Research Perspectives of Initial Geometry Education

Olivera J. Đokić

University of Belgrade, Faculty of Education, Serbia

Article Info

Article History
Submitted: 3 June 2018
Revised: 20 July 2018
Published: 4 August 2018

Keywords
Geometry education
Teaching approach
RME
Innovative mathematics textbook
Serbia

Abstract

The paper discusses the perspectives of teaching geometry based on the study The International Commission on Mathematical Instruction ICMI. Bearing in mind the current needs in geometry teaching, we considered that the topics of space and spatial reasoning should be part of it as key issues, as well as teaching approaches, RME, learning through (re)discovering mathematical ideas, the role of textbooks in the initial mathematics education, and so on. Therefore, this research paper offers a general insight into our need to deal with these topics, and invites researchers from the broader mathematical community, who we herewith familiarize with the results of our research in the sphere of initial geometry teaching in Serbia.

1. Introduction

Analyzing the perspectives of 21st century geometry teaching, Villani (1998) clearly points out in the International Commission on Mathematical Instruction (ICMI) study that primary school geometry teaching should not only be reduced to the introduction of mathematical concepts/procedures, but should, instead, rather help students to:

I. Improve the ability of *spatial reasoning* and
II. Improve their experience in *measuring length, surface* and *volume*, especially in initial education.

Exercises that are based on the use of a ruler, compasses and a protractor are always desirable, despite the possibility of using computer tools.

The great Russian mathematician Lobachevsky (N. I. Lobachevsky 1792-1856) emphasized the important role of the mathematics teaching methodology (according to Manturov, Solncev, Sorkin & Fedin, 1969). It was exactly Manturov and his associates who gave the important concepts a central position in mathematics teaching methodology, in nine points. In this paper, we will mention those concepts that are directly related to our research. The first is the concept of a boundary value on the basis of which a further one is developed – the concept of derivatives, integrals and measuring geometric quantities – *length, surface* and *volume*. The second concept is developing the *ability to represent objects and their interrelations in space*, and based on this, developing *abstract thinking in students*. If one wants to create a good basis for progressing through a spiral curriculum in mathematics teaching for the two concepts mentioned above, the main stronghold for these concepts is to be found in primary school, that is, in initial education (Đokić & Zeljić, 2017; Romano, 2009).

Address of Corresponding Author

Olivera J. Đokić, PhD, Department of Didactics of Mathematics, Faculty of Education, University of Belgrade, 11000 Belgrade, Serbia.

olivera.djokic@uf.bg.ac.rs

0000-0002-2025-3596

How to cite: Đokić, O. J. (2018). Research perspectives of initial geometry education. *Journal of Pedagogical Research, 2*(2), 102-111

Geometry teaching should not be solely based on two-dimensional forms. Moreover, it should not only deal with 'micro-space' on a textbook or workbook page. On the contrary. Initial geometry teaching should be based on a careful observation of the three-dimensional reality of the environment. Later, as students mature, the observation of the three-dimensional situation should be extended to (and at the same time enriched with) other activities, in particular – focusing on the relationship between the three-dimensional space and its two-dimensional plane representation. Namely, objects should be presented exactly as they are, i.e. the way they appear on the retina of our eyes, on a sheet of paper, or on a computer or TV screen, just as one of the greatest mathematicians, Poincaré (H. Poincaré 1854-1912), wrote in his *Science and hypothesis* (Poincaré, 1905). Dealing with the concept of space (visual, tactile, and motoric entities), Poincaré ponders upon forming the image of an object from a 'real environment' (on the retina of our eyes). In such a context, it is unnatural for geometry teaching to be reduced to the mere metric aspect, but should also include the affine properties of a plane, as well as a parallel space projection. Some students can be engaged in dealing with the central projection, at least in its initial form (Đokić, 2007; 2017).

Bearing in mind the current needs in geometry teaching, we considered that the topics of space and spatial reasoning should be part of it as key issues, as well as teaching approaches, real environment, learning through (re)discovering mathematical ideas, the role of textbooks in the initial mathematics education, and so on. Therefore, this research paper offers a general insight into our need to deal with these topics (Đokić, 2017).

2. Space and Spatial Reasoning

One of the aims of geometry teaching is developing the ability to interpret figure-related information and visual processing – *spatial visualization*, as well as a feeling about spatial shapes, properties and their interactions – *spatial orientation*. In geometry, these two spatial elements – *spatial orientation* and *spatial visualization/observation* are explained by Clements and Battista in the following way (Clements & Battista, 1992). *Spatial orientation* refers to the ability to spot the position of an object in relation to other objects, for example, finding someone's way in a building, while *spatial visualization/observation* refers to the ability to understand and visualize the consequences of changes, that is, of (imaginary) movement of objects from a two- and three-dimensional space. It implies, therefore, understanding, interpretation and a verbal description of visual figure representation.

