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# Double Roman domination and domatic numbers of graphs

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**Abstract:** A double Roman dominating function on a graph $G$ with vertex set $V(G)$ is defined in [4] as a function $f : V(G) \to \{0,1,2,3\}$ having the property that if $f(v) = 0$, then the vertex $v$ must have at least two neighbors assigned 2 under $f$ or one neighbor $w$ with $f(w) = 3$, and if $f(v) = 1$, then the vertex $v$ must have at least one neighbor $u$ with $f(u) \geq 2$. The weight of a double Roman dominating function $f$ is the sum $\sum_{v \in V(G)} f(v)$, and the minimum weight of a double Roman dominating function on $G$ is the double Roman domination number $\gamma_{dR}(G)$ of $G$.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct double Roman dominating functions on $G$ with the property that $\sum_{i=1}^{d} f_i(v) \leq 3$ for each $v \in V(G)$ is called in [12] a double Roman dominating family (of functions) on $G$. The maximum number of functions in a double Roman dominating family on $G$ is the double Roman domatic number of $G$.

In this note we continue the study of the double Roman domination and domatic numbers. In particular, we present a sharp lower bound on $\gamma_{dR}(G)$, and we determine the double Roman domination and domatic numbers of some classes of graphs.

**Keywords:** Domination; Double Roman domination number; Double Roman domatic number

**AMS Subject classification:** 05C69

## 1. Terminology and introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [7]. Specifically, let $G$ be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the *order* and the *size* of the graph $G$, respectively. The *open neighborhood* of vertex $v$ is $N_G(v) = N(v) = \{u \in V(G) | uv \in E(G)\}$, and the *closed neighborhood* of $v$ is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v$ is $d_G(v) = d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph $G$ are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively. The *complement* of a graph $G$ is denoted by $\overline{G}$. Let $K_n$ be the complete graph of order $n$ and $K_{p,q}$ the complete bipartite graph with partite sets $X$ and $Y$, where $|X| = p$ and $|Y| = q$. Recall that the *join* $G + H$ of two graphs $G$ and $H$ is a graph formed from disjoint copies of $G$ and $H$ by connecting each vertex of $G$ to each vertex of $H$.

In this paper, we continue the study of Roman dominating functions and Roman domatic numbers in graphs (see, for example, [4–6, 9–12]). A *double Roman dominating function* (DRD function) on a graph $G$ is defined by Beeler, Haynes and Hedetniemi in [4] as a function $f : V(G) \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then the vertex $v$ must have at least two neighbors assigned 2 under $f$ or one neighbor $w$ with $f(w) = 3$, and if $f(v) = 1$, then the vertex $v$ must have at least one neighbor $u$ with $f(u) \geq 2$. The weight of a DRD function $f$ is the value $\omega(f) = f(V(G)) = \sum_{v\in V(G)} f(v)$. The *double Roman domination number* $\gamma_{dR}(G)$ equals the minimum weight of a double Roman dominating function on $G$, and a double Roman dominating function of $G$ with weight $\gamma_{dR}(G)$ is called a $\gamma_{dR}(G)$-*function* of $G$. Further results on the double Roman domination number can be found in [1–3, 8].

A set $\{f_1, f_2, \dots, f_d\}$ of distinct double Roman dominating functions on $G$ with the property that $\sum_{i=1}^{d} f_i(v) \leq 3$ for each $v \in V(G)$ is called in [12] a *double Roman dominating family* (of functions) on $G$. The maximum number of functions in a double Roman dominating family (DRD family) on $G$ is the *double Roman domatic number* of $G$, denoted by $d_{dR}(G)$. The double Roman domatic number is well-defined and $d_{dR}(G) \geq 1$ for each graph $G$ since the set consisting of any DRD function forms a DRD family on $G$.

In this work, we study the double Roman domination and domatic numbers. In particular, we prove the lower bound $\gamma_{dR}(G) \geq \left\lceil \frac{3n(G)}{\Delta(G)+1} \right\rceil$ for each graph $G$ with $\Delta(G) \geq 1$. Furthermore, we present some Nordhaus-Gaddum type results on the double Roman domatic number. In addition, we determine the double Roman domination and domatic numbers for some special classes of graphs.

## 2. A lower bound on $\gamma_{dR}(G)$

In this section, we present a lower bound on the double Roman domination number and a consequence.

