Neutrosophic Units of Neutrosophic Rings and Fields

T. Chalapathi[1], and R. V. M. S. S. Kiran Kumar[2]

[1]Department of Mathematics, Sree Vidyanikethan Eng.College Tirupati,-517502, Andhra Pradesh, India.
E-mail: chalapathi.tekuri@gmail.com.

[2]Research Scholar, Department of Mathematics, S.V.University, Tirupati,-517502, Andhra Pradesh, India.
E-mail: kksaisiva@gmail.com.

Abstract. Let $N(R, I)$ be a commutative Neutrosophic ring with unity. Then the set of all Neutrosophic group units of $N(R, I)$ is denoted by $N(R^{\times}, I)$. In this paper, we studied concrete properties of $N(R^{\times}, I)$ and presented some standard examples with construction of different illustrations and also examine properties of $N(R^{\times}, I)$ satisfied by certain general collections of classical rings and fields. Further, we proved an important result $N(Z_m^{\times} \times Z_n^{\times}, I) \ncong N(Z_m^{\times}, I) \times N(Z_n^{\times}, I)$ for all positive integers m and n.

Keywords: Classical ring, group units, Neutrosophic ring, Neutrosophic units, Neutrosophic isomorphism.

1. Introduction

In recent years, the inter connection between classical structures and Neutrosophic structures is studied by few researchers. For such kind of study, researchers defined new algebraic structures whose elements are generated by elements in classical algebraic set and indeterminate of the real world problem with respect to algebraic operations on the well defined Neutrosophic elements.

The idea of associating a Neutrosophic structure to a classical structure first appears in [1, 2]. For the elements of the Neutrosophic set, Vasantha Kandasamy and Smarandache takes all elements of a classical ring R together with indeterminate I. The notion $N(R, I)$ of Neutrosophic ring was introduced by Vasantha Kandasamy and Smarandache in 2006 and the Neutrosophic element in $N(R, I)$ is denoted by $a+bI$ if for all $a, b \in R$ and $I^2 = I$. Basically, they specify that $N(R, I)$ is not a classical ring with respect to Neutrosophic addition and Neutrosophic multiplication. Further investigation of Neutrosophic rings was done by Agboola, Akinola and Oyebolain [3, 4]. Recently, Chalapathi and Kiran studied the enumeration of Neutrosophic self additive inverse elements of Neutrosophic rings and fields in [5].

Neutrosophic rings are additive Neutrosophic groups with a new binary operation of Neutrosophic multiplication. This new kind of Neutrosophic multiplication operation constrains the new generated algebraic structures of classical rings and makes it more benefit than classical rings to obtained elementary structural theorems of indeterminacy modeled situations. So, the use of Neutrosophic algebraic theory becomes inevitable when a real world problem contains indeterminacy.

In this paper, we study some concepts of Neutrosophic units of Neutrosophic rings and fields explained with suitable examples, and examine properties satisfied by certain general collections of classical rings. The classical rings of primary interest are finite, so many of the results about classical groups and Neutrosophic groups will be helpful fundamentally.

2. Definitions and notations

In this section, we discuss the terminology used when working with the two Neutrosophic operations, namely Neutrosophic addition and Neutrosophic multiplication, in an abstractly given Neutrosophic rings. Before going to the abstract definition of a Neutrosophic ring, we get some definitions and notations by considering the classical rings from [6].

Let R be a ring. If there is an element $1 \in R$ such that $1 \neq 0$ and $1a = a = a1$ for each element $a \in R$, we say that R is a ring with unity. The ring R is commutative if $ab = ba$ for all $a, b \in R$. Suppose R has unity 1. Then $R^{\times}$ denote the units of R. So, an element $u \in R^{\times}$ is a unit of R if there exist $u' \in R^{\times}$ such that $uu' = u'u = 1$, and $R^{\times}$ forms an abelian group under usual multiplication of R. Next the ring F is a field if its multiplication is commutative and if every non zero element of F is a unit. Now we recall that the following well known results about $R^{\times}$ and $F^{\times}$ from [6].

