the line Grassmannian of a projective 4-space as was noted in the proof of Lemma 9.2.21, which is connected and residually connected.
- If S contains a member M of $\mathscr{M}$, then R is connected since $\mathsf{Res}(M)$ is a projective 5-space, which is connected and residually connected.
- If S is just a point p, then R is connected since Ω is locally connected (cf. Lemma 9.2.6).

This covers all possibilities for R and hence $\mathscr{G}$ is residually connected indeed.

The last assertion follows from a result of Brouwer and Cohen building on fundamental work of Tits in his paper on the local approach to buildings ([8]). □

A (−1)-lacunary parapolar space of minimum symplectic rank $d = 5$ containing at least one sympthick line is a Lie incidence geometry of type $\mathsf{E}_{6,1}(\mathbb{K})$.

This completes the classification of the (−1)-lacunary parapolar spaces with at least one sympthick line when the minimum symplectic rank is at least 3.

9.3 Conclusion

We have shown that a (−1)-lacunary parapolar space Ω of minimum symplectic rank d, in case $d = 2$ and Ω strong, is a Lie incidence geometry of type $\mathsf{A}_{1,1}(*) \times \mathsf{A}_{2,1}(*)$ or $\mathsf{A}_{2,1}(*) \times \mathsf{A}_{2,1}(*)$; and, if $d \geq 3$ and if Ω contains a sympthick line, it is a Lie incidence geometry of type $\mathsf{A}_{4,2}(\mathbb{L})$, $\mathsf{A}_{5,2}(\mathbb{L})$ or $\mathsf{E}_{6,1}(\mathbb{K})$.

It also follows that:

Lemma 9.3.1. *A* (−1)*-lacunary parapolar space* Ω *of minimum symplectic rank* $d \geq 3$ *is strong if and only if it has a sympthick line if and only if it is one of* $\mathsf{A}_{4,2}(\mathbb{L})$*,* $\mathsf{A}_{5,2}(\mathbb{L})$ *or* $\mathsf{E}_{6,1}(\mathbb{K})$*.*

Proof. Suppose first that Ω is strong. As $d \geq 3$, strongness implies that Ω is locally connected and hence by Lemma 9.2.7, Ω has a sympthick line. Consequently, Ω is of type $\mathsf{A}_{4,2}(\mathbb{L})$, $\mathsf{A}_{5,2}(\mathbb{L})$ or $\mathsf{E}_{6,1}(\mathbb{K})$. Now assume that Ω has a sympthick line. Then the above classification shows that Ω is strong. □

Another consequence is the following:

Corollary 9.3.2. *The strong* (−1)*-lacunary parapolar spaces* Ω *all have diameter 2.*

Proof. This follows immediately from the previous lemma and a direct verification in each of the five listed geometries. □

A description of the (−1)-lacunary parapolar spaces of symplectic rank at least 3 in which no line is sympthick is given in Chapter 11 on locally disconnected parapolar spaces of symplectic rank at least 3.

CHAPTER

10

LOCALLY CONNECTED K-LACUNARY PARAPOLAR SPACES OF SYMPLECTIC RANK AT LEAST $K+3 \geq 3$

We show that the locally connected k-lacunary parapolar spaces of symplectic rank at least $k+3$ are exactly those in Table 10.1. This table also contains additional information: the symplectic rank and the singular ranks, whether the parapolar space is strong or not (the non-strong ones are in white text) and the diameter (the ones with diameter >2 are in gray cells).

d	singular ranks	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$k+3$	$\{k+2,k+3\}$	$\mathsf{A}_{1,1}(*) \times \mathsf{A}_{2,1}(*)$	$\mathsf{A}_{4,2}(\mathbb{L})$	$\mathsf{D}_{5,5}(\mathbb{K})$	$\mathsf{E}_{6,1}(\mathbb{K})$	$\mathsf{E}_{7,7}(\mathbb{K})$	$\mathsf{E}_{8,8}(\mathbb{K})$
	$\{k+3\}$	$\mathsf{A}_{2,1}(*) \times \mathsf{A}_{2,1}(*)$	$\mathsf{A}_{5,3}(\mathbb{L})$	$\mathsf{E}_{6,2}(\mathbb{K})$			
$k+4$	$\{k+3,k+4\}$	$\mathsf{A}_{4,2}(\mathbb{L})$	$\mathsf{D}_{5,5}(\mathbb{K})$	$\mathsf{E}_{6,1}(\mathbb{K})$	$\mathsf{E}_{7,7}(\mathbb{K})$	$\mathsf{E}_{8,8}(\mathbb{K})$	
	$\{k+3,k+5\}$	$\mathsf{A}_{5,2}(\mathbb{L})$	$\mathsf{D}_{6,6}(\mathbb{K})$	$\mathsf{E}_{7,1}(\mathbb{K})$			
$k+6$	$\{k+5,k+6\}$	$\mathsf{E}_{6,1}(\mathbb{K})$	$\mathsf{E}_{7,7}(\mathbb{K})$	$\mathsf{E}_{8,8}(\mathbb{K})$			

Table 10.1: The k-lacunary parapolar spaces with symplectic rank $d \geq k+3$.

Recall that the point-residuals of a locally connected k-lacunary parapolar space are strong $(k-1)$-lacunary parapolar spaces (cf. Lemma 8.2.2). Before heading off, we give some more explanation on this table. First of all, we explain why this table is complete:

— For locally connected 0-lacunary parapolar spaces of symplectic rank at least 3, we show that the point-residuals always have a sympthick line and hence, by the previous chapter, these are precisely the parapolar spaces contained in column $k = -1$ of Table 10.1.

— We moreover show that a 0-lacunary parapolar space of symplectic rank at least 3 is always locally connected and therefore, the point-residual of a locally connected 1-lacunary parapolar space of symplectic rank at least 4, are those that can be found in column $k = 0$ in Table 10.1.

— For locally connected k-lacunary parapolar spaces with $k \geq 2$, we can show that the point-residual is also locally connected and as such to be found in the corresponding column of Table 10.1.

The fact that the k-lacunary parapolar space occurring in Table 10.1 for a fixed k all differ in at least their symplectic rank or singular ranks, allows to recognise the point-residuals. Moreover, in each locally connected k-lacunary parapolar space, we can show that the point-residuals Ω_p and Ω_q for each pair of points p,q are isomorphic. It suffices to know the structure of the point-residuals to geometrically recover the structure of Ω, but it requires quite some work and needs to be done for each geometry separately. This is the right time to invoke other classification theorems (and then we only need to do the cases $k = 0$, $k = 1$ and $k \geq 2$ separately).

As mentioned before, all sequences (i.e., rows of the table) end, because the rightmost parapolar spaces is non-strong and as such cannot occur as a point-residual of a locally connected parapolar spaces. Moreover, a non-strong geometry of course has diameter bigger than 2, but it is also preceded by a (strong) parapolar space of diameter at least 3, as follows from Lemma 2.1.45.

We start with $k = 0$.

10.1 The case $k = 0$

Let Ω be a 0-lacunary parapolar space of minimum symplectic rank d with $d \geq 3$. We first show that Ω is locally connected.

Lemma 10.1.1. *Ω is strong and locally connected, and has bounded singular rank.*

Proof. By Axiom (PPS1), there exists a point p contained in at least two symps. Since any two symps through p have rank at least 3 and share a line by 0-lacunarity, Ω_p is connected (even by planes which are contained in symps). Hence, if we show that each point of Ω is contained in two symps, then Ω is locally connected. Similarly as in Lemma 8.2.2, it follows that Ω_p is a strong (-1)-lacunary parapolar space. As Ω_p is strong, each of its points is contained in at least two symps by Lemma 9.3.1. We obtain that also each point collinear to p is contained in at least two symps and as such, we can interchange its role with that of p. By connectivity, Ω_p is a strong (-1)-lacunary parapolar space for each point p. In particular, Ω is locally connected.

Moreover, for each point p, we have that Ω_p is one of the first five geometries mentioned in Main Result 8.2.1 (as those are precisely the strong ones by Lemma 9.3.1); and hence

$\operatorname{Diam}\Omega_p = 2$. It then follows from Lemma 2.1.45 and the fact that $d \geq 3$ that Ω is strong. Since Ω_p has bounded singular rank for each point p, the same holds for Ω. □

The following lemma (Exercise 13.26 in [42]) bounds the diameter. We provide a proof for completeness.

Lemma 10.1.2. *The diameter of Ω is either 2 or 3.*

Proof. Suppose for a contradiction that $p_0 \perp p_1 \perp p_2 \perp p_3 \perp p_4$ are points of Ω with $\delta(p_0, p_4) = 4$. By the previous lemma, this gives us two symps $\xi(p_0, p_2)$ and $\xi(p_2, p_4)$ through p_2, which by 0-lacunarity have a line L in common. Let $i = 0, 4$. Then in the symp $\xi(p_2, p_i)$, there is a unique point q_i on L collinear to p_i, and hence $p_0 \perp q_0 \perp q_4 \perp p_4$ is a path of length 3 between p_0 and p_4, a contradiction. This proves the lemma. □

We now consider diameters 2 and 3 separately. In both cases, we make use of the fact that we know the exact structure of the point-residuals (as these are strong (-1)-lacunary parapolar spaces). We could use these residuals to recover the structure of Ω in a geometric way (case by case), yet it is faster to make use of Theorems 8.4.2 and 8.4.1.

10.1.1 Diameter 2

If $\operatorname{Diam}\Omega = 2$, we show that Ω satisfies the assumptions of Theorem 8.4.2.

Lemma 10.1.3. *If $\operatorname{Diam}\Omega = 2$, then Ω has uniform symplectic rank $d \geq 3$ and satisfies Condition $(CC)_{d-2}$.*

Proof. By Lemma 10.1.1, the point-residual Ω_p, for each $p \in X$, is a strong parapolar space with lacunary index -1. These all have maximal singular subspaces of finite projective dimension by our classification, see Table 10.1, whence the bounded singular rank. By the Main Result 8.2.1, it is one of $\mathsf{A}_{1,1}(*) \times \mathsf{A}_{2,1}(*)$, $\mathsf{A}_{2,1}(*) \times \mathsf{A}_{2,1}(*)$, $\mathsf{A}_{4,2}(\mathbb{L})$, $\mathsf{A}_{5,2}(\mathbb{L})$, $\mathsf{E}_{6,1}(\mathbb{K})$. Clearly, if $p \perp q$ in Ω, then the parameters (singular ranks, symplectic rank; see Table 10.1) of Ω_p and Ω_q coincide, which implies, given the above list, that Ω_p and Ω_q are isomorphic. By connectivity we conclude that all point-residuals all have uniform symplectic rank $d-1 \geq 2$. Hence Ω has uniform symplectic rank $d \geq 3$.

By Lemma 8.4.5, it suffices to check $(CC)_{d-3}$ in the point-residuals. This is a straightforward verification, given the list of strong (-1)-lacunary parapolar spaces. □

So, if Ω has diameter 2, Lemmas 10.1.1 and 10.1.3 imply that Ω is among the parapolar spaces listed in Theorem 8.4.2. The 0-lacunary parapolar spaces in this list are exactly those whose point-residuals are strong (-1)-lacunary parapolar spaces.

> A 0-lacunary parapolar space of symplectic rank at least 3 with diameter 2 is a Lie incidence geometries of type $\mathsf{A}_{4,2}(\mathbb{L})$ or $\mathsf{D}_{5,5}(\mathbb{K})$, where $\mathbb{L}$ is a skew field and $\mathbb{K}$ a commutative field.

Note that the parapolar spaces $\mathsf{A}_{5,3}(\mathbb{L})$, or $\mathsf{D}_{6,6}(\mathbb{K})$ and $\mathsf{E}_{7,7}(\mathbb{K})$ are 0-lacunary but omitted, for they have diameter 3. We obtained exactly the parapolar spaces in the white cells of the $k=0$ column of Table 10.1. We now continue with diameter 3.

10.1.2 Diameter 3

If $\mathrm{Diam}\,\Omega = 3$, we verify that the conditions of Theorem 8.4.1 are fulfilled.

