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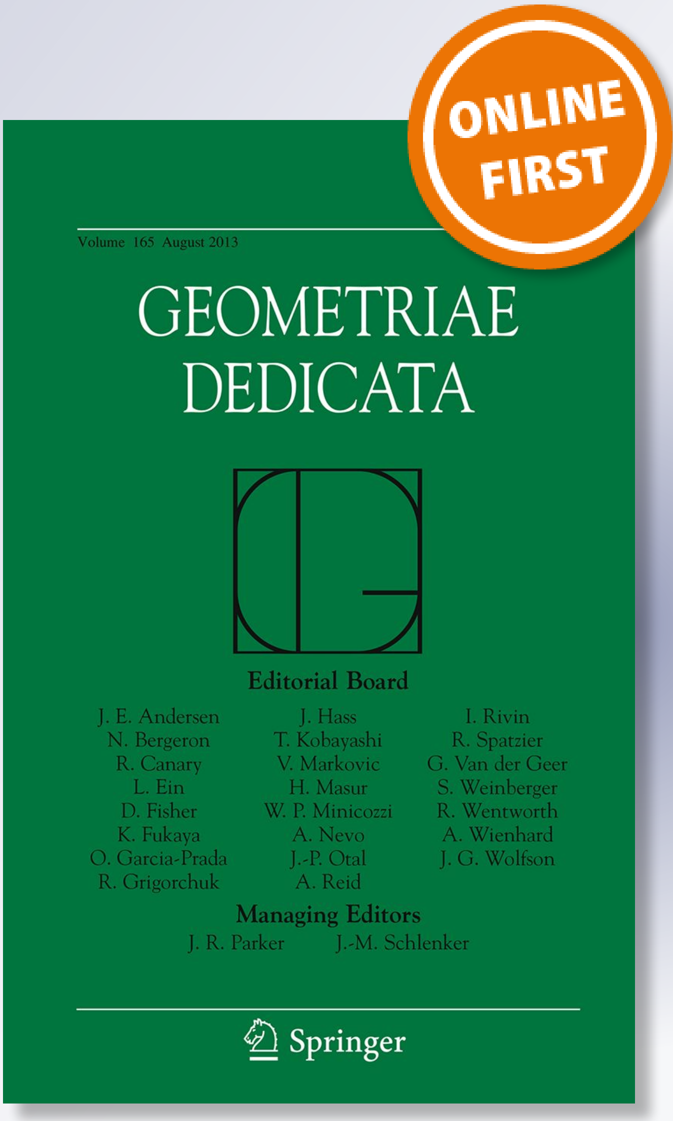
Geometriae Dedicata

ISSN 0046-5755

Geom Dedicata

DOI 10.1007/s10711-019-00423-7



Springer

Geometriae Dedicata
https://doi.org/10.1007/s10711-019-00423-7

ORIGINAL PAPER

Open subgroups of the automorphism group of a right-angled building

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Received: 15 February 2018 / Accepted: 4 January 2019

Abstract

We study the group of type-preserving automorphisms of a right-angled building, in particular when the building is locally finite. Our aim is to characterize the proper open subgroups as the finite index closed subgroups of the stabilizers of proper residues. One of the main tools is the new notion of firm elements in a right-angled Coxeter group, which are those elements for which the final letter in each reduced representation is the same. We also introduce the related notions of firmness for arbitrary elements of such a Coxeter group and n-flexibility of chambers in a right-angled building. These notions and their properties are used to determine the set of chambers fixed by the fixator of a ball. Our main result is obtained by combining these facts with ideas by Pierre-Emmanuel Caprace and Timothée Marquis in the context of Kac–Moody groups over finite fields, where we had to replace the notion of root groups by a new notion of root wing groups.

Keywords Right-angled buildings · Right-angled Coxeter groups · Totally disconnected locally compact groups · open subgroups

Mathematics Subject Classification 51E24 · 22D05 · 20F65

1 Introduction

A Coxeter group is right-angled if the entries of its Coxeter matrix are all equal to 1, 2 or ∞ (see Definition 2.1 below for more details). A right-angled building is a building for which the underlying Coxeter group is right-angled. The most prominent examples of right-angled buildings are trees. To some extent, the combinatorics of right-angled Coxeter groups and right-angled buildings behave like the combinatorics of trees, but in a more complicated and therefore in many aspects more interesting fashion.

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Published online: 14 January 2019

Right-angled buildings have received attention from very different perspectives. One of the earlier motivations for their study was the connection with lattices, starting with the important contributions of Burger and Mozes [2] on lattices in products of trees; see, for instance [8,12, 16,17,19]. On the other hand, the automorphism groups of locally finite right-angled buildings are totally disconnected locally compact (t.d.l.c.) groups, and their full automorphism group was shown to be an abstractly simple group by Pierre-Emmanuel Caprace [5], making these groups valuable in the study of t.d.l.c. groups. Caprace's work also highlighted important combinatorial aspects of right-angled buildings; in particular, his study of parallel residues and his notion of wings (see Definition 3.10 below) are fundamental tools. From this point of view, we have, in a joint work with Koen Struyve, introduced and investigated universal groups for right-angled buildings; see [10]. Another interesting aspect is their connection with spaces of non-positive curvature, beginning with the work of Bourdon [4] and including the profound result of Mike Davis that buildings (with the appropriate metric realization) are always CAT(0) [9]. More recently, Andreas Baudisch, Amador Martin-Pizarro and Martin Ziegler have studied right-angled buildings from a model-theoretic point of view; see [3].

In this paper, we continue the study of right-angled buildings in a combinatorial and topological fashion. In particular, we introduce some new tools in right-angled Coxeter groups and we study the (full) automorphism group of right-angled buildings. Our main goal is to characterize the proper open subgroups of the automorphism group of a locally finite semi-regular right-angled building as the closed finite index subgroups of the stabilizer of a proper residue. We prove our result in Theorem 4.28 below.

Main Theorem *Let Δ be a thick irreducible semi-regular locally finite right-angled building of rank at least* 2*. Then any proper open subgroup of* $\mathrm{Aut}(\Delta)$ *is contained with finite index in the stabilizer in* $\mathrm{Aut}(\Delta)$ *of a proper residue.*

The first tool we introduce is the notion of *firm elements* in a right-angled Coxeter group: these are the elements with the property that every possible reduced representation of that element ends with the same letter (see Definition 2.10 below), i.e., the last letter cannot be moved away by elementary operations. If an element of the Coxeter group is not firm, then we define its *firmness* as the maximal length of a firm prefix.

This notion will be used to define the concepts of *firm chambers* in a right-angled building and of *n-flexibility* of chambers with respect to another chamber; this then leads to the notion of the *n-flex* of a given chamber. See Definition 3.13 below.

A second new tool is the concept of a *root wing group*, which we define in Definition 4.6. Strictly speaking, this is not a new definition since the root wing groups are defined as wing fixators, and as such they already appear in the work of Caprace [5]. However, we associate such a group to each *root* (i.e., a half-apartment) of the building, and we explore the fact that they behave very much like root subgroups in groups of a more algebraic nature, such as automorphism groups of Moufang spherical buildings or Kac–Moody groups.

Outline of the paper In Sect. 2, we provide the necessary tools for right-angled Coxeter groups. In Sect. 2.1, we recall the notion of a poset $\prec_w$ that we can associate to any word w in the generators, introduced in [10]. Section 2.2 introduces the concepts of firm elements and the firmness of elements in a right-angled Coxeter group. Our main result in that section is the fact that long elements cannot have a very low firmness; see Theorem 2.18.

Section 3 collects combinatorial facts about right-angled buildings. After recalling the important notions of parallel residues and wings, due to Caprace [5], in Sect. 3.1, we proceed in Sect. 3.2 to introduce the notion of chambers that are n-flexible with respect to another chamber and the notion of the square closure of a set of chambers (which is based on results from [10]); see Definitions 3.13 and 3.16. Our main result in Sect. 3 is Theorem 3.17, showing that the square closure of a ball of radius n around a chamber c_0 is precisely the set of chambers that are n-flexible with respect to c_0.

In Sect. 4, we study the automorphism group of a semi-regular right-angled building. We begin with a short Sect. 4.1 that uses the results of the previous sections to show that the set of chambers fixed by a ball fixator is bounded; see Theorem 4.4. In Sect. 4.2, we associate a root wing group U_α to each root (Definition 4.6), we show that U_α acts transitively on the set of apartments through α (Proposition 4.7) and we adopt some facts from [6] to the setting of root wing groups.

We then continue towards our characterization of the open subgroups of the full automorphism group of a semi-regular locally finite right-angled building. Our final result is Theorem 4.28 showing that every proper open subgroup is a finite index subgroup of the stabilizer of a proper residue. We follow, to a very large extent, the strategy taken by Pierre-Emmanuel Caprace and Timothée Marquis [6] in their study of open subgroups of Kac–Moody groups over finite fields. In particular, we show that an open subgroup of $\mathrm{Aut}(\Delta)$ contains sufficiently many root wing groups, and much of the subtleties of the proof go into determining precisely the types of the root groups contained in the open subgroup; this will, in turn, pin down the residue, the stabilizer of which contains the given open subgroup as a finite index subgroup.

In the final Sect. 5, we mention two applications of our main theorem. The first is a rather immediate corollary, namely the fact that the automorphism group of a semi-regular locally finite right-angled building is a Noetherian group; see Proposition 5.3. The second application shows that every open subgroup of the automorphism group is the reduced envelope of a cyclic subgroup; see Proposition 5.6.

2 Right-angled Coxeter groups

We begin by recalling some basic definitions and facts about Coxeter groups.

Definition 2.1 (i) A *Coxeter group* is a group W with generating set $S = \{s_1, \ldots, s_n\}$ and with presentation

$$W = \langle s \in S \mid (st)^{m_{st}} = 1 \rangle$$

where $m_{ss} = 1$ for all $s \in S$ and $m_{st} = m_{ts} \geq 2$ for all $i \neq j$. It is allowed that $m_{st} = \infty$, in which case the relation involving st is omitted. The pair (W, S) is called a *Coxeter system* of *rank* n. The matrix $M = (m_{s_i s_j})$ is called the *Coxeter matrix* of (W, S). The Coxeter matrix is often conveniently encoded by its *Coxeter diagram*, which is a labeled graph with vertex set S where two vertices are joined by an edge labeled m_{st} if and only if $m_{st} \geq 3$.