Theorem.2.1 Let R and S be finite commutative rings. Then $(R \times S)^{\times} \cong R^{\times} \times S^{\times}$ as groups. Also, $\left|R^{\times} \times S^{\times}\right| = \left|R^{\times}\right|\left|S^{\times}\right|$.

Theorem.2.2 Let Z_n be the ring of integers modulo n. Then $(Z_m \times Z_n)^{\times} \cong Z_m^{\times} \times Z_n^{\times}$ if and only if $\gcd(m, n) = 1$.

Theorem.2.3 Let R be a finite Boolean ring .Then $\left|R^{\times}\right| = 1$.

Theorem.2.4 Let F be a finite field of order $n > 1$. Then its unit group $F^{\times}$ is a cyclic group of order $n-1$.

Now define the Neutrosophic group and these groups in general do not have classical group structure, which are defined specifically with respect to Neutrosophic multiplication as follows.

Definition.2.5 Let $(G, \cdot)$ be a multiplicative group. Then the set $\langle G \cup I \rangle = \langle a, aI : a \in G, I^2 = I \rangle$ is called a Neutrosophic group generated by G and I under the operation on G, where I is the Neutrosophic element. Based on this definition we have the following.

1. Neutrosophic group $\langle G \cup I \rangle$ of G is also denoted by $N(G, I)$.
2. $N(G, I) = G \cup GI$, where $G \cap GI = \phi$ and $GI = \{aI : a \in G\}$.
3. $G \subset N(G, I)$ and $N(G, I) \not\subset G$.
4. Let $n \geq 1$ be a positive integer. Then $(aI)^n = a^n I$ for every $a \in G$.

Now we proceed on to define the Neutrosophic ring and consider their basic properties from [2].

Definition.2.6 Let $(R, +, \cdot)$ be a ring. Then the Neutrosophic set $N(R, I) = \{a + bI : a, b \in R, I^2 = I\}$ is called Neutrosophic ring generated by R and I under the following Neutrosophic addition and Neutrosophic multiplication operations.

1. $(a+bI)+(c+dI) = (a+c)+(b+d)I$.

2. $(a+bI)(c+dI) = ac+(bc+ad+bd)I$.

Properties of $N(R,\ I)$ **.2.7**

1. R is a commutative ring with unity $1 \Leftrightarrow N(R,\ I)$ is a commutative Neutrosophic ring with unity 1 and Neutrosophic unity I .
2. If R is finite ring then $|N(R,\ I)| = |R|^2$.
3. In general, $I+I \neq I$ and $-I \neq I$, where $-I$ exist in $N(R,\ I)$. In particular, $-I = I$ if and only if $N(R,\ I) \cong N(Z_2,\ I)$.
4. $I^n = I$ for each $n > 1$

For further details about Neutrosophy and Neutrosophic rings the reader should refer [7, 8].

3. Neutrosophic units

In this section we define Neutrosophic units of finite commutative rings, fields and study its concrete properties which are comparing the group units of classical rings and fields.

Definition.3.1 Let $R^{\times}$ be the set of group units of the commutative ring R . Then the set

$$N(R^{\times},\ I) = \langle u, uI : u \in R^{\times},\ \ I^2 = I \rangle$$

is called Neutrosophic group units or simply Neutrosophic units generated by $R^{\times}$ and I under the operations of $R^{\times}$, where I^{-1} does not exist.

Examples.3.2

1. $N(Z_3^{\times},\ I) = \{1,\ 2,\ I,\ 2I\}$.
2. $N(Z_6^{\times},\ I) = \{1,\ 5,\ I,\ 5I\}$.

Properties of $N(R^{\times},\ I)$ **.3.3**

1. $N(R^{\times},\ I)$ is a Neutrosophic group but not a classical group.
2. $R^{\times} \subset N(R^{\times},\ I) \subset N(R,\ I)$.
3. $R^{\times}I \subset N(R^{\times},\ I) \subset N(R,\ I)$.
4. $R^{\times} \cap R^{\times}I = \phi$ and $N(R^{\times},\ I) = R^{\times} \cup R^{\times}I$.
5. For any $u, u' \in R$, the Neutrosophic element $u+u'I$ is a Neutrosophic unit if and only if either $u = 0$ or $u' = 0$.
6. Let $N(R,\ I)$ be a Neutrosophic ring without zero devisors. Then for any $u, u' \in R^{\times}$, $uI = vI \Leftrightarrow uI - vI = 0 \Leftrightarrow (u-v)I = 0 \Leftrightarrow u = v$, since $I \neq 0$.