Lemma 10.1.4. *If* $\mathrm{Diam}\,\Omega = 3$*, then*

(i) *For every point-symplecton pair (p,ξ), we have $p^\perp \cap \xi \neq \emptyset$;*

(ii) *for each point p, the set $\delta_{\leq 2}(p)$ of points at distance at most 2 from p is a geometric hyperplane of Ω (possibly $\delta_{\leq 2}(p) = \Omega$).*

Proof. (i) Consider a point q not in ξ, for which there is a path of length two, say $q \perp r \perp s$, for $r,s \in X$, with $s \in \xi$. We may assume $q \notin s^\perp$. By assumption the symps $\xi(q,s)$ and ξ intersect in a line L. But then q is collinear with a point on L, which also lies in ξ. So we can keep shortening the path from p to ξ, which exists by connectivity, and hence we have proved the first part of the lemma.

(ii) Let L be a line containing at least two points $x,y \in \delta_{\leq 2}(p)$. We show that $\delta(p,L)=1$, from which follows that $L \subseteq \delta_{\leq 2}(p)$. Assume for a contradiction that $\delta(p,L) > 1$, so in particular, $\delta(p,x)=\delta(p,y)=2$. Since Ω is strong, we have the symps $\xi(p,x)$ and $\xi(p,y)$, which intersect in a line L_p. Let x_p and y_p be the unique respective points on L_p collinear to x and y. If $x_p \neq y_p$, then p and L are contained in the symp $\xi(y,x_p)=\xi(x,y_p)$ and hence $\delta(p,L)=1$, contradicting our assumption. So $x_p = y_p$ and hence $x \perp y \perp x_p \perp x$; then, by Axiom (PPS1), x_p is collinear to each point of L, again a contradiction. We conclude that $\delta_{\leq 2}(p)$ is a subspace of Ω.

To see that $\delta_{\leq 2}(p)$ is a geometric hyperplane, take an arbitrary line L and a symp ξ through L. By (i), there is a point $q \in \xi$ with $q \perp p$, and so any point on L collinear to q belongs to $\delta_{\leq 2}(p)$. This lemma is proven. $\square$

So, if Ω has diameter 3, Lemmas 10.1.3 and 10.1.4 imply that Ω is among the parapolar spaces listed in Theorem 8.4.1. As above, the 0-lacunary parapolar spaces in this list are exactly those whose point-residuals are strong (-1)-lacunary parapolar spaces.

A 0-lacunary parapolar space of symplectic rank at least 3 with diameter 3 is a Lie incidence geometry of type $\mathsf{A}_{5,3}(\mathbb{L})$, $\mathsf{D}_{6,6}(\mathbb{K})$ or $\mathsf{E}_{7,7}(\mathbb{K})$, where $\mathbb{K}$ is a (commutative) field and $\mathbb{L}$ a skew field.

We obtained exactly the parapolar spaces in the gray cells of the $k=0$ column of Table 10.1.

10.2 The case $k = 1$

Assume now that Ω is a locally connected 1-lacunary parapolar space of minimum symplectic rank d with $d \geq 4$. Then, since the point-residuals are 0-lacunary parapolar spaces, we can distinguish between the diameters 2 and 3 in these point-residuals. To that end we first show that all such residuals are of the same type.

Lemma 10.2.1. *For each two points p and q of Ω, the point-residuals Ω_p and Ω_q are Lie incidence geometries of the same Coxeter type. Moreover, the singular rank of Ω is finite.*

Proof. Take any point p and consider the point-residual Ω_p. Since Ω is locally connected, Proposition 2.1.43 and Lemma 8.2.2 imply that Ω_p is a strong 0-lacunary parapolar space of symplectic rank at least 3. Consequently, Ω_p is as in Tables 10.1 (column corresponding to $k = 0$); in particular, Ω_p is a Lie incidence geometry.

Now let $q \in X$ be collinear with p. We claim that Ω_q has the same symplectic and singular ranks as Ω_p. Indeed, each symp through the line pq corresponds with a unique symp of Ω_p through q and with a unique symp Ω_q through p, and vice versa. Hence there is a bijective correspondence between the symps of Ω_p through q and of Ω_q through p; likewise for the maximal singular subspaces. Since one can distinguish the 0-lacunary parapolar spaces of symplectic rank at least 3 symplectic and singular ranks, it follows that Ω_p and Ω_q have the same Coxeter type indeed. By connectivity, this holds for all pairs p, q. Since the singular subspaces of each Ω_p are finite-dimensional, the same holds for Ω. □

By the previous Section, $\operatorname{Diam}\Omega_p \in \{2, 3\}$ for each point p of Ω. We now split up the proof in those two cases.

10.2.1 Point-residue of diameter 3

Proposition 10.2.2. *If* $\operatorname{Diam}\Omega_q = 3$ *for some point $q \in X$, then Ω satisfies the property*

(NP) *Given a point p not incident with a symp ξ, the intersection $p^\perp \cap \xi$ is never just a point.*

Proof. By Lemma 10.2.1, Ω_q has diameter 3, for any point $q \in X$. It suffices to check that property (NP) holds in every point-residual. Since these are 0-lacunary and have diameter 3 by assumption, this follows from Lemma 10.1.4. □

By the previous two lemmas, the requirements of Theorem 8.4.4 are met. We conclude that Ω is among the parapolar spaces in the statement. Again, we only keep those parapolar spaces from the list whose point-residuals are strong 0-lacunary parapolar spaces of diameter 3.

A locally connected 1-lacunary parapolar space with point-residues of diameter 3 is a Lie incidence geometry of type $\mathsf{E}_{6,2}(\mathbb{K})$, $\mathsf{E}_{7,1}(\mathbb{K})$ or $\mathsf{E}_{8,8}(\mathbb{K})$, where $\mathbb{K}$ is a commutative field.

We obtained exactly the parapolar spaces in the gray cells of the $k = 1$ column of Table 10.1, which are preceded by a gray cell.

10.2.2 Point-residue of diameter 2

Proposition 10.2.3. *If* $\mathrm{Diam}\,\Omega_p = 2$ *for some point* $p \in X$*, then* Ω *is strong, has uniform symplectic rank* $d \geq 3$ *and satisfies Condition* $(CC)_{d-2}$*.*

Proof. For any $p \in X$, consider the point-residual Ω_p. As in the proof of Lemma 10.2.1, Ω_p is a strong 0-lacunary parapolar space, each of whose singular subspaces are projective. Hence Table 10.1 and our assumption on $\mathrm{Diam}\,\Omega_p$ imply that Ω_p is either $\mathsf{A}_{1,1}(*) \times \mathsf{A}_{n,1}(\mathbb{L})$, $\mathsf{A}_{4,2}(\mathbb{L})$ or $\mathsf{D}_{5,5}(\mathbb{K})$. Since these all have uniform symplectic rank, Lemma 10.2.1 implies that Ω has uniform symplectic rank 3, 4 or 5, respectively; and since their singular rank is finite, so is Ω's.

Furthermore, as the point-residuals have diameter 2 and $d \geq 3$, it follows from Lemma 2.1.45 that Ω is strong. Finally, we verify Condition $(\mathsf{CC})_{d-2}$. By Lemma 8.4.5, it suffices to verify Condition $(\mathsf{CC})_{d-3}$ (or (H)) in the point-residuals. For $d = 4, 5$, this follows from the last statement of Theorem 8.4.6. □

It then follows from Theorem 8.4.2 that Ω is among the parapolar spaces listed there. Keeping those parapolar spaces which have point-residuals which are strong 0-lacunary parapolar spaces of diameter 2, we obtain:

> A locally connected 1-lacunary parapolar space with point-residues of diameter 2 is a Lie incidence geometry of type $\mathsf{D}_{5,5}(\mathbb{K})$ or $\mathsf{E}_{6,1}(\mathbb{K})$, where $\mathbb{K}$ is a commutative field.

We obtained exactly the parapolar spaces in the white cells of the $k = 1$ column of Table 10.1, which are also preceded by a white cell.

10.3 The case $k \geq 2$

Let Ω be a locally connected k-lacunary parapolar space for $k \geq 2$ of minimum symplectic rank d with $d \geq k+3$.

We verify the assumptions of Shult's Haircut Theorem (cf. Theorem 8.4.6).

Lemma 10.3.1. *We have that:*

(i) *For each point* p*, the point-residual* Ω_p *is locally connected;*
(ii) Ω *has bounded singular rank and uniform symplectic rank;*
(iii) *the Haircut Axiom is satisfied.*

Proof. Since Ω is locally connected, Lemma 8.2.2 says that, that for each point $p \in X$, the point-residual Ω_p is a strong $(k-1)$-lacunary parapolar space of minimum symplectic rank $d-1 \geq k+2 \geq 4$. In particular, Ω_p is strong and has no symps of rank 2; so by Lemma 2.1.45 it follows, for each line pq through p, that Ω_{pq} is connected. Hence Ω_p is locally connected, showing (i).

We now take subsequent point-residuals until we obtain a 1-lacunary parapolar space, i.e., a K-residual of Ω for some singular subspace K of Ω of dimension $k-2$. Such a k-residual is locally connected by the previous paragraph. By the results of the previous section, these residuals are precisely those written in black (these are the strong ones) in the $k=1$ column of Table 10.1, i.e., $\mathsf{D}_{5,5}(\mathbb{K})$ and $\mathsf{E}_{6,1}(\mathbb{K})$. As in Lemma 10.2.1, we obtain that for each two $(k-2)$-spaces K and K', the residuals Ω_K and $\Omega_{K'}$ are isomorphic. Since those residues all have uniform symplectic rank and bounded singular rank, the same holds for Ω. This shows (ii).

By Lemma 8.4.5, it suffices to check the Haircut Axiom (H) in each K-residual of Ω, i.e., in $\mathsf{D}_{5,5}(\mathbb{K})$ and $\mathsf{E}_{6,1}(\mathbb{K})$. Since both geometries appear in the conclusion of Theorem 8.4.6, they satisfy (H) and hence so does Ω. This shows (iii). □

We can now use Shult's Haircut Theorem and an induction on the lacunary index k (starting with $k=2$), concluding the proof of Main Theorem 8.2.5 in the locally connected case.

Theorem 10.3.2. *Let $\Omega = (X, \mathscr{L})$ be a locally connected parapolar space with lacunary index k, $k \geq 2$, and uniform symplectic rank $d \geq k+3$. Then Ω is one of the parapolar spaces mentioned in Table* 10.1 *in the columns corresponding to $k = 2, 3, 4, 5$.*

Proof. We already know by Lemma 10.3.1 that Ω satisfies all assumptions of Theorem 8.4.6. Hence we can apply the latter theorem and, for increasing values of $k \geq 2$, single out the k-lacunary parapolar spaces with point-residuals isomorphic to the black entries in Tables 10.1 in the columns corresponding to lacunary index $k-1 \geq 1$. This gives us the desired parapolar spaces. □

We continue with the locally disconnected case.

CHAPTER

11

LOCALLY DISCONNECTED (K-LACUNARY) PARAPOLAR SPACES OF SYMPLECTIC RANK $D \geq 3$

In this chapter, we show that each disconnected parapolar space of symplectic rank at least 3 can be constructed as a certain (non-disjoint) union of locally connected (para)polar spaces (note that we do not restrict ourselves to lacunary parapolar spaces). Our approach has an analogue for graphs, see Exercise 1.17 in [42].

Using the obtained classification of locally connected k-lacunary parapolar spaces of symplectic rank at least $k+3$, this yields a universal construction of all k-lacunary locally disconnected parapolar spaces with symplectic rank at least $k+3$.

Remark 11.0.1. Recall that we classified the *strong* (-1)-lacunary parapolar spaces of minimum symplectic rank 2 without any assumption on local connectedness (the latter's definition becoming of no use in this case) and that we only considered k-lacunary parapolar spaces of symplectic rank at least $k+3$. This allows us to restrict ourselves to parapolar spaces of symplectic rank at least 3.

