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Transmutations for Strings

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Abstract. We investigate the existence and representation of transmutations, also known as transformation operators, for strings. Using measure theory and functional analytic methods we prove their existence and study their representation. We show that in general they are not close to unity since their representation does not involve a Volterra operator but rather the eigenvalue parameter. We also obtain conditions under which the transmutation is either a bounded or a compact operator. Explicit examples show that they cannot be reduced to Volterra type operators.

Key words : Transmutation, Transformation Operator, String, Sturm-Liouville Problem
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1. Introduction

We are concerned with the existence and representation of transmutation operators between two strings $\mathbb{S}_1$ and $\mathbb{S}_2$ which are respectively defined by

$$\begin{cases} \mathbb{S}_1(f) = -\frac{d}{dM_1(x)}\frac{d^+}{dx^+}f(x), & 0 \le x < L \\ bf'(0) - af(0) = 0 \end{cases} \tag{1.1}$$

and

$$\begin{cases} \mathbb{S}_2(f) = -\frac{d}{dM_2(x)}\frac{d^+}{dx^+}f(x) & 0 \le x < L \\ bf'(0) - af(0) = 0 \end{cases} \tag{1.2}$$

where $dM_i(x)$, for $i = 1, 2$, are Stieltjes measures, i.e. $M_i(x)$ is a real valued function, continuous from the right, nondecreasing and normalized by $M_i(0+) = 0$. The string $\mathbb{S}_i$ models the vibration of a string and $M_i(x)$ can be seen as its mass between 0 and x, while L is its total length. The constants $a,\ b$ are real with $a^2 + b^2 \neq 0$, and describe how the strings are tied down at the origin. Observe that M_i can include jumps and $\frac{d^+}{dx^+}f(x)$ denotes the usual right derivative at a point x. Recall that $\mathbb{S}_i$, defined by (1.1) and (1.2), are symmetric operators, acting in the Hilbert spaces, see [**15, 13**]

$$L^2_{M_i} = \left\{ f \text{ measurable: } ||f||^2_{M_i} = \int_0^L |f(x)|^2\, dM_i(x) < \infty \right\}.$$

Let us denote by φ and y the normalized solutions, which we call eigensolutions, of the initial value problems

$$\begin{cases} \mathbb{S}_1(\varphi(x,\lambda)) = \lambda\varphi(x,\lambda) \\ \varphi(0,\lambda) = b,\ \varphi'(0,\lambda) = a \end{cases} \quad \text{and} \quad \begin{cases} \mathbb{S}_2(y(x,\lambda)) = \lambda y(x,\lambda) \\ y(0,\lambda) = b,\ y'(0,\lambda) = a. \end{cases} \tag{1.3}$$

If $\varphi(.,\lambda) \in L^2_{M_1}$, then λ is an eigenvalue of $\mathbb{S}_1$ and $\varphi(.,\lambda)$ is an eigenfunction and in case λ belongs to the continuous spectrum then $\varphi(.,\lambda)$ is an eigenfunctional, see [**10**]. Since an eigenfunction is not unique, the initial condition in (1.3) provides a simple normalization.

We recall that an operator $\mathbb{V} : \mathbb{L}^2_{\mathbb{M}_1} \to \mathbb{L}^2_{\mathbb{M}_2}$ is said to transmute the strings $\mathbb{S}_1$ and $\mathbb{S}_2$, see [**3, 4, 17**], if

$$\mathbb{S}_2\mathbb{V} = \mathbb{V}\mathbb{S}_1. \tag{1.4}$$

Applying φ to both sides of (1.4) formally yields

$$\begin{aligned} \mathbb{S}_2\mathbb{V}\varphi(x,\lambda) &= \mathbb{V}\mathbb{S}_1\varphi(x,\lambda) \\ &= \lambda\mathbb{V}\varphi(x,\lambda). \end{aligned}$$

which means that $\mathbb{V}\varphi$ is an eigensolution of $\mathbb{S}_2$, and so if it is unique, we should also have

$$y(x,\lambda) = \mathbb{V}\varphi(x,\lambda). \tag{1.5}$$

One should emphasize that (1.4) does not imply that $\mathbb{S}_1$ and $\mathbb{S}_2$ are similar operators, since their spectra may be different and usually $\varphi(x,\lambda) \notin L^2_{M_1}$ and $y(x,\lambda) \notin L^2_{M_2}$, see[**10**]. Nevertheless the Gelfand-Levitan theory uses (1.5) and (1.4), to compare and express one operator in terms of the other. This simple idea is at the heart of the Gelfand-Levitan inverse spectral theory, which we now briefly outline. The relation (1.5) in the 1951 Gelfand and Levitan theory, [**9**], reads

$$y(x,\lambda) = \cos\left(x\sqrt{\lambda}\right) + \int_0^x K(x,t)\cos\left(t\sqrt{\lambda}\right)dt, \tag{1.6}$$

where $y(x,\lambda)$ is the eigensolution of the Sturm-Liouville problem

$$\begin{cases} -y''(x,\lambda) + q(x)y(x,\lambda) = \lambda y(x,\lambda), \qquad 0 \le x < \infty, \\ y(0,\lambda) = 1,\ y'(0,\lambda) = h. \end{cases} \tag{1.7}$$

A necessary condition for the existence of the transmutation in (1.6) is the asymptotic behavior of the spectral function at infinity, [**9, 17**]

$$\Gamma(\lambda) \approx \frac{2}{\pi}\sqrt{\lambda_+} \ \text{ as } \ \lambda \to \infty. \tag{1.8}$$

A year later, M. G. Krein came up with a completely new direct method for the inverse spectral theory for the string. Surprisingly, it used no transmutations or perturbation techniques, but function theory, continued fractions and moments problem, [**6**]. To reconstruct the mass M of a string, the spectral function is required to satisfy

$$\int_0^\infty \frac{1}{1+\lambda}d\Gamma(\lambda) < \infty. \tag{1.9}$$

Observe that (1.9) covers a larger class of spectral functions than (1.8). Using a set of rules, on how simple operations on M would affect the spectral function, M.G. Krein could in some special cases reconstruct the mass of the string explicitly. In 1966, at the Moscow international congress, M.G. Krein mentioned the open problem regarding the uniqueness of the inverse spectral problem for the string which was then solved few years later by de Branges using Hilbert spaces of entire functions. In 1976, Dym and McKean summarized the inverse spectral theory for the string in their book [**6**].

How to extend the Gelfand–Levitan theory to strings? As a first step, this paper is to address the question of existence and representation of the transmutation between two strings. In [**7, 8**], Dym and Kravitsky studied the existence of the transmutation under the assumption of a small perturbation of the mass. Their method starts with the spectral functions and then solves a nonlinear integral equation through the Gokhberg-Krein special factorization theorem, see also [**1, 11, 24**]. Basically for the transmutation to be close to unity, one needs the measures to be close enough.

Here the treatment is different. We use direct methods, where only the transforms of the strings are used, which avoids the heavy machinery of partial differential equations used in the Gelfand-Levitan theory [**9**].

Note also that in general, a string such as $\mathbb{S}_1$ cannot be reduced to a Sturm-Liouville equation such as (1.7), for the simple reason that the growth condition (1.8) may not be valid, see [**13**]. Also the Liouville transformation cannot be used unless M_i is C^3 and is strictly increasing. For applications and numerical methods of the string we refer to [**2, 4, 5, 6, 16, 17, 21, 23**].