(ii) A Coxeter system (W, S) is called *right-angled* if all entries of the Coxeter matrix are 1, 2 or ∞. In this case, we call the Coxeter diagram Σ of W a *right-angled Coxeter diagram*; all its edges have label ∞.

Definition 2.2 Let (W, S) be a Coxeter system and let $J \subseteq S$.

(i) We define $W_J := \langle s \mid s \in J \rangle \leq W$ and we call this a *standard parabolic subgroup* of W. It is itself a Coxeter group, with Coxeter system (W_J, J). Any conjugate of a standard parabolic subgroup W_J is called a *parabolic subgroup* of W.

(ii) The subset $J \subseteq S$ is called a *spherical subset* if W_J is finite. When (W, S) is right-angled, J is spherical if and only if $|st| \leq 2$ for all $s, t \in J$.
(iii) The subset $J \subseteq S$ is called *essential* if each irreducible component of J is non-spherical. In general, the union J_0 of all irreducible non-spherical components of J is called the *essential component* of J.

If P is a parabolic subgroup of W conjugate to some W_J, then the *essential component* P_0 of P is the corresponding conjugate of W_{J_0}, where J_0 is the essential component of J. Observe that P_0 has finite index in P.
(iv) Let $E \subseteq W$. We define the *parabolic closure* of E, denoted by $\mathrm{Pc}(E)$, as the smallest parabolic subgroup of W containing E.

Lemma 2.3 ([6, Lemma 2.4]) *Let $H_1 \leq H_2$ be subgroups of W. If H_1 has finite index in H_2, then* $\mathrm{Pc}(H_1)$ *has finite index in* $\mathrm{Pc}(H_2)$.

2.1 A poset of reduced words

Let $\Sigma = (W, S)$ be a right-angled Coxeter system and let M_S be the free monoid over S, the elements of which we refer to as *words*. Notice that there is an obvious map $M_S \to W$ denoted by $w \mapsto \overline{w}$; if $w \in M_S$ is a word, then its image $\overline{w}$ under this map is called the element *represented by* w, and the word w is called a *representation of* $\overline{w}$. For $w_1, w_2 \in M_S$, we write $w_1 \sim w_2$ when $\overline{w_1} = \overline{w_2}$. By some slight abuse of notation, we also say that w_2 is a representation of w_1 (rather than a representation of $\overline{w_1}$).

Definition 2.4 A Σ-*elementary operation* on a word $w \in M_S$ is an operation of one of the following two types:

(1) Delete a subword of the form ss, with $s \in S$.
(2) Replace a subword st by ts if $m_{st} = 2$.

A word $w \in M_S$ is called *reduced* (with respect to Σ) if it cannot be shortened by a sequence of Σ-elementary operations.

Clearly, applying elementary operations on a word w does not alter its value in W. Conversely, if $w_1 \sim w_2$ for two words $w_1, w_2 \in M_S$, then w_1 can be transformed into w_2 by a sequence of Σ-elementary operations. The number of letters in a reduced representation of $\overline{w} \in W$ is called the *length* of $\overline{w}$ and is denoted by $l(\overline{w})$. Tits proved in [18] (for arbitrary Coxeter systems) that two *reduced* words represent the same element of W if and only if one can be obtained from the other by a sequence of elementary operations of type (2) (or rather its generalization to all values for m_{st}).

Definition 2.5 Let $w = s_1 s_2 \cdots s_\ell \in M_S$. If $\sigma \in \mathrm{Sym}(\ell)$, then we let $\sigma.w$ be the word obtained by permuting the letters in w according to the permutation σ, i.e.,

$$\sigma.w := s_{\sigma(1)} s_{\sigma(2)} \cdots s_{\sigma(\ell)}.$$

In particular, if w' is obtained from w by applying an elementary operation of type (2) replacing $s_i s_{i+1}$ by $s_{i+1} s_i$, then $\sigma.w = w'$ for $\sigma = (i\ i+1) \in \mathrm{Sym}(\ell)$. In this case, s_i and s_{i+1} commute and we call $\sigma = (i\ i+1)$ a w-*elementary transposition*.

In this way, we can associate an elementary transposition to each Σ-elementary operation of type (2). It follows that two reduced words w and w' represent the same element of W if and only if

$$w' = (\sigma_k \cdots \sigma_1).w, \text{ where each } \sigma_i \text{ is a}$$
$$(\sigma_{i-1} \cdots \sigma_1).w\text{-elementary transposition,}$$

i.e., if w' is obtained from w by a sequence of elementary transpositions.

Definition 2.6 If $w \in M_S$ is a reduced word of length ℓ, then we define

$$\mathrm{Rep}(w) := \{\sigma \in \mathrm{Sym}(\ell) \mid \sigma = \sigma_k \cdots \sigma_1, \text{ where each } \sigma_i \text{ is a } (\sigma_{i-1} \cdots \sigma_1).w\text{-elementary transposition}\}.$$

In other words, the set $\mathrm{Rep}(w)$ consists of the permutations of ℓ letters which give rise to reduced representations of w.

We now define a partial order $\prec_w$ on the letters of a reduced word w in M_S with respect to Σ.

Definition 2.7 ([10, Definition 2.6]) Let $w = s_1 \cdots s_\ell$ be a reduced word of length ℓ in M_S and let $I_w = \{1, \ldots, \ell\}$. We define a partial order "$\prec_w$" on I_w as follows:

$$i \succ_w j \iff \sigma(i) < \sigma(j) \text{ for all } \sigma \in \mathrm{Rep}(w).$$

Note that $i \succ_w j$ implies that $i < j$. As a mnemonic, one can regard $i \succ_w j$ as "$i \to j$", i.e., the generator s_j comes always after the generator s_i regardless of the reduced representation of w.

We point out a couple of basic but enlightening consequences of the definition of this partial order.

Observation 2.8 Let $w = s_1 \cdots s_i \cdots s_j \cdots s_\ell$ be a reduced word in M_S with respect to a right-angled Coxeter diagram Σ.

(i) If $|s_i s_j| = \infty$, then $i \succ_w j$.

The converse is not true. Indeed, suppose there is $i < k < j$ such that $|s_i s_k| = \infty$ and $|s_k s_j| = \infty$. Then $i \succ_w j$, independently of whether $|s_i s_j| = 2$ or ∞.

(ii) If $i \not\succ_w j$, then by (i), it follows that $|s_i s_j| = 2$ and, moreover, for each $k \in \{i + 1, \ldots, j - 1\}$, either $|s_i s_k| = 2$ or $|s_k s_j| = 2$ (or both).

(iii) On the other hand, if s_j and s_{j+1} are consecutive letters in w, then $|s_j s_{j+1}| = \infty$ if and only if $j \succ_w j + 1$.

Lemma 2.9 ([10, Lemma 2.8]) *Let $w = w_1 \cdot s_i \cdots s_j \cdot w_2 \in M_S$ be a reduced word. If $i \not\succ_w j$, then there exist two reduced representations of w of the form*

$$w_1 \cdots s_i s_j \cdots w_2 \quad \textit{and} \quad w_1 \cdots s_j s_i \cdots w_2,$$

i.e., the positions of s_i and s_j can be exchanged using only elementary operations on the generators $\{s_i, s_{i+1}, \ldots, s_{j-1}, s_j\}$, without changing the prefix w_1 and the suffix w_2.

2.2 Firm elements of right-angled Coxeter groups

In this section we define firm elements in a right-angled Coxeter group W and we introduce the concept of firmness to measure "how firm" an arbitrary elements of W is. This concept will be used over and over throughout the paper. See, in particular, Definition 3.13, Theorems 3.17, 4.4 and Proposition 4.7. Our main result in this section is Theorem 2.18, showing that the firmness of elements cannot drop below a certain value once they become sufficiently long.

Definition 2.10 Let $\overline{w} \in W$ be represented by some reduced word $w = s_1 \cdots s_\ell \in M_S$.

(i) We say that $\overline{w}$ is *firm* if $i \succ_w \ell$ for all $i \in \{1, \ldots, \ell - 1\}$. In other words, $\overline{w}$ is firm if its final letter s_ℓ is in the final position in each possible reduced representation of $\overline{w}$. Equivalently, $\overline{w}$ is firm if and only if there is a unique $r \in S$ such that $l(\overline{w}r) < l(\overline{w})$.

(ii) Let $F^\#(\overline{w})$ be the largest k such that $\overline{w}$ can be represented by a reduced word in the form

$$s_1 \cdots s_k t_{k+1} \cdots t_\ell, \text{ with } s_1 \cdots s_k \text{ firm.}$$

We call $F^\#(\overline{w})$ the *firmness* of $\overline{w}$. We will also use the notation $F^\#(w) := F^\#(\overline{w})$.

Lemma 2.11 *Let $w = s_1 \cdots s_k t_{k+1} \cdots t_\ell$ be a reduced word such that $s_1 \cdots s_k$ is firm and $F^\#(w) = k$. Then*

(i) $|s_k t_i| = 2$ *for all* $i \in \{k + 1, \ldots, \ell\}$.
(ii) $i \succ_w k$ *for all* $i \in \{1, \ldots, k - 1\}$.
(iii) *Let* $r \in S$. *If* $l(\overline{w}r) > l(\overline{w})$, *then* $F^\#(wr) \geq F^\#(w)$.

Proof (i) Assume the contrary and let j be minimal such that $|s_k t_j| = \infty$. Using elementary operations to swap t_j to the left in w as much as possible, we rewrite

$$w \sim s_1 \cdots s_k t'_1 \cdots t'_p t_j \cdots$$

as a word with $s_1 \cdots s_k t'_1 \cdots t'_p t_j$ firm, which is a contradiction to the maximality of k.
(ii) The fact that the prefix $p = s_1 \cdots s_k$ is firm tells us that $i \succ_p k$ for all $i \in \{1, \ldots, k-1\}$. By Lemma 2.9, this implies that also $i \succ_w k$ for all $i \in \{1, \ldots, k - 1\}$.
(iii) Since $l(\overline{w}r) > l(\overline{w})$, firm prefixes of w are also firm prefixes of wr, hence the result. ☐

The following definition will be a useful tool to identify which letters of the word appear in a firm subword.