11.1 The unbuttoning of a locally disconnected parapolar space

Henceforth, let $\Omega = (X, \mathscr{L})$ be an arbitrary parapolar space with minimum symplectic rank d for $d \geq 3$. For each point $p \in X$, we denote by $\mathfrak{C}_p$ the set of connected components of Ω_p

(see Definition 2.1.42 and Fact 2.1.43). Since $d \geq 3$, Ω is locally connected if and only if Ω_p is connected.

The following construction introduces a copy of a point p for each connected component of Ω_p.

Construction 1. *The* unbuttoning of Ω *is defined as the following point-line geometry* $\widetilde{\Omega} = (\widetilde{X}, \widetilde{\mathscr{L}})$*:*

- $\widetilde{X} = \{(p, \Upsilon) : p \in X$ *and* $\Upsilon \in \mathfrak{C}_p\}$*;*
- *for each line* $L \in \mathscr{L}$*, we define* $\widetilde{L} = \{(p, \Upsilon) \in \widetilde{X} : p \in L \in \Upsilon\}$*,*
- $\widetilde{\mathscr{L}} = \{\widetilde{L} : L \in \mathscr{L}\}$.

So two points (p_1, Υ_1) and (p_2, Υ_2), with $\Upsilon_i \in \Omega_{p_i}$ for $i = 1, 2$, are collinear in $\widetilde{\Omega}$ if and only if $p_1 \perp p_2$ and the line $p_1 p_2$ is an element of both Υ_1 and Υ_2.

We now have the following result.

Proposition 11.1.1. *The unbuttoning* $\widetilde{\Omega}$ *of* Ω *is a disjoint union of locally connected (para)polar spaces.*

Proof. We verify the axioms of a parapolar space except that in Axiom (PPS1) we do not require that $\widetilde{\Omega}$ is connected (instead we will in the end consider its connected components), nor do we require that there is a point-line pair (p, L) such that no point of L is collinear to p (this allows that some connected components are polar spaces).

(PPS1) Suppose $(p, \Upsilon) \in \widetilde{X}$ and $\widetilde{L} \in \widetilde{\mathscr{L}}$ are such that $(p, \Upsilon) \notin \widetilde{L}$ is collinear to at least two points of $\widetilde{L}$. Let (x^*, Υ^*) be any point of $\widetilde{L}$. In Ω, at least two points of L are collinear to p, so $\langle p, L \rangle$ is a plane π. This means that each line of π through p belongs to Υ (in particular, $px^* \in \Upsilon$) and likewise each line of π through x^* is contained in Υ^* (in particular, $px^* \in \Upsilon^*$). Consequently, (p, Υ) and (x^*, Υ^*) are contained in $\widetilde{px^*}$ and as such they are indeed collinear in $\widetilde{\Omega}$.

(PPS2) Let $(p_i, \Upsilon_i) \in \widetilde{X}$, $i = 1, 2$, be two non-collinear points of $\widetilde{\Omega}$ collinear to at least one common point (x_1, Σ_1) of $\widetilde{\Omega}$. We claim that p_1 and p_2 are not collinear in Ω. Indeed, suppose they are. Since (p_1, Υ_1) is collinear to (x_1, Σ_1), the line $p_1 x_1$ belongs to Υ_1. As x_1 is collinear to p_2, the line $p_1 p_2$ lies in Υ_1 too. Likewise we obtain $p_1 p_2 \in \Upsilon_2$. But then the points (p_i, Υ_i), $i = 1, 2$, both belong to $\widetilde{p_1 p_2}$, a contradiction. Our claim follows. Now suppose that both (p_i, Υ_i), $i = 1, 2$, are collinear to a second point (x_2, Σ_2), with $(x_1, \Sigma_1) \neq (x_2, \Sigma_2)$. Since $x_i p_1 \in \Sigma_i$ for $i = 1, 2$ and $\Sigma_1 \cap \Sigma_2 = \{x_1\}$ if $x_1 = x_2$, we deduce $x_1 \neq x_2$.

We now show that the convex closure C of (p_1, Υ_1) and (p_2, Υ_2) is a polar space canonically isomorphic to the symp $\xi := \xi(p_1, p_2)$. To that aim, we have to show two claims:

Claim 1: If $x \in \xi(p_1, p_2)$ *and if we denote by* $\Sigma_{x,\xi}$ *the component of* Ω_x *containing the lines of* ξ *through* x*, then* $(x, \Sigma_{x,\xi})$ *belongs to* C.

By Fact 7.4.5, it suffices to show that $(x, \Sigma_{x,\xi}) \in C$ for all points x which are contained in a line joining p_1 or p_2 with a point of $p_1^\perp \cap p_2^\perp$. Suppose first that $x \in p_1^\perp \cap p_2^\perp$. Then, firstly, xp_1 and xp_2 belong to $\Sigma_{x,\xi}$ by our assumption on $\Sigma_{x,\xi}$. Secondly, $p_i x$ belongs to Υ_i,

$i = 1,2$, because $p_i x$ lies in the same connected component of Ω_{p_i} as $p_i x_1$ and $p_i x_2$ (these lines all lie in ξ), $i = 1,2$. This shows that the point $(x, \Sigma_{x,\xi})$ is collinear to (p_i, Υ_i), $i = 1,2$, and hence belongs to C indeed. Similarly one can now show that each point x' on the line $p_i x$ is such that $(x', \Sigma_{x',\xi})$ is on the line joining (p_i, Υ_i) and $(x, \Sigma_{x,\xi})$, $i = 1,2$, and hence $(x', \Sigma_{x',\xi}) \in C$ too. This shows Claim 1. The second claim is the following.

Claim 2: If $(y, \Upsilon) \in C$, then $y \in \xi$ and Υ is the component of Ω_y containing the lines of ξ through y.

Let C' denote the set of points $(x, \Sigma_{x,\xi})$, with $x \in \xi$. Let ρ be the projection map $C' \to \xi$: $(x, \Sigma) \mapsto x$. Then ρ is an isomorphism of point-line geometries: the first paragraph implies that ρ preserves collinearity and is injective; surjectivity follows by definition of C'. Hence C' is a polar space containing (p_1, Υ_1) and (p_2, Υ_2) and therefore $C' = C$, as claimed.

This concludes the verification of Axiom (PPS2).

(PPS3) Let $\widetilde{L}$ be a line of $\widetilde{\Omega}$. Then $L \in \mathscr{L}$ is contained in some symp ξ. We consider two points $p_1, p_2 \in \xi$ at distance 2 and with $p_1 \in L$. We showed above that $(p_1, \Sigma_{p_1,\xi})$ and $(p_2, \Sigma_{p_2,\Sigma})$ determine a symp $\widetilde{\xi}$ in $\widetilde{\Omega}$, which contains precisely the points $(x, \Sigma_{x,\xi})$ with $x \in \xi$, so in particular those with $x \in L$. Since $L \subseteq \xi$, we have that $L \subseteq \Sigma_{x,\xi}$ for all $x \in L$, showing that $\widetilde{L}$ belongs to $\widetilde{\xi}$.

This shows that each connected component ω of $\widetilde{\Omega}$ is a (para)polar space. The fact that ω is locally connected follows immediately from the definition of $\widetilde{\Omega}$. This proves the proposition. □

11.2 Buttoning a family of locally connected (para)polar spaces

We are now interested in a reverse procedure. Which parapolar spaces can we obtain by collecting connected locally (para)polar spaces and identifying certain points? As before we may restrict to the case of symplectic rank at least 3.

The following lemma is necessary to make the construction universal. Basically it says that, in Ω, you cannot walk from a point p to itself in less than five steps using two different components of Ω_p to start and come back in.

Lemma 11.2.1. *Let $\Omega = (X, \mathscr{L})$ be a not necessarily locally connected parapolar space with symplectic rank at least 3. Let $\widetilde{\Omega}$ be its unbuttoning. Let $p \in X$ be such that Ω_p is disconnected and let $\Upsilon_1^{(p)}$ and $\Upsilon_2^{(p)}$ be two distinct connected components of Ω_p. Let $q, r, s \in X \setminus \{p\}$ be arbitrary (not necessarily distinct) and let $\Upsilon_1^{(q)}, \Upsilon_2^{(q)}, \Upsilon_1^{(r)}, \Upsilon_2^{(r)}, \Upsilon_1^{(s)}, \Upsilon_2^{(s)}$ be the not necessarily distinct respective connected components (with self-explaining notation) of $\Omega_q, \Omega_r, \Omega_s$. Then*

$$\ell := \delta((p, \Upsilon_2^{(p)}), (q, \Upsilon_1^{(q)})) + \delta((q, \Upsilon_2^{(q)}), (r, \Upsilon_1^{(r)})) + \delta((r, \Upsilon_2^{(r)}), (s, \Upsilon_1^{(s)})) + \delta((s, \Upsilon_2^{(s)}), (p, \Upsilon_1^{(p)})) \geq 5$$

(points in different components of $\widetilde{\Omega}$ have distance ∞, which is by definition larger than any positive number).

Proof. Suppose for a contradiction that $\ell \leq 4$. We examine the case $\ell = 4$, leaving the easier cases $\ell = 1,2,3$ to the interested reader. The assumption $\ell = 4$ allows us to also assume that

$$\delta((p,\Upsilon_2^{(p)}),(q,\Upsilon_1^{(q)})) = \delta((q,\Upsilon_2^{(q)}),(r,\Upsilon_1^{(r)})) = \delta((r,\Upsilon_2^{(r)}),(s,\Upsilon_1^{(s)})) = \delta((s,\Upsilon_2^{(s)}),(p,\Upsilon_1^{(p)})) = 1,$$

since, if some of these distances would be 0, then another distance must be at least 2 and we can insert a chain of points consecutively at distance 1, rename, and get the above assumption back.

By the definition of lines in $\widetilde{\Omega}$ we then obtain $p \perp q \perp r \perp s \perp p$. First note that the lines pq and ps belong to $\Upsilon_2^{(p)}$ and $\Upsilon_1^{(p)}$, respectively. By assumption, $\Upsilon_1^{(p)} \neq \Upsilon_2^{(p)}$. This already implies $q \neq s$. It also implies that q cannot be collinear to s, for then $\langle p,q,s\rangle$ would be a projective plane (cf. Fact 2.1.38), yielding $\Upsilon_1^{(p)} = \Upsilon_2^{(p)}$ after all. However, if q and s are not collinear, they determine a symp ξ since $p \neq r$, clearly containing the lines pq and ps, which again leads to $\Upsilon_1^{(p)} = \Upsilon_2^{(p)}$. This contradiction proves the lemma. □

11.3 Locally disconnected k-lacunary parapolar spaces of symplectic rank at least $\max\{k+3,3\}$

We now present a construction of the class of locally disconnected k-lacunary parapolar spaces with symplectic rank at least $\max\{k+3,3\}$ and with lacunary index $k \geq -1$, $k \neq 0$ (recall that 0-lacunary spaces are automatically locally connected by Lemma 10.1.1 and that -1-lacunary parapolar spaces of minimum symplectic rank 2 are classified without any assumption on locally connectedness).

For convenience we shall call a polar space *k-lacunary* whenever its rank is at least $k+1$ and refer to its rank as its symplectic rank.

Construction 2. *Let $\mathscr{F} = \{\Omega_i = (X_i,\mathscr{L}_i) : i \in I\}$ be a family of (disjoint) k-lacunary locally connected (para)polar spaces of symplectic rank at least 3, over some nonempty index set I, $0 \neq k \geq -1$. If $k = -1$, then we additionally require that $\mathscr{F}$ only consists of polar spaces of rank at least* $\max\{k+3,3\}$. *Let $\mathscr{R}$ be an equivalence relation on the union $\widetilde{X} = \bigcup_{i\in I} X_i$ of the sets of points of all members of $\mathscr{F}$, satisfying the following two conditions (C1) and (C2).*

(C1) Let $\widetilde{p},\widetilde{q},\widetilde{r},\widetilde{s}$ be four (not necessarily distinct, but $\widetilde{p} \notin \{\widetilde{q},\widetilde{r},\widetilde{s}\}$) equivalence classes with respect to $\mathscr{R}$, and let $p_1,p_2 \in \widetilde{p}$, with $p_1 \neq p_2$. If $q_1,q_2 \in \widetilde{q}$, $r_1,r_2 \in \widetilde{r}$ and $s_1,s_2 \in \widetilde{s}$, then

$$\delta(p_2,q_1) + \delta(q_2,r_1) + \delta(r_2,s_1) + \delta(s_2,p_1) \geq 5.$$

(C2) The graph with vertex set $\mathscr{F}$, where two vertices Ω_i and Ω_j, $i,j \in I$, are adjacent if some point of Ω_i is contained in the same equivalence class as some point of Ω_j, is connected. If $k = -1$, we additionally require that this graph is a complete graph.