Many researchers have discussed these spatial elements (Bruce et al., 2015; Clements & Battista, 1992; Sinclair & Bruce, 2014; 2015; Sinclair et al., 2016). Thus, the following two spatial components are proposed, which are believed to be particularly important for teaching mathematics. The first is the ability to interpret figure-related information and it includes understanding the visual representation and vocabulary that is being formed. The second is the ability of visual processing, including the manipulation and translation of visual representations and images, as well as translation of abstract relationships into visual representation. On the other hand, mathematical curricula are becoming increasingly focused on the development of the sense of space, by means of geometric instructions supported by the mathematics textbook. In reference literature, this sense of space is marked by numerous researchers as *spatial reasoning*, which we will be using in this paper as well. Therefore, for the research mathematical community we propose the first important question – the question of *the concept of space and spatial reasoning*.

3. Teaching Approaches

Evaluation of student achievement has been carried out in two directions in the last two decades (Kuzmanović & Pavlović Babić, 2011). One is formative evaluation, i.e. evaluation in the service of learning, and the other is standardized testing of knowledge, the aim of which is to obtain data that will serve the educational authorities when planning the education policy. Thus, domestic testing is carried out (what is examined is the degree of achievement of the educational standards of student achievement) (Ministarstvo prosvete, 2006; Zavod za vrednovanje kvaliteta obrazovanja i vaspitanja, 2011) and international testing (what is evaluated is the effectiveness of educational systems) (TIMSS, PISA, etc.).

The approaches to learning developed in the mid-20th century offered a conceptual framework for developing new ways (models) of evaluating student achievement. Thus, according to the cognitive theory, learning is a complex cognitive activity, and the acquisition of knowledge cannot be reduced to the accumulation of factual information and routine procedures, but it rather implies the ability to integrate a range of knowledge, skills and procedures in ways which enable efficient problem solving (Kuzmanović & Pavlović Babić, 2011). One of the main principles of the cognitive theory is students' active knowledge building, based on understanding and linking new information with previously acquired knowledge. The focus is on types or kinds of knowledge, and the purpose of assessment is not only to determine what a student knows, but also to judge how and under what conditions he/she can apply that knowledge. Therefore, there is a paradigm shift in the measuring of students' academic achievement, whereby researchers propose that it is rather drawing conclusions based on what we perceive than measuring that should be spoken about (according to Kuzmanović & Pavlović Babić, 2011). This paradigm shift treats evaluation as a system with interconnected elements of cognition (the theory of what students know and how they build competencies in certain domains), observation (tasks and situations by means of which performance data are collected), and interpretation (method of drawing conclusions based on observations).

Mathematical knowledge becomes an indispensable part of contemporary institutionalized education of any individual, regardless of the level and type of education that the individual acquires. There is more and more discussion about the direct or indirect *application of mathematical knowledge*, in various areas of human life, the application of the mathematics methodology, the mathematical way of thinking, and the different forms of gaining mathematical knowledge that are applied in everyday life. Mathematical knowledge and skills are used in many concrete situations and in everyday life and are extremely important for the intellectual development of the individual, on the one hand, and they are also important for the technological development of contemporary society, on the other. In the European Commission's paper (EACEA, 2011), one of the important tasks in the current reform of education systems is an increase in interest in mathematics and achievement in mathematics. In order to enable better preparation and implementation of appropriate changes within the educational system, international research on student achievement is of great importance. In their research, Dindyal, and Cai with associates (Cai et al., 2016; Dindyal, 2014) state that the decision to leave a great deal of space to mathematics in the international TIMSS and PISA research proves the importance of teaching mathematics for any society and individual in it. Their role is becoming more and more important and there is an increasing number of countries participating in such research. Therefore, it is necessary to analyze from both mathematical and methodological points of view the official reports of these studies on student achievement in mathematics for Serbia (Antonijević, 2007; Božić, 2012; Jelić & Đokić, 2017; Ministarstvo prosvete, 2006; Zavod za vrednovanje kvaliteta obrazovanja i vaspitanja, 2011). They fall within the ranks of international studies of multiple significance, both locally and internationally.

In the above mentioned international and national research reports, we recognize and emphasize, and also describe and methodologically analyze the teaching approaches which have been perceived as the ones providing both better effects in terms of mathematical achievement and better motivation for learning in geometry classes.