Definition 2.12 Let $w = s_1 \cdots s_\ell \in M_S$ be a reduced word and consider the poset $(I_w, \prec_w)$ as in Definition 2.7. For any $i \in \{1, \ldots, \ell\}$, we define

$$I_w(i) = \left\{ j \in \{1, \ldots, \ell\} \mid j \succ_w i \right\}.$$

In words, $I_w(i)$ is the set of indices j such that s_j comes at the left of s_i in any reduced representation of the element $w \in W$.

Observation 2.13 Let $w = s_1 \cdots s_\ell \in M_S$ be a reduced word.

(i) Let $i \in \{1, \ldots, \ell\}$ and write $I_w(i) = \{j_1, \ldots, j_k\}$ with $j_p < j_{p+1}$ for all p. Then we can perform elementary operations on w so that

$$w \sim s_{j_1} \cdots s_{j_k} s_i t_1 \cdots t_q$$

and the word $s_{j_1} \cdots s_{j_k} s_i$ is firm.

In particular, if $I_w(i) = \emptyset$, then we can rewrite w as $s_i w_1$.
(ii) If $i \succ_w j$, then $I_w(i) \subsetneq I_w(j)$.
(iii) It follows from (i) that $F^\#(w) = \max_{i \in \{1,\ldots,\ell\}} |I_w(i)| + 1$.

Remark 2.14 If the Coxeter system (W, S) is spherical, then $F^\#(\overline{w}) = 1$ for all $\overline{w} \in W$. Indeed, as each pair of distinct generators commute, we always have $I_w(i) = \emptyset$.

The next definition will allow us to deal with possibly infinite words.

Definition 2.15 (i) A (finite or infinite) sequence $(r_1, r_2, \dots)$ of letters in S will be called a *reduced increasing sequence* if $l(r_1 \cdots r_i) < l(r_1 \cdots r_i r_{i+1})$ for all $i \geq 1$.
(ii) Let $w \in M_S$. A sequence $(r_1, r_2, \dots)$ of letters in S will be called a *reduced increasing w-sequence* if $l(wr_1 \cdots r_i) < l(wr_1 \cdots r_i r_{i+1})$ for all $i \geq 0$.

Lemma 2.16 *Let $\alpha = (r_1, r_2, \dots)$ be a reduced increasing sequence in S. Assume that each subsequence of α of the form*

$$(r_{a_1}, r_{a_2}, \dots) \text{ with } |r_{a_i} r_{a_{i+1}}| = \infty \text{ for all } i$$

has $\leq b$ elements. Then there is some positive integer $f(b)$ depending only on b and on the Coxeter system (W, S), such that α has $\leq f(b)$ elements.

Proof We will prove this result by induction on $|S|$; the case $|S| = 1$ is trivial.

Suppose now that $|S| \geq 2$. If (W, S) is a spherical Coxeter group, then the result is obvious since the length of any reduced increasing sequence is bounded by the length of the longest element of W. We may thus assume that there is some $s \in S$ that does not commute with some other generator in $S \backslash \{s\}$.

Since the sequence α is a reduced increasing sequence, we know that between any two s's, there must be some t_i such that $|st_i| = \infty$. Consider the subsequence of α given by

$$(s, t_1, s, t_2, \dots).$$

This subsequence has $\leq b$ elements by assumption, and between any two generators s in the original sequence α, we only use letters in $S \backslash \{s\}$. The result now follows from the induction hypothesis. □

Lemma 2.17 *Let $\overline{w} \in W$. Then there is some $k(\overline{w}) \in \mathbb{N}$, depending only on $\overline{w}$, such that for every reduced increasing w-sequence $(r_1, r_2, \dots)$ in S, we have*

$$F^{\#}(wr_1 \cdots r_{k(\overline{w})}) > F^{\#}(w).$$

Proof Assume that there is a reduced increasing w-sequence $\alpha = (r_1, r_2, \dots)$ in S such that

$$F^{\#}(wr_1 \cdots r_i) = F^{\#}(w) \text{ for all } i. \qquad (*)$$

Let $w_0 = w$, $w_i = w_{i-1} r_i$ and denote $I_i = I_{w_i}(l(\overline{w}) + i)$ for all i. Let $b = F^{\#}(w)$. By assumption $(*)$ and Observation 2.13(iii), we have $|I_i| \leq b - 1$ for all i. Moreover, if $i < j$ with $|r_i r_j| = \infty$, then $I_i \subsetneq I_j$ by Observations 2.8(i) and 2.13(ii); it follows that each subsequence of α of the form

$$(r_{a_1}, r_{a_2}, \dots) \text{ with } |r_{a_i} r_{a_{i+1}}| = \infty \text{ for all } i$$

has at most b elements. By Lemma 2.16, this implies that the sequence α has at most $f(b)$ elements. We conclude that every reduced increasing w-sequence $(r_1, r_2, \dots, r_{k(\overline{w})})$ in S with $k(\overline{w}) := f(F^{\#}(w)) + 1$ must have strictly increasing firmness. □

Theorem 2.18 *Let (W, S) be a right-angled Coxeter system. For all $n \geq 0$, there is some $d(n) \in \mathbb{N}$ depending only on n, such that $F^{\#}(\overline{w}) > n$ for all $\overline{w} \in W$ with $l(\overline{w}) > d(n)$.*

Proof This follows by induction on n from Lemma 2.17 since there are only finitely many elements in W of any given length. □

3 Right-angled buildings

We will start by recalling the procedure of "closing squares" in right-angled buildings from [10] and we define the square closure of a set of chambers. Our goal in this section is to describe the square closure of a ball in the building and to show that this is a bounded set, i.e., it has finite diameter; see Theorem 3.17.

3.1 Preliminaries

We regard buildings as chamber systems, following the notation in [20]. We briefly recall the basic notions and refer the reader to *loc. cit.* for more details. Recall the notation from Sect. 2.1.

Definition 3.1 (i) Let Δ be an edge-colored graph with color set S. Let $J \subseteq S$. A J-*residue* of Δ is a connected component of the subgraph of Δ obtained from Δ by discarding all the edges whose color is not in J. A residue of Δ is a J-residue for some $J \subseteq S$. If $s \in S$ then an $\{s\}$-residue is called an s-*panel*.

(ii) A chamber system is an edge-colored graph Δ with color set S such that for each $s \in S$, all s-panels of Δ are complete graphs with at least two vertices. We will refer to the vertices of Δ as *chambers* and we will denote the vertex set by $\mathrm{Ch}(\Delta)$. The cardinality of S is called the *rank* of Δ.

(iii) Two chambers c_1, c_2 are called s-*adjacent* if they are connected by an edge with color s, and we denote this by $c_1 \overset{s}{\sim} c_2$.

(iv) A chamber system is *thin* if every panel contains exactly two chambers and is *thick* if every panel contains at least three chambers.

(v) A *gallery* in a chamber system Δ is a sequence $(v_0, v_1, \ldots, v_k)$ of chambers such that v_{i-1} is adjacent to v_i for all i. We then call this a gallery *from* v_0 *to* v_k; the number k is the *length* of the gallery. If for each i, v_{i-1} is s_i-adjacent to v_i, then the word $w = s_1 s_2 \cdots s_k \in M_S$ is called the *type* of the gallery.

Definition 3.2 Let (W, S) be a Coxeter system. A *building of type* (W, S) is a pair (Δ, δ), where Δ is a chamber system with index set S and δ is a map

$$\delta\colon\ \mathrm{Ch}(\Delta) \times \mathrm{Ch}(\Delta) \to W$$

such that for each reduced word $w \in M_S$ and for each pair of chambers $c_1, c_2 \in \mathrm{Ch}(\Delta)$, we have

$$\delta(c_1, c_2) = \overline{w} \iff \text{there is a gallery in } \Delta \text{ of type } w \text{ from } c_1 \text{ to } c_2.$$

We call the group W the *Weyl group* and the map δ the *Weyl distance*.

Remark 3.3 Notice that with our combinatorial setup, the basic objects are chambers, and panels contain chambers. There exist various other realizations of buildings (that are nevertheless equivalent) in which the containment is the other way around. We refer, for instance, to the introduction of [1] for a good overview.

Definition 3.4 Let $\Sigma = (W, S)$ be a Coxeter system.

(i) We define a thin building Δ_Σ of type (W, S) by taking $\mathrm{Ch}(\Delta_\Sigma) = W$ as the set of chambers, declaring $x \overset{s}{\sim} y$ for $s \in S$ if and only if $x^{-1}y = s$, and defining a Weyl distance $\delta(x, y) :=$ for all $x, y \in \mathrm{Ch}(\Delta_\Sigma)$.

(ii) Let Δ be an arbitrary building of type (W, S). An *apartment* in Δ is a subbuilding of Δ that is δ-isometric to Δ_Σ.

(iii) Let Δ_Σ be as in (i). A *reflection* of Δ_Σ is a non-trivial element $r \in W$ fixing an edge (i.e., a panel) of Δ_Σ; such an element r is always an involution of W. The set of edges fixed by r is called the *wall* of r.

To each reflection r, we can associate a partition of the chamber set into two parts, as follows. Let $\{x, y\}$ be a panel in the wall of r. Then each chamber of Δ_Σ is either nearer to x than to y or nearer to y than to x, so we get two parts $\{c \in W \mid \mathrm{dist}(c, x) < \mathrm{dist}(c, y)\}$ and its complement $\{c \in W \mid \mathrm{dist}(c, y) < \mathrm{dist}(c, x)\}$. These two parts are called the *roots* associated to r and they are interchanged by r. (They are independent of the choice of $\{x, y\}$ in the wall of r. See [20, Proposition 3.11].) In particular, if α is a root, then its complement is again a root and is denoted by $-\alpha$.

(iv) Let Δ be an arbitrary building of type (W, S). A *root* of Δ is then defined to be a root in one of its apartments. (Roots are also sometimes referred to as *half-apartments*.)

From now on, let (W, S) be a right-angled Coxeter system with Coxeter diagram Σ and let Δ be a right-angled building of type (W, S).