*Set $X = \widetilde{X}/\mathscr{R}$. For each line L contained in some member of $\mathscr{F}$, we put $\widetilde{L} := \{\widetilde{p} \mid p \in L\}$ and define $\mathscr{L}$ as $\{\widetilde{L} \mid L \in \mathscr{L}_i$ for some $i \in I\}$. Then we denote the geometry $\Omega = (X,\mathscr{L})$ by $\Omega(\mathscr{F},\mathscr{R})$. If $\mathscr{R}$ is non-trivial, then we call Ω a k-*buttoned geometry. ■

Remark 11.3.1. We claim that distinct lines L and L' define distinct sets $\widetilde{L}$ and $\widetilde{L'}$. Indeed, suppose $\widetilde{L} = \widetilde{L'}$ and take points $\widetilde{p},\widetilde{q} \in \widetilde{L} = \widetilde{L'}$ with $\widetilde{p} \neq \widetilde{q}$. Let p_1, q_2 be (distinct) points on L and p_2, q_1 (distinct) points on L' with $p_1, p_2 \in \widetilde{p}$ and $q_1, q_2 \in \widetilde{q}$. Then Condition $(C1)$, with $\widetilde{r} = \widetilde{s} = \widetilde{q}$, $r_1 = s_2 = q_2$ and $r_2 = s_1 = q_1$ implies $p_1 = p_2$. Since $\widetilde{p} \in \widetilde{L}$ was arbitrary, we obtain $L = L'$. The claim follows.

Remark 11.3.2. Note that (C1) implies that an equivalence class of $\mathscr{R}$ cannot contain two points of the same member of $\mathscr{F}$ if that member's diameter is smaller than 5. This situation in particular applies if all members of $\mathscr{F}$ are ordinary polar spaces; which is always the case when $k = -1$ (by assumption) and when $k \geq 6$ (as follows from our classification of locally connected k-lacunary parapolar spaces).

We have the following crucial result.

Proposition 11.3.3. *For every $k \geq -1$, $k \neq 0$, every k-buttoned parapolar space is a locally disconnected k-lacunary parapolar space. More exactly, let $\mathscr{F} = \{\Omega_i = (X_i, \mathscr{L}_i) : i \in I\}$ be a family of (disjoint) k-lacunary locally connected (para)polar spaces of symplectic rank at least 3, over some nonempty index set I (if $k = -1$, we only allow polar spaces of arbitrary rank at least 3). Let $\mathscr{R}$ be a non-trivial equivalence relation on the union $\widetilde{X} = \bigcup_{i \in I} X_i$ of the sets of points of all members of $\mathscr{F}$, satisfying Conditions* (C1) *and* (C2)*. Then the geometry $\Omega(\mathscr{F}, \mathscr{R})$ is a locally disconnected k-lacunary parapolar space.*

Proof. We verify the axioms of a parapolar space for $\Omega := \Omega(\mathscr{F}, \mathscr{R})$.

(PPS1) Condition (C2) implies immediately that Ω is connected. Let $\widetilde{L} \in \mathscr{L}$ and $\widetilde{p} \in X$ with $\widetilde{p} \notin \widetilde{L}$ be such that $\widetilde{p}$ is collinear to at least two points $\widetilde{q}, \widetilde{r} \in \widetilde{L}$. The definition of $\mathscr{L}$ yields (unique) points $p_1, p_2 \in \widetilde{p}$, $q_1, q_2 \in \widetilde{q}$, $r_1, r_2 \in \widetilde{r}$ with $p_1 \perp q_2$, $q_1 \perp r_2$, $r_1 \perp p_2$. By Condition (C1), $p_1 = p_2$, $q_1 = q_2$ and $r_1 = r_2$. It follows that $q_1 r_1 = L$ and hence p_1 is collinear to each point of L. This implies that $\widetilde{p}$ is collinear to each point of $\widetilde{L}$. Condition (C1) implies that there exist $p \in \widetilde{p}$, $q \in \widetilde{q}$ and $r \in \widetilde{r}$ with $p \perp q \perp r \perp p$ and so all of p, q, r lie in a common member of $\mathscr{F}$, implying that p is collinear to all points of the line qr. Hence $\widetilde{p}$ is collinear to all points of L.

Now assume first that $\mathscr{F}$ contains a parapolar space. Let Ω_i, $i \in I$ be such a member, let $p \in X_i$ be arbitrary and let $L \in \mathscr{L}_i$ be arbitrary but such that $\delta(p, L) = 2$. Suppose for a contradiction that $\widetilde{p}$ is collinear to some point $\widetilde{q}$ on $\widetilde{L}$. Let $p \perp s \perp r \in L$. Then $\widetilde{p} \perp \widetilde{q} \perp \widetilde{r} \perp \widetilde{s} \perp \widetilde{p}$, which implies by (C1) that all of $\widetilde{p}, \widetilde{q}, \widetilde{r}, \widetilde{s}$ contain representatives in Ω_i, and hence p is collinear to some point of L after all, a contradiction.

Next assume all members of $\mathscr{F}$ are polar spaces. Then there exist two members Ω_i, Ω_j, $i, j \in I$, and points $p_i \in X_i$ and $p_j \in X_j$ such that p_i and p_j are contained in the same equivalence class $\widetilde{p}$. Choose a point $x \in X_i$ collinear to p_i and a line L in Ω_j not incident with p_j. One now easily checks that, with self-explaining notation, $\widetilde{x}$ is not collinear to any point of $\widetilde{L}$.

This completes the proof of (PPS1).

(PPS2) Let $\widetilde{p} \perp \widetilde{q} \perp \widetilde{r} \perp \widetilde{s} \perp \widetilde{p}$ be a quadrangle in Ω with $\widetilde{p}$ not collinear to $\widetilde{r}$. Condition (C1) implies that there are unique representatives p,q,r,s of $\widetilde{p},\widetilde{q},\widetilde{r},\widetilde{s}$, respectively, contained in a common member Ω_i of $\mathscr{F}$, for a unique $i \in I$. Clearly, $p \perp q \perp r \perp s$ and p and r are not collinear. It follows that the image in X of the unique symp $\xi(p,r)$ of Ω_i containing p and r is part of the convex closure C of $\widetilde{p}$ and $\widetilde{r}$. But C does not contain any further points since this would yield a circuit of length 4 and again a contradiction to Condition (C1). In particular this shows that the map $p \mapsto \widetilde{p}$ is bijective when restricted to the point set of symps of $\mathscr{F}$ and hence symps of $\mathscr{F}$ correspond bijectively with symps in Ω.

(PPS3) In view of the above, this follows immediately from the fact that every line of any member of $\mathscr{F}$ is contained in a symp of that member.

The k-lacunarity for $k \geq 1$ now follows easily since symps of $\Omega(\mathscr{F},\mathscr{R})$ that intersect in at least a line are contained in the same member of $\mathscr{F}$. For $k=-1$, this follows directly from Condition (C2). Also, the relation $\mathscr{R}$ is non-trivial and so there exists a class $\widetilde{p}$ with at least two elements, say p,p'. By Condition (C1) $p^\perp \cap p'^\perp = \emptyset$ and moreover, no pair of points in $p^\perp \cup p'^\perp$ is in the same equivalence class. This implies that $p^\perp$ and $p'^\perp$ induce two connected components of $\Omega_{\widetilde{p}}$, so Ω is locally disconnected at $\widetilde{p}$. □

This implies the following classification of locally disconnected lacunary parapolar spaces of symplectic rank at least $\max\{k+3,3\}$.

Theorem 11.3.4. *Let $\Omega = (X,\mathscr{L})$ be a k-lacunary parapolar space with symplectic rank at least* $\max\{k+3,3\}$ *and $k \geq -1$. Then either Ω is locally connected (and hence is one of the parapolar spaces of rank at least* 3 *in* Table 10.1*) or Ω is a k-buttoned parapolar space.*

Proof. If Ω is locally connected, then this follows from the previous sections. If $k=0$, then by Lemma 10.1.1, Ω is automatically locally connected. If Ω is not locally connected, then let $\widetilde{\Omega}$ be its unbuttoning. Let $\mathscr{F}$ be the family of connected components of $\widetilde{\Omega}$ and let $\mathscr{R}$ be the equivalence relation on the point set of $\widetilde{\Omega}$ defined by "sharing the first component" (remember points of $\widetilde{\Omega}$ are pairs (p,Υ) with $p \in X$ and Υ a connected component of Ω_p). If $\mathscr{R}$ satisfies Conditions (C1) and (C2), it is clear that Ω is isomorphic to the k-buttoned geometry $\Omega(\mathscr{F},\mathscr{R})$ arising from $\mathscr{F}$ and $\mathscr{R}$. Now, by Lemma 11.2.1, $\mathscr{R}$ satisfies Condition (C1), and if $k \neq -1$, the connectivity of Ω implies Condition (C2).

So suppose $k=-1$. Since we also assume that Ω is not locally connected, Lemma 9.2.6 implies that every line of Ω is contained in a unique symp. A moment's thought reveals that all connected components of each point-residual are ordinary polar spaces. Consequently, the unbuttoning of Ω only contains polar spaces, and then (-1)-lacunarity implies Condition (C2). The theorem is completely proved. □

This completes our classification of the (possibly locally disconnected) k-lacunary parapolar spaces Ω of symplectic rank at least $\max\{k+3, 3\}$, and hence our classification of all k-lacunary parapolar spaces Ω, only assuming strongness if the minimum symplectic rank is 2.

APPENDIX

A ENGLISH SUMMARY

This thesis is divided into two parts. Part 1 is the core of my Ph.D. and its results can also be found in [16] and [19] as well. Part 2 is based on a joint work, the results of which can also be found in [18]. Both parts are related to geometries occurring in the Freudenthal-Tits magic square, a 4×4 array containing both classical and exceptional Lie incidence geometries.

Essentially, this thesis contains three classification results. Below, we describe the objects that we intend to classify; the classification results are in the grey boxes. Some essential information to understand this is given, but it is probably not entirely self-contained. It should give an accurate idea though of what can be found (in more detail) in this thesis.

A.1 Part 1: A characterisation of the dualised version of the second row of the Freudenthal-Tits magic square (FTMS)

Let $\mathbb{K}$ be a field.

A.1.1 Introduction

The varieties in the second row of the FTMS are, in the non-split case, the projective planes coordinatised over quadratic alternative $\mathbb{K}$-algebras; and in the split case, a projective plane $\mathsf{A}_2(\mathbb{K})$, the direct product geometry $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$, the line Grassmannian $\mathsf{A}_{5,2}(\mathbb{K})$ and the $\mathsf{E}_{6,1}(\mathbb{K})$ variety. The Veronese representations of these geometries have been characterised axiomatically ([29], [39]).

In this thesis we consider the second row of the FTMS defined over a certain class of *degenerate* quadratic alternative algebras, which we will call the *generalised dual numbers*. This gives rise to Veronese varieties over these generalised dual numbers, which we then characterise axiomatically.

A.1.2 Quadratic alternative algebras and generalised dual numbers

Let $\mathbb{A}$ be a quadratic alternative $\mathbb{K}$-algebra. Then $\mathbb{A}$ comes with a multiplicative norm form N. The *radical* R of $\mathbb{A}$ is defined as the set $\{r \in \mathsf{rad}(f) \mid \mathsf{N}(r) = 0\}$, or equivalently as the nil ideal of $\mathbb{A}$. We say that $\mathbb{A}$ is non-degenerate if $R = \{0\}$.