2. Notation

Recall that dM_i is in fact a Lebesgue-Stieltjes measure [**20**], which vanishes when M_i is constant and its support supp $dM_i \subset [0, L]$. The differential expression $-\frac{d}{dM_i(x)}\frac{d^+}{dx^+}$ defines then a symmetric operator in the Hilbert space $L^2_{M_i}$. To avoid any ambiguity about the division by zero, M.G. Krein interpreted the initial value problem $\frac{-d}{dM_i(x)}\frac{d^+}{dx^+}y(x) = f(x),\ y(0) = b,\ y'(0) = a$, when $f \in L^2_{M_i}$, as an integral equation

$$y(x) = ax + b - \int_0^x \int_0^t f(\xi) dM_i(\xi) dt. \tag{2.1}$$

For a self-adjoint extension, we need to examine the right end point. In case the length is infinite, $L = \infty$, it is well known that operator $\mathbb{S}_i$ is in the limit point case at $x = \infty$ if and only if $\int_0^\infty x^2 dM_i(x) = \infty$, see [**15**, p. 70]. In that follows we assume that we are in the limit point case, otherwise we must add a boundary condition at $x = \infty$ to make $\mathbb{S}_1$ in (1.1) self-adjoint. In case the length is finite, $L < \infty$, the type of a boundary condition to be added at $x = L$ depends on the presence of a jump of the mass at $x = L$, which is called "heavy mass", see [**6**]. M.G. Krein allowed the boundary condition at $x = L$ to be a function of λ, which led to a family of spectral functions. He then defined a principal spectral function when the associated transform was onto, [**15**]. Since finite length strings can be extended to the right, without loss of generality we can assume that $L = \infty$ and that $\mathbb{S}_i$ are self-adjoint in $L^2_{M_i}$.

When $\mathbb{S}_1$ is self-adjoint, its eigensolutions, (1.3), form the kernel of the transform associated with $\mathbb{S}_1$

$$L^2_{M_1} \overset{\mathbf{F}_\varphi}{\to} L^2_{\Gamma_1}$$

where

$$\mathbf{F}_\varphi(f)(\lambda) = \int_0^\infty f(x)\varphi(x,\lambda) dM_1(x).$$

The inverse transform is given by

$$f(x) = \int \mathbf{F}_\varphi(f)(\lambda)\varphi(x,\lambda) d\Gamma_1(\lambda)$$

where the spectral function Γ_1 is non decreasing, right continuous, supp $d\Gamma_1$ is the spectrum of $\mathbb{S}_1$ and the Parseval relation, for any $f,\ g \in L^2_{M_1}$ yields

$$\int_0^\infty f(x)\overline{g(x)} dM_1(x) = \int \mathbf{F}_\varphi(f)(\lambda)\overline{\mathbf{F}_\varphi(g)(\lambda)} d\Gamma_1(\lambda).$$

We now introduce a notation used to compare Stieltjes measures

$$d\Gamma_1(\lambda) = O\left(d\Gamma_2\left(\lambda\right)\right) \text{ as } \quad \lambda \to \infty,$$

if for all measurable functions with respect to $d\Gamma_1$, and $d\Gamma_2$

$$\int_N^\infty |f(\lambda)|\, d\Gamma_1(\lambda) \le c \int_N^\infty |f(\lambda)|\, d\Gamma_2(\lambda) \text{ holds for large } N.$$

The fact that $d\Gamma_1$ is absolutely continuous with respect to $d\Gamma_2$, is denoted by $d\Gamma_1 \ll d\Gamma_2$ and means there exists $g \in L^{1,loc}_{\Gamma_2}$ such that $d\Gamma_1(\lambda) = g(\lambda) d\Gamma_2(\lambda)$. Similarly $d\Gamma_1 \ll^{2,loc} d\Gamma_2$ means that $g \in L^{2,loc}_{\Gamma_2}$ while $d\Gamma_1 \ll^\infty d\Gamma_2$ means $\underset{\lambda \in \mathrm{supp} d\Gamma_2}{\text{esssup}}\ g(\lambda) < \infty$ and finally the cut-off function is defined by $x_+ = \begin{cases} x & \text{if } \; x \geq 0 \\ 0 & \text{if } \; x < 0 \end{cases}$. When integrating functions of two variables with respect to one of the variable, we shall indicate it by labeling the measure. For example $f(x,t) \in L^2_{M_1(t)}$ means

$$||f(x,t)||^2_{M_1(t)} = \int_0^\infty |f\,(x,t)|^2\, dM_1(t) < \infty.$$

The measures Γ_i are always associated transforms and so with the variable λ.

In all that follows we assume that the strings in (1.1) and (1.2) have infinite lengths, $M_i(0+) = 0$, $L = \infty$, and are self-adjoint. To this end we need either

$$\int_0^\infty x^2 dM_i(x) = \infty \quad \text{for} \quad i = 1, 2 \text{ (LP case at } x = \infty)$$

or $\int_0^\infty x^2 dM_i(x) < \infty$, limit circle case at $x = \infty$, but then we must add a boundary condition there.

3. Preliminaries

The normalized eigenfunctions of $\mathbb{S}_1$, see (1.1) and (2.1), satisfy the integral equation

$$\varphi(x,\lambda) - ax - b = -\lambda \int_0^\infty (x-t)_+ \varphi(t,\lambda) dM_1(t).$$

For any fixed x, we have $(x-t)_+ \in L^2_{M_1(t)}$, and so $-\frac{1}{\lambda}\left(\varphi(x,\lambda) - ax - b\right)$, as its $\mathbf{F}_\varphi$ transform, belongs to $L^2_{\Gamma_1}$. Therefore by the Parseval relation we get

$$\int \frac{1}{\lambda^2}\left|\varphi(x,\lambda) - ax - b\right|^2 d\Gamma_1(\lambda) = \int_0^x (x-t)^2 dM_1(t) \quad \text{ for } \quad 0 \leq x. \tag{3.1}$$

Similar relations hold for the transform $\mathbf{F}_y$ associated with operator $\mathbb{S}_2$ and its spectral function Γ_2. Using the above relation we have

Proposition 1. *For all $x \geq 0$ we deduce*

(i) $\frac{1}{\lambda}\left(\varphi(x,\lambda) - ax - b\right) \in L^2_{\Gamma_1}$
(ii) $\int \frac{1}{\lambda^2}\left|\varphi(x,\lambda) - ax - b\right|^2 d\Gamma_1(\lambda) = \int_0^x (x-t)^2 dM_1(t)$
(iii) The set $\frac{1}{\lambda}\left(\varphi(x,\lambda) - ax - b\right)$ is complete in $L^2_{\Gamma_1}$

Proof. (i) and (ii) follow from (3.1). To see (iii), we first show that the set $\left\{(x-t)_+\right\}_{x\geq 0}$ is complete in $L^2_{M_1}$ that is for any $g \in L^2_{M_1}$, if

$$G(x) := \int_0^x (x-t)\, g(t) dM_1(t) = 0 \quad \text{ for all } \quad x \geq 0 \quad \text{then} \quad g = 0 \quad dM_1 - \text{a.e.} \tag{3.2}$$

Note that the function $G(x)$ is continuous and differentiable and so

$$0 = \frac{d^+}{dx^+} G(x) = \int_0^x g(t) dM_1(t).$$

Differentiation with respect to M_1 leads to

$$0 = \frac{d}{dM_1(x)} \frac{d^+}{dx^+} G(x) = g(x),$$

which means that the family $\left\{(x-t)_+\right\}_{x\geq 0}$ is complete in $L^2_{M_1(t)}$. The image of a complete set in $L^2_{M_1}$ by the $F_\varphi -$ transform remains complete in $L^2_{\Gamma_1}$ and so $\{\mathbf{F}_\varphi\left((x-t)_+\right)\}_{0\leq x<\infty} = \{\frac{1}{\lambda}\left(\varphi(x,\lambda) - ax - b\right)\}_{0\leq x<\infty}$ is also complete in $L^2_{\Gamma_1}$.$\square$

Similar result can be also stated for the string $\mathbb{S}_2$. We now prove the existence of a transmutation between eigensolutions of $\mathbb{S}_1$ and $\mathbb{S}_2$

PROPOSITION 2. *Assume that* $d\Gamma_1(\lambda) = O\left(d\Gamma_2\left(\lambda\right)\right)$ *as* $\lambda \to \infty$, *then for each* $x > 0$ *there exists* $H(x,.) \in L^2_{M_1}$ *such that*

$$y(x,\lambda) = \varphi(x,\lambda) + \lambda \int_0^\infty H(x,t)\varphi(t,\lambda)dM_1(t). \tag{3.3}$$