Theorem. 3.4 For any non-trivial integral domain R we have $|R^{\times}| = |R^{\times}I|$.

Proof. Define a map $f: R^{\times} \to R^{\times}I$ by the relation $f(u) = u^{-1}I$ for every $u \in R^{\times}$, $I^2 = I$ and I^{-1} does not exist. Trivially, $f(1) = I$. Further, for any $u, v \in R^{\times}$, $f(uv) = (uv)^{-1}I = u^{-1}v^{-1}I = (u^{-1}I)(v^{-1}I) = f(u)f(v)$ this implies that f is a group homomorphism. Also, for each $u \in R^{\times}$, there exist unique $u^{-1} \in R^{\times}$ such that $f(u^{-1}) = (u^{-1})^{-1} I = uI$, f is onto. Finally, $f(u) = f(v)$ implies that $u^{-1}I = v^{-1}I$.

Therefore, $(1 - uv^{-1})I = 0$ implies $u = v$ because $N(R,\ I)$ has no zero devisors and $I \neq 0$. This proves that there is a one-one correspondence between $R^{\times}$ and $R^{\times}I$, and hence $|R^{\times}| = |R^{\times}I|$.

Theorem.3.5 If $|R| = 1$, then $N(R^{\times},\ I)$ is empty.

Proof. Follows from well-known result that $R = \{0\}$ if and only if $N(R,\ I) = \{0\}$.

Theorem. 3.6 For any finite non-trivial commutative ring R we have $2 \leq |N(R^{\times},\ I)| \leq 2|R^{\times}|$.

Proof. Suppose $|R| = 2$. Then $R^{\times} = \{1\}$ and $N(R^{\times},\ I) = \{1, I\}$. Therefore, $|N(R, I)| = 2$, it is one extremity of the required inequality. Further, if $|R| > 2$, then by the definition of $N(R^{\times}, I)$, we have $N(R^{\times},\ I) = R^{\times} \cup R^{\times}I$ and $R^{\times} \cap R^{\times}I = \phi$. Thus, by the Theorem [3.4], $|N(R^{\times}, I)| = |R^{\times}| + |R^{\times}I| = 2|R^{\times}|$ **,** which are maximum number of elements in $N(R^{\times},\ I)$. This completes the proof.

In what follows here onward, $\varphi(n)$ denotes the well known Euler-Totient function of the integer $n \geq 1$, which gives the number of positive integers less than n that are relatively prime to n . For more details of $\varphi(n)$ we refer [9]. The immediate results are consequences of the Theorem [3.6].

Corollary. 3.7 Let $n > 1$ be a positive integer. Then the maximum number of elements in $N(Z_n^{\times},\ I)$ is $2\varphi(n)$. Moreover, this bound is sharp.

Proof. We know that $Z_n^{\times}$ is the group of units of the ring Z_n of integers modulo n . Then clearly, in view of Theorem [3.6], $|N(Z_n^{\times},\ I)| = 2|Z_n^{\times}| = 2\varphi(n)$.

Corollary.3.8 Let $n \geq 1$. If R is a Boolean ring of order 2^n , then $|N(R^{\times},\ I)| = 2$.

Proof. By the Theorem [2.3], we know that R is a finite Boolean ring if and only if $R^{\times} = \{1\}$. Hence $|N(R^{\times},\ I)| = 2$.

Let F be a finite field of order $|F| > 1$. Then $F^* = F - \{0\} = F^{\times}$ is a cyclic group with respect to multiplication on F . But $N(F^{\times},\ I)$ is not a cyclic group with respect to either multiplication or Neutrosophic multiplication. However, $F^{\times}I$ is a Neutrosophic semigroup and it is generated by uI where u generator of $F^{\times}$. In this connection we have to prove that the following results and for further information of fields and Neutrosophic field's reader refer [10] and [5], respectively.