It is a well-known fact that the non-degenerate quadratic alternative $\mathbb{K}$-algebras are precisely on of the following:

($d = 1$) $\mathbb{A} = \mathbb{K}$;
($d = 2$) $\mathbb{A}$ is either a quadratic Galois extension $\mathbb{L}$ of $\mathbb{K}$ or $\mathbb{K} \times \mathbb{K}$;
($d = 4$) $\mathbb{A}$ is either a quaternion division algebra $\mathbb{H}$ over $\mathbb{K}$ or the 2×2 matrices over $\mathbb{K}$;
($d = 8$) $\mathbb{A}$ is a Cayley-Dickson algebra $\mathbb{O}$ with center $\mathbb{K}$ (either division or split);
(insep) $\mathbb{A}$ is a purely inseparable extension of $\mathbb{K}$ with $\mathbb{A}^2 \subseteq \mathbb{K}$, and if d is finite, it is a power of 2 (this case only occurs if $\mathrm{char}(\mathbb{K}) = 2$).

In particular, a non-degenerate quadratic alternative $\mathbb{K}$-algebra is either division (also called *non-split*) or split, which corresponds to its norm form either being anisotropic or of maximal Witt index.

We show the following theorem.

Theorem A.1.1. *Let $\mathbb{A}$ be a degenerate quadratic alternative algebra whose radical R is generated by a single element $t \in \mathbb{A} \setminus \{0\}$. Then $\mathbb{A}$ has a non-degenerate quadratic associate algebra $\mathbb{B}$ such that $\mathbb{A} = \mathbb{B} \oplus t\mathbb{B}$. Moreover*

(i) *If $\mathbb{B}$ is division, then $\mathbb{A}$ is isomorphic to* $\mathsf{CD}(\mathbb{B}, 0)$*;*

(ii) *If $\mathbb{B}$ is split, then either $\mathbb{A}$ is isomorphic to* $\mathsf{CD}(\mathbb{B}, 0)$ *or* $\dim_{\mathbb{K}}(\mathbb{A}) \in \{3, 6\}$ *and in the latter case, $\mathbb{A}$ is isomorphic to the following respective quotients of* $\mathsf{CD}(\mathbb{B}, 0)$*:*

(a) the upper triangular 2×2-matrices over $\mathbb{K}$ (which we shall refer to as $\mathbb{T}'$);
(b) $\{M(a, b, c, d, 0, y, z, 0) \mid a, b, c, d, y, z \in \mathbb{K}\}$ (which we shall refer to as $\mathbb{S}'$).

Lastly, if $\dim_{\mathbb{K}}(\mathbb{A}) < 8$*, then $\mathbb{A}$ is isomorphic to a sub-algebra of the split octonions $\mathbb{O}'$.*

The algebras $\mathbb{A} = \mathbb{B} \oplus t\mathbb{B}$ of the above theorem we call *generalised dual numbers*, (non-)split if $\mathbb{B}$ is (non-)split.

A.1.3 Veronese varieties associated to generalised dual numbers

We now associate certain geometries to generalised dual numbers. They come into two subcategories (corresponding to non-split and split):

(*i*) $\mathsf{CD}(\mathbb{B},0)$, where $\mathbb{B}$ is a quadratic associative division algebra;

(*ii*) $\mathbb{T}'$, $\mathsf{CD}(\mathbb{L}',0)$, $\mathbb{S}'$, $\mathsf{CD}(\mathbb{H}',0)$.

We do not consider the case $\mathsf{CD}(\mathbb{K},0)$ twice, and we will not associate a Veronese variety to $\mathsf{CD}(\mathbb{H}',0)$, as it appears not to fit in our framework.

The non-split generalised dual numbers

In the first case, the associated Veronese varieties are defined by first considering the following ring projective plane, which has the structure of a *Hjelmslev projective plane of level 2*:

Definition A.1.2. The point-line geometry $\mathsf{G}_2(\mathbb{K},\mathbb{A}) := (\mathscr{P},\mathscr{L})$ is defined as follows:

- $\mathscr{P} = \{(x,y,1) \mid x,y \in \mathbb{A}\} \cup \{(1,y,tz_1) \mid z_1 \in \mathbb{B}, y \in \mathbb{A}\} \cup \{(tx_1,1,tz_1) \mid x_1,z_1 \in \mathbb{B}\}$;
- $\mathscr{L} = \{[a,1,c] \mid a,c \in \mathbb{A}\} \cup \{[1,tb_1,c] \mid b_1 \in \mathbb{B}, c \in \mathbb{A}\} \cup \{[ta_1,tb_1,1] \mid a_1,b_1 \in \mathbb{B}\}$;
- A point (x,y,z) is *incident* with a line $[a,b,c]$ if and only if $ax+by+cz=0$.

Afterwards one then applies the Veronese map on these ring geometries:

Definition A.1.3. The Veronese representation $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ of $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ is the point-subspace structure (X,Ξ) defined by means of the Veronese map

$$\rho : \mathsf{G}_2(\mathbb{K},\mathbb{A}) \to \mathsf{PG}(3d+2,\mathbb{K}) : (x,y,z) \mapsto \mathbb{K}(x\overline{x}, y\overline{y}, z\overline{z}; y\overline{z}, z\overline{x}, x\overline{y})$$

by setting $X = \{\rho(p) \mid p \in \mathscr{P}\}$ and $\Xi = \{\langle \rho(L) \rangle \mid L \in \mathscr{L}\}$, where $\rho(L)$ is defined as $\{\rho(p) \mid p \in L\}$ and incidence is given by containment made symmetric.

The split generalised dual numbers

Here we need a different approach, as it is no longer possible to list all triples of points. We use the following definition, which requires $|\mathbb{K}| > 2$.

Definition A.1.4. Let $\mathbb{A}$ be one of $\mathbb{L}'$, $\mathbb{H}'$, $\mathbb{O}'$, $\mathbb{T}'$, $\mathsf{CD}(\mathbb{L}',0)$, $\mathbb{S}'$ and put $d := \dim_{\mathbb{K}}(\mathbb{A})$. Then we define the following map ρ, called the *partial Veronese map*,

$$\rho : \mathbb{A} \times \mathbb{A} \to \mathsf{PG}(3d+2,\mathbb{K}) : (A,B) \mapsto (1, A\overline{A}, B\overline{B}, AB, B, A)$$

and define $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ as the projective closure of $\mathrm{im}(\rho)$.

In [51], it has been shown that the geometries $\mathscr{V}_2(\mathbb{K},\mathbb{L}')$, $\mathscr{V}_2(\mathbb{K},\mathbb{H}')$ and $\mathscr{V}_2(\mathbb{K},\mathbb{O}')$ are isomorphic to the Segre variety $\mathsf{S}_{2,2}(\mathbb{K})$, the line Grassmannian $\mathsf{G}_{5,1}(\mathbb{K})$ and the $\mathsf{E}_{6,1}(\mathbb{K})$ variety, respectively. We will give a description and characterisation for $\mathscr{V}_2(\mathbb{K},\mathbb{T}')$, $\mathscr{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{L}',0))$ and $\mathscr{V}_2(\mathbb{K},\mathbb{S}')$.

A.1.4 Hjelmslevean Veronese sets

We now characterise the Veronese varieties $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ for the non-split generalised dual numbers $\mathbb{A}$ over $\mathbb{K}$.

Definition A.1.5. In a projective space $\mathsf{PG}(d+v+2,\mathbb{K})$, we consider a v-space V and an ovoid O in a $(d+1)$-space complementary to V. The union of lines joining all points of V with all points of O is called a (d,v)-*cone* with *base* O and *vertex* V. The cone without its vertex is called a (d,v)-*tube* (with base O).

Consider a spanning point set X of $\mathsf{PG}(N,\mathbb{K})$, $N > d+v+2$, together with a collection Ξ of $(d+v+2)$-dimensional projective subspaces of $\mathsf{PG}(N,\mathbb{K})$, called the *tubic spaces* of X, such that, for any $\xi \in \Xi$, the intersection $\xi \cap X$ is a (d,v)-tube $X(\xi)$ in ξ with base O. The union of all vertices of those tubes is denoted by Y; so $(X \cup Y) \cap \xi$ is the unique (d,v)-cone containing $X(\xi)$, denoted by $\overline{X(\xi)}$.

The pair (X,Ξ), or simply X, is called a **Hjelmslevean Veronesean set (of type (d,v))** if the following properties hold.

(H1) Any two distinct points x_1 and x_2 of X lie in at least one element of Ξ,

(H2*) Any two distinct elements ξ_1 and ξ_2 of Ξ intersect in points of $X \cup Y$, i.e., $\xi_1 \cap \xi_2 = \overline{X(\xi_1)} \cap \overline{X(\xi_2)}$. Moreover, $\xi_1 \cap \xi_2 \cap X$ is non-empty.

The main theorem states that the geometries satisfying those axioms are essentially the Veronese varieties $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ where $\mathbb{A}$ is either a quadratic alternative division algebra or one of the non-split generalised dual numbers.

Main Theorem A.1.6. *Suppose (X,Ξ) is a Hjelmslevean Veronesean set of type (d,v) such that X generates $\mathsf{PG}(N,\mathbb{K})$, where $\mathbb{K}$ is a field with $|\mathbb{K}| > 2$. Then d is a power of 2, with $d \le 8$ if $\mathrm{char}(\mathbb{K}) \neq 2$, and one of the following holds.*

- (i) *There is only one vertex V and projected from V, the resulting point-subspace geometry (X',Ξ') is projectively equivalent to $\mathscr{V}_2(\mathbb{K},\mathbb{B})$, where $\mathbb{B}$ is a quadratic alternative division algebra over $\mathbb{K}$ and, in particular, $N = 3d+v$ and $d = \dim_{\mathbb{K}}(\mathbb{A})$;*
- (ii) *There is a quadratic associative division algebra $\mathbb{B}$ over $\mathbb{K}$ and two complementary subspaces U and W of $\mathsf{PG}(N,\mathbb{K})$, where U is possibly empty and $\dim W = 6d+2$, with $d = v - \dim(U) = 2\dim_{\mathbb{K}}(\mathbb{B})$, such that the intersection of every pair of distinct vertices is U, and the structure of (X,Ξ) induced in W is projectively equivalent to $\mathscr{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{B},0))$.*

In particular, the basis of the tube $X \cap \xi$, for each $\xi \in \Xi$, is a quadric.

Note that the “ordinary” Veronese varieties $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ where $\mathbb{A}$ is a quadratic alternative division algebra over $\mathbb{K}$ are also captured. This we also did for fields of size 2:

Main Theorem A.1.7. *Suppose (X,Ξ) is a Hjelmslevean Veronesean set of type $(d,-1)$ such that X generates $\mathsf{PG}(N,\mathbb{K})$, where $\mathbb{K}$ is a field. Then, as a point-line geometry, (X,Ξ) (with natural incidence) is isomorphic to $\mathsf{PG}(2,\mathbb{A})$ where $\mathbb{A}$ is a quadratic alternative division algebra over $\mathbb{K}$ with $\dim_{\mathbb{K}}(\mathbb{A})=d$. Moreover,*

- *If $|\mathbb{K}|>2$, (X,Ξ) is projectively equivalent to $\mathscr{V}_2(\mathbb{K},\mathbb{A})$, so $N=3d+1$;*
- *If $|\mathbb{K}|=2$, then either $d=1$ or $d=2$.*
  - *If $d=1$, then $N\in\{5,6\}$. If $N=5$, there are two projectively non-isomorphic examples, among which $\mathscr{V}_2(\mathbb{F}_2,\mathbb{F}_2)$; if $N=6$, there is a unique possibility.*
  - *If $d=2$, then $N\in\{8,9,10\}$. If $N=10$, then there is precisely one example; in the other two cases there are precisely two projectively unique examples, among which is $\mathscr{V}_2(\mathbb{F}_2,\mathbb{F}_4)$, if $N=8$.*

A.1.5 Split Veronese sets

We now characterise the varieties $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ for the split generalised dual numbers $\mathbb{A}$ over $\mathbb{K}$, except for $\mathbb{A}=\mathsf{CD}(\mathbb{H}',0)$.