3.1. Realistic Mathematics Education

Different *teaching approaches* and their effects can be important starting points in mathematical education. One of the teaching approaches in the initial geometry teaching which aroused great research interest is based on the 'real environment' and Freudenthal's (F. Freudenthal 1905-1990) didactic phenomenology and the concept of mathematical education. The 'real environment' as a source of mathematical concepts is the basic starting point for *Realistic Mathematics Education* (RME), launched in the Netherlands at the Institute in Utrecht in the 1970s, when Freudenthal recommended linking mathematics to real situations close to children and relevant to society (Freudenthal, 1968).

3.1.1. Freudenthals' Work Continue

The process of learning mathematics should be *a guided process of discovering mathematical ideas* with the basic aim of understanding the process of mathematization rather than mastering a closed system of facts. Mathematization, according to Treffers and Freudenthal, can be vertical and horizontal (Freudenthal, 1968; Treffers, 1987). In horizontal mathematization, one moves from the realistic world into the world of mathematical symbols through a mathematical apparatus that allows solving problems. Vertical mathematization implies the movement within the world of mathematical symbols and implies a reorganization of mathematical knowledge by establishing connections between mathematical concepts and procedures. The basis of mathematization lies in the idea of finding problems and ways of solving them, which would be like (re)discovery of mathematical ideas. Realistic mathematical education uses real, everyday nonstandard, problem situations for *motivating* one to learn mathematical contents. According to Van den Heuvel-Panhuizen, they are the basis for the so-called *guided discovery of mathematical concepts or procedures* (Van den Heuvel-Panhuizen, 2002). Van den Heuvel-Panhuizen and Gravemeijer take *the problems in the context* to be the problems in which a problem situation seems to children to be experientially possible, and which is close to them (Gravemeijer, 1994; Van den Heuvel-Panhuizen, 2002). The educational process with this kind of an approach is opposite to the mechanistic approach to teaching because it encourages learning with understanding. Models play an important role in mathematics teaching, and for RME they are particularly important and their role is different. In the teaching of mathematics, a model represents the concretization of a mathematical concept. In contrast, Streefland and Treffers state in their studies that one moves in RME from a model as an approximation of an abstract phenomenon to a model as a means of solving the problem (therefore, it can also be a formal mathematical expression) (according to Treffers, 1991). In RME teaching, the tasks in the context are used primarily to gain new knowledge – *knowledge acquired in a different way* in the process of attainment of formal knowledge, and in order to recognize the possibility of *applying the knowledge*. By solving problems, students develop a mathematical apparatus and understand mathematical concepts and procedures, as opposed to a mechanistic approach which is oriented to practising procedures. The RME insists on a complex meaningful conceptualization of learning. In RME, students are not mere recipients of information, but actively participate in the teaching process, revealing the mathematical ideas themselves. Hence, it can be said that the basic characteristics of RME are the use of a context, the use of models, the active participation of students in the learning process, the interactive nature of teaching, the combination of different learning methods (Fauzan, 2002). Different activities provide children with the opportunity to establish themselves *the links between the observed objects and events and abstract ideas that explain the relationships between these objects and events*. The power of this approach to learning, according to House and Coxford, lies in the activities that arise from children's need – their interest in a particular phenomenon – and not from externally imposed isolated tasks (House & Coxford, 1995).

Many international and domestic projects, as we have presented in our previous papers, use the 'real environment' in their concepts of mathematical education as the basic starting point (Milinković, Đokić, & Dejić, 2008). These projects started in the 1990s, and most of them propose, as one of the conclusions, curricula harmonized with the philosophical and pedagogical-psychological orientation of the projects. Projects rely on constructivist theories of psychic children development (Piaget's theory and Vygotsky's theory). They start from student's intuition and previous knowledge, through a gradual formation of abstract concepts, in order to finally reach a symbolic way of expression. What connects the given projects is the context of the 'real environment' for the concepts formation – from one example to another, in a guided conversation in situations close to children's experience, to generalization and abstraction. It is worth stressing that these projects have been recognized: 1) by the good methodological approach to learning mathematical contents (as shown in the results of the international TIMSS testing) and 2) by the high achievement in solving mathematical problems (as shown by the results of the international PISA testing). Van den Heuvel-Panhuizen states that the TIMSS results show the top positions in the list belong to the countries which foster the approach to learning mathematics based on a realistic context (Van den Heuvel-Panhuizen, 1996; 2002). As a result of these projects, there were textbooks created. That is why the analysis of the reports of the mentioned research projects, their mathematical curricula and teaching approaches represent the pivotal point of many studies. Therefore, *the second* important issue for research is the *'real environment' as a source of geometric concepts and the point of knowledge application*.