Proof. We only need to prove that for any given $x > 0$

$$\int_0^\infty \left| \frac{1}{\lambda}\left(y(x,\lambda) - \varphi(x,\lambda)\right)\right|^2 d\Gamma_1(\lambda) < \infty. \tag{3.4}$$

Observe that since $\frac{1}{\lambda}\left(y(x,\lambda) - \varphi(x,\lambda)\right)$ is continuous in λ, and $y(x,0) - \varphi(x,0) = 0$, then

$$\int_0^N \left| \frac{1}{\lambda}\left(y(x,\lambda) - \varphi(x,\lambda)\right)\right|^2 d\Gamma_1(\lambda) < \infty$$

for any finite N. For large N, use the fact that $d\Gamma_1(\lambda) = O\left(d\Gamma_2\left(\lambda\right)\right)$ as $\lambda \to \infty$ to write

$$\int_N^\infty \left| \frac{1}{\lambda}\left(y(x,\lambda) - ax - b\right)\right|^2 d\Gamma_1(\lambda) \leq c\int_N^\infty \left| \frac{1}{\lambda}\left(y(x,\lambda) - ax - b\right)\right|^2 d\Gamma_2(\lambda) < \infty,$$

thus (3.4) holds. Using the inverse $\mathbf{F}_\varphi -$ transform, (3.4) implies the existence of $H(x,t) \in L^2_{M_1(t)}$ such that

$$\frac{1}{\lambda}\left(y(x,\lambda) - \varphi(x,\lambda)\right) = \int_0^\infty H(x,t)\varphi(t,\lambda)dM_1(t).$$

i.e.

$$y(x,\lambda) = \varphi(x,\lambda) + \lambda \int_0^\infty H(x,t)\varphi(t,\lambda)dM_1(t). \square \tag{3.5}$$

So far the transmutation operator (3.5) has been defined over a family of solutions only, namely $\varphi(.,\lambda)$ and its range is also a family of solutions $y(.,\lambda)$. Unfortunately these solutions cannot be in the Hilbert space $L^2_{M_i}$, when λ is not an eigenvalue, see rigged spaces, [**10**]. Next in order for (3.5) to define a linear operator, its action must be independent of λ, thus we must remove the spectral parameter λ. To this end use the fact that

$$d\varphi^+(x,\lambda) = -\lambda\varphi(x,\lambda)dM_1(x)$$

to recast (3.5) into an operator form

$$y(x,\lambda) = \varphi(x,\lambda) - \int_0^\infty H(x,t)d\varphi^+(t,\lambda). \tag{3.6}$$

To find the domain of the integral operator in (3.6) that maps $\varphi(.,\lambda) \to y(.,\lambda)$, we need to examine the integrability of the kernel H. For that purpose we prove the following proposition, which by itself is of independent interest.

PROPOSITION 3. *Let* Γ_1 *and* Γ_2 *be two nondecreasing functions defining Stieltjes measures. Then*

$$d\Gamma_1 \ll^\infty d\Gamma_2 \ \textit{ if and only if } \ L^2_{\Gamma_2} \subset L^2_{\Gamma_1}.$$

Proof. It is enough to prove the converse, that is the assumption $L^2_{\Gamma_2} \subset L^2_{\Gamma_1}$ leads to boundedness of the identity mapping $L^2_{\Gamma_2} \to L^2_{\Gamma_1}$. Otherwise there exists a sequence $\{f_n\}_{n\geq 1} \subset L^2_{\Gamma_2}$ such that $f_n \overset{L^2_{\Gamma_2}}{\to} 0$ but $f_n \overset{L^2_{\Gamma_1}}{\nrightarrow} 0$. Thus there exists $\varepsilon > 0$ and a subsequence of f_n, which we denote again by f_n, such that

$$\|f_n\|^2_{\Gamma_2} \leq \frac{1}{n^2} \quad \text{but} \quad \|f_n\|^2_{\Gamma_1} \geq \varepsilon > 0.$$

Define by $\psi=\sqrt{\sum\limits_{n\geq 1}|f_n|^2}$. It is readily seen that $\psi\in L^2_{\Gamma_2}$ since $\|\psi\|^2_{\Gamma_2}=\sum\limits_{n\geq 1}\|f_n\|^2_{\Gamma_2}\leq\sum\limits_{n\geq 1}\frac{1}{n^2}<\infty$. However

$$\|\psi\|^2_{\Gamma_1}=\sum_{n\geq 1}\|f_n\|^2_{\Gamma_1}\geq\sum_{n\geq 1}\varepsilon=\infty,$$

which means that there is $\psi\in L^2_{\Gamma_2}$ such that $\psi\notin L^2_{\Gamma_1}$, which contradicts $L^2_{\Gamma_2}\subset L^2_{\Gamma_1}$. Thus the identity mapping $L^2_{\Gamma_2}\longrightarrow L^2_{\Gamma_1}$ must be bounded, i.e.

$$\int|f(\lambda)|^2\,d\Gamma_1(\lambda)\leq c\int|f(\lambda)|^2\,d\Gamma_2(\lambda). \tag{3.7}$$

Thus a negligible set with respect to $d\Gamma_2$ is also negligible with respect to $d\Gamma_1(\lambda)$ by (3.7). By the Radon-Nikodym theorem [**20**] $d\Gamma_1$ is absolutely continuous with respect to $d\Gamma_2$, i.e. there exists a locally $d\Gamma_2$ integrable function $g(\lambda)$ such that

$$d\Gamma_1(\lambda)=g(\lambda)d\Gamma_2(\lambda)\qquad d\Gamma_2\text{ a.e.}$$

We now show that g is essentially bounded. To this end use (3.7) to obtain

$$\int|f(\lambda)|^2\,g(\lambda)d\Gamma_2(\lambda)\leq c\int|f(\lambda)|^2\,d\Gamma_2(\lambda).$$

Thus the mapping $h\rightarrow\int h(\lambda)g(\lambda)d\Gamma_2(\lambda)$ is a bounded functional on $L^1_{d\Gamma_2}$ which implies that $g\in\left(L^1_{d\Gamma_2}\right)'=L^\infty_{d\Gamma_2}$.$\square$

We now prove:

THEOREM 1. *Let $d\Gamma_1\ll^\infty d\Gamma_2$ then*

$$\|H(x,t)\|_{M_1(t)}\leq c\left\|(x-t)_+\right\|_{M_1(t)+M_2(t)}. \tag{3.8}$$

In all that follows by c we denote a universal constant, that can be distinct in different places. Proof. Observe that (3.8) is a uniform bound on the kernel H. ¿From (3.3) it follows that

$$\begin{aligned}\|H(x,t)\|_{M_1(t)}&=\left\|\frac{\varphi(x,\lambda)-y(x,\lambda)}{\lambda}\right\|_{\Gamma_1}\\&\leq\left\|\frac{\varphi(x,\lambda)-ax-b}{\lambda}\right\|_{\Gamma_1}+\left\|\frac{y(x,\lambda)-ax-b}{\lambda}\right\|_{\Gamma_1}.\end{aligned}$$

The fact $d\Gamma_1\ll^\infty d\Gamma_2$, see Proposition 3, implies

$$\left\|\frac{y(x,\lambda)-ax-b}{\lambda}\right\|_{\Gamma_1}\leq c\left\|\frac{y(x,\lambda)-ax-b}{\lambda}\right\|_{\Gamma_2}$$

and Proposition 1, but stated for $\mathbb{S}_2$, then yields

$$\begin{aligned}\|H(x,t)\|_{M_1(t)}&\leq\left\|(x-t)_+\right\|_{M_1(t)}+c\left\|\frac{y(x,\lambda)-ax-b}{\lambda}\right\|_{\Gamma_2}\\&\leq\left\|(x-t)_+\right\|_{M_1(t)}+c\left\|(x-t)_+\right\|_{M_2(t)}\\&\leq c\left\|(x-t)_+\right\|_{M_2(t)+M_1(t)}.\square\end{aligned}$$