Theorem.3.9 The Neutrosophic group $N(F^{\times},\ I)$ is not a cyclic group.

Proof. By characterization of finite fields, it is well known that F be a finite field of order n if and only if $F^{\times}$ is a cyclic group of order $n-1$ with respect to multiplication defined on F . Therefore, for a generator $u \in F^{\times}$ we have $F^{\times} = \langle u \rangle$. To complete the proof, it is enough to show that the Neutrosophic group $N(F^{\times},\ I)$ is not a cyclic. If possible assume that $N(F^{\times},\ I)$ generated by its Neutrosophic unit uI , then

$|N(F^{\times},\ I)| = |uI| \ \Rightarrow (n-1)^2 = |uI|$

$\Rightarrow (uI)^{(n-1)^2} = 1$

$\Rightarrow u^{(n-1)^2} I^{(n-1)^2} = 1$

$\Rightarrow u^{(n-1)^2} I = 1$,

which is not possible because $I \neq 1$, $u^{(n-1)^2} \neq 1$ and $u^{(n-1)^2}$ is not multiplicative inverse of I.

The above theorem proves that the following result, which is of fundamental importance of Neutrosophic rings and fields.

Theorem. 3.10 $F^{\times} = \langle u \rangle$ if and only if $F^{\times} I = \langle uI \rangle$.

Proof. Let $u \in F^{\times}$. Then

$F^{\times} = \langle u \rangle \Leftrightarrow u^{n-1} = 1$ where $n = |F|$

$\Leftrightarrow u^{n-1} I^{n-1} = I^{n-1}$

$\Leftrightarrow (uI)^{n-1} = I$

$\Leftrightarrow \langle uI \rangle = F^{\times} I$.

We usually write $u1 = u = 1u$ for every u in $R^{\times}$ and $uI \neq u \neq Iu$ for every $u \neq 1$ in $R^{\times}$. So, the element 1 is unity and I is not unity but it is Neutrosophic unit because $I^2 = I$ and I^{-1} does not exist. The most familiar examples of infinite Neutrosophic units of infinite rings Z and $Z[i]$, respectively, are $N(Z^{\times},\ I) = \{1, -1, I, -I\}$ and $N(Z[i]^{\times},\ I) = \{1, -1, i, -i, I, -I, iI, -iI\}$ where $i^2 = -1$ and $I^2 = I$. These examples support our claim that the sum of elements in $N(R^{\times},\ I)$ is zero. However, the following important results showing that the sum of elements of a Neutrosophic ring is zero when $\text{char}(R) \neq 2$. This is one of similar result of classical rings.

Theorem. 3.11 If $\text{char}(R) = 2$ then the sum of elements of $N(R^{\times},\ I)$ is not zero.

Proof. It is obvious because $1, I \in N(R^{\times},\ I)$ implies $1 + I \neq 0$.

Theorem. 3.12 Let $N(R^{\times},\ I)$ be a commutative Neutrosophic ring whose characteristic is not equal to 2, then $bI \neq -bI$ for every $b \in R$.

Proof. Suppose $bI = -bI \Leftrightarrow 2bI = 0$ and

$2a = 0 \Leftrightarrow 2(a + bI) = 0 \Leftrightarrow 2(a + bI) = 0 \Leftrightarrow \text{char}\left(N(R,\ I)\right) = 2$ because $a + bI \in N(R,\ I)$.

Theorem. 3.13 Let F be a finite field. If $|F| > 2$ then the sum of the elements of $N(F^{\times},\ I)$ is zero.

Proof. Suppose that $|F| = n > 2$. Then the Neutrosophic units group $N(F^{\times},\ I)$ is the disjoint union of $F^{\times}$ and $F^{\times} I$. By the Theorem [3.9] and Theorem [3.10], we have

$u^n = 1$ and $(uI)^n = I \Rightarrow 1 - u^n = 0$ and $I - (uI)^n = 0$

$\Rightarrow (1-u)\ (1 + u + u^2 + \cdots + u^{n-1}) = 0$ and

$(I - uI)\ (I + uI + (uI)^2 + \cdots + (uI)^{n-1}) = 0$.