Definition 3.5 (i) Let $\delta\colon \Delta \times \Delta \to W$ be the Weyl distance of the building Δ. The *gallery distance* between the chambers c_1 and c_2 is defined as

$$\mathsf{d}_W(c_1, c_2) := l(\delta(c_1, c_2)),$$

i.e., the length of a minimal gallery between the chambers c_1 and c_2.

(ii) For a fixed chamber $c_0 \in \mathrm{Ch}(\Delta)$ we define the *spheres* at a fixed gallery distance from c_0 as

$$\mathsf{S}(c_0, n) := \{c \in \mathrm{Ch}(\Delta) \mid \mathsf{d}_W(c_0, c) = n\}$$

and the *balls* as

$$\mathsf{B}(c_0, n) := \{c \in \mathrm{Ch}(\Delta) \mid \mathsf{d}_W(c_0, c) \le n\}.$$

Definition 3.6 (i) Let c be a chamber in Δ and $\mathcal{R}$ be a residue in Δ. The *projection* of c on $\mathcal{R}$ is the unique chamber in $\mathcal{R}$ that is closest to c and it is denoted by $\mathrm{proj}_\mathcal{R}(c)$.

(ii) If $\mathcal{R}_1$ and $\mathcal{R}_2$ are two residues, then the set of chambers

$$\mathrm{proj}_{\mathcal{R}_1}(\mathcal{R}_2) := \{\mathrm{proj}_{\mathcal{R}_1}(c) \mid c \in \mathrm{Ch}(\mathcal{R}_2)\}$$

is again a residue and the rank of $\mathrm{proj}_{\mathcal{R}_1}(\mathcal{R}_2)$ is bounded above by the ranks of both $\mathcal{R}_1$ and $\mathcal{R}_2$; see [5, Section 2].

(iii) The residues $\mathcal{R}_1$ and $\mathcal{R}_2$ are called *parallel* if $\mathrm{proj}_{\mathcal{R}_1}(\mathcal{R}_2) = \mathcal{R}_1$ and $\mathrm{proj}_{\mathcal{R}_2}(\mathcal{R}_1) = \mathcal{R}_2$.

In particular, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are two parallel panels, then the chamber sets of $\mathcal{P}_1$ and $\mathcal{P}_2$ are mutually in bijection under the respective projection maps (see again [5, Section 2]).

Definition 3.7 Let $J \subseteq S$. We define the set

$$J^\perp = \{t \in S \setminus J \mid ts = st \text{ for all } s \in J\}.$$

If $J = \{s\}$, then we write the set $J^\perp$ as $s^\perp$.

Proposition 3.8 ([5, Proposition 2.8]) *Let Δ be a right-angled building of type* (W, S).

(i) *Any two parallel residues have the same type.*
(ii) *Let $J \subseteq S$. Given a residue $\mathcal{R}$ of type J, a residue $\mathcal{R}'$ is parallel to $\mathcal{R}$ if and only if $\mathcal{R}'$ is of type J, and $\mathcal{R}$ and $\mathcal{R}'$ are both contained in a common residue of type $J \cup J^{\perp}$.*

Proposition 3.9 ([5, Corollary 2.9]) *Let Δ be a right-angled building. Parallelism of residues of Δ is an equivalence relation.*

Another very important notion in right-angled buildings is that of *wings*, introduced in [5, Section 3]. For our purposes, it will be sufficient to consider wings with respect to panels.

Definition 3.10 Let $c \in \mathrm{Ch}(\Delta)$ and $s \in S$. Denote the unique s-panel containing c by $\mathcal{P}_{s,c}$. Then the set of chambers

$$X_s(c) = \{x \in \mathrm{Ch}(\Delta) \mid \mathrm{proj}_{\mathcal{P}_{s,c}}(x) = c\}$$

is called the *s-wing* of c.

Notice that if $\mathcal{P}$ is any s-panel, then the set of s-wings of each of the different chambers of $\mathcal{P}$ forms a partition of $\mathrm{Ch}(\Delta)$ into equally many combinatorially convex subsets (see [5, Proposition 3.2]).

3.2 Sets of chambers closed under squares

We start by presenting two results proved in [10, Lemmas 2.9 and 2.10] that can be used in right-angled buildings to modify minimal galleries using the commutation relations of the Coxeter group. We will refer to these results as the "Closing Squares Lemmas" (see also Fig. 1 below).

Lemma 3.11 (Closing squares 1) *Let c_0 be a fixed chamber in Δ. Let $c_1, c_2 \in S(c_0, n)$ and $c_3 \in S(c_0, n+1)$ such that*

$$c_1 \overset{t}{\sim} c_3 \quad \textit{and} \quad c_2 \overset{s}{\sim} c_3$$

for some $s \neq t$. Then $|st| = 2$ in Σ and there exists $c_4 \in S(c_0, n-1)$ such that

$$c_1 \overset{s}{\sim} c_4 \quad \textit{and} \quad c_2 \overset{t}{\sim} c_4.$$

Lemma 3.12 (Closing Squares 2) *Let c_0 be a fixed chamber in Δ. Let $c_1, c_2 \in S(c_0, n)$ and $c_3 \in S(c_0, n-1)$ such that*

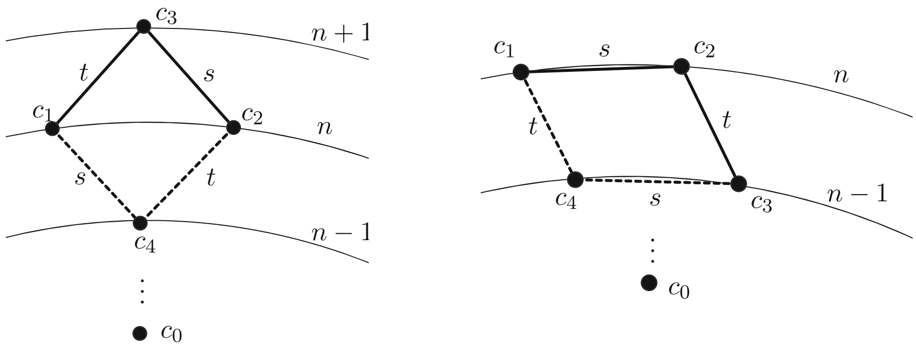
$$c_1 \overset{s}{\sim} c_2 \quad \textit{and} \quad c_2 \overset{t}{\sim} c_3$$

for some $s \neq t$. Then $|st| = 2$ in Σ and there exists $c_4 \in S(c_0, n-1)$ such that

$$c_1 \overset{t}{\sim} c_4 \quad \textit{and} \quad c_3 \overset{s}{\sim} c_4.$$

Definition 3.13 Let c_0 be a fixed chamber of Δ and let $n \in \mathbb{N}$.

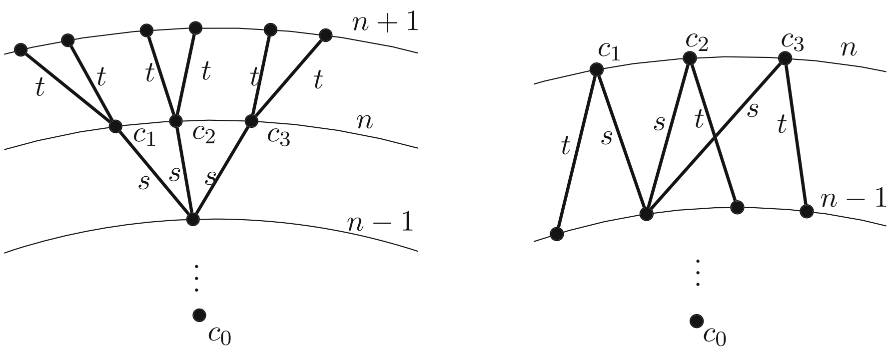
(i) Let $c \in \mathrm{Ch}(\Delta)$. Then we call c *firm with respect to c_0* if and only if $\delta(c_0, c) \in W$ is firm (as in Definition 2.10(i)).



(a) Lemma 3.11

(b) Lemma 3.12

Fig. 1 Closing squares lemmas



(a) c_i firm: for all $t \neq s$, $l(\delta(c_0, c_i)t) > l(\delta(c_0, c_i))$.

(b) c_i not firm: for some $t \neq s$, $l(\delta(c_0, c_i)t) < l(\delta(c_0, c_i))$.

Fig. 2 Partition of $\mathsf{S}(c_0, n)$

(ii) We will create a partition of the sphere $\mathsf{S}(c_0, n)$ by defining

$$A_1(n) = \{c \in \mathsf{S}(c_0, n) \mid c \text{ is firm}\},$$
$$A_2(n) = \{c \in \mathsf{S}(c_0, n) \mid c \text{ is not firm}\},$$

as in Fig. 2. Notice that this is equivalent to the definition given in [10, Definition 4.3].

(iii) Let $c \in \mathsf{S}(c_0, k)$ for some $k > n$. We say that c is *n-flexible with respect to* c_0 if for each minimal gallery $\gamma = (c_0, c_1, \ldots, c_{n+1}, \ldots, c_k = c)$ from c_0 to c, none of the chambers $c_{n+1}, \ldots, c_k$ is firm. By convention, all chambers of $\mathsf{B}(c_0, n)$ are also n-flexible with respect to c_0.
Observe that a chamber c is n-flexible with respect to c_0 if and only if $F^{\#}(\delta(c_0, c)) \leq n$. In particular, if c is n-flexible, then so is any chamber on any minimal gallery between c_0 and c.

(iv) We define the *n-flex of* c_0, denoted by $\mathrm{Flex}(c_0, n)$, to be the set of all chambers of Δ that are n-flexible with respect to c_0.

We also record the following result, which we rephrased in terms of firm chambers; its Corollary 3.15 will be used several times in Sect. 4.