Our research was also inspired by Freudenthal's speech at the Fourth International Congress of Mathematical Education. On this occasion, this leading 20th-century mathematician presented a series of problems encountered by mathematical education researchers (Freudenthal, 1983), some of them being:

I. How to use progressive schematics and formalizations in teaching students about some mathematical concept (or in a broader sense about a mathematical object);

II. How to arouse students' interest in (motivation for) mathematics;

III. How mathematical instruction is structured by level and how to use these structures for differentiated teaching;

IV. How to encourage one's thoughts about physical, mental and mathematical activities;

V. How to create a suitable context for teaching mathematization of contextual problems;

VI. Can one be taught geometry if one relies on their personal reflections and their own intuitive understanding of space?

VII. What is the role of textbooks and textbook materials, i.e. teaching materials?

Freudenthal believes that contextual learning goes in the following direction: from the 'real environment' to mathematization, and by no means the other way round. In mathematics teaching, the 'real' refers to the problem in the context, whose solution search has a certain meaning for students, just as mathematical problems do. It is recommended that even the most abstract mathematical objects should be introduced through contextually based problems. In all of this, of course, both the teacher's work and the textbooks are important, too. It is, therefore, important to consider their role in the teaching process.

Freudenthal, as it turns out, puts before us tasks for improving the teaching process in the field of mathematics teaching, taking into account all students' characteristics (age, mental, physical, intellectual). A successful completion of these tasks requires good methods (the way in which the teacher does the teaching). There are numerous recommendations on how to teach and how to learn.

In our paper, attention is further paid to learning *with understanding*, which is also discussed by Hiebert and Carpenter (1992). One form of learning with understanding is *learning by discovery* (through discovery). Through their independent activities students reveal the basic rules and principles and understand their way of creation. Bruner (1966) advocates in his research the so-called *guided learning by discovery*, that is, learning guided by the teacher, who gives instructions, recommendations, etc. Resnick and Ford (1981), as well as Mićić (1999) and others, have spoken about *learning through discovery* in the field of mathematics teaching. Therefore, an interesting research question is also the relation between learning with understanding and learning by discovery, i.e., *the third* important research question is *learning by (re)discovering mathematical ideas*.

4. The Role of Textbooks

One of the key issues for the didactic theory of mathematics and the teaching practice is the effectiveness of the organization and realization of mathematics teaching. Numerous factors are taken into account, with some of the most frequently mentioned (and analyzed) ones being teaching methods and interaction patterns, organization of teaching, and the teacher's personality. These issues were dealt with by Adler and Sfard (2016), Tarr et al., (2008), and Fan, Zhu, and Miao (2013), who emphasized two essential elements – the first one is *the curriculum* and the other is the *textbook*. The importance of the curriculum is also reflected in the fact that educational reform is generally and most often reduced to a call for the reform of curricula. Our interest is directed at the second element – often neglected – which is the *textbook* (Milinković et al., 2008). Thereby, we first have in mind the didactic-methodological curricular instructions and their implementation through a textbook, that is, we have in mind a general overview of programme activities. It can be said that there is a widespread (implicitly present) view that a *quality* textbook can and must provide adequate support in the teaching process, regardless of the chosen approach to mathematics teaching.

In didactic theory, the textbook represents a primary teaching resource; it is the basic, main and obligatory school book (Trebješanin, 2001). The textbook, therefore, potentially has a huge impact on student's knowledge and development. The textbook is supposed to provide support in the process of achieving the educational, developmental and socialization goals of formal education and teaching. Van den Heuvel-Panhuizen states that in the countries which are extremely successful in mathematical education according to international research, the textbook is one of the key reformer bearers, while the teaching approach applied is the approach of the 'real environment' (Van den Heuvel-Panhuizen, 2002).

Adler and Sfard (2016), Villani (1998), Törnroos (2005), Fan el al., (2013), and Đokić (2017) all mention in their research that there is valuable initiative for innovative textbook models in many countries. However, it has been noticed that curricular innovations are often set up in such a way as to relate to former traditional topics, thus generating repetition and lacking in comprehensive connectivity. The textbook authors are expected to avoid accumulation in the preparation of texts and to be more selective in their choice of the parts to be interrelated, as well as to be careful when interrelating them, since they must be equally clear and understandable to both teachers and students. Finally, for most innovative topics, according to the old didactic tradition, appropriate professional literature is recommended in the form of *a handbook for teachers* (which is already done in some countries) and in the form of special publications intended for students/future teachers for their training during the studies through the subject teaching methodology (with the important role of faculties in the professional development being emphasized).