In terms of integrals (3.8) means that

$$\int_0^\infty|H(x,t)|^2\,dM_1(t)\leq c\int_0^x(x-t)^2\,d\left[M_1+M_2\right](t).$$

COROLLARY 1. *If $d\Gamma_1\ll^\infty d\Gamma_2$ and $dM_1\ll^\infty dM_2$ then*

$$\|H(x,t)\|_{M_1(t)}\leq c\,\|(x-t)_+\|_{M_2(t)}. \tag{3.9}$$

Proof. Observe that if $\operatorname*{esssup}_{x>0}\frac{dM_1}{dM_2}<\infty$ then $\left\|(x-t)_+\right\|_{M_1(t)}\le c\left\|(x-t)_+\right\|_{M_2(t)}$ and (3.8) yields (3.9).□

Since $M_2(0+)=0$ we have

$$\|(x-t)_+\|^2_{M_2(t)}=\int_0^x (x-t)^2\,dM_2(t)\le x^2\int_0^x dM_2(t)=x^2M_2(x).$$

Thus the norm of $H(x,\cdot)$ satisfies the inequality

$$\|H(x,t)\|_{M_1(t)}\le cx\sqrt{M_2(x)}.$$

We now prove the converse of Theorem 1.

Theorem 2. *Assume that*
i) $y(x,\lambda)=\varphi(x,\lambda)+\lambda\int_0^\infty H(x,t)\varphi(t,\lambda)dM_1(t),$
ii) $\|H(x,t)\|_{M_1(t)}\le c\,\|(x-t)_+\|_{M_2(t)}$
iii) $dM_1\ll^\infty dM_2$

then $\qquad d\Gamma_1\ll^\infty d\Gamma_2.$

Proof: From i) it follows that

$$\frac{1}{\lambda}\left(y(x,\lambda)-\varphi(x,\lambda)\right)=\int_0^\infty H(x,t)\varphi(t,\lambda)dM_1(t)$$

then by $\mathbf{F}_\varphi$-transform $\frac{1}{\lambda}\left(y(x,\lambda)-\varphi(x,\lambda)\right)\in L^2_{\Gamma_1}$ since by *ii)* $H(x,\cdot)\in L^2_{M_1}$. Recall that from Proposition 1 we also have

$$\frac{1}{\lambda}\left(\varphi(x,\lambda)-ax-b\right)\in L^2_{\Gamma_1}$$

and so we deduce that

$$\frac{1}{\lambda}\left(y(x,\lambda)-ax-b\right)\in L^2_{\Gamma_1}.$$

Hence,

$$\begin{aligned}
(3.10)\quad \left\|\frac{1}{\lambda}\left(y(x,\lambda)-ax-b\right)\right\|_{\Gamma_1} &\le \left\|\frac{1}{\lambda}\left(y(x,\lambda)-\varphi(x,\lambda)\right)\right\|_{\Gamma_1}+\left\|\frac{1}{\lambda}\left(\varphi(x,\lambda)-ax-b\right)\right\|_{\Gamma_1}\\
&= \|H(x,t)\|_{M_1(t)}+\|(x-t)_+\|_{M_1(t)}\\
&\le c\,\|(x-t)_+\|_{M_2(t)}+c\,\|(x-t)_+\|_{M_2(t)}\\
&\le c\left\|\frac{1}{\lambda}\left(y(x,\lambda)-ax-b\right)\right\|_{\Gamma_2}
\end{aligned}$$

Let C_o be the set of continuous functions with compact support. Since $C_o\subset L^2_{\Gamma_2}\cap L^2_{\Gamma_1}$ the identity operator $L^2_{\Gamma_2}\to L^2_{\Gamma_1}$ is always densely defined. Observe that (3.10) means that the identity operator $L^2_{\Gamma_2}\to L^2_{\Gamma_1}$ being uniformly bounded on the complete set $\left\{\frac{1}{\lambda}\left(y(x,\lambda)-ax-b\right)\right\}_{\lambda\in\sigma_2}$, whose convex hull is dense in $L^2_{\Gamma_2}$, is therefore bounded. This means that $\|f\|_{\Gamma_1}\le c\,\|f\|_{\Gamma_2}$ for all $f\in L^2_{\Gamma_2}$, or $L^2_{\Gamma_2}\subset L^2_{\Gamma_1}$. Thus by Proposition 3, $d\Gamma_1\ll^\infty d\Gamma_2$.

Combining Proposition 2, Theorems 1, 2 and Corollary 1 we arrive at

Theorem 3. *Let* $dM_1\ll^\infty dM_2$ *then*

$$y(x,\lambda)=\varphi(x,\lambda)+\lambda\int_0^\infty H(x,t)\varphi(t,\lambda)\,dM_1(t),\quad \textit{with } \|H(x,t)\|_{M_1(t)}\le c\|(x-t)_+\|_{M_2(t)},$$

if and only if $d\Gamma_1\ll^\infty d\Gamma_2$.

Under the assumption that Γ_1 grows slowly at $\lambda=0$, we can prove the square integrability of $H(\cdot,t)$ with respect to $dM_1(x)$.

Proposition 4. *Let* $d\Gamma_1 \ll^{\infty} d\Gamma_2$ *and* $dM_1 \ll^{\infty} dM_2$. *If moreover,* $\int_0^{\varepsilon} \frac{1}{\lambda^2} d\Gamma_1(\lambda) < \infty$ *for some* $\epsilon > 0$, *then* $\int_0^{\infty} |H(x,t)|^2 \, dM_1(t) \in L^2_{M_1(x)}$

Proof. From (3.5) and the inverse $\mathbf{F}_{\varphi}$-transform we deduce that

$$\begin{aligned} H(x,t) &= \int \frac{1}{\lambda} \left(y(x,\lambda) - \varphi(x,\lambda) \right) \varphi(t,\lambda) d\Gamma_1(\lambda) \\ &= \int y(x,\lambda) \frac{\varphi(t,\lambda)}{\lambda} d\Gamma_1(\lambda) - \int \frac{\varphi(t,\lambda)}{\lambda} \varphi(x,\lambda) d\Gamma_1(\lambda) \\ &= I_1(x,t) - I_2(x,t). \end{aligned}$$

To examine $I_2(x,t)$, recall that any spectral function Γ_1 of the string satisfies $\int_0^{\infty} \frac{1}{1+\lambda} d\Gamma_1(\lambda) < \infty$, see [**6**]. The assumption $\int_0^{\varepsilon} \frac{1}{\lambda^2} d\Gamma_1(\lambda)$ implies that $\int_0^{\infty} \frac{1}{\lambda^2} d\Gamma_1(\lambda) < \infty$. Since $\mathbb{S}_1$ has at most a finite number of negative eigenvalues, then $\int_{-\infty}^{\infty} \frac{1}{\lambda^2} d\Gamma_1(\lambda) < \infty$ which means that

$$\frac{1}{\lambda} (at + b) \in L^2_{\Gamma_1}, \tag{3.11}$$

which combined with $\frac{1}{\lambda} \left(\varphi(t,\lambda) - at - b \right) \in L^2_{\Gamma_1}$ see Proposition 1, imply that $\frac{1}{\lambda} \varphi(t,\lambda) \in L^2_{\Gamma_1}$, i.e.

$$\int \left| \frac{1}{\lambda} \varphi(t,\lambda) \right|^2 d\Gamma_1(\lambda) < \infty. \tag{3.12}$$

For every fixed $t > 0$, $I_2(\cdot,t)$ being the inverse $\mathbf{F}_{(}\varphi)$-transform of $\frac{1}{\lambda}\varphi(t,\lambda) \in L^2_{\Gamma_1}$ belongs to $L^2_{M_1(x)}$.