As $uI \neq I$ and $u \neq 1$, these relations becomes

$$1+u+u^2+\cdots+u^{n-1}=0 \text{ and}$$

$$I+uI+(uI)^2+\cdots+(uI)^{n-1}=0 .$$

This implies that the sum of elements in the Neutrosophic units group $N(F^{\times},\ I)$ is zero. Hence the result.

The following table illustrates the main differences between classical field and their Neutrosophic filed.

	Classical filed	Neutrosophic filed.
1	$\lvert F\rvert = p^n$	$\lvert N(F,\ I)\rvert = p^{2n}$
2	$F^{\times}$ is a group of order p^n-1	$N(F^{\times},\ I)$ is a Neutrosophic group of order $2(p^n-1)$
3	$F^{\times}$ is a cyclic group	$N(F^{\times},\ I)$ is not a cyclic group
4	$F^{\times}=\langle u\rangle$	$F^{\times}I=\langle uI\rangle$
5	$1\in F^{\times}$	$1\notin F^{\times}I$
6	$Z_2^{\times}=\{1\}$	$N(Z_2^{\times},\ I)=\{1,\ I\}$

4. Isomorphic properties of Neutrosophic units

Isomorphism of finite groups is central to the study of point symmetries and geometric symmetries of any object in the nature. They also provide abundant relations of abelian and non-abelian groups. If the group $R^{\times}$ is isomorphic to the group $S^{\times}$, we write $R^{\times}\cong S^{\times}$, the map $f: R^{\times}\to S^{\times}$ is an isomorphism if there exist a one-one and onto map such that the group operation preserved. The concept of isomorphism of groups is analogues to the concept of Neutrosophic isomorphism of Neutrosophic groups. For this reason the authors Agboola et al. [3, 4] and Chalapathi and Kiran [5] define Neutrosophic group isomorphism as follows.

Definition.4.1 Two Neutrosophic groups $N(R^{\times},\ I)$ and $N(S^{\times},\ I)$ are Neutrosophic isomorphic if there exist a well-defined map $\phi: N(R^{\times},\ I)\to N(S^{\times},\ I)$ such that

1. $\phi(1)=1$ and $\phi(I)=I$,
2. ϕ is a group homomorphism,
3. ϕ is one-one correspondence.

If $N(R^{\times},\ I)$ is Neutrosophic isomorphic to $N(S^{\times},\ I)$, we write $N(R^{\times},\ I)\cong N(S^{\times},\ I)$.

Theorem.4.2 [6]. Let R and S be any two non-trivial finite commutative rings. Then $R\cong S$ if and only if $R^{\times}\cong S^{\times}$.

An important consequence of above theorem is the following immediate in Neutrosophic rings which we state as a theorem in view of its importance throughout our study of Neutrosophic ring theory.

Theorem.4.3 If $R^{\times}\cong S^{\times}$ then $N(R^{\times},\ I)\cong N(S^{\times},\ I)$.

Proof. Let $R^{\times}$ and $S^{\times}$ be the set of units of the rings R and S respectively. Suppose, $R^{\times} \cong S^{\times}$. Then there exist a group isomorphism $f : R^{\times} \to S^{\times}$ such that $f(1) = 1$. Now define a map $\phi : N(R^{\times},\ I) \to N(S^{\times},\ I)$ by setting

$$\phi(x) = \begin{cases} f(x) & if\ \ x \in R^{\times} \\ f(x)I & if\ \ x \in R^{\times}I \end{cases}$$

for all $x \in N(R^{\times},\ I) = R^{\times} \cup R^{\times}I$. Because f is a group isomorphism, we get ϕ is well defined. For $I \in R^{\times}I$, we have $\phi(I) = \phi(1I) = f(1)I = 1I = I$. Next, we show that ϕ is a homomorphism. Writing x for uI and y for $u'I$, where $u, u' \in R^{\times}$, $\phi(xy) = \phi\big((uI)(u'I)\big) = \phi(uu'I) = f(uu')I = f(u)f(u')I = (f(u)I)(f(u')I)$ $= \phi(x)\phi(y)$. Clearly, ϕ is onto, since f is onto. Finally, we show that ϕ is one-one. For this let $\phi(x) = \phi(y)$, then $f(u)I = f(u')I \Rightarrow \big(f(u) - f(u')\big)I = 0$