Lemma 3.14 ([10, Lemma 2.15]) *Let c_0 be a fixed chamber of Δ and let $s \in S$. Let $d \in S(c_0, n)$ and $e \in B(c_0, n+1) \setminus \mathrm{Ch}(\mathcal{P}_{s,d})$. If $c := \mathrm{proj}_{\mathcal{P}_{s,d}}(e) \in S(c_0, n+1)$, then c is not firm with respect to c_0.*

Corollary 3.15 *Let $c_0 \in \mathrm{Ch}(\Delta)$ and $c \in S(c_0, n+1)$ such that c is firm with respect to c_0. Let d be the unique chamber of $S(c_0, n)$ adjacent to c and let $s = \delta(d, c) \in S$. Then $B(c_0, n) \subset X_s(d)$.*

Proof Let $e \in \mathrm{B}(c_0, n)$. If $e = d$, then of course $e \in X_s(d)$, so assume $e \neq d$; then $e \in \mathrm{B}(c_0, n+1) \setminus \mathrm{Ch}(\mathcal{P}_{s,d})$. Notice that all chambers of $\mathcal{P}_{s,d} \setminus \{d\}$ have the same Weyl distance from c_0 as c and hence are firm. By Lemma 3.14, this implies that the projection of e on $\mathcal{P}_{s,d}$ must be equal to d, so by definition of the s-wing $X_s(d)$, we get $e \in X_s(d)$. □

We now come to the concept of the square closure of a set of chambers of Δ.

Definition 3.16 (i) We say that a subset $T \subseteq W$ is *closed under squares* if the following holds:

If ws_i and ws_j are contained in T for some $w \in T$ with $|s_i s_j| = 2$, $s_i \neq s_j$ and $l(ws_i) = l(ws_j) = l(w) + 1$, then also $ws_i s_j = ws_j s_i$ is an element of T.

(ii) Let c_0 be a fixed chamber of Δ. A set of chambers $\mathscr{C} \subseteq \mathrm{Ch}(\Delta)$ is *closed under squares* with respect to c_0 if for each $n \in \mathbb{N}$, the following holds (see Fig. 1a):

If $c_1, c_2 \in \mathscr{C} \cap \mathrm{S}(c_0, n)$ and $c_4 \in \mathscr{C} \cap \mathrm{S}(c_0, n-1)$ such that $c_4 \overset{s_i}{\sim} c_1$ and $c_4 \overset{s_j}{\sim} c_2$ for some $|s_i s_j| = 2$ with $s_i \neq s_j$, then the unique chamber $c_3 \in \mathrm{S}(c_0, n+1)$ such that $c_3 \overset{s_j}{\sim} c_1$ and $c_3 \overset{s_i}{\sim} c_2$ is also in $\mathscr{C}$.

In particular, if $\mathscr{C}$ is closed under squares with respect to c_0, then the set of Weyl distances $\{\delta(c_0, c) \mid c \in \mathscr{C}\} \subseteq W$ is closed under squares.

(iii) Let $c_0 \in \mathrm{Ch}(\Delta)$ and let $\mathscr{C} \subseteq \mathrm{Ch}(\Delta)$. We define the *square closure* of $\mathscr{C}$ with respect to c_0 to be the smallest subset of $\mathrm{Ch}(\Delta)$ containing $\mathscr{C}$ and closed under squares with respect to c_0.

Theorem 3.17 *Let $c_0 \in \mathrm{Ch}(\Delta)$ and let $n \in \mathbb{N}$. The square closure of $B(c_0, n)$ with respect to c_0 is $\mathrm{Flex}(c_0, n)$. Moreover, the set $\mathrm{Flex}(c_0, n)$ is bounded.*

Proof We will first show that $\mathrm{Flex}(c_0, n)$ is indeed closed under squares. Let c be a chamber in $\mathrm{Flex}(c_0, n)$ at Weyl distance w from c_0 and let c_1 and c_2 be chambers in $\mathrm{Flex}(c_0, n)$ adjacent to c, at Weyl distance ws_i and ws_j from c_0, respectively, such that $|s_i s_j| = 2$ and $l(ws_i) = l(ws_j) = l(w) + 1$. Let c_3 be the unique chamber at Weyl distance $ws_i s_j$ from c_0 that is s_j-adjacent to c_1 and s_i-adjacent to c_2.
Our aim is to show that also c_3 is an element of $\mathrm{Flex}(c_0, n)$. If $l(ws_i s_j) \leq n$, then this is obvious, so we may assume that $l(ws_i s_j) > n$.

Let $\gamma = (c_0 = v_0, \ldots, v_{n+1}, \ldots, v_k = c_3)$ be an arbitrary minimal gallery between c_0 and c_3, as in Fig. 3 (so $k = l(w) + 2 > n$). We have to show that none of the chambers $v_{n+1}, \ldots, v_k$ is firm with respect to c_0. This is clear for $v_k = c_3$.

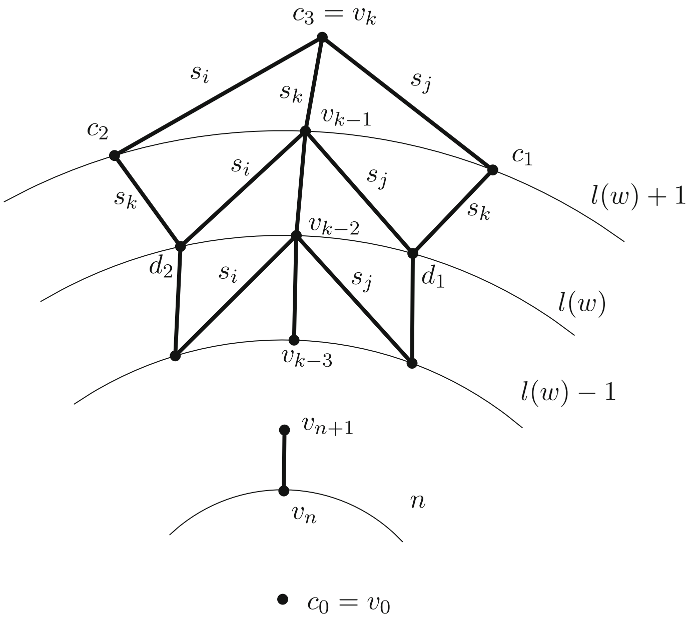


Fig. 3 Proof of Theorem 3.17

If $k = n+1$, then there is nothing left to show, so assume $k \geq n+2$. If $v_{k-1} \in \{c_1, c_2\}$, then v_{k-1} is n-flexible by assumption, and since $k-1 > n$ it is not firm. (In fact, this shows immediately that in this case, none of the chambers $v_{n+1}, \ldots, v_{k-1}$ is firm). So assume that v_{k-1} is distinct from c_1 and c_2; then v_{k-1} is s_k-adjacent to c_3 for some s_k different from s_i and s_j. Then by closing squares (Lemma 3.11), we have $|s_j s_k| = 2$ and there is a chamber $d_1 \in \mathsf{S}(c_0, l(w))$ such that $d_1 \overset{s_j}{\sim} v_{k-1}$ and $d_1 \overset{s_k}{\sim} c_1$. Similarly, there is a chamber $d_2 \in \mathsf{S}(c_0, l(w))$ such that $d_2 \overset{s_i}{\sim} v_{k-1}$ and $d_2 \overset{s_k}{\sim} c_2$. Hence v_{k-1} is not firm with respect to c_0.

Continuing this argument inductively (see Fig. 3), we conclude that none of the chambers $v_{n+1}, \ldots, v_k$ is firm with respect to c_0. Hence c_3 is n-flexible; we conclude that $\mathrm{Flex}(c_0, n)$ is closed under squares with respect to c_0.

Conversely, let $\mathscr{C}$ be a set of chambers closed under squares that contains $\mathsf{B}(c_0, n)$; we have to prove that $\mathrm{Flex}(c_0, n) \subseteq \mathscr{C}$. So let $c \in \mathrm{Flex}(c_0, n)$ be arbitrary; we will show by induction on $k := \mathsf{d}_W(c_0, c)$ that $c \in \mathscr{C}$. This is obvious for $k \leq n$, so assume $k > n$. Then c is not firm, hence there exist $c_1, c_2 \in \mathsf{S}(c_0, k-1)$ such that $c_1 \overset{s_1}{\sim} c$ and $c_2 \overset{s_2}{\sim} c$ for some $s_1 \neq s_2 \in S$. By Lemma 3.11 we have $|s_1 s_2| = 2$ and there is $d \in \mathsf{S}(c_0, k-2)$ such that $d \overset{s_2}{\sim} c_1$ and $d \overset{s_1}{\sim} c_2$.

Since c is n-flexible and c_1, c_2 and d all lie on some minimal gallery between c_0 and c, it follows that also c_1, c_2 and d are n-flexible. By the induction hypothesis, all three elements are contained in $\mathscr{C}$. Since $\mathscr{C}$ is assumed to be closed under squares, however, we immediately deduce that also $c \in \mathscr{C}$.

We conclude that $\mathrm{Flex}(c_0, n)$ is the square closure of $\mathsf{B}(c_0, n)$ with respect to c_0.

We finally show that $\mathrm{Flex}(c_0, n)$ is a bounded set. Recall that a chamber c is contained in $\mathrm{Flex}(c_0, n)$ if and only if $F^{\#}(\delta(c_0, c)) \leq n$. By Theorem 2.18, there is a constant $d(n)$ such that $F^{\#}(\overline{w}) > n$ for all $\overline{w} \in W$ with $l(\overline{w}) > d(n)$. This shows that $\text{Flex}(c_0, n) \subseteq \mathsf{B}(c_0, d(n))$ is indeed bounded. □

Remark 3.18 In fact, the square closure of $\mathsf{B}(c_0, n)$ with respect to c_0 is precisely the convex hull of $\mathsf{B}(c_0, n)$. Indeed, the square closure is clearly contained in the convex hull. For the reverse inclusion, the crucial fact is that for any two chambers c_1, c_2 in $\mathsf{B}(c_0, n)$, there always exists a minimal gallery from c_1 to c_2 completely contained in $\mathsf{B}(c_0, n)$, and any two minimal galleries can be transformed into each other by a sequence of closing square operations; see [10, Lemma 2.11] and its proof.

4 The automorphism group of a right-angled building

In this section, we study the group $\text{Aut}(\Delta)$ of type-preserving automorphisms of a thick semi-regular right-angled building Δ. We will first study the action of a ball fixator and introduce root wing groups. Then, when the building is locally finite, we will show that any proper open subgroup of $\text{Aut}(\Delta)$ is a finite index subgroup of the stabilizer of a proper residue; see Theorem 4.28.