Definition A.1.8. Let R,V be integers with $V\geq -1$ and $R\geq 1$. An (R,V)-*cone* C is a cone with a V-dimensional vertex and as basis a hyperbolic quadric of rank $R+1$ (i.e., a non-degenerate quadric of maximal Witt index in $\mathsf{PG}(2R+1,\mathbb{K})$); C without its vertex is called an (R,V)-*tube*.

Let r,v,r',v',N be integers which are at least -1 with $r'>r\geq 1$. Suppose that $X\cup Z$ is a spanning point set of $\mathsf{PG}(N,\mathbb{K})$ and we define Y as the subspace spanned by the points of Z. Put $d:=2r+v+1$ and $d':=2r'+v'+1$. Let Ξ be a collection of $(d+1)$-dimensional subspaces of $\mathsf{PG}(N,\mathbb{K})$ with $|\Xi|>1$ and Θ a possibly empty collection of $(d'+1)$-dimensional subspaces of $\mathsf{PG}(N,\mathbb{K})$ such that:

- For each $\xi\in\Xi$, the intersection $XY(\xi):=(X\cup Y)\cap\xi$ is an (r,v)-cone C_ξ, $X(\xi):=X\cap\xi$ is a (r,v)-tube T_ξ and $Y(\xi):=Y\cap\xi$ is the vertex of C_ξ;
- for each $\theta\in\Theta$, the intersection $XY(\theta):=(X\cup Y)\cap\theta$ is an (r',v')-cone C_θ, $Y(\theta):=Y\cap\theta$ is precisely a generator M of the quadric C_θ (which in particular contains the vertex V_θ of C_θ), and then $Z(\theta):=Z\cap\theta$ is the (disjoint) union of V_θ and some r'-space of M complementary to it; lastly $X(\theta):=X\cap\theta$ is $C_\theta\setminus M$.

For each point $x\in X$ we denote by T_x the subspace spanned by all singular lines through x.

A quadruple (X,Z,Ξ,Θ) is called a **split Veronese set** with parameters (r,v,r',v') if the following axioms are satisfied:

(S1) Each pair of distinct points $p_1,p_2\in X\cup Z$ is contained in a member of $\Xi\cup\Theta$;
(S2) for each pair of distinct members $\zeta_1,\zeta_2\in\Xi\cup\Theta$, the intersection $\zeta_1\cap\zeta_2$ belongs to $X\cup Y$;

(S3) for each point $x \in X$, there exists ξ_1, ξ_2 in Ξ such that $T_x = \langle T_x(\xi_1), T_x(\xi_2)\rangle$;

If (X, Z, Ξ, Θ) satisfies (S1) and (S2), then we call it a **split pre-Veronese set**. A split Veronese set is called **mono-symplectic** if Θ is empty, in which case the parameters are just (r, v).

The main theorem states that the geometries satisfying those axioms are essentially Veronese varieties $\mathscr{V}_2(\mathbb{K}, \mathbb{A})$ for the non-degenerate split quadratic alternative algebras $\mathbb{A}$ or the split generalised dual numbers $\mathbb{A}$ over $\mathbb{K}$, except for $\mathbb{A} = \mathsf{CD}(\mathbb{H}', 0)$.

Theorem A.1.9. *Let (X, Z, Ξ, Θ) be a split Veronese set with parameters (r, v, r', v') where $\langle X, Z\rangle = \mathsf{PG}(N, \mathbb{K})$ for some arbitrary field $\mathbb{K}$ with $|\mathbb{K}| > 2$. If mono-symplectic, then X is projectively equivalent to a cone with a vertex of dimension v^* (possibly, $v^* = -1$) over one of the following geometries:*

- (i) *A Segre variety $\mathsf{S}_{1,2}(\mathbb{K})$ or $\mathsf{S}_{2,2}(\mathbb{K})$, a line Grassmannian $\mathsf{G}_{4,1}(\mathbb{K})$ or $\mathsf{G}_{5,1}(\mathbb{K})$, or the variety $\mathsf{E}_{6,1}(\mathbb{K})$; in this case $v = v' = v^*$.*

If duo-symplectic, then X is either projectively equivalent to a cone with a vertex of dimension v^* (possibly, $v^* = -1$) over one of the following geometries:

- (ii) *A half dual Segre variety $\mathsf{HDS}_{2,k}(\mathbb{K})$, where $k \in \{1, 2\}$, which is a split Veronese set with parameters $(1, 0, 2, -1)$;*
- (iii) *A dual line Grassmannian variety $\mathsf{DG}_{5,1}(\mathbb{K})$, which is a split Veronese set with parameters $(2, 1, 4, -1)$,*

or projectively equivalent to the following geometry:

- (iv) *A dual Segre variety $\mathsf{DS}_{2,2}(\mathbb{K})$, with parameters $(1, 1, 2, 1)$.*

In particular, the varieties in (i) up to (iv) are subvarieties of the Veronese variety $\mathscr{V}_2(\mathbb{K}, \mathbb{O}')$ over the split octonions $\mathbb{O}'$, and apart from $\mathsf{S}_{1,2}(\mathbb{K})$ and $\mathsf{G}_{4,1}(\mathbb{K})$, all of them are a Veronese variety $\mathscr{V}_2(\mathbb{K}, \mathbb{A})$ for some split quadratic alternative algebra $\mathbb{A}$ whose radical is either empty or generated by a single element t.

A.2 Lacunary parapolar spaces

It turns out that many of the interesting exceptional Lie incidence geometries (some Grassmannians are more interesting than others, often these with the smallest diameter, etc.) have certain gaps in the spectrum of the dimensions of the singular subspaces that occur as intersections of two symplecta. Such parapolar spaces we will call *lacunary*; more precisely:

Let k be an integer with $k \geq -1$. We say that a parapolar space is k-**lacunary** if k never occurs as the dimension of the intersection of two symplecta, and the symplectic rank is at least $k+1$.

The following theorem captures what is shown in this part of the thesis.

Main Theorem A.2.1. *Let $\Omega = (X, \mathscr{L})$ be a k-lacunary parapolar space of minimum symplectic rank d with $d \geq k+3$, which is strong in case there are symplecta of rank 2. Then, if locally connected, Ω is one of the Lie incidence geometries occurring in Table A.1. If Ω is not locally connected, then Ω arises from a collection of locally connected k-lacunary parapolar spaces and polar spaces of rank at least $k+1$ as described in* Construction 2 *(called a k-*buttoned parapolar space*).*

$k=-1$		$k=0$		$k=1$		$k=2$		$k=3$		$k=4$
$\mathsf{A}_{1,1}(*) \times \mathsf{A}_{2,1}(*)$	$\leftarrow$	$\mathsf{A}_{4,2}(\mathbb{L})$	$\leftarrow$	$\mathsf{D}_{5,5}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{6,1}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{7,7}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{8,8}(\mathbb{K})$
$\mathsf{A}_{2,1}(*) \times \mathsf{A}_{2,1}(*)$	$\leftarrow$	$\mathsf{A}_{5,3}(\mathbb{L})$	$\leftarrow$	$\mathsf{E}_{6,2}(\mathbb{K})$						
$\mathsf{A}_{4,2}(\mathbb{L})$	$\leftarrow$	$\mathsf{D}_{5,5}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{6,1}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{7,7}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{8,8}(\mathbb{K})$		
$\mathsf{A}_{5,2}(\mathbb{L})$	$\leftarrow$	$\mathsf{D}_{6,6}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{7,1}(\mathbb{K})$						
$\mathsf{E}_{6,1}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{7,7}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{8,8}(\mathbb{K})$						

Table A.1: The locally connected k-lacunary parapolar spaces of minimum rank at least $k+3$.

The inductive nature of this is explained by the fact that, if a parapolar space Ω is a locally connected and k-lacunary of minimum symplectic rank $d \geq 3$, then for each point p of Ω, the point-residual Ω_p is a strong $(k-1)$-lacunary parapolar space of minimum symplectic rank $d-1$.

In [18], we also classify the k-lacunary parapolar space of minimum symplectic rank d with $d < k+3$ (i.e., $d \in \{k+1, k+2\}$). The distinction between these two classes is that the ones with $d \geq k+3$ have a residue which is (-1)-lacunary, whereas the others turn out to have a residue which has minimum symplectic rank 2 and is 0-lacunary. Since the split geometries of the second row of the FTMS are (-1)-lacunary parapolar spaces, it is this class that fits in this thesis.

APPENDIX

B

NEDERLANDSTALIGE SAMENVATTING

Deze thesis bestaat uit twee delen. Deel 1 omvat de kern van mijn doctoraat en de resultaten hierin staan ook beschreven in [16] en [19]. Deel 2 is gebaseerd op een gezamenlijk werk, waarvan de resultaten ook kunnen gevonden worden in [18]. Beide delen zijn gerelateerd aan de meetkundes die voorkomen in het Freudenthal-Tits magisch vierkant, een 4×4 vierkant dat zowel klassieke als exceptionele Lie incidentie meetkundes bevat.

Hieronder geven we de drie grote (classificatie) resultaten die deze dissertatie bevat. De resultaten zijn omkaderd. Ook de nodige informatie om de inhoud van deze kaders te begrijpen wordt meegegeven, maar voor de details verwijzen we naar de inhoud van dit werk. Desalniettemin zou deze samenvatting een goed idee moeten geven van de resultaten die kunnen gevonden worden in deze thesis.

B.1 Deel 1: Een karakterisatie van de duale versie van de tweede rij van het Freudenthal-Tits magisch vierkant (FTMV)

Zij $\mathbb{K}$ een veld.

B.1.1 Inleiding

De meetkundes van de tweede rij van het FTMV zijn, in het niet-gespleten geval, de projectieve vlakken die gecoördinatiseerd worden over kwadratische alternatieve $\mathbb{K}$-algebra's; in het gespleten geval krijgen we een projectief vlak $\mathsf{A}_2(\mathbb{K})$, de direct product meetkunde $\mathsf{A}_{2,1}(\mathbb{K}) \times \mathsf{A}_{2,1}(\mathbb{K})$, de lijngrassmanniaan $\mathsf{A}_{5,2}(\mathbb{K})$ en de $\mathsf{E}_{6,1}(\mathbb{K})$ variëteit. De Veronese representaties van deze meetkundes werden reeds axiomatisch gekarakteriseerd ([29], [39]).

In deze thesis beschouwen we de tweede rij van het FTMV, maar dan gedefinieerd over een bepaalde klasse van *ontaarde* kwadratische alternative algebras, die we de *veralgemeende duale getallen* zullen noemen. Het zijn de Veronese representaties geassocieerd aan deze veralgemeende duale getallen die we zullen karakteriseren.

B.1.2 Kwadratische alternatieve algebra's en veralgemeende duale getallen

Zij $\mathbb{A}$ een kwadratische alternatieve $\mathbb{K}$-algebra. Dan komt $\mathbb{A}$ met een multiplicatieve norm vorm N. Het *radicaal* R van $\mathbb{A}$ wordt gedefinieerd als de verzameling $\{r \in \mathsf{rad}(f) \mid \mathsf{N}(r) = 0\}$, wat equivalent is met het nil ideaal van $\mathbb{A}$. We zeggen dat $\mathbb{A}$ niet-ontaard is als $R = \{0\}$.

Het is een bekend resultaat dat, als $\mathbb{A}$ niet-ontaard is, dat het dan een van de volgende algebra's is.