$\Rightarrow f(u) - f(u') = 0$, since $I \neq 0$ and f is one-one. Hence, $N(R^{\times},\ I) \cong N(S^{\times},\ I)$.

In view of the Theorem [2.2] and Theorem [4.3], the proof of the following result is obvious.

Theorem. 4.4 Let m and n be two positive integers such that $m > 1$ and $n > 1$. Then the following are equivalent.

1. $\gcd(m,\ n) = 1$,
2. $Z_{mn}^{\times} \cong Z_m^{\times} \times Z_n^{\times}$,
3. $N(Z_{mn}^{\times},\ I) \cong N(Z_m^{\times},\ I) \times N(Z_n^{\times},\ I)$.

Theorem. 4.5 Let $m > 1$ and $n > 1$ be any two positive integers. Then

$$N(Z_m^{\times} \times Z_n^{\times},\ I) \ncong N(Z_m^{\times},\ I) \times N(Z_n^{\times},\ I).$$

Proof. Let $m > 1$ and $n > 1$ be any two positive integers. By Theorem [2.1] and Corollary [3.7] we have $\left|Z_m^{\times}\right| = \phi(m)$, $\left|Z_n^{\times}\right| = \phi(n)$ and $\left|N(Z_m^{\times} \times Z_n^{\times},\ I)\right| = 2\varphi(m)\varphi(m)$. But

$\left|N(Z_m^{\times},\ I) \times N(Z_n^{\times},\ I)\right| = \left|N(Z_m^{\times},\ I)\right|\left|N(Z_n^{\times},\ I)\right|$

$= 4\varphi(m)\varphi(m)$. Hence the result.

Acknowledgments

The authors wish to express their cordial thanks to Prof. L. Nagamuni Reddy and Prof. S.Vijaya Kumar Varma for valuable suggestions to improve the presentation of this paper.

References

[1] W.B. Vasantha Kandasamy and Florentin Smarandache, Basic Neutrosophic Algebraic Structures and their Applications to Fuzzy and Neutrosophic Models, Hexis, Church Rock (2004).

[2] W.B. Vasantha Kandasamy and F. Smarandache, Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures, Hexis, Phoenix, Arizona (2006).

[3] A.A.A. Agboola, A.D. Akinola and O.Y.Oyebola, Neutrosophic Rings I, Int. J. of Math. Comb, 4(2011)1-14.

[4] A.A.A. Agboola, E.O. Adeleke and S.A. Akinleye, Neutrosophic Rings II, International J.Math. Combin, 2(2012) 1-8.

[5] T.Chalapathi and R.V.M.S.S.Kiran Kumar, Self Additive Inverse Elements of Neutrosophic Rings and Fields, Annals of Pure and Applied Mathematics, 13 (2017) 63-72.

[6] L. Gilbert, Elements of Modern Algebra, 8th edition, Cengage Learning, (2014).

[7] P. Majumdar, Neutrosophic Sets and Its Applications to Decision Making,Computational Intelligence for Big Data Analysis, Springer International Publishing, Switzerland, 19(2015)97-115.

[8] T Chalapathi and R.V.M.S.S. Kiran Kumar, Neutrosophic Graphs of Finite Groups,Neutrosophic Sets and Systems,15 (2017) 22-30.

[9] Tom M. Apostol, Introduction to Analytic Number Theory, Springer Science & Business Media, (1998).

[10] Gary L. Mullen, Daniel Panario, Handbook of Finite Fields, Taylor & Francis, CRC Press, (2013).

Received: May 15, 2018. Accepted: June 4, 2018.