Definition 4.1 Let Δ be a right-angled building of type (W, S). Then Δ is called *semi-regular* if for each s, all s-panels of Δ have the same cardinality q_s of chambers. In this case, the building is said to have *prescribed thickness* $(q_s)_{s\in S}$ in its panels.

By [11, Proposition 1.2], there is a unique right-angled building of type (W, S) of prescribed thickness $(q_s)_{s\in S}$ for any choice of cardinal numbers $q_s \geq 1$.

Theorem 4.2 ([12, Theorem B], [5, Theorem 1.1]) *Let Δ be a thick semi-regular building of right-angled type (W, S). Assume that (W, S) is irreducible and non-spherical. Then the group* $\text{Aut}(\Delta)$ *of type-preserving automorphisms of Δ is abstractly simple and acts strongly transitively on Δ.*

Recall that a group G acts *strongly transitively* on a building Δ if it acts transitively on the pairs (A, c) of an apartment A and a chamber c contained in A. The strong transitivity has first been shown by Kubena and Thomas [12] and has been reproved by Pierre-Emmanuel Caprace in the same paper where he proved the simplicity [5]. In our proof of Proposition 4.7 below, we will adapt Caprace's proof of the strong transitivity to a more specific setting.

The following extension result is very powerful and will be used in the proof of Theorem 4.4 below.

Proposition 4.3 ([5, Proposition 4.2]) *Let Δ be a semi-regular right-angled building. Let $s \in S$ and $\mathcal{P}$ be an s-panel. Given any permutation $\theta \in \text{Sym}(\text{Ch}(\mathcal{P}))$, there is some $g \in \text{Aut}(\Delta)$ stabilizing $\mathcal{P}$ satisfying the following two conditions:*

(a) *$g|_{\text{Ch}(\mathcal{P})} = \theta$;*
(b) *g fixes all chambers of Δ whose projection on $\mathcal{P}$ is fixed by θ.*

4.1 The action of the fixator of a ball in Δ

In this section we study the action of the fixator K in $\text{Aut}(\Delta)$ of a ball $\mathsf{B}(c_0, n)$ of radius n around a chamber c_0. Our goal will be to prove that the fixed point set Δ^K coincides with the square closure of the ball $\mathsf{B}(c_0, n)$ with respect to c_0, which is $\text{Flex}(c_0, n)$, and which we know is bounded by Theorem 3.17.

Theorem 4.4 *Let Δ be a thick semi-regular right-angled building. Let c_0 be a fixed chamber of Δ and let $n \in \mathbb{N}$. Consider the pointwise stabilizer $K = \mathrm{Fix}_{\mathrm{Aut}(\Delta)}(B(c_0, n))$ in $\mathrm{Aut}(\Delta)$ of the ball $B(c_0, n)$.*

Then the fixed-point set Δ^K is equal to the bounded set $\mathrm{Flex}(c_0, n)$.

Proof Recall from Theorem 3.17 that $\mathrm{Flex}(c_0, n)$ is precisely the square closure of $\mathsf{B}(c_0, n)$ with respect to c_0. First, notice that the fixed point set of any automorphism fixing c_0 is square closed with respect to c_0 because the chamber "closing the square" is unique [see Definition 3.16(ii)]. It immediately follows that $\mathrm{Flex}(c_0, n) \subseteq \Delta^K$.

We will now show that if c is a chamber not in $\mathrm{Flex}(c_0, n)$, then there exists a $g \in K$ not fixing c. Since c is not n-flexible, there exists a chamber d on some minimal gallery between c_0 and c with $k := \mathsf{d}_W(c_0, d) > n$ such that d is firm. Notice that any automorphism fixing c_0 and c fixes every chamber on any minimal gallery between c_0 and c, so it suffices to show that there exists a $g \in K$ not fixing d.

Since d is firm, there is a unique chamber $e \in \mathsf{S}(c_0, k-1)$ such that $e \overset{s}{\sim} d$ for some $s \in S$. By Corollary 3.15, $\mathsf{B}(c_0, n) \subseteq X_s(e)$, where $X_s(e)$ is the s-wing of Δ corresponding to e.

Now take any permutation θ of $\mathcal{P}_{s,e}$ fixing e and mapping d to some third chamber d'' different from d and e (which exists because Δ is thick). By Proposition 4.3, there is an element $g \in \mathrm{Aut}(\Delta)$ fixing $X_s(e)$ and mapping d to d''. In particular, g belongs to K and does not fix d, as required.

We conclude that $\Delta^K = \mathrm{Flex}(c_0, n)$. The fact that this set is bounded was shown in Theorem 3.17. □

4.2 Root wing groups

In this section we define groups that resemble root groups, using the partition of the chambers of a right-angled building by wings; we call these groups *root wing groups*.

We show that a root wing group acts transitively on the set of apartments of Δ containing the given root. We also prove that the root wing groups corresponding to roots disjoint from a ball $\mathsf{B}(c_0, n)$ are contained in the fixator of that ball in the automorphism group.

We first fix some notation for the rest of this section. Recall the notions from Definition 3.4.

Notation 4.5 (i) Fix a chamber $c_0 \in \mathrm{Ch}(\Delta)$ and an apartment A_0 containing c_0 (which can be considered as the fundamental chamber and the fundamental apartment). Let Φ denote the set of roots of A_0. For each $\alpha \in \Phi$, we write $-\alpha$ for the root opposite α in A_0.

(ii) We will write $\mathscr{A}_0$ for the set of all apartments containing c_0. For any $A \in \mathscr{A}_0$, we will denote its set of roots by Φ_A.

(iii) For any $k \in \mathbb{N}$, we write $K_r := \mathrm{Fix}_{\mathrm{Aut}(\Delta)}(\mathsf{B}(c_0, r))$.

Definition 4.6 (i) When $\alpha \in \Phi_A$ is a root in an apartment A, its *wall* $\partial\alpha$ consists of the panels of Δ having chambers in both α and $-\alpha$. Since the building is right-angled, these panels all have the same type $s \in S$, which we refer to as the *type of* α and write as $\mathrm{type}(\alpha) = s$. Notice that the s-wings of A are precisely the roots of A of type s.

(ii) Let $\alpha \in \Phi_A$ of type s and let $c \in \alpha$ be such that $\mathcal{P}_{s,c} \in \partial\alpha$. Then we define the *root wing group* U_α as

$$U_\alpha := U_s(c) := \mathrm{Fix}_{\mathrm{Aut}(\Delta)}(X_s(c)).$$

Observe that U_α does not depend of the choice of the chamber c as all panels in the wall $\partial\alpha$ are parallel [see Definition 3.6(iii)] and hence determine the same s-wings in Δ.

The fact that these groups behave, to some extent, like root groups in Moufang spherical buildings or Moufang twin buildings, is illustrated by the following fact.

Proposition 4.7 *Let $\alpha \in \Phi_A$ be a root. The root wing group U_α acts transitively on the set of apartments of Δ containing α.*

Proof We carefully adapt the proof of the strong transitivity of $\mathrm{Aut}(\Delta)$ from [5, Proposition 6.1]. Let c be a chamber of α on the boundary and let A and A' be two apartments of Δ containing α. The strategy in *loc. cit.* (where A and A' are arbitrary apartments containing c) is to construct an infinite sequence of automorphisms $g_0, g_1, g_2, \ldots$ such that

(a) each g_n fixes the ball $\mathsf{B}(c, n-1)$ pointwise;
(b) let $A_n := g_n g_{n-1} \cdots g_0(A)$; then $A_n \cap A' \supseteq \mathsf{B}(c, n) \cap A'$.

We will show that the elements g_i constructed in *loc. cit.* are all contained in U_α; the result then follows because U_α is a closed subgroup of $\mathrm{Aut}(\Delta)$.

To construct the element g_{n+1}, we consider the set E of chambers in $\mathsf{B}(c, n+1) \cap A'$ that are not contained in A_n (as in *loc. cit.*). The crucial observation now is that by Theorem 4.4, the chambers of E are firm with respect to c. Hence, for each $x \in E$, there is a unique chamber $y \in \mathsf{S}(c, n)$ that is s-adjacent to x (for some $s \in S$). The element g_{n+1} constructed in *loc. cit.* is then contained in the group generated by the subgroups $U_s(y)$ for such pairs (y, s) corresponding to the various elements of E. However, because the elements of E are firm, the root α is contained in each root corresponding to a pair (y, s) in A'; [5, Lemma 3.4(b)] now implies that each such group $U_s(y)$ is contained in U_α. □

Remark 4.8 The group U_α does *not*, in general, act sharply transitively on the set of apartments containing α. This is clear already in the case of trees: an automorphism fixing a half-tree and an apartment need not be trivial.

Corollary 4.9 *Let $\alpha \in \Phi_A$ be a root of type s and let c, c' be two s-adjacent chambers of A with $c \in \alpha$ and $c' \in -\alpha$. Then there exists an element in $\langle U_\alpha, U_{-\alpha} \rangle$ stabilizing A and interchanging c and c'.*

Proof Let A' be an apartment different from A containing α (which exists because Δ is thick) and let β be the root opposite α in A'. By Proposition 4.7, there is some $g \in U_\alpha$ mapping $-\alpha$ to β. Similarly, there is some $h \in U_{-\alpha}$ mapping β to α. Let $\gamma := h.\alpha$; then there exists a third automorphism $g' \in U_\alpha$ mapping γ to $-\alpha$. The composition $g'hg \in U_\alpha U_{-\alpha} U_\alpha$ is the required automorphism. □

Next we present a property similar to the FPRS ("Fixed Points of Root Subgroups") property introduced in [7] for groups with a twin root datum. It is the analogous statement of [6, Lemma 3.8], but in the case of right-angled buildings, we can be more explicit.

Lemma 4.10 *For every root $\alpha \in \Phi$ with $\mathrm{dist}(c_0, \alpha) > r$, the group $U_{-\alpha}$ is contained in $K_r = \mathrm{Fix}_{\mathrm{Aut}(\Delta)}(B(c_0, r))$.*

Proof Let α be a root at distance $n > r$ from c_0 and let s be the type of α. Let c be a chamber of α at distance n from c_0 and let c' be the other chamber in $\mathcal{P}_{s,c} \cap A_0$; notice that $c' \in \mathsf{S}(c_0, n-1)$. We will show that $\mathsf{B}(c_0, r) \subseteq X_s(c')$, which will then of course imply that $U_{-\alpha} = U_s(c') \subseteq K_r$.