($d = 1$) $\mathbb{A} = \mathbb{K}$;
($d = 2$) $\mathbb{A}$ is ofwel een kwadratische Galois uitbreiding $\mathbb{L}$ van $\mathbb{K}$, ofwel $\mathbb{K} \times \mathbb{K}$;
($d = 4$) $\mathbb{A}$ is ofwel een quaternionen delingsalgebra $\mathbb{H}$ over $\mathbb{K}$, ofwel de 2×2 matrices over $\mathbb{K}$;
($d = 8$) $\mathbb{A}$ is een Cayley-Dickson algebra $\mathbb{O}$ met centrum $\mathbb{K}$ (ofwel een delingsalgebra ofwel gespleten);
(insep) $\mathbb{A}$ is een zuiver inseparabele uitbreiding van $\mathbb{K}$ met $\mathbb{A}^2 \subseteq \mathbb{K}$, en als d eindig is, is het een macht van 2 (dit geval doet zich slechts voor als $\mathrm{char}(\mathbb{K}) = 2$).

In het bijzonder hebben we dat een niet-ontaarde kwadratische alternative $\mathbb{K}$ algebra $\mathbb{A}$ steeds ofwel een delingsalgebra is of een gespleten algebra, wat overeenkomt met de norm vorm die ofwel anisotroop is ofwel maximale Witt index heeft. In het eerste geval noemen we $\mathbb{A}$ ook wel eens *niet-gespleten*. De volgende stelling karakteriseert de algebra's waarin we geïnteresseerd zijn.

Stelling B.1.1. *Zij $\mathbb{A}$ een ontaarde kwadratische alternatieve algebra wiens radicaal R voortgebracht wordt door één element $t \in \mathbb{A} \setminus \{0\}$. Dan heeft $\mathbb{A}$ een niet-ontaarde kwadratische alternatieve deelalgebra $\mathbb{B}$ zodat $\mathbb{A} = \mathbb{B} \oplus t\mathbb{B}$. Bovendien*

(*i*) *Als $\mathbb{B}$ niet-gespleten is, dan is $\mathbb{A}$ isomorf aan $\mathsf{CD}(\mathbb{B}, 0)$;*

(*ii*) *Als $\mathbb{B}$ gespleten is, dan is $\mathbb{A}$ ofwel isomorf aan $\mathsf{CD}(\mathbb{B}, 0)$, of $\dim_{\mathbb{K}}(\mathbb{A}) \in \{3, 6\}$ en in dat laatste geval is $\mathbb{A}$ isomorf aan een van de volgende quotiënten van $\mathsf{CD}(\mathbb{B}, 0)$:*

  (a) de 2×2 bovendriehoeksmatrices over $\mathbb{K}$ (deze zullen we noteren met $\mathbb{T}'$);
  (b) $\{M(a, b, c, d, 0, y, z, 0) \mid a, b, c, d, y, z \in \mathbb{K}\}$ (deze zullen we noteren met $\mathbb{S}'$).

  Ten slotte, als $\dim_{\mathbb{K}}(\mathbb{A}) < 8$, dan is $\mathbb{A}$ isomorf aan een deelalgebra van de gespleten octonen $\mathbb{O}'$.

De algebras $\mathbb{A} = \mathbb{B} \oplus t\mathbb{B}$ van de stelling hierboven noemen we *veralgemeende duale getallen*, (niet-)gespleten als $\mathbb{B}$ (niet-)gespleten is.

B.1.3 Veronese variëteiten geassocieerd aan veralgemeende duale getallen

We hechten nu bepaalde meetkundes aan de veralgemeende duale getallen. Deze komen in twee categorieën, afhankelijk van het al dan niet gespleten zijn van de algebra $\mathbb{B}$.

(i) $\mathsf{CD}(\mathbb{B},0)$, waar $\mathbb{B}$ een kwadratische alternatieve delingsalgebra is;

(ii) $\mathbb{T}'$, $\mathsf{CD}(\mathbb{L}',0)$, $\mathbb{S}'$, $\mathsf{CD}(\mathbb{H}',0)$.

We beschouwen het geval $\mathsf{CD}(\mathbb{K},0)$ geen twee keer, en we zullen geen Veronese variëteit hechten aan $\mathsf{CD}(\mathbb{H}',0)$, omdat het zal blijken niet in onze meetkundige setting te passen.

De niet-gespleten veralgemeende duale getallen

In het eerste geval starten we met het beschouwen van de volgende ring projectief vlak, dat de structuur heeft van een *Hjelsmslev projectief vlak van level 2*.

Definitie B.1.2. De punt-rechte meetkunde $\mathsf{G}_2(\mathbb{K},\mathbb{A}) := (\mathscr{P},\mathscr{L})$ wordt als volgt gedefinieerd:

— $\mathscr{P} = \{(x,y,1) \mid x,y \in \mathbb{A}\} \cup \{(1,y,tz_1) \mid z_1 \in \mathbb{B}, y \in \mathbb{A}\} \cup \{(tx_1,1,tz_1) \mid x_1,z_1 \in \mathbb{B}\}$;
— $\mathscr{L} = \{[a,1,c] \mid a,c \in \mathbb{A}\} \cup \{[1,tb_1,c] \mid b_1 \in \mathbb{B}, c \in \mathbb{A}\} \cup \{[ta_1,tb_1,1] \mid a_1,b_1 \in \mathbb{B}\}$;
— Een punt (x,y,z) is *incident* met een rechte $[a,b,c]$ als en slechts als $ax+by+cz=0$.

Hierna laten we dan de Veronese afbeelding los op deze meetkundes.

Definitie B.1.3. De Veronese representatie $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ van $\mathsf{G}_2(\mathbb{K},\mathbb{A})$ is de punt-deelruimte structuur (X,Ξ) gedefinieerd door de Veronese afbeelding:

$$\rho : \mathsf{G}_2(\mathbb{K},\mathbb{A}) \to \mathsf{PG}(3d+2,\mathbb{K}) : (x,y,z) \mapsto \mathbb{K}(x\overline{x}, y\overline{y}, z\overline{z}; y\overline{z}, z\overline{x}, x\overline{y}).$$

Hierbij stellen we $X = \{\rho(p) \mid p \in \mathscr{P}\}$ en $\Xi = \{\langle \rho(L) \rangle \mid L \in \mathscr{L}\}$, waar $\rho(L)$ gedefinieerd wordt als $\{\rho(p) \mid p \in L\}$, incidentie wordt gegeven door de gesymmetriseerde inclusie.

De gespleten veralgemeende duale getallen

Deze moeten we anders aanpakken, want het is niet doenbaar om alle drietallen van punten en rechten op te noemen en hier een steekhoudende definitie mee te geven. We gebruiken de volgende definitie, die vereist dat $|\mathbb{K}| > 2$.

Definitie B.1.4. Zij $\mathbb{A}$ één van $\mathbb{L}'$, $\mathbb{H}'$, $\mathbb{O}$', $\mathbb{T}'$, $\mathsf{CD}(\mathbb{L}',0)$, $\mathbb{S}'$ en stel $d := \dim_{\mathbb{K}}(\mathbb{A})$. Dan voeren we de afbeelding ρ in, als een *partiële Veronese afbeelding*,

$$\rho : \mathbb{A} \times \mathbb{A} \to \mathsf{PG}(3d+2,\mathbb{K}) : (A,B) \mapsto (1, A\overline{A}, B\overline{B}, AB, B, A)$$

en definiëren we $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ als de projectieve sluiting van $\mathrm{im}(\rho)$.

Er is aangetoond in [51] dat de meetkundes $\mathscr{V}_2(\mathbb{K},\mathbb{L}')$, $\mathscr{V}_2(\mathbb{K},\mathbb{H}')$ and $\mathscr{V}_2(\mathbb{K},\mathbb{O}')$ isomorf zijn aan de Segre variëteit $\mathsf{S}_{2,2}(\mathbb{K})$, de lijngrassmanniaan $\mathsf{G}_{5,1}(\mathbb{K})$ en de $\mathsf{E}_{6,1}(\mathbb{K})$ variëteit, respectievelijk. De drie andere meetkundes zijn aan ons om te beschrijven en klasseren.

B.1.4 Hjelmslevische Veronese verzamelingen

We karakteriseren nu de Veronese varietiëteiten $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ voor de niet-gespleten veralgemeende duale getallen $\mathbb{A}$ over $\mathbb{K}$.

Definitie B.1.5. In een projectieve ruimte $\mathsf{PG}(d+v+2,\mathbb{K})$, beschouwen we een v-ruimte V en een ovoïde O in a $(d+1)$-space complementair aan V. De unie van de rechten die punten van V verbinden met punten van O zullen we een (d,v)-*kegel* noemen met *basis* O en *top* V. De kegel zonder zijn top noemen we een (d,v)-*tube* (met basis O).

Beschouw een puntenverzameling X van $\mathsf{PG}(N,\mathbb{K})$, $N > d+v+2$, tezamen met een collectie Ξ of $(d+v+2)$-dimensionale projectieve deelruimten van $\mathsf{PG}(N,\mathbb{K})$, die we de *tube ruimtes* van X noemen, zo dat, voor elke $\xi\in\Xi$, de doorsnede $\xi\cap X$ een (d,v)-tube $X(\xi)$ is in ξ met basis een zekere ovoïde O. De unie van alle toppen van deze tubes noteren we met Y; dus $(X\cup Y)\cap\xi$ is de unieke (d,v)-kegel die $X(\xi)$ bevat, en deze duiden we aan met $\overline{X(\xi)}$.

Het paar (X,Ξ), of simpelweg X, wordt een **Hjelmslevische Veronese verzameling (van type (d,v))** genoemd als de volgende voorwaarden voldaan zijn.

(H1) Elk tweetal punten x_1,x_2 uit X behoort tot minstens één element van Ξ,

(H2*) Elk tweetal ξ_1,ξ_2 uit Ξ snijdt in punten van $X\cup Y$, m.a.w. $\xi_1\cap\xi_2=\overline{X(\xi_1)}\cap\overline{X(\xi_2)}$. Bovendien is $\xi_1\cap\xi_2\cap X$ niet ledig.

De hoofdstelling zegt dat de meetkundes die aan deze axioma's voldoen in essentie de Veronese variëteiten $\mathscr{V}_2(\mathbb{K},\mathbb{A})$ zijn, waar $\mathbb{A}$ ofwel een kwadratische alternative delingsalgebra is of één van de niet-gespleten veralgemeende duale getallen.

Hoofdstelling B.1.6. *Stel dat (X,Ξ) een Hjelmslevische Veronese verzameling van type (d,v) is zodanig dat X de ruimte $\mathsf{PG}(N,\mathbb{K})$ voortbrengt, waarbij $\mathbb{K}$ een veld is met $|\mathbb{K}|>2$. Dan is d een macht van 2, met $d\le 8$ als $\mathrm{char}(\mathbb{K})\neq 2$, en zitten we in één van de volgende situaties.*

(i) Er is slechts één top V en hieruit geprojecteerd krijgen we een punt-deelruimte meetkunde (X',Ξ') projectief equivalent aan $\mathscr{V}_2(\mathbb{K},\mathbb{B})$, met $\mathbb{B}$ een kwadratische alternatieve delingsalgebra over $\mathbb{K}$. I.h.b. is $N=3d+v$ en $d=\dim_{\mathbb{K}}(\mathbb{A})$;

(ii) Er is een kwadratische associatieve delingsalgebra $\mathbb{B}$ over $\mathbb{K}$ en twee complementaire deelruimtes U en W van $\mathsf{PG}(N,\mathbb{K})$, waar U eventueel ledig is en $\dim W=6d+2$, met $d=v-\dim(U)=2\dim_{\mathbb{K}}(\mathbb{B})$, zo dat de doorsnede van elk paar van toppen van tubes U is en dat de structuur geïnduceerd door (X,Ξ) in W projectief equivalent is aan $\mathscr{V}_2(\mathbb{K},\mathsf{CD}(\mathbb{B},0))$.

In het bijzonder is de basis van de tube $X\cap\xi$, voor elke $\xi\in\Xi$, een kwadriek.