Now use the fact that $d\Gamma_1 \ll^{\infty} d\Gamma_2$ to recast the function I_1 in the form

$$\begin{aligned} I_1(x,t) &= \int y(x,\lambda) \frac{1}{\lambda} \varphi(t,\lambda) d\Gamma_1(\lambda) \\ &= \int y(x,\lambda) \frac{1}{\lambda} \varphi(t,\lambda) \frac{d\Gamma_1}{d\Gamma_2}(\lambda) d\Gamma_2(\lambda). \end{aligned}$$

It remains to see $I_1(\cdot,t)$ as the $\mathbf{F}_y-$inverse transform of a function from $L^2_{\Gamma_2}$. To this end we estimate the function $\frac{\varphi(t,\lambda)}{\lambda} \frac{d\Gamma_1}{d\Gamma_2}(\lambda)$ using (3.12)

$$\int \left| \frac{\varphi(t,\lambda)}{\lambda} \frac{d\Gamma_1}{d\Gamma_2}(\lambda) \right|^2 d\Gamma_2(\lambda) \le \operatorname*{esssup}_{\lambda} \frac{d\Gamma_1}{d\Gamma_2}(\lambda) \int \left| \frac{1}{\lambda} \varphi(t,\lambda) \right|^2 d\Gamma_1(\lambda) < \infty.$$

Thus $I_1(x,t) \in L^2_{M_2(x)} \subset L^2_{M_1(x)}$ which means that $H(x,t) = I_1(x,t) - I_2(x,t) \in L^2_{M_1(x)}$.□

4. The transmutation operator $\mathbb{H}$

¿From Proposition 3, we have the existence of the kernel H. Now observe that

$$f(x) \to \int_0^{\infty} H(x,t) f(t) dM_1(t)$$

defines a continuous functional on $L^2_{M_1}$,

$$\begin{aligned} \left| \int_0^{\infty} H(x,t) f(t) dM_1(t) \right| &\le \sqrt{\int_0^{\infty} |H(x,t)|^2 \, dM_1(t)} \sqrt{\int_0^{\infty} |f(t)|^2 \, dM_1(t)} \\ &\le \|H(x,t)\|_{M_1(t)} \|f\|_{M_1}. \end{aligned}$$

Thus we have proved

Proposition 5. *Assume that* $d\Gamma_1 = O(d\Gamma_2)$ *as* $\lambda \to \infty$ *then for each fixed* $x > 0$, $f \to \int_0^{\infty} H(x,t) f(t) dM_1(t)$ *defines a bounded functional on* $L^2_{M_1}$.

We now show that (3.3) can be used to define an integral operator in $L^2_{M_1}$. More precisely we have

Proposition 6. *Assume* $d\Gamma_1 = O(d\Gamma_2)$ *as* $\lambda \to \infty$, $\frac{d\Gamma_1}{d\Gamma_2} \in L^{2,loc}_{\Gamma_2}$, $dM_1 = O(dM_2)$ *as* $x \to \infty$ *then the operator* $L^2_{M_1} \to L^2_{M_1}$

$$g \to \int_0^\infty H(x,t)\mathbb{S}_1 g(t)dM_1(t)$$

is densely defined in $L^2_{M_1}$.

Proof. From Proposition 2, $H(x,t) \in L^2_{M_1(t)}$ and Parseval identity and (3.3) imply that for any $f \in L^2_{M_1}$

$$\begin{aligned}\int_0^\infty H(x,t)f(t)dM_1(t) &= \int_0^\infty \mathbf{F}_\varphi\left(H(x,\cdot)\right)(\lambda)\mathbf{F}_\varphi\left(f\right))(\lambda)d\Gamma_1(\lambda) \\ &= \int_0^\infty \left(y(x,\lambda)-\varphi(x,\lambda)\right)\frac{1}{\lambda}\mathbf{F}_\varphi\left(f\right)(\lambda)d\Gamma_1(\lambda).\end{aligned} \tag{4.1}$$

Now choose $f = \mathbb{S}_1 g$, which means

$$\mathbf{F}_\varphi\left(f\right)(\lambda) = \mathbf{F}_\varphi\left(\mathbb{S}_1 g\right)(\lambda) = \lambda\mathbf{F}_\varphi\left(g\right)(\lambda).$$

Thus (4.1) can be written as

$$\begin{aligned}\int_0^\infty H(x,t)\mathbb{S}_1 g(t)dM_1(t) &= \int_0^\infty \left(y(x,\lambda)-\varphi(x,\lambda)\right)\mathbf{F}_\varphi\left(g\right)(\lambda)d\Gamma_1(\lambda) \\ &= \int_0^\infty y(x,\lambda)\mathbf{F}_\varphi\left(g\right)(\lambda)d\Gamma_1(\lambda) - \int_0^\infty \varphi(x,\lambda)\mathbf{F}_\varphi\left(g\right)(\lambda)d\Gamma_1(\lambda) \\ &= \int_0^\infty y(x,\lambda)\mathbf{F}_\varphi\left(g\right)(\lambda)d\Gamma_1(\lambda) - g(x),\end{aligned}$$

which holds for all $g \in D_{\mathbb{S}_1}$, that is when $\mathbf{F}_\varphi\left(g\right), \lambda\mathbf{F}_\varphi\left(g\right) \in L^2_{\Gamma_1}$.
The range of the operator $g \to \int_0^\infty H(x,t)\mathbb{S}_1\left(g\right)(t)dM_1(t)$ depends on the set of functions

$$\gamma(g)(x) := \int_0^\infty y(x,\lambda)\mathbf{F}_\varphi\left(g\right)(\lambda)d\Gamma_1(\lambda).$$

In order to see $\gamma(g)$ as an inverse $\mathbf{F}_y-$transform we first restrict γ to the following dense set in $L^2_{M_1}$:

$$\left\{g \in L^2_{M_1} : \mathbf{F}_\varphi\left(g\right) \in C_o\right\} \subset D_{\mathbb{S}_1}.$$

We deduce that $\gamma(g)$ is a continuous function and so $\gamma(g) \in L^{2,loc}_{M_1}$. Furthermore since $\frac{d\Gamma_1}{d\Gamma_2} \in L^{2,loc}_{\Gamma_2}$ we have $\mathbf{F}_\varphi\left(g\right)(\lambda)\frac{d\Gamma_1}{d\Gamma_2}(\lambda) \in L^{2,}_{\Gamma_2}$ and

$$\gamma(g)(x) = \int_0^\infty y(x,\lambda)\mathbf{F}_\varphi\left(g\right)(\lambda)\frac{d\Gamma_1}{d\Gamma_2}(\lambda)d\Gamma_2(\lambda) \in L^2_{M_2}.$$

¿From the condition $dM_1 = O(dM_2)$ as $x \to \infty$ it follows that for large N

$$\int_N^\infty \left|\gamma(g)(x)\right|^2 dM_1(x) \leq c\int_N^\infty \left|\gamma(g)(x)\right|^2 dM_2(x)$$

and since $\gamma(g) \in L^{2,loc}_{M_1}$ we deduce that $\gamma(g) \in L^2_{M_1}$ which means that

$$\int_0^\infty H(x,t)\mathbb{S}_1 g(t)dM_1(t) = \gamma(g)(x) - g(x) \in L^2_{M_1}.$$

Thus the operation $g \to \int_0^\infty H(x,t)\mathbb{S}_1 g(t)dM_1(t)$ is densely defined in the space $L^2_{M_1}$.□

We now obtain a sufficient condition for the operator $\mathbb{H}$ to be compact.