We also analyzed the importance of textbooks in terms of the development of thinking. At the Fourth Congress of the European Society for Research in Mathematics Education, Gutierrez, Kuzniak and Straesser emphasized, from the perspective of studying geometrical thinking, the importance of the textbook as a crucial instrument in learning and teaching, even in the era of new technologies (Gutierrez, Kuzniak, & Straesser, 2005). Hence, for us, an important question is how to use the textbook to encourage the development of geometrical thinking, i.e. how to accommodate the child's mind to *geometrical reasoning which leads to systemic thinking* (Diezmann, Watters, & English, 2002; Đokić & Zeljić, 2017; Hershkowitz, 1998; Prenger, 2005; Steen, 1999).

Since the educational system is multifaceted, it can hardly be changed, arranged or corrected. The measure of disorder in such a system is entropy, which is highly present in the education system. A resource which can substantially influence the change of such a state of the system is the textbook. That is why *the fourth* important question for researchers is the question of *the textbook in initial mathematics teaching*.

5. Motivation for Learning

Woolfolk believes that the student can be encouraged to learn and to engage in intellectual activity only by creating a cognitive conflict, that is, by creating a situation where the student is aware of where he/she currently is in the intellectual sense, and where he/she wants to be (Woolfolk, 1998). In this way, the student is enabled to spontaneously expand, correct, link and upgrade his/her knowledge.

The motivation for learning mathematics, both external and internal, is seen as a significant component in achieving the goal of *learning with understanding*, which Skemp (Skemp, 1986) spoke about. A serious question is how to encourage most students to feel interested in geometry and show better achievement in this field of mathematics. According to Glejzer, it is necessary to change the understanding of the leading goals of teaching geometry at school (Glejzer, 1997). However, according to how they are organized and what they contain, it is noticeable that the main goal of teaching geometry, including the initial teaching of geometry, in school textbooks and traditional teaching is developing logical thinking among students. However, at an early stage of education, this turns out to be unachievable.

Since learning through (re)discovery corresponds to the nature of the learning process, and to the nature of science (not being mere transfer and adopting of ready-made knowledge, but active participation of students in building knowledge based on problem solving), it is interesting for our research to find out how learning through discovery stimulates the learning of geometric contents. That is why *the fifth* important question for researchers is *motivation for learning*.

What is the situation with textbooks like in Serbia? In the conducted analysis of innovative textbooks, which appeared for the first three grades of initial mathematics teaching in the period from 2003 to 2006, it was observed that *memorization* was the dominant type of intellectual activation of students (Korać et al., 2007). It is fostered in various ways, and most often a) by introducing new concepts in a "ready-made" form; and b) by long-term insisting on the application of matrixes in solving tasks (mechanistic approach in problem solving without looking into the mathematical concept/procedure).

A different model of teaching was also noticed. New concepts are introduced through problem situations, where *problematization* is one of the ways of acquiring new knowledge or improving the existing one at a more complex level that allows students to gain knowledge independently, often guided by teachers' recommendations, advice, etc. In this way, students get to understand and comprehend the mathematical procedure themselves.

Textbooks often contain contents that are *associated* with the child's experience, with the concepts and contents from other subjects, and are thus used as a context (framework) in which mathematical requirements are placed. There is a visible effort to place as many tasks as possible in a realistic life context. However, these attempts turn out to futile at times. The problem is, namely, in the choice of situations from everyday life (the task frame) which give themselves to mathematical language expressions and representations through mathematical symbols. This aspect of teaching mathematics is increasingly present in newer textbooks, but insufficiently tested (Đokić, 2017).

6. Some Research Recommendations

Our recommendation is to conduct research studies that focus on teaching approaches in initial mathematics teaching, especially those based on the 'real-life'. In Serbia, this approach has been investigated through research for the first time (Đokić, 2014a; Djokic, 2014b; Đokić, 2015; Đokić, 2017). An empirical study of the 'real environment' teaching approach has been done, supported by an appropriate innovative model of early mathematics teaching textbooks, and the effects on student achievement and motivation for learning have been monitored. Students in the first three grades of primary school were not included in the survey, as the classic teaching approach does not exclude the 'real environment'. On the contrary, it contains it as the basic source of mathematical concepts in the process of their formation. As the age increases, students get increasingly 'accustomed' to the world of abstract ideas and 'detached' from the 'real environment' as a source of concepts (Đokić, 2007). We were interested in what effects the 'return' to the 'real environment' would have in mathematics teaching.

References

Adler, J., & Sfard, A. (2016). *Research for educational change - transforming researchers' insights into improvement in mathematics teaching and learning*. London and New York: Routledge, Taylor & Francis Group.

Antonijević, R. (2007). Student attainment research areas: TIMSS 2007 and PISA 2006. *Nastava i vaspitanje*, *56*(4), 373-386. [Антонијевић, Р. (2007). Области истраживања постигнућа ученика: TIMSS 2007 и PISA 2006. *Настава и васпитање*, *56*(4), 373-386.]

