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# ON THE BEST APPROXIMATION OF THE INFINITESIMAL GENERATOR OF A CONTRACTION SEMIGROUP IN A HILBERT SPACE[1]

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**Abstract:** Let $A$ be the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space $H$. We give an upper estimate for the best approximation of the operator $A$ by bounded linear operators with a prescribed norm in the space $H$ on the class $Q_2 = \{x \in \mathcal{D}(A^2) : \|A^2x\| \le 1\}$, where $\mathcal{D}(A^2)$ denotes the domain of $A^2$.

**Key words:** Contraction semigroup, Infinitesimal generator, Stechkin's problem.

## 1. Introduction

Let $H$ be a Hilbert space with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$, and let $A$ be the infinitesimal generator of a strongly continuous contraction semigroup in $H$. For the definition and properties of the infinitesimal generator of a semigroup in a Banach space see, e.g., [6, §14.2]. Note that a strongly continuous contraction semigroup is also called a contraction semigroup of the class $C_0$ ([8, 9]). For an operator $F$ on the space $H$, $\mathcal{D}(F)$ denotes the domain of $F$. We denote by $I$ the identity operator.

In this paper, we study the so-called Stechkin's problem of the best approximation of the operator $A$ by bounded linear operators with a prescribed norm on the class of elements $x \in \mathcal{D}(A^2)$ such that $\|A^2x\| \le 1$. We give an upper estimate for the best approximation of the operator $A$.

The problem we consider is a special case of the general problem of the best approximation of an unbounded operator by linear bounded ones on a certain class of elements in a Banach space. This problem first appeared in Stechkin's work in 1965–1967 [11]. The problem was studied by a number of authors (see surveys [1], [2], monograph [4], paper [3], and the bibliography therein).

Stechkin formulated this problem in a general setting as follows. Let $X$, $Y$ be two Banach spaces, let $A$ be a linear operator (in general, unbounded) from $X$ to $Y$, and let $Q \subseteq \mathcal{D}(A)$ be a certain class of elements from the domain $\mathcal{D}(A)$ of the operator $A$. We denote by $\mathscr{B}(N)$ the set of

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linear bounded operators from $X$ to $Y$ with the norm $\|T\|_{X\to Y} \le N$. The best approximation of the operator $A$ by linear bounded operators $T \in \mathscr{B}(N)$ on the class $Q$ is

$$E_N(A;Q) = \inf\{U(A,T,Q) : T \in \mathscr{B}(N)\},$$

where

$$U(A,T,Q) = \sup\{\|Ax - Tx\|_Y : x \in Q\}$$

is the deviation of the operator $T$ from the operator $A$ on the class $Q$.

One of the most important cases of the problem formulated above is when the class $Q$ is defined in the following way. Let $Z$ be a Banach space and $B$ be a linear operator from $X$ to $Z$ such that $\mathcal{D}(B) \subseteq \mathcal{D}(A)$. The class $Q$ is then defined as $Q = \{x \in X\ : \|Bx\|_Z \le 1\}$.

Stechkin [11] suggested an estimate from below for the best approximation $E_N(A;Q)$ in terms of the modulus of continuity of the operator $A$ on the class $Q$ defined by

$$\Phi(\delta) = \sup\{\|Ax\|_Y : x \in Q,\ \|x\|_X \le \delta\}, \quad \delta > 0.$$

Namely, Stechkin showed that

$$E_N(A;Q) \ge \sup\{\Phi(\delta) - N\delta :\ \delta > 0\}. \tag{1.1}$$

In particular, when $B = A^n$, the problem $E_N(A^k;Q)$ turned out to be closely connected to the exact constants in the Kolmogorov-type inequalities of the form

$$\|A^k x\| \le C\|x\|^{\frac{n-k}{n}}\|A^n x\|^{\frac{k}{n}}, \quad x \in \mathcal{D}(A^n), \tag{1.2}$$

with $n, k \in \mathbb{N}$, $0 < k < n$, and a certain constant $C$ that depends on $n$ and $k$.

If $A$ is the differentiation operator, inequalities (1.2) are inequalities between the norms of the derivatives of a function. Such inequalities have been studied by a large number of authors (see [1], [2], [4] and the bibliography therein). Here we only mention that Hardy, Littlewood and Pólya [7, Chapter VII, §7.8] obtained the exact inequality

$$\|f'\|^2 \le 2\|f\|\|f''\| \tag{1.3}$$

in the space $L_2(0,\infty)$ on the class of functions $f \in L_2(0,\infty)$ such that $f'$ is locally absolutely continuous on $(0,\infty)$, and $f'' \in L_2(0,\infty)$.