Proposition 7. *Assume that* $d\Gamma_1 \ll^{\infty} d\Gamma_2$, *then* $\mathbb{H}$ *is a compact operator from* $L^2_{M_1}$ *into* $L^2_{M_1}$ *if*

$$\int_0^\infty \int_0^x (x-t)^2\, dM_2(t)dM_2(x) < \infty \text{ and } \int_0^\infty \int_0^x (x-t)^2\, dM_1(t)dM_2(x) < \infty. \tag{4.2}$$

Proof. From Theorem 1 it follows

$$\int_0^\infty |H(x,t)|^2\, dM_1(t) \le 2 \operatorname*{esssup}_{\lambda} \frac{d\Gamma_1}{d\Gamma_2}(\lambda) \int_0^x (x-t)^2\, dM_2(t) + 2\int_0^x (x-t)^2\, dM_1(t),$$

which leads to

$$\begin{aligned}\int_0^\infty \int_0^\infty |H(x,t)|^2\, dM_1(t)dM_2(x) \quad &\le \quad 2 \operatorname*{esssup}_{\lambda} \frac{d\Gamma_1}{d\Gamma_2}(\lambda). \int_0^\infty \int_0^x (x-t)^2\, dM_2(t)dM_2(x)\\ &\quad +2\int_0^\infty \int_0^x (x-t)^2\, dM_1(t)dM_2(x).\square\end{aligned}$$

A sufficient condition for the realization of (4.2) would be

$$\int_0^\infty x^2 M_2(x)dM_2(x) < \infty \quad \text{and} \quad \int_0^\infty x^2 M_1(x)dM_2(x) < \infty.$$

Once an operator is densely defined, we can ask for its adjoint.

5. The adjoint operator $\mathbb{H}^*$.

Proposition 8. *Assume that* $dM_1 \ll^{2,loc} dM_2$ *and* $d\Gamma_1(\lambda) = O(d\Gamma_2)$ *as* $\lambda \to \infty$. *Then* $\mathbb{H}^* : L^2_{M_1} \to L^2_{M_1}$, *which is defined by*

$$\mathbb{H}^*(f)(t) = \int_0^\infty H(x,t)f(x)dM_1(x),$$

is densely defined, and for any $f \in C_o$, *we have*

$$\mathbf{F}_\varphi(f)(\lambda) + \lambda \mathbf{F}_\varphi\left(H^* f\right)(\lambda) = \mathbf{F}_y\left(f(x)\frac{dM_1}{dM_2}(x)\right)(\lambda). \tag{5.1}$$

Proof. Multiply (3.3) by $f \in C_o$, supp $f \subset [a,\ b]$ say, to obtain

(5.2)

$$\lambda \int_a^b f(x) \int_0^\infty H(x,t)\varphi(t,\lambda)dM_1(t)dM_1(x) = \int_a^b f(x)y(x,\lambda)dM_1(x) - \int_a^b f(x)\varphi(x,\lambda)dM_1(x)$$

Observe that from proposition 4, $\int_0^\infty H(x,t)\varphi(t,\lambda)dM_1(t) \in L^2_{M_1}$ and so

$$f(x)\int_0^\infty H(x,t)\varphi(t,\lambda)dM_1(t) \in L^1_{M_1},$$

from which it follows that

$$\begin{aligned}\int_a^b f(x)\int_0^\infty H(x,t)\varphi(t,\lambda)dM_1(t)dM_1(x) \quad &= \quad \int_a^b \int_0^\infty f(x)H(x,t)\varphi(t,\lambda)dM_1(t)dM_1(x)\\ &= \quad \int_0^\infty \int_a^b f(x)H(x,t)dM_1(x)\varphi(t,\lambda)dM_1(t)\\ &= \quad \mathbf{F}_\varphi\left(\int_a^b f(x)H(x,t)dM_1(x)\right)(\lambda).\end{aligned} \tag{5.3}$$

On the other hand the right-hand side of (5.2) is equal to

$$\int_a^b f(x)y(x,\lambda)dM_1(x) - \int_a^b f(x)\varphi(x,\lambda)dM_1(x) = \int_a^b f(x)y(x,\lambda)\frac{dM_1}{dM_2}(x)dM_2(x) - \mathbf{F}_\varphi(f)(\lambda)$$

$$(5.4) \qquad = \mathbf{F}_y\left(f(x)\frac{dM_1}{dM_2}(x)\right)(\lambda) - \mathbf{F}_\varphi(f)(\lambda).$$

Since $f \in C_o$ and $dM_1(x) \ll^{2,loc} dM_2$, then $f(x)\frac{dM_1}{dM_2}(x) \in L^2_{M_2}$ and thus $\mathbf{F}_y\left(f(x)\frac{dM_1}{dM_2}(x)\right) \in L^2_{\Gamma_2}$. Because $d\Gamma_1(\lambda) = O\left(d\Gamma_2\right)$ as $\lambda \to \infty$, then

$$(5.5) \qquad \int_N^\infty \left|\mathbf{F}_y\left(f(x)\frac{dM_1}{dM_2}(x)\right)(\lambda)\right|^2 d\Gamma_1(\lambda) \leq c\int_N^\infty \left|\mathbf{F}_y\left(f(x)\frac{dM_1}{dM_2}(x)\right)(\lambda)\right|^2 d\Gamma_2(\lambda) < \infty.$$

But

$$\mathbf{F}_y\left(f(x)\frac{dM_1}{dM_2}(x)\right)(\lambda) = \int_a^b f(x)y(x,\lambda)dM_1(x)$$

is continuous with respect to λ, therefore, for any N

$$(5.6) \qquad \int_0^N \left|\mathbf{F}_y\left(f(x)\frac{dM_1}{dM_2}(x)\right)(\lambda)\right|^2 d\Gamma_1(\lambda) < \infty.$$

We deduced from (5.5) and (5.6) that $\mathbf{F}_y\left(f(x)\frac{dM_1}{dM_2}(x)\right) \in L^2_{d\Gamma_1}$ and since $\mathbf{F}_\varphi(f) \in L^2_{\Gamma_1}$, it follows

$$\mathbf{F}_y\left(f(x)\frac{dM_1}{dM_2}(x)\right) - \mathbf{F}_\varphi(f) \in L^2_{\Gamma_1}.$$

Observe

$$\frac{1}{\lambda}\left(\mathbf{F}_y\left(f(x)\frac{dM_1}{dM_2}(x)\right) - \mathbf{F}_\varphi(f)\right) = \int_a^b f(x)\frac{1}{\lambda}\left(y(x,\lambda) - \varphi(x,\lambda)\right)dM_1(x)$$

is bounded as $\lambda \to 0$, it follows that

$$\frac{1}{\lambda}\left(\mathbf{F}_y\left(f(x)\frac{dM_1}{dM_2}(x)\right) - \mathbf{F}_\varphi(f)\right) \in L^2_{\Gamma_1}$$

and so $\int_a^b f(x)H(x,t)dM_1(x) \in L^2_{M_1}$ for any $f \in C_o$. As C_o is dense in $L^2_{M_1}$, the operator $\mathbb{H}^* : L^2_{M_1} \to L^2_{M_1}$ defined by

$$\mathbb{H}^*(f)(t) = \int_a^b H(x,t)f(x)dM_1(x)$$

is densely defined. Furthermore from (5.3) and (5.4) it follows (5.1).$\square$

6. Adding a potential

We now extend the above construction to include operators such as

$$(6.1) \qquad \begin{cases} \mathbb{S}_{1q}(\phi)(x) := -\frac{d}{dM_1(x)}\frac{d^+}{dx^+}\phi(x,\lambda) + q_1(x)\phi(x,\lambda) = \lambda\phi(x,\lambda), & 0 \leq x < \infty \\ \phi(0,\lambda) = b,\ \phi'(0,\lambda) = a. \end{cases}$$

where the potential $q_1 \in L^{2,loc}_{M_1}(0,\infty)$. The classical Sturm-Liouville problem corresponds to particular case when $M_1(x) = x$, i.e. $\mathbb{S}_{1q}(\phi)(x) := -\frac{d^2}{dx^2}\phi(x,\lambda) + q(x)\phi(x,\lambda) = \lambda\phi(x,\lambda)$. Similarly define the second generalized string by

$$\begin{cases} \mathbb{S}_{2q}(\psi)(x) := -\frac{d}{dM_2(x)}\frac{d^+}{dx^+}\psi(x,\lambda) + q_2(x)\psi(x,\lambda) = \lambda\psi(x,\lambda), & 0 \leq x < \infty \\ \psi(0,\lambda) = b,\ \psi'(0,\lambda) = a. \end{cases}$$

where the potential $q_2 \in L^{2,loc}_{M_2}(0,\infty)$. Equation (6.1) is equivalent to

$$\phi(x,\lambda) = ax + b + \int_0^x (x-t)\, q_1(t)\phi(t,\lambda)dM_1(t) + \lambda \int_0^x (x-t)\, \phi(t,\lambda)dM_1(t). \tag{6.2}$$

The addition of a potential changes dramatically the spectrum from being mainly positive to possibly covering the whole real line. Thus the support of the spectral function is a subset of the real line.