Božić, M. (2012). PISA Tests and What Shall We Do. *Nastava matematike*, *LVII*(1-2), 1-11. [Божић, М. (2012). ПИСА тестови и шта нам је чинити. *Настава математике*, *LVII*(1-2), 1-11.]

Bruce, C. D., Sinclair, N., Moss, J., Hawes, Z., and Caswell, B. (2015). Spatializing the curriculum. In B. Davis and Spatial Reasoning Study Group (Eds.), *Spatial Reasoning in the Early Years. Principles, Assertions, and Speculations* (pp. 85-106). London and New York: Routledge, Taylor & Francis Group.

Bruner, J.S. (1966). *Toward a theory of instruction*. Cambridge, MA, US: Harvard University Press.

Cai, J., Mok, I. A. C., Reddy, V., & Stacey, K. (2016). International comparative studies in mathematics: Lessons for improving students' learning. In G. Kaiser, (Series Ed.), *Part of the series ICME-13* (pp. 1-36). Hamburg: Springer Open. doi: 10.1007/978-3-319-42414-9_1

Clements, D.H. & Battista, M.T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 420-464). New York: Macmillan Publishing Copmany.

Diezmann, C. M., Watters, J., & English, L. (2002). Teacher behaviours that influence young children's reasoning. In A. D. Cockburn, & E. Nardi (Eds.). *Learning from Learners*. Proceedings 27th Annual Conference of the International Group for the Psychology of Mathematics, Norwich: University of East Anglia, UK, July 21-26, 2002 (pp. 289-296). Norwick, UK.

Dindyal, J. (2014). International Comparative Studies in Mathematics: An Overview. In S. **Lerman** (Ed.), *Encyclopedia of Mathematics Education* (pp. 320-325). Dordrecht, Netherland: Springer Reference. doi: 10.1007/978-94-007-4978-8

Đokić, O. (2007). *Formation of Concept of Line in Elementary School Geometry*. Belgrade: Teacher Education Faculty. [Ђокић, О. (2007). *Појам линије у почетној настави геометрије*. Београд: Учитељски факултет.]

Đokić, O. (2014a). Realistic mathematics education in teaching and learning elementary geometry. *Teaching Innovations*, *27*(2), 7-21. doi:10.5937/inovacije1402007D [Ђокић, О. (2014a). Реално окружење у почетној настави геометрије. *Иновације у настави*, *27*(2), 7-21.]

Djokic, O. (2014b). RME as a Teaching Approach – A case study of elementary geometry in Serbian innovative 4th-grade textbook. In K. Jones, C. Bokhove, & L. Fan (Eds.), *Mathematics Textbook Research and Development.* Proceedings of the International Conference on Mathematics Textbook Research and Development (ICMT-2014), University of Southampton, Southampton, GB, 29-31 July 2014 (pp. 203-208). Southampton: University of Southampton.

Đokić, O. (2015). The effects of RME and innovative textbook model on 4th grade pupils' reasoning in geometry. In J. Novotná, & H. Moraová (Eds.), *Developing mathematical language and reasoning*. Proceedings of International Symposium Elementary Mathematics Teaching SEMT-2015, Charles University, Prague, Czech Republic, 16-21 August 2015 (pp. 107-117). Prague, Czech Republic: Charls University.

Đokić, O. (2017). *Realistic mathematics education in teaching and learning elementary geometry*. Belgrade: Teacher Education Faculty. [Ђокић, О. (2017). *Реално окружење у почетној настави геометрије*. Београд: Учитељски факултет.]

Đokić, O. & Zeljić, M. (2017). Theoretical frameworks of development geometrical thinking according to Van Hiele, Fischbein and Houdement-Kuzniak. *Teme*, *XLI*(3), 623-637. doi: 10.22190/TEME1703623D [Ђокић, О. и Зељић, М. (2017). Теорије развоја геометријског мишљења према Ван Хилу, Фишбајну и Удемон-Кузниаку. *Теме*, *XLI*(3), 623-637.]

Education, Audiovisual and Culture Executive Agency [EACEA]. (2011). *Mathematics education in Europe: common challenges and national policies*. Retrieved from https://doi.org/10.2797/72660

Fan, L., Zhu, Y. & Miao, Z. (2013). Textbook research in mathematics education: development status and directions. *ZDM Mathematics Education*. *45*(5), 633-646. doi: 10.1007/s11858-013-0539-x

Fauzan, A. (2002). *Applying Realistic Mathematics Education (RME) in Teaching Geometry in Indonesian primary schools*. Enschede: Universiteit Twente.