In 1971, Kato [9] proved the following result which can be considered as a generalization of (1.3). Let $A$ be the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space $H$. Then

$$\|Ax\|^2 \le 2\|x\|\|A^2x\|, \quad x \in \mathcal{D}(A^2).$$

In this paper, we study Stechkin's problem of the best approximation of the infinitesimal generator $A$ of a strongly continuous contraction semigroup by bounded linear operators on the class

$$Q_2 = \{x \in \mathcal{D}(A^2) :\ \|A^2x\| \le 1\} \tag{1.4}$$

in a Hilbert space. Namely, we estimate

$$E_N(A;Q_2) = \inf\{U(T) :\ T \in \mathscr{B}(N)\}, \tag{1.5}$$

where

$$U(T) = U(A,T,Q_2) = \sup\{\|Ax - Tx\| :\ x \in Q_2\}. \tag{1.6}$$

## 2. The main result

The main result of the paper is the following statement.

**Theorem 1.** *The best approximation* (1.5) *of the infinitesimal generator $A$ of a strongly continuous contraction semigroup in a Hilbert space on the class $Q_2$ defined in* (1.4) *satisfies the inequality*

$$E_N(A; Q_2) \le \frac{1}{N}.$$

It is known that the infinitesimal generator $A$ of a strongly continuous contraction semigroup in a Banach space possesses the following properties:

1) The domain $\mathcal{D}(A)$ of the operator $A$ is dense (see, e.g., [6, Lemma 14.5, p. 411]).

2) The resolvent set $\rho(A)$ of the operator $A$ contains the right half-plane $\{\lambda \in \mathbb{C} | \ \Re\lambda > 0\}$. Moreover, $\|(A - \lambda I)^{-1}\| \le (\Re\lambda)^{-1}$ for all $\lambda \in \mathbb{C}$ with $\Re\lambda > 0$ (e.g., [6, Theorem 14.7, p. 412]).

Furthermore, if $A$ is the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space, we have additionally:

3) The operator $A$ is upper semibounded, with the upper bound 0, i.e.,

$$\Re(Ax, x) \le 0$$

for $x \in \mathcal{D}(A)$ [6, Lemma 14.9, p. 416].

The following lemma is not new. However, we will formulate and prove it for the sake of completeness.

**Lemma 1.** *Let $A$ be the infinitesimal generator of a strongly continuous contraction semigroup in a Hilbert space $H$ and $c > 0$. Then the operator*

$$B_c = (cI + A)(cI - A)^{-1}$$

*is densely defined and bounded* (*and thus can be extended to the whole space $H$ by continuity*). *Moreover,*

$$\|B_c\| \le 1.$$

*Remark.* The operator $B_c$ is the Cayley transform of the operator $A$ in the terminology of Kato [9], see also [10, p. 545].

P r o o f. Since $c > 0$, the operator $(cI - A)^{-1}$ is defined everywhere on $H$ and bounded. Since $A$ is the infinitesimal generator of a strongly continuous contraction semigroup, the operator $-A$ is $m$-accretive (see [10, Chapter IX, §1.4 as well as Problem 1.18, both p. 485]). Therefore, the domain $\mathcal{D}(A)$ of the operator $A$ is equal to the range $\mathcal{R}((cI - A)^{-1})$ of the operator $(cI - A)^{-1}$ which is dense in $H$ (see [10, Chapter V, §3.10, p. 279]). Thus, $B_c$ is densely defined.

Now we estimate the norm of $B_c$. For $x \in \mathcal{D}(A)$ we have

$$\|cx + Ax\|^2 = c^2\|x\|^2 + \|Ax\|^2 + 2c\Re(Ax, x),$$

$$\|cx - Ax\|^2 = c^2\|x\|^2 + \|Ax\|^2 - 2c\Re(Ax, x).$$

It follows immediately that

$$\|(cI + A)x\| \le \|(cI - A)x\|. \tag{2.1}$$

Now take $y \in \mathcal{D}((cI - A)^{-1})$. Applying (2.1) to $x = (cI - A)^{-1}y \in \mathcal{D}(A)$, we obtain

$$\|(cI + A)(cI - A)^{-1}y\| \le \|y\|,$$

and thus $\|B_c\| \le 1$. □

Now we are ready to prove Theorem 1.

P r o o f. We will construct a concrete approximating operator $T$ in problem (1.5) and estimate its norm and its deviation (1.6) from the operator $A$ on the class $Q_2$.

Note that all the operators we consider commute on the set $\mathcal{D}(A^2)$.

The restriction of the operator $A$ to the set $\mathcal{D}(A^2)$ (which we will denote by the same symbol) can be represented as

$$A = \frac{N}{2}(B_N - I) - \frac{1}{2N}(B_N + I)A^2.$$

Put $T : H \to H$,

$$T = \frac{N}{2}(B_N - I).$$

Then, for the restriction of the operator $A - T$ to $\mathcal{D}(A^2)$, we have

$$A - T = -\frac{1}{2N}(B_N + I)A^2.$$

We estimate the norm of the operator $T$ as follows:

$$\|T\| = \frac{N}{2}\|B_N - I\| \le \frac{N}{2}(\|B_N\| + \|I\|) = N. \tag{2.2}$$

For the deviation $U(T)$ of the operator $T$ from the operator $A$, we obtain that

$$U(T) = \sup_{x\in Q_2} \|(A - T)x\| \le \sup_{x\in Q_2} \frac{1}{2N}\|B_N + I\| \cdot \|A^2x\| \le \frac{1}{N}. \tag{2.3}$$