We now show that a transmutation between the strings $\mathbb{S}_{1q}$ and $\mathbb{S}_{2q}$ exists under minimal conditions $d\Gamma_1(\lambda) = O\left(d\Gamma_2(\lambda)\right)$ as $\lambda \to \pm\infty$.

Proposition 9. *Assume that $d\Gamma_1(\lambda) = O\left(d\Gamma_2(\lambda)\right)$ as $\lambda \to \pm\infty$, then there exists $H_q(x,t) \in L^2_{M_1(t)}$ such that for $x \geq 0$*

$$\psi(x,\lambda) = \phi(x,\lambda) + \lambda \int_0^\infty H_q(x,t)\phi(t,\lambda)dM_1(t).$$

Proof. For $\lambda \to \pm\infty$, we can recast (6.2) into

$$\frac{1}{\lambda}\left(\phi(x,\lambda) - ax - b\right) = \frac{1}{\lambda}\int_0^x (x-t)\, q_1(t)\phi(t,\lambda)dM_1(t) + \int_0^x (x-t)\, \phi(t,\lambda)dM_1(t)$$

and since $(x-t)_+\, q_1(t),\ (x-t)_+ \in L^2_{M_1(t)}$, we use Parseval equality to deduce successively that

$$\int_0^x (x-t)\, q_1(t)\phi(t,\lambda)dM_1(t),\ \int_0^x (x-t)\, \phi(t,\lambda)dM_1(t) \in L^2_{\Gamma_1}(N,\infty)$$

and since the mapping $\mathbf{F} \to \frac{1}{\lambda}\mathbf{F}$ is bounded in $L^2_{\Gamma_1}(N,\infty)$, it also follows that

$$\frac{1}{\lambda}\int_0^x (x-t)\, q_1(t)\phi(t,\lambda)dM_1(t) \in L^2_{\Gamma_1}(N,\infty)$$

and so

$$\frac{1}{\lambda}\left(\phi(x,\lambda) - ax - b\right) \in L^2_{\Gamma_1}(N,\infty). \tag{6.3}$$

We repeat the same argument for the second solution ψ to obtain

$$\frac{1}{\lambda}\left(\psi(x,\lambda) - ax - b\right) \in L^2_{\Gamma_2}(N,\infty)$$

and since $d\Gamma_1(\lambda) = O\left(d\Gamma_2(\lambda)\right)$ as $\lambda \to \pm\infty$, we deduce that

$$\frac{1}{\lambda}\left(\psi(x,\lambda) - ax - b\right) \in L^2_{\Gamma_2}(N,\infty) \subset L^2_{\Gamma_1}(N,\infty). \tag{6.4}$$

Subtract (6.3) from (6.4) then yields

$$\frac{1}{\lambda}\left(\psi(x,\lambda) - \phi(x,\lambda)\right) \in L^2_{\Gamma_1}(N,\ \infty).$$

Repeat all the above arguments using $L^2_{\Gamma_1}(-\infty,-N)$ instead of $L^2_{\Gamma_1}(N,\ \infty)$ leads to $\frac{1}{\lambda}\left(\psi(x,\lambda) - \phi(x,\lambda)\right) \in L^2_{\Gamma_1}(-\infty,-N)$ which means

$$\frac{1}{\lambda}\left(\psi(x,\lambda) - \phi(x,\lambda)\right) \in L^2_{\Gamma_1}\left[(-\infty,-N) \cup (N,\ \infty)\right]. \tag{6.5}$$

To finish we observe that

$$\frac{1}{\lambda}\left(\psi(x,\lambda) - \phi(x,\lambda)\right)$$

as a function of λ is continuous for $\lambda \in [-N,\ N]$ for any large $N > 0$, and so

$$\frac{1}{\lambda}\left(\psi(x,\lambda) - \phi(x,\lambda)\right) \in L^2_{\Gamma_1}(-N,\ N). \tag{6.6}$$

Combining (6.5) and (6.6) leads to

$$\frac{1}{\lambda}\left(\psi(x,\lambda)-\phi(x,\lambda)\right)\in L^2_{\Gamma_1}\left(-\infty,\ \infty\right),$$

and the inverse $\mathbf{F}_\phi$−transform implies the existence of $H_q(x,t)\in L^2_{M_1(t)}$

$$\frac{1}{\lambda}\left(\psi(x,\lambda)-\phi(x,\lambda)\right)=\int_0^\infty H_q(x,t)\phi(t,\lambda)dM_1(t).\Box$$

In order to proceed further we rewrite the transmutation operator as

$$\begin{aligned}\psi(x,\lambda) &= \phi(x,\lambda)+\int_0^\infty H_q(x,t)\lambda\phi(t,\lambda)dM_1(t)\\ &= \phi(x,\lambda)+\int_0^\infty H_q(x,t)\mathbb{S}_{1q}\phi(t,\lambda)dM_1(t)\\ &= \phi(x,\lambda)+\int_0^\infty H_q(x,t)d^+\phi(t,\lambda).\end{aligned}$$

The kernel H_q contains all information about the solution ψ and so contains the information about M_2 and q_2.
In practice, we usually do not have the spectral function explicitly. In the next proposition we use an important fact that asymptotics of the spectral function Γ as $\lambda\to\infty$ depend on the behavior of M as $x\to 0$.

PROPOSITION 10. *Assume that* $\lim_{x\to 0}\frac{DM_1}{Dx^{\alpha_1}}=k_1\neq 0,\ \lim_{x\to 0}\frac{DM_2}{Dx^{\alpha_2}}=k_2\neq 0,$ *and spectrum of* $\mathbb{S}_1$ *is bounded from below. If* $0<\alpha_1\leq\alpha_2$ *then there exists* $H_q(x,\cdot)\in L^2_{M_1}(0,x)$ *such that for* $x\geq 0$

$$\psi(x,\lambda)=\phi(x,\lambda)+\lambda\int_0^\infty H_q(x,t)\phi(t,\lambda)dM_1(t).$$

Proof. We only need to show that conditions of Proposition 9 hold. First when $\lambda\to-\infty$, $d\Gamma_1(\lambda)=0$ and so the condition $d\Gamma_1(\lambda)=O\left(d\Gamma_2(\lambda)\right)$ is trivially verified. However when $\lambda\to\infty$, we have two separate cases, see [**14**]: if

$$\begin{aligned} a &\neq 0 \quad \text{then}\ \ \Gamma_i(\lambda)=c_1\lambda^{\frac{\alpha_i}{1+\alpha_i}}+o\left(\lambda^{\frac{\alpha_i}{1+\alpha_i}}\right)\ \ \text{as}\quad \lambda\to\infty,\\ a &= 0 \quad\ \ \text{then}\ \ \Gamma_i(\lambda)=c_1\lambda^{\frac{2+\alpha_i}{1+\alpha_i}}+o\left(\lambda^{\frac{2+\alpha_i}{1+\alpha_i}}\right)\ \ \text{as}\quad \lambda\to\infty.\end{aligned}$$

Since $0<\alpha_1\leq\alpha_2$ implies $\frac{\alpha_1}{1+\alpha_1}\leq\frac{\alpha_2}{1+\alpha_2}$ i.e. $\frac{d\Gamma_1}{d\Gamma_2}\approx c\lambda^{\frac{\alpha_1-\alpha_2}{(1+\alpha_1)(1+\alpha_2)}}<\infty$ as $\lambda\to\infty$.$\Box$

We now present explicit examples which show that the representation of the transmutations cannot be in general triangular or close to unity, see [**11**].