Freudenthal, H. (1968). Why to teach mathematics so as to be useful. *Educational Studies in Mathematics*, *1*(1-2), 3-8.

Freudenthal, H. (1983). Major problems of mathematics education. In M. Zweng, T. Green, J. Kilpatrick, H. Pollak, & M. Suydam (Eds.), *Fourth International Congress on Mathematical Education.* Proceeedings of the Fourth International Congress on Mathematical Education, Berkeley, California, USA, August 10-16, 1980 (pp. 1-7). Boston, USA: Birkähuser Boston Inc.

Glejzer, G. D. (1997). *School Geometry*. *Nastava matematike*, *42*(1-2), 1-6. [Глејзер, Г. Д. (1997). Геометрија у школи. *Настава математике*, *42*(1-2), 1-6.]

Gravemeijer, K.P.E. (1994). *Developing realistic mathematics education*. Utrecht, Netherlands: CD-β Press / Freudenthal Institute.

Gutierrez, A., Kuzniak, A., & Straesser, R. (2005). Research on Geometrical Thinking. Retrieved from CERME 4 conference, WORKING GROUP 7 website: http://www.mathematik.uni-dortmund.de/~erme/CERME4/CERME4_WG7.pdf

Hershkowitz, R. (1998). About reasoning in geometry. In: C. Mammana and V. Villani (Eds.), *Perspectives on the Teaching of Geometry for the 21st Century* - An ICMI Study (pp. 29-36). Springer Netherlands: Kluwer Academic Publishers.

Hiebert, J. & Carpenter, T. (1992). Learning and Teaching with Understanding. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 65-97). New York: Macmillan Publishing Copmany.

House, P. A. & Coxford, A. F. (Eds.) (1995). *Connectig Mathematics across the Curriculum* - NCTM Yearbook. Reston, Virginia: National Council of Teachers of Mathematics.

Jelić, M. & Đokić, O. (2017). Towards coherent structure of Mathematics textbooks – analysis of textbooks according to structural blocks of TIMSS research. *Teaching Innovations*, *30*(1), 67-81. DOI: 10.5937/inovacije1701067J [Јелић, М. и Ђокић, О. (2017). Ка кохерентној структури уџбеника математике – анализа уџбеника према структурним блоковима ТИМСС истраживања. *Иновације у настави*, *30*(1), 67-81.]

Korać, N. et al., (2007). *The Quality of Textbooks for Younger School Age*. Belgrade: Institute of Psychology. [Кораћ, Н. и сар. (2007). *Квалитет уџбеника за млађи школски узраст*. Београд: Институт за психологију.]

Kuzmanović, D. & Pavlović Babić, D. (2011). Approaches to Assessment of Student Academic Achievements: A Critical Review. *Zbornik Instituta za pedagoška istraživanja*, *43*(1), 63-85. [Кузмановић, Д. и Павловић Бабић, Д. (2011). Приступи процењивању образовних постигнућа ученика - критички осврт. *Зборник Института за педагошка истраживања*, *43*(1), 63-85.]

Manturov, O.V., Solncev, J.K., Sorkin, J.I., & Fedin, N.G. (1969). *A Dictionary of Mathematical Terms with Interpretations*. Belgrade: Naučna knjiga. [Manturov, O.V., Solncev, J.K., Sorkin, J.I., Fedin, N.G. (1969). *Rečnik matematičkih termina sa tumačenjima*. Beograd: Naučna knjiga.]

Mićić, V. (2005). Discovery learning - probably a new approach. *Nastava matematike*, *L*(4), 13-21. [Мићић, В. (2005). Учење откривањем – можда нови приступ. *Настава математике*, *L*(4), 13-21.]

Milinković, J., Đokić, O. & Dejić, M. (2008). Textbook Model as the Basis for Active Learning in Teaching Mathematics. *Teaching Innovations*, *21*(1), 70–79. [Милинковић, Ј., Ђокић, О. и Дејић, М. (2008). Модел уџбеника као основе активног учења у настави математике. *Иновације у настави*, *21*(1), 70–79.]

Ministarstvo prosvete (2006). *Third Grade Students' Academic Achievement – The National 2004 Testing.* Belgrade, Serbia: Ministarstvo prosvete i sporta Republike Srbije i Zavod za vrednovanje kvaliteta obrazovanja i vaspitanja. [Министарство просвете (2006). *Образовна постигнућа ученика трећег разреда - национално тестирање 2004*. Београд, Србија: Министарство просвете и спорта Републике Србије и Завод за вредновање квалитета образовања и васпитања.]

Poincaré, H. (1905). *Science and hypothesis*. London and Newcastle-on-Cyne: The Walter Scott Publishing Co., Ltd New York.