It follows immediately from (2.2) and (2.3) that

$$E_N(A; Q_2) \le U(T) \le \frac{1}{N}.$$

□

## 3. Approximation of the differentiation operator in the space $L_2(0, \infty)$

An important concrete case of problem (1.5) is the problem of the best approximation of the differentiation operator $Df = f'$ by bounded linear operators in the Hilbert space $L_2(0, \infty)$ of real-valued functions whose squares are integrable on $(0, \infty)$ on the class $Q^{(2)}$ defined as follows:

$Q^{(2)}$ is the class of functions $f \in L_2(0,\infty)$ such that $f'$ is locally absolutely continuous on $[0,\infty)$, $f'' \in L_2(0,\infty)$, and $\|f''\| \le 1$. Problem (1.5) takes in this case the form

$$E_N(D;Q^{(2)}) = \inf_{T\in\mathscr{B}(N)} \sup_{f\in Q^{(2)}} \|f' - Tf\|. \tag{3.1}$$

It took about 20 years of research to solve the problem completely. Stechkin's inequality (1.1) and inequality (1.3) of Hardy, Littlewood and Pólya provide the lower bound

$$E_N(D;Q^{(2)}) \ge \frac{1}{2N}.$$

One of the first upper bounds for (3.1)

$$E_N(D;Q^{(2)}) \le \frac{1}{\sqrt{3}N}$$

was obtained by using a concrete approximating operator by the first named author in 1996 [5]. Problem (3.1) was fully solved only in 2014 by Arestov and the second named author [3] . Namely, they showed that

$$E_N(D;Q^{(2)}) = \frac{1}{2N}.$$

In this section, we discuss what the statement of Theorem 1 means in the concrete case (3.1) of problem (1.5). The approximating operator $T$ used in Theorem 1 is

$$T = \frac{N}{2}(B_N - I) = NA(NI - A)^{-1}. \tag{3.2}$$

Below we will describe this operator in the special case. We consider and calculate its norm $\|T\|$ and its deviation $U(T)$ from the operator $A = D$ on the class $Q^{(2)}$.

It is not difficult to see that the operator $T$ in the concrete case can be represented as follows. Let $W$ be the class of functions $y \in L_2(0,\infty)$ such that $y$ is locally absolutely continuous on $[0,\infty)$ and $y' \in L_2(0,\infty)$. For $f \in L_2(0,\infty)$, we consider the differential equation

$$-y' + Ny = f, \quad y \in W. \tag{3.3}$$

For each function $f \in L_2(0,\infty)$, equation (3.3) has a unique solution which is a real-valued function from $L_2(0,\infty)$. The operator $T$ is defined as

$$Tf = Ny', \tag{3.4}$$

where $y$ is the solution of the differential equation (3.3).

Integrating by parts and taking into account that $\lim_{t\to\infty} y(t) = 0$, we obtain (see [3] for details) that

$$\|f\|^2 = \int_0^\infty (-y'(t) + Ny(t))^2 dt = \int_0^\infty (y'(t))^2 dt + N^2 \int_0^\infty (y(t))^2 dt + Ny^2(0).$$

It follows from (3.4) that $\|Tf\|^2 = N^2 \int_0^\infty (y'(t))^2 dt$. Thus, we immediately obtain

$$\|Tf\|^2 \le N^2 \|f\|^2, \tag{3.5}$$

which gives the estimate $\|T\| \le N$. Now we show that indeed $\|T\| = N$. Consider the family of functions $y_K = e^{-Kt}$, $K > 0$. Let $f_K$ be the corresponding right-hand side of equation (3.3). Take an arbitrary $0 < \alpha < 1$. We have

$$\alpha N^2 \|f_K\|^2 - \|Tf_K\|^2 = \alpha N^2 \int_0^\infty (-y_K'(t) + Ny_K(t))^2 dt - N^2 \int_0^\infty (y_K'(t))^2 dt$$
$$= \frac{N^2}{2K}(\alpha(K+N)^2 - K^2).$$

This expression is negative for all $0 < \alpha < \dfrac{K^2}{(N+K)^2}$ which yields $\|Tf_K\|^2 > \alpha N^2 \|f_K\|^2$. Letting $K$ go to infinity (with fixed $N$) we let $\alpha$ approach 1, and thus obtain $\|T\| \ge N$. Consequently, $\|T\| = N$.

Note that inequality (3.5) is a strict inequality if $y \ne 0$ and, consequently, $f \ne 0$. In other words, the norm of the operator $T$ is not attained.

It can be shown similarly that the norm of the operator $V = -\dfrac{1}{2N}(B_N + I)$ is equal to $1/N$. Since the domain $\mathcal{D}(D^2)$ of the operator $D^2$ is dense in $L_2(0,\infty)$, it follows that the deviation of the operator $T$ from the differentiation operator $D$ on the class $Q^{(2)}$ is equal to $1/N$.

Thus, the approximating operator (3.2) gives the estimate $E_N(D; Q^{(2)}) \le \dfrac{1}{N}$ in the general case (1.5) as well as in the concrete case (3.1).

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