Merk op dat we ook de "gewone" Veronese variëteiten $\mathscr{V}_2(\mathbb{K},\mathbb{A})$, waar $\mathbb{A}$ een kwadratische alternatieve delingsalgebra is, uitkomen als resultaat. Dit hebben we in feite ook kunnen doen in het geval dat $|\mathbb{K}|=2$:

Hoofdstelling B.1.7. *Stel dat* (X,Ξ) *een Hjelmslevische Veronese verzameling van type* $(d,-1)$ *is zodanig dat* X *de ruimte* $\mathsf{PG}(N,\mathbb{K})$ *voortbrengt, waar* $\mathbb{K}$ *een veld is. Dan hebben we dat* (X,Ξ)*, als punt-rechte meetkunde met natuurlijke incidentie, isomorf is aan* $\mathsf{PG}(2,\mathbb{A})$ *waar* $\mathbb{A}$ *een kwadratische alternatieve delingsalgebra is over* $\mathbb{K}$ *with* $\dim_{\mathbb{K}}(\mathbb{A})=d$*. Bovendien,*

- *Als* $|\mathbb{K}|>2$*, dan is* (X,Ξ) *projectief equivalent met* $\mathcal{V}_2(\mathbb{K},\mathbb{A})$*, dus* $N=3d+1$*;*
- *Als* $|\mathbb{K}|=2$*, dan is ofwel* $d=1$ *ofwel* $d=2$*.*
  - *Als* $d=1$*, dan is* $N\in\{5,6\}$*. Als* $N=5$ *dan zijn er twee niet-isomorfe voorbeelden, waaronder* $\mathcal{V}_2(\mathbb{F}_2,\mathbb{F}_2)$*; als* $N=6$*, dan is er een unieke mogelijkheid.*
  - *Als* $d=2$*, dan is* $N\in\{8,9,10\}$*. als* $N=10$*, dan is er één projectief uniek voorbeeld; in de twee andere gevallen zijn er precies twee voorbeelden, waaronder* $\mathcal{V}_2(\mathbb{F}_2,\mathbb{F}_4)$ *als* $N=8$*.*

B.1.5 Gespleten Veronese verzamelingen

Nu karakteriseren we de meetkundes $\mathcal{V}_2(\mathbb{K},\mathbb{A})$ voor de gespleten veralgemeende duale getallen $\mathbb{A}$ over $\mathbb{K}$, behalve dan voor $\mathbb{A}=\mathsf{CD}(\mathbb{H}',0)$.

Definitie B.1.8. Zij R,V gehele getallen met $V\geq -1$ en $R\geq 1$. Een (R,V)*-kegel* C is een kegel met een V-dimensionale top en als basis een hyperbolische kwadriek van rang $R+1$ (m.a.w., een niet-ontaarde kwadriek van maximale Witt index in $\mathsf{PG}(2R+1,\mathbb{K})$); de kegel zonder zijn top noemen we een (R,V)*-tube*.

Zij r,v,r',v',N gehele getallen groter of gelijk aan -1 met $r'>r\geq 1$. Zij $X\cup Z$ een puntenverzameling die de ruimte $\mathsf{PG}(N,\mathbb{K})$ voortbrengt en definieer Y als de deelruimte voortgebracht door de punten van Z. Stel $d:=2r+v+1$ en $d':=2r'+v'+1$. Zij Ξ een collectie van $(d+1)$-dimensionale deelruimten van $\mathsf{PG}(N,\mathbb{K})$ met $|\Xi|>1$ en Θ een eventueel ledige collectie van $(d'+1)$-dimensionale deelruimten van $\mathsf{PG}(N,\mathbb{K})$ zo dat:

- Voor elke $\xi\in\Xi$ is de doorsnede $XY(\xi):=(X\cup Y)\cap\xi$ een (r,v)-kegel C_ξ, $X(\xi):=X\cap\xi$ is een (r,v)-tube T_ξ en $Y(\xi):=Y\cap\xi$ is de top van C_ξ;
- voor elke $\theta\in\Theta$ is de doorsnede $XY(\theta):=(X\cup Y)\cap\theta$ een (r',v')-kegel C_θ, $Y(\theta):=Y\cap\theta$ is een generator M van de kwadriek C_θ (die in het bijzonder de top V_θ van C_θ bevat), $Z(\theta):=Z\cap\theta$ de (disjuncte) unie van V_θ en een zekere r'-ruimte van M complementair aan V_θ; en $X(\theta):=X\cap\theta$, wat precies $C_\theta\setminus M$ is.

Voor elke $x\in X$ duiden we met T_x de deelruimte aan die voortgebracht wordt door de rechten door x die volledig in $X\cup Y$ bevat zijn.

We noemen een viertal (X,Z,Ξ,Θ) een **gespleten Veronese verzameling** met parameters (r,v,r',v') als aan volgende voorwaarden voldaan is:

(S1) Elk tweetal punten p_1,p_2 uit $X\cup Z$ is bevat in ten minste een element van $\Xi\cup\Theta$;

(S2) Elk tweetal elementen ζ_1, ζ_2 uit $\Xi \cup \Theta$ is zodat $\zeta_1 \cap \zeta_2$ singulier is;
(S3) voor elke $x \in X$ bestaan er ξ_1, ξ_2 uit Ξ zodat $T_x = \langle T_x(\xi_1), T_x(\xi_2) \rangle$;

Als (X, Z, Ξ, Θ) aan (S1) en (S2) voldoet dan noemen we het een **gespleten pre-Veronese verzameling**. Een gespleten Veronese verzameling noemen we **mono-symplectisch** als Θ ledig is, en in dit geval noteren we de parameters met (r, v).

De hoofdstelling zegt dat de meetkundes die voldoen aan bovenstaande axioma's in essentie de Veronese variëteiten $\mathcal{V}_2(\mathbb{K}, \mathbb{A})$ zijn waarbij $\mathbb{A}$ ofwel een niet-ontaarde gespleten kwadratische alternatieve algebra is ofwel één van de gespleten veralgemeende duale getallen over $\mathbb{K}$, behalve dan $\mathbb{A} = \mathsf{CD}(\mathbb{H}', 0)$.

Stelling B.1.9. *Zij (X, Z, Ξ, Θ) een gespleten Veronese verzameling met parameters (r, v, r', v') waar $\langle X, Z \rangle = \mathsf{PG}(N, \mathbb{K})$ voor een veld $\mathbb{K}$ met $|\mathbb{K}| > 2$. Als X mono-symplectisch is, dan is X projectief equivalent aan een kegel met een top van dimensie v^* (eventueel met $v^* = -1$) over een van de volgende meetkundes:*

(i) *Een Segre variëteit $\mathsf{S}_{1,2}(\mathbb{K})$ of $\mathsf{S}_{2,2}(\mathbb{K})$, een lijngrassmanniaan $\mathsf{G}_{4,1}(\mathbb{K})$ of $\mathsf{G}_{5,1}(\mathbb{K})$, of de $\mathsf{E}_{6,1}(\mathbb{K})$ variëteit; in dit geval is $v = v' = v^*$.*

Als X duo-symplectisch is, dan is X is ofwel projectief equivalent aan een kegel met een top van dimensie v^* (eventueel is $v^* = -1$) over een van de volgende meetkundes

(ii) *Een half duale Segre variëteit $\mathsf{HDS}_{2,k}(\mathbb{K})$, waar $k \in \{1, 2\}$, wat een gespleten Veronese verzameling is met parameters $(1, 0, 2, -1)$;*

(iii) *Een duale lijngrassmaniaan $\mathsf{DG}_{5,1}(\mathbb{K})$, wat een gespleten Veronese verzameling is met parameters $(2, 1, 4, -1)$,*

of projectief equivalent met de volgende meetkunde:

(iv) *Een duale Segre variëteit $\mathsf{DS}_{2,2}(\mathbb{K})$, met parameters $(1, 1, 2, 1)$.*

De variëteiten in (i) tot (iv) zijn deelvariëteiten van de Veronese variëteit $\mathcal{V}_2(\mathbb{K}, \mathbb{O}')$ over de gespleten octonen $\mathbb{O}'$, en behalve $\mathsf{S}_{1,2}(\mathbb{K})$ en $\mathsf{G}_{4,1}(\mathbb{K})$, zijn het allemaal Veronese variëteiten $\mathcal{V}_2(\mathbb{K}, \mathbb{A})$ voor een zekere gespleten kwadratische alternatieve algebra $\mathbb{A}$ waarvoor het radicaal ofwel leeg is ofwel voortgebracht wordt door één element $t \in \mathbb{A} \setminus \{0\}$.

B.2 Lacunaire parapolaire ruimten

Het blijkt dat vele van de interessante exceptionele Lie incidentie meetkundes lacunes vertonen in het spectrum van de dimensies van de singuliere deelruimten die voorkomen als de doorsnijding van twee symplecta. Zulke parapolaire ruimten zullen we *lacunair* noemen:

Zij k een geheel getal groter of gelijk aan -1. We zeggen dat een parapolaire ruimte **k-lacunair** is als de dimensie van de doorsnede van twee symplecta nooit exact k is en als de symplecta allemaal rang ten minste $k+1$ hebben.

De volgende stelling bevat het resultaat aangaande k-lacunaire parapolaire ruimten dat we in deze thesis zullen bewijzen.

Hoofdstelling B.2.1. *Zij $\Omega = (X, \mathscr{L})$ een k-lacunaire parapolaire ruimte van minimum symplectische rang d met $d \geq k+3$, waarvan we vereisen dat ze sterk is als er symplecta zijn van rang 2. Als Ω lokaal samenhangend is, dan is het een van de Lie incidentie meetkundes uit Tabel B.1. Als Ω niet lokaal samenhangend is, dan ontstaat Ω uit een collectie van lokaal samenhangende k-lacunaire parapolaire ruimten en polaire ruimten van rang ten minste $d+1$, op een manier die beschreven staat in* Constructie 2 *(we noemen Ω dan een k-*geknoopte parapolaire ruimte*).*

$k=-1$		$k=0$		$k=1$		$k=2$		$k=3$		$k=4$
$\mathsf{A}_{1,1}(*)\times\mathsf{A}_{2,1}(*)$	$\leftarrow$	$\mathsf{A}_{4,2}(\mathbb{L})$	$\leftarrow$	$\mathsf{D}_{5,5}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{6,1}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{7,7}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{8,8}(\mathbb{K})$
$\mathsf{A}_{2,1}(*)\times\mathsf{A}_{2,1}(*)$	$\leftarrow$	$\mathsf{A}_{5,3}(\mathbb{L})$	$\leftarrow$	$\mathsf{E}_{6,2}(\mathbb{K})$						
$\mathsf{A}_{4,2}(\mathbb{L})$	$\leftarrow$	$\mathsf{D}_{5,5}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{6,1}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{7,7}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{8,8}(\mathbb{K})$		
$\mathsf{A}_{5,2}(\mathbb{L})$	$\leftarrow$	$\mathsf{D}_{6,6}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{7,1}(\mathbb{K})$						
$\mathsf{E}_{6,1}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{7,7}(\mathbb{K})$	$\leftarrow$	$\mathsf{E}_{8,8}(\mathbb{K})$						

Table B.1: De lokaal samenhangende k-lacunaire parapolaire ruimten met minimum symplectische rang ten minste $k+3$.

De inductieve natuur wordt verklaard door het feit dat, als een parapolaire ruimte Ω lokaal samenhangend en k-lacunair is en minimum symplectische rang $d \geq 3$ heeft, dan hebben we voor elk van diens punten p dat het punt-residue Ω_p een sterke $(k-1)$-lacunaire parapolaire ruimte is van minimum symplectische rank $d-1$.

In [18] worden ook de k-lacunaire parapolaire ruimten van minimum symplectische rang d met $d < k+3$ (dus $d \in \{k+1, k+2\}$) geklasseerd. Het verschil tussen deze twee klassen is dat degenen met $d \geq k+3$ een residue hebben dat (-1)-lacunair is, terwijl die andere uiteindelijk blijken terug te gaan tot een residue van minimum symplectische rank 2 dat 0-lacunair is. Daar de meetkundes van de gespleten versie van de tweede rij van het FTMS (-1)-lacunair zijn, is het de eerste klasse die in deze thesis past, vandaar de restrictie.

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