Prenger, J. (2005). *Construction of meaning in the mathematics classroom*. Unpublished PhD thesis, University of Groningen, Groningen, Netherlands.

Resnick, L.B. & Ford, W.W. (1981). *The psychology of mathematics instruction*. Hillsdale, New Jersey: Lawrence Erlbaum Associates Publishers.

Romano, D. A. (2009). About Geometrical Thinking. *Nastava matematike*, *54*(2-3), 1-11. [Романо, Д. А. (2009). О геометријском мишљењу. *Настава математике*, *54*(2-3), 1-11.]

Sinclair, N., Bartolini Bussi, M. G., Villieres, M., Jones, K., Kortenkamp, U., Leung, A., and Owens, K. (2016). Recent Research on geometry education: an ICME-13 survey team report. *ZDM Mathematics Education*, *48*(5), 691–719. doi 10.1007/s11858-016-0796-6

Sinclair, N., & Bruce, C. D. (2015). New opportunities in geometry education at the primary school. *ZDM Mathematics Education*, *47*(3), 319–329. doi: 10.1007/s11858-015-0693-4

Sinclair, N., & Bruce, C. D. (2014). Spatial Reasoning for Young Learners. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.). Proceedings of the Joint Meeting of PME 38 and PME-NA 36, Vancouver, Canada, July 15-20, 2014 (Vol. 1, pp. 173-203). Vancouver, Canada: PME.

Skemp, R. (1986). *The Psychology of Learning Mathematics*. London: Penguin Books.

Steen, L. A. (1999). The concluding chapter. In V. L. Stiff (Ed.), *Developing Mathematical Reasoning in Grades K-12* - NCTM's Yearbook (pp. 270-285). Reston, VA: National Council of Teachers of Mathematics.

Tarr, J. E., Reys, R. E., Reys, B. J., Chavez, O., Shih, J. & Osterlind, S. J. (2008). The Impact of Middle-Grades Mathematics Curricula and Classroom Learning Environment of Student Achievement. *Journal for Research in Mathematics Education*, *39*(3), 247-280.

Törnroos, J. (2005). Mathematics textbook, opportunity to learn and student achievement. *Studies in Educational Evaluation*, *31*(4), 315–327. doi:10.1016/j.stueduc.2005.11.005

Trebješanin, B. (2001). Types and levels of knowledge in a contemporary textbook. In B. Trebješanin and D. Lazarević, (Eds.), *A Contemporary Elementary School Textbook* (pp. 69-78). Belgrade: Zavod za udžbenike i nastavna sredstva. [Требјешанин, Б. (2001). Врсте и нивои знања у савременом уџбенику. У Б. Требјешанин и Д. Лазаревић (ур.), *Савремени основношколски уџбеник* (стр. 69-78). Београд: Завод за уџбенике и наставна средства.]

Treffers, A. (1987). *Three dimensions: A model of goal and theory description in mathematics education*. Dordrecht, Netherlands: Reidel.

Treffers, A. (1991). Realistic mathematics education in the Netherlands 1980-1990. In L. Streefland (Ed.), *Realistic Mathematics Education in Primary School* (pp. 11-20). Utrecht Netherlands: CD-β Press / Freudenthal Institute, Utrecht University.

Van den Heuvel-Panhuizen, M. (1996). *Assessment and realistic mathematics education*. Utrecht, Netherlands: CD-β Press / Freudenthal Institute, Utrecht University.

Van den Heuvel-Panhuizen, M. (2002). Realistic mathematics education as work in progress. In F. L. Lin (Ed.), *Common Sense in Mathematics Education*. Proceedings of 2001 The Netherlands and Taiwan Conference on Mathematics Education, Taipei, Taiwan, 19-23 November 2001 (pp. 1-43). Taipei, Taiwan: National Taiwan Normal University.

Villani, V. (1998). The way ahead. In C. Mammana and V. Villani (Eds.), *Perspectives on the Teaching of Geometry for the 21st Century* (pp. 319-327). Springer Netherlands: Kluwer Academic Publishers.

Woolfolk, A.E. (1998). *Educational Psychology*. Boston, MA: Allyn & Bacon.

Zavod za vrednovanje kvaliteta obrazovanja i vaspitanja (2011). *General Achievement Standards - Educational Standards for the End of the First Cycle of Compulsory Education for the Subject Mathematics.* Belgrade: Zavod za vrednovanje kvaliteta obrazovanja i vaspitanja. [Завод за вредновање квалитета образовања и васпитања (2011). *Општи стандарди постигнућа – образовни стандарди за крај првог циклуса обавезног образовања за предмет Математика*. Београд: Завод за вредновање квалитета образовања и васпитања.]