The chamber c is firm with respect to c_0 because if c would be t-adjacent to some chamber at distance $n-1$ from c_0 for some $t \neq s$, then $\partial\alpha$ would contain panels of type s and of type t, which is impossible. Corollary 3.15 now implies that $\mathsf{B}(c_0, n-1) \subseteq X_s(c')$, so in particular $\mathsf{B}(c_0, r) \subseteq X_s(c')$. □

Following the idea of [6, Lemmas 3.9 and 3.10], we present two variations on the previous lemma that allow us to transfer the results to other apartments containing the chamber c_0.

Lemma 4.11 *Let* $g \in \mathrm{Aut}(\Delta)$ *and let* $A \in \mathscr{A}_0$ *containing the chamber* $d = gc_0$*. Let* $b \in \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(c_0)$ *such that* $A = bA_0$*, and let* $\alpha = b\alpha_0$ *be a root of* A *with* $\alpha_0 \in \Phi$*.*

If $\mathrm{dist}(d, -\alpha) > r$*, then* $bU_{\alpha_0}b^{-1} \subseteq gK_r g^{-1}$*.*

Proof Analogous to the proof of [6, Lemma 3.9]. □

Definition 4.12 ([6, Section 2.4]) Let $w \in W$.

(i) A root $\alpha \in \Phi$ is called *w-essential* if there is an $n \in \mathbb{Z}$ such that $w^n\alpha \subsetneq \alpha$.
(ii) A wall is called *w-essential* if it is the wall $\partial\alpha$ of some w-essential root α.

Lemma 4.13 *Let* $A \in \mathscr{A}_0$ *and let* $b \in \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(c_0)$ *such that* $A = bA_0$*. Also, let* $\alpha = b\alpha_0$ *(with* $\alpha_0 \in \Phi$*) be a w-essential root for some* $w \in \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(A)/\mathrm{Fix}_{\mathrm{Aut}(\Delta)}(A)$*. Let* $g \in \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(A)$ *be a representative of w.*

Then there exists some $n \in \mathbb{Z}$ *such that*

$$U_{\alpha_0} \subseteq b^{-1}g^n K_r g^{-n} b \quad \text{and}$$
$$U_{-\alpha_0} \subseteq b^{-1}g^{-n} K_r g^n b.$$

Proof The proof can be copied ad verbum from [6, Lemma 3.10]. □

4.3 Open subgroups of Aut(Δ)

We now focus on the description of open subgroups of the automorphism group of Δ. The main result of this section will be that any proper open subgroup of the automorphism group of a locally finite thick semi-regular right-angled building Δ is contained with finite index in the setwise stabilizer in $\mathrm{Aut}(\Delta)$ of a proper residue of Δ (see Theorem 4.28 below).

We will split the proof in the cases where the open subgroup is compact and non-compact. The compact case is easy:

Proposition 4.14 *Let H be an open subgroup of* $\mathrm{Aut}(\Delta)$*. Then H is compact if and only if it is a finite index subgroup of the stabilizer of a spherical residue of* Δ*.*

Proof This follows immediately from the fact that the maximal compact open subgroups of $\mathrm{Aut}(\Delta)$ are precisely the stabilizers of a maximal spherical residue of Δ; see, for instance, [10, Proposition 4.2]. □

> From now on, we assume that H is a *non-compact* open subgroup of $\mathrm{Aut}(\Delta)$.

Definition 4.15 We continue to use the conventions from Notation 4.5 and we will identify the apartment A_0 with W.

(i) Given a root $\alpha \in \Phi$, let r_α denote the unique reflection of W setwise stabilizing the panels in $\partial\alpha$ and let U_α be the root wing group introduced in Definition 4.6. By Corollary 4.9, the reflection $r_\alpha \in W$ lifts to an automorphism $n_\alpha \in \langle U_\alpha, U_{-\alpha}\rangle \leq \mathrm{Aut}(\Delta)$ stabilizing A_0.

(ii) For each $c \in \mathrm{Ch}(\Delta)$ and each subset $J \subseteq S$, we write $\mathcal{R}_{J,c}$ for the residue of Δ of type J containing c. We use the shorter notation $\mathcal{R}_J := \mathcal{R}_{J,c_0}$ when $c = c_0$. Moreover, we write $P_J := \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_J)$, and we call this a *standard parabolic subgroup* of $\mathrm{Aut}(\Delta)$. Any conjugate of P_J, i.e., any stabilizer of an arbitrary residue, is then called a *parabolic subgroup*.

(iii) Let $J \subseteq S$ be minimal such that there is a $g \in \mathrm{Aut}(\Delta)$ such that $H \cap g^{-1} P_J g$ has finite index in H. In particular, J is essential [see Definition 2.2(iii)]. See also [6, Lemma 3.4].

For such a g, we set $H_1 = gHg^{-1} \cap P_J$. Thus H_1 stabilizes $\mathcal{R}_J$ and it is an open subgroup of $\mathrm{Aut}(\Delta)$ contained in gHg^{-1} with finite index; since H is non-compact, so is H_1. Hence we may assume without loss of generality that $g = 1$ and hence $H_1 = H \cap P_J$ has finite index in H.

(iv) Let $\mathscr{A}_0$ be the set of apartments of Δ containing c_0. For $A \in \mathscr{A}_0$ we let

$$N_A := \mathrm{Stab}_{H_1}(A) \quad \text{and} \quad W_A := N_A / \mathrm{Fix}_{H_1}(A),$$

which we identify with a subgroup of W. For $h \in N_A$, let $\overline{h}$ denote its image in $W_A \leq W$.

The idea will be to prove that H_1 contains a hyperbolic element h such that the chamber c_0 achieves the minimal displacement of h. Moreover, we can find the element h in the stabilizer in H_1 of an apartment A_1 containing c_0. Thus we can identify it with an element $\overline{h}$ of W and consider its parabolic closure [see Definition 2.2(iv)]. The key point will be to prove that the type of $\mathrm{Pc}(\overline{h})$ is J, which will be achieved in Lemma 4.23.

We will also show that H_1 acts transitively on the chambers of $\mathcal{R}_J$; this will allow us to conclude that any open subgroup of $\mathrm{Aut}(\Delta)$ containing H_1 as a finite index subgroup is contained in the stabilizer of $\mathcal{R}_{J \cup J'}$ for some spherical subset J' of $J^{\perp}$ (Proposition 4.25).

This strategy is analogous (and, of course, inspired by) [6, Section 3]. As the arguments of *loc. cit.* are of a geometric nature, we will be able to adapt them to our setting. The root groups associated with the Kac–Moody group in that paper can be replaced by the root wing groups defined in Sect. 4.2. It should not come as a surprise that many of our proofs will simply consist of appropriate references to arguments in [6].

Lemma 4.16 *For all $A \in \mathscr{A}_0$, there exists a hyperbolic automorphism $h \in N_A$ such that*

$$\mathrm{Pc}(\overline{h}) = \langle r_\alpha \mid \alpha \textit{ is an } \overline{h}\textit{-essential root of } \Phi \rangle$$

and is of finite index in $\mathrm{Pc}(W_A)$.

Proof Using the fact that the reflections r_α lift to elements $n_\alpha \in \langle U_\alpha, U_{-\alpha} \rangle$ [see Definition 4.15(i)], the proof is the same as for [6, Lemma 3.5]. Notice that by [6, Lemma 2.7], the type of the parabolic subgroup $\mathrm{Pc}(\overline{h})$ is always essential [in the sense of Definition 2.2(iii)]. □

Lemma 4.17 *There exists an apartment $A \in \mathscr{A}_0$ such that the orbit $N_A.c_0$ is unbounded. In particular, the parabolic closure in W of W_A is non-spherical.*

Proof The proofs of [6, Lemmas 3.6 and 3.7] continue to hold without a single change. Notice that this depends crucially on the fact that H_1 is non-compact. □

Definition 4.18 (i) Let $A_1 \in \mathscr{A}_0$ be an apartment such that the essential component of $\mathrm{Pc}(W_{A_1})$ is non-empty and maximal with respect to this property [see Definition 2.2(iii)]; such an apartment exists by Lemma 4.17. Choose $h_1 \in N_{A_1}$ as in Lemma 4.16. In particular, h_1 is a hyperbolic element of H_1.

(ii) Up to conjugating H_1 by an element of $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_J)$, we can assume without loss of generality that $\mathrm{Pc}(\overline{h_1})$ is a standard parabolic subgroup that is non-spherical and has essential type I ($\neq \emptyset$). Moreover, the type I is maximal in the following sense: if $A \in \mathscr{A}_0$ is such that $\mathrm{Pc}(W_A)$ contains a parabolic subgroup of essential type I_A with $I \subseteq I_A$, then $I = I_A$.

Definition 4.19 Recall that Φ is the set of roots of the apartment A_0. For each $T \subseteq S$, let

$$\Phi_T := \{\alpha \in \Phi \mid \mathcal{R}_T \text{ contains at least one panel of } \partial\alpha\}$$

and

$$L_T^+ := \langle U_\alpha \mid \alpha \in \Phi_T \rangle,$$

where U_α is the root wing group introduced in Definition 4.6.

Our next goal is to prove that H_1 contains L_J^+, where J is as in Definition 4.15(iii); as we will see in Lemma 4.21 below, this fact is equivalent to H_1 being transitive on the chambers of $\mathcal{R}_J$.

We will need the results in Sect. 4.2 regarding fixators of balls and root wing groups.

Notation 4.20 Since H_1 is open, we fix, for the rest of the section, some $r \in \mathbb{N}$ such that $\mathrm{Fix}_{\mathrm{Aut}(\Delta)}(\mathsf{B}(c_0, r)) \subseteq H_1$.

The next lemma corresponds to [6, Lemma 3.11], but some care is needed because of our different definition of the groups U_α.