7. Examples

Many examples of all kinds of spectra, for various potentials, can be found in [**6, 22**]. The purpose of the following examples is to illustrate two essential facts of transmutations for strings, which make all the difference from the usual transmutations for Sturm-Liouville operators (1.6). The first is that one should have the parameter λ in the integral. Secondly the upper bound may not be just x as in the Gelfand–Levitan theory.

Example 1. Let $M_1(x)=x$ and $M_2(x)=\rho^2x,\ a=1,\ b=0,\ \rho>1$. Then for $x\geq 0$, we have

$$[\mathbb{S}_1]\begin{cases}-\frac{d}{dx}\frac{d^+}{dx^+}\varphi(x,\lambda)=\lambda\varphi(x,\lambda),\\ \varphi(0,\lambda)=1,\ \varphi'(0,\lambda)=0\end{cases}\qquad\text{and}\qquad[\mathbb{S}_2]\begin{cases}-\frac{d}{\rho^2dx}\frac{d^+}{dx^+}y(x,\lambda)=\lambda y(x,\lambda)\\ y(0,\lambda)=1,\ y'(0,\lambda)=0.\end{cases}$$

The eigenfunctionals of the strings are

$$\varphi(x,\lambda)=\cos(x\sqrt{\lambda})\ \text{ and }\ y(x,\lambda)=\cos(\rho x\sqrt{\lambda})\qquad\text{for } x\geq 0,$$

and their spectral functions, [**19**], are given by

$$d\Gamma_1(\lambda) = \frac{1}{\pi}\frac{d\lambda}{\sqrt{\lambda_+}} \quad \text{and} \quad d\Gamma_2(\lambda) = \frac{\rho}{\pi}\frac{d\lambda}{\sqrt{\lambda_+}}.$$

Thus by Proposition 2, the transmutation exists

$$\cos(\rho x\sqrt{\lambda}) = \cos(x\sqrt{\lambda}) + \lambda \int_0^\infty H(x,t)\cos(t\sqrt{\lambda})dt.$$

Computing the kernel H, we obtain

$$\begin{aligned} H(x,t) &= \frac{1}{\pi}\int_0^\infty \frac{1}{\lambda}\left(\cos\left(\rho x\sqrt{\lambda}\right) - \cos(x\sqrt{\lambda})\right)\cos\left(t\sqrt{\lambda}\right)\frac{d\lambda}{\sqrt{\lambda}} \\ &= \min\left\{\frac{\rho+1}{2}x,\ \left|\frac{\rho-1}{2}x+t\right|\right\}\operatorname{sign}\left[\frac{\rho-1}{2}x+t\right] \\ &\quad + \min\left\{\frac{\rho+1}{2}x,\ \left|\frac{\rho-1}{2}x-t\right|\right\}\operatorname{sign}\left[\frac{\rho-1}{2}x-t\right]. \end{aligned}$$

One can verify that $H(x,t) = 0$ if $t > \rho x$, but $H(x,t) = \rho x - t \neq 0$ if $t < \rho x$, but close to ρx. Therefore we deduce an explicit form of the type

$$\cos(\rho x\sqrt{\lambda}) = \cos(x\sqrt{\lambda}) + \lambda \int_0^{\rho x} H(x,t)\cos(t\sqrt{\lambda})dt. \tag{7.1}$$

Here we notice that the multiplier λ is needed simply because for any fixed $x > 0$ we have $\cos(\rho x\sqrt{\lambda}) - \cos(x\sqrt{\lambda}) \notin L^2_{\sqrt{\lambda}}$ while the integral on the right hand side

$$\int_0^{\rho x} H(x,t)\cos(t\sqrt{\lambda})dt \in L^2_{\sqrt{\lambda_+}}.$$

Thus the role of λ is to ensure that $\frac{1}{\lambda}\left(\cos(\rho x\sqrt{\lambda}) - \cos(x\sqrt{\lambda})\right) \in L^2_{\sqrt{\lambda_+}}$.

As for the integral upper bound in (7.1), it cannot be just x as in the transmutation used by Gelfand-Levitan. It is easily seen that since the growth type of $\cos(\rho x\sqrt{\lambda}) - \cos(x\sqrt{\lambda})$ as a function of $\sqrt{\lambda}$ is ρx, when $\rho > 1$ the Paley-Wiener theorem implies that the support of the transform must be included in $[-\rho x,\ \rho x]$ and the fact that the transform is even, reduces it to $[0,\ \rho x]$.

Example 2. Consider transmuting the strings when $\alpha, \beta > 0$,

$$[\mathbb{S}_1]\begin{cases} -x^{-\alpha}\varphi''(x,\lambda) = \lambda\varphi(x,\lambda), & 0 \le x \\ \varphi(0,\lambda) = 1 \quad \varphi'(0,\lambda) = 0 \end{cases} \qquad [\mathbb{S}_2]\begin{cases} -x^{-\beta}y''(x,\lambda) = \lambda y(x,\lambda), & 0 \le x \\ y(0,\lambda) = 1 \quad y'(0,\lambda) = 0. \end{cases}$$

Their eigensolutions are the well known Bessel functions

$$\varphi(x,\lambda) = c_1(\lambda)\sqrt{x}Y_{\frac{1}{2+\alpha}}\left(\frac{2\sqrt{\lambda}}{2+\alpha}x^{\frac{2+\alpha}{2}}\right) \quad \text{and} \quad y(x,\lambda) = c_2(\lambda)\sqrt{x}Y_{\frac{1}{2+\beta}}\left(\frac{2\sqrt{\lambda}}{2+\beta}x^{\frac{2+\beta}{2}}\right).$$

The spectral functions are [**14**],

$$\Gamma_1(\lambda) \approx C_1\lambda^{\frac{\alpha+1}{\alpha+2}} \ \text{and}\ \Gamma_2(\lambda) \approx C_2\lambda^{\frac{\beta+1}{\beta+2}} \quad \text{as } \lambda \to \infty.$$

This leads to the following conclusion.

Corollary 2. *If $\beta \ge \alpha > 0$ then there exists a transmutation such that*

$$y(x,\lambda) = \varphi(x,\lambda) + \lambda\int_0^\infty H(x,t)\varphi(t,\lambda)t^\alpha dt.$$

In terms of Bessel functions the above relation becomes

$$
\begin{aligned}
c_2(\lambda)\sqrt{x}Y_{\frac{1}{2+\beta}}\left(\frac{2\sqrt{\lambda}}{2+\beta}x^{\frac{2+\beta}{2}}\right) &= c_1(\lambda)\sqrt{x}Y_{\frac{1}{2+\alpha}}\left(\frac{2\sqrt{\lambda}}{2+\alpha}x^{\frac{2+\alpha}{2}}\right)\\
&\quad +\lambda c_1(\lambda)\int_0^\infty H(x,t)\sqrt{t}Y_{\frac{1}{2+\alpha}}\left(\frac{2\sqrt{\lambda}}{2+\alpha}t^{\frac{2+\alpha}{2}}\right)t^\alpha dt
\end{aligned}
$$

An interesting particular case is when $\alpha = 1$, i.e. $\varphi(x,\lambda) = \cos\left(x\sqrt{\lambda}\right)$, thus for $\beta \geq 1$ we have

$$
c_2(\lambda)\sqrt{x}Y_{\frac{1}{2+\beta}}\left(\frac{2\sqrt{\lambda}}{2+\beta}x^{\frac{2+\beta}{2}}\right) = \cos\left(x\sqrt{\lambda}\right) + \lambda\int_0^\infty H(x,t)\cos\left(t\sqrt{\lambda}\right)dt.
$$

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