Lemma 4.21 *Let $T \subseteq S$ be essential and let $A \in \mathscr{A}_0$. Then the following are equivalent:*

(a) *H_1 contains L_T^+;*
(b) *H_1 is transitive on $\mathcal{R}_T$;*
(c) *N_A is transitive on $\mathcal{R}_T \cap A$;*
(d) *W_A contains the standard parabolic subgroup W_T of W.*

Proof It is clear that (c) and (d) are equivalent.

We first show that (a) implies (c). It suffices to show that for each chamber c_1 of A that is s-adjacent to c_0 for some $s \in T$, there is an element of N_A mapping c_0 to c_1. Let α be the root of A_0 containing c_0 but not the chamber c_2 in A_0 that is s-adjacent to c_0; notice that U_α and $U_{-\alpha}$ are contained in L_T^+. By Proposition 4.7, there is some $g \in U_\alpha$ fixing c_0 and mapping c_1 to c_2. Now the element $n_\alpha \in \langle U_\alpha, U_{-\alpha}\rangle$ stabilizes A_0 and interchanges c_0 and c_2; it follows that the conjugate $g^{-1}n_\alpha g$ stabilizes A and interchanges c_0 and c_1, as required.

The proofs of the implications (d) $\Rightarrow$ (b) $\Rightarrow$ (a) are exactly as in [6, Lemma 3.11]. □

The next statement is the analogue of [6, Lemma 3.12].

Lemma 4.22 *Let $A \in \mathscr{A}_0$. There exists $I_A \subseteq S$ such that W_A contains a parabolic subgroup P_{I_A} of W of type I_A as a finite index subgroup.*

Proof The proof can be copied ad verbum from [6, Lemma 3.12]. □

For each $A \in \mathscr{A}_0$, we fix such an $I_A \subseteq S$; without loss of generality, we may assume that I_A is essential. We also consider the corresponding parabolic subgroup P_{I_A} contained in W_A. Observe that $P_{I_{A_1}}$ has finite index in $\mathrm{Pc}(W_{A_1})$ by Lemma 2.3, where A_1 is as in Definition 4.18(i). Therefore $I = I_{A_1}$.

The next task in the process of showing that H_1 contains L_J^+ is to prove that $J = I$, which is achieved by the following sequence of steps, each of which follows from the previous ones and which are analogues of results in [6].

Lemma 4.23 *Let $A \in \mathscr{A}_0$ and let I and J be as in Definition 4.18(ii) and 4.15(iii), respectively. Then:*

(i) *H_1 contains L_I^+;*
(ii) *$I_A \subset I$;*
(iii) *W_A contains W_I as a subgroup of finite index;*
(iv) *$I = J$.*

Proof (i) This follows from the fact that $I = I_{A_1}$ and $P_I = W_I$; the conclusion follows from Lemma 4.21.
(ii) See [6, Lemma 3.14].
(iii) See [6, Lemma 3.15].
(iv) See [6, Lemma 3.16]. □

Corollary 4.24 *The group H_1 acts transitively on the chambers of $\mathcal{R}_J$.*

Proof This follows by combining Lemmas 4.21 and 4.23. □

We are approaching our main result; the following proposition already shows, in particular, that H is contained in the stabilizer of a residue, and it will only require slightly more effort to show that it is a *finite index* subgroup of such a stabilizer.

Proposition 4.25 *Every subgroup of $\mathrm{Aut}(\Delta)$ containing H_1 as a subgroup of finite index is contained in a stabilizer $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J\cup J'})$, where J' is a spherical subset of $J^\perp$.*

Proof The proof is exactly the same as in [6, Lemma 3.19]. □

Notice that since Δ is irreducible, the index set $J \cup J'$ is only equal to S if already $J = S$.

Lemma 4.26 *The group H_1 is a finite index subgroup of $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_J)$.*

Proof Let $G := \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_J)$. We already know that H_1 stabilizes $\mathcal{R}_J$ [see Definition 4.15(iii)] and acts transitively on the set of chambers of $\mathcal{R}_J$ (see Corollary 4.24). Notice that the stabilizer in G of a chamber of $\mathcal{R}_J$ is compact, hence H_1 is a cocompact subgroup of G. Since H_1 is also open in G, we conclude that H_1 is a finite index subgroup of G. □

Lemma 4.27 *For every spherical $J' \subseteq J^\perp$, the index of $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_J)$ in $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J\cup J'})$ is finite.*

Proof By [5, Lemma 2.2], we have $\mathrm{Ch}(\mathcal{R}_{J\cup J'}) = \mathrm{Ch}(\mathcal{R}_J) \times \mathrm{Ch}(\mathcal{R}_{J'})$. As J' is spherical, the chamber set $\mathrm{Ch}(\mathcal{R}_{J'})$ is finite; the result follows. □

We are now ready to prove our main theorem.

Theorem 4.28 *Let Δ be a thick irreducible semi-regular locally finite right-angled building of rank at least 2. Then any proper open subgroup of $\mathrm{Aut}(\Delta)$ is contained with finite index in the stabilizer in $\mathrm{Aut}(\Delta)$ of a proper residue.*

Proof Let H be a proper open subgroup of $\mathrm{Aut}(\Delta)$. If H is compact, then the result follows from Proposition 4.14.

So assume that H is not compact. By Definition 4.15(iii), we may assume that H contains a finite index subgroup H_1 which, by Corollary 4.24, acts transitively on the chambers of some residue $\mathcal{R}_J$. By Proposition 4.25, H is a subgroup of $G := \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J\cup J'})$ for some spherical $J' \subseteq J^{\perp}$. On the other hand, Lemmas 4.26 and 4.27 imply that H_1 is a finite index subgroup of G; since H_1 is a finite index subgroup of H, it follows that also H has finite index in G.

It only remains to show that $\mathcal{R}_{J \cup J'}$ is a proper residue. If not, then $G = \mathrm{Aut}(\Delta)$, but since G is simple (Theorem 4.2) and infinite, it has no proper finite index subgroups. Since H is a proper open subgroup of G, the result follows. □

5 Two applications of the main theorem

In this last section we present two consequences of Theorem 4.28, both of which were suggested to us by Pierre-Emmanuel Caprace. The first states that the automorphism group of a locally finite thick semi-regular right-angled building Δ is Noetherian (see Definition 5.1); the second deals with reduced envelopes in $\mathrm{Aut}(\Delta)$.

Definition 5.1 We call a topological group *Noetherian* if it satisfies the ascending chain condition on open subgroups.

We will prove that the group $\mathrm{Aut}(\Delta)$ is Noetherian by making use of the following characterization.

Lemma 5.2 ([6, Lemma 3.22]) *Let G be a locally compact group. Then G is Noetherian if and only if every open subgroup of G is compactly generated.*

Proposition 5.3 *Let Δ be a locally finite thick semi-regular right-angled building. Then the group* $\mathrm{Aut}(\Delta)$ *is Noetherian.*

Proof By Lemma 5.2, we have to show that every open subgroup of $\mathrm{Aut}(\Delta)$ is compactly generated. By Theorem 4.28, every open subgroup of $\mathrm{Aut}(\Delta)$ is contained with finite index in the stabilizer of a residue of Δ.

Stabilizers of residues are compactly generated, since they are generated by the stabilizer of a chamber c_0 (which is a compact open subgroup) together with a choice of elements mapping c_0 to each of its (finitely many) neighbors. Since a closed cocompact subgroup of a compactly generated group is itself compactly generated (see [13]), we conclude that indeed every open subgroup of $\mathrm{Aut}(\Delta)$ is compactly generated and hence $\mathrm{Aut}(\Delta)$ is Noetherian. □

Our next application deals with reduced envelopes, a notion introduced by Colin Reid [15] in the context of arbitrary totally disconnected locally compact (t.d.l.c.) groups.

Definition 5.4 (i) Two subgroups H_1 and H_2 of a group G are called *commensurable* if $H_1 \cap H_2$ has finite index in both H_1 and H_2.

(ii) Let G be a totally disconnected locally compact (t.d.l.c.) group and let $H \leq G$ be a subgroup. An *envelope* of H in G is an open subgroup of G containing H. An envelope E of H is called *reduced* if for any open subgroup E_2 with $[H : H \cap E_2] < \infty$ we have $[E : E \cap E_2] < \infty$.

Not every subgroup of G has a reduced envelope, but clearly any two reduced envelopes of a given group are commensurable.

Theorem 5.5 ([14, Theorem B]) *Let G be a t.d.l.c. group and let H be a (not necessarily closed) compactly generated subgroup of G. Then there exists a reduced envelope for H in G.*

We will apply Reid's result to show the following.

Proposition 5.6 *Every open subgroup of* $\mathrm{Aut}(\Delta)$ *is commensurable with the reduced envelope of a cyclic subgroup.*

Proof Let H be an open subgroup of $\mathrm{Aut}(\Delta)$ and assume without loss of generality that $J \subseteq S$ and $H_1 = H \cap \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_J)$ are as in Definition 4.15(iii). Let h_1 be the hyperbolic element of H_1 as in Definition 4.18, so that $\mathrm{Pc}(\overline{h_1}) = W_J$.

By Theorem 5.5, the group $\langle h_1 \rangle$ has a reduced envelope E in $\mathrm{Aut}(\Delta)$. In particular, $[E : E \cap H_1]$ is finite.

On the other hand, $H_2 := E \cap \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_J)$ is an open subgroup of G containing $\langle h_1 \rangle$, hence Lemma 4.26 applied on H_2 shows that H_2 is a finite index subgroup of $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_J)$ for the same subset $J \subseteq S$, i.e.,

$$[\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_J) : \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_J) \cap E] < \infty.$$

Since also H_1 has finite index in $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_J)$ by Lemma 4.26 again, it follows that also $[H_1 : H_1 \cap E]$ is finite. We conclude that H_1, and hence also H, is commensurable with E, which is the reduced envelope of a cyclic subgroup. □

Acknowledgements This paper would never have existed without the help of Pierre-Emmanuel Caprace. Not only did he suggest the study of open subgroups of the automorphism group of right-angled buildings to us; we also benefited a lot from discussions with him. Two anonymous referees did a great job in pointing out inaccuracies and suggesting improvements for the exposition. We also thank the Research Foundation in Flanders (F.W.O.-Vlaanderen) for their support through the project "Automorphism groups of locally finite trees" (G011012N).

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