**Theorem 1.** If $G$ is a graph of order $n$ and maximum degree $\Delta \geq 1$, then

$$\gamma_{dR}(G) \geq \left\lceil \frac{3n}{\Delta + 1} \right\rceil .$$

*Proof.* If $\Delta = 1$, then $G = pK_2 \cup qK_1$ with $p \geq 1$ and so $\gamma_{dR}(G) = 3p + 2q$. Since

$n = 2p + q$, we obtain

$$\gamma_{dR}(G) = 3p + 2q \geq \left\lceil \frac{6p+3q}{2} \right\rceil = \left\lceil \frac{3n}{\Delta+1} \right\rceil.$$

Assume now that $\Delta \geq 2$, and let $f$ be a $\gamma_{dR}(G)$-function. According to [4], we can assume, without loss of generality, that $f(x) \in \{0, 2, 3\}$ for each vertex $x \in V(G)$. If $V_i$ is the set of vertices assigned $i$ by the function $f$, then $\gamma_{dR}(G) = 2|V_2| + 3|V_3|$ and $n = |V_0| + |V_2| + |V_3|$. Since each vertex of $V_0$ is adjacent to at least one vertex of $V_3$ or to at least two vertices of $V_2$, we deduce that

$$|V_0| \leq \frac{\Delta}{2}|V_2| + \Delta|V_3|.$$

It follows that

$$\begin{aligned}
(\Delta+1)\gamma_{dR}(G) &= (\Delta+1)(2|V_2| + 3|V_3|) \\
&= 3\Delta|V_3| + \frac{3\Delta}{2}|V_2| + 3|V_3| + \left(\frac{\Delta}{2} + 2\right)|V_2| \\
&\geq 3|V_0| + 3|V_3| + 3|V_2| + \left(\frac{\Delta}{2} - 1\right)|V_2| \\
&= 3n + \left(\frac{\Delta}{2} - 1\right)|V_2| \geq 3n,
\end{aligned}$$

and this leads to the desired bound. □

For the following corollary, we use the next proposition, which can be found in [3].

**Proposition 1.** Let $G$ be a connected graph of order $n \geq 3$. Then

(1) $\gamma_{dR}(G) = 3$ if and only if $\Delta(G) = n - 1$.

(2) $\gamma_{dR}(G) = 4$ if and only if $G = \overline{K_2} + H$, where $H$ is a graph with $\Delta(H) \leq |V(H)| - 2$.

(3) $\gamma_{dR}(G) = 5$ if and only if $\Delta(G) = n - 2$ and $G \neq \overline{K_2} + H$ for any graph $H$ of order $n - 2$.

**Corollary 1.** Let $G = K_{n_1,n_2,\ldots,n_r}$ be the complete $r$-partite graph with $r \geq 2$ and $n_1 \leq n_2 \leq \ldots \leq n_r$.

(a) If $n_1 = 1$, then $\gamma_{dR}(G) = 3$.

(b) If $n_1 = 2$, then $\gamma_{dR}(G) = 4$.

(c) If $n_1 \geq 3$, then $\gamma_{dR}(G) = 6$.

*Proof.* Statement (a) follows from Proposition 1 (1), and Statement (b) follows from Proposition 1 (2).
(c) Assume now that $n_1 \geq 3$. Proposition 1 (3) implies that $\gamma_{dR}(G) \geq 6$. Let $X_1, X_2, \ldots, X_r$ be the partite sets of $G$, and let $v_1 \in X_1$ and $v_2 \in X_2$. Define the function $f$ by $f(v_1) = f(v_2) = 3$ and $f(x) = 0$ for $x \in V(G) \setminus \{v_1, v_2\}$. Then $f$ is a DRD function on $G$ of weight 6 and hence $\gamma_{dR}(G) \leq 6$ and thus $\gamma_{dR}(G) = 6$. $\square$

If $G = K_{n_1,n_2,\ldots,n_r}$ with $r \geq 2$ and $2 = n_1 \leq n_2 \leq \ldots \leq n_r$, then

$$\left\lceil \frac{3n(G)}{\Delta(G)+1} \right\rceil = \left\lceil \frac{3(n(G)-1)+3}{n(G)-1} \right\rceil = 4,$$

and thus Corollary 1 (b) shows that Theorem 1 is sharp.

## 3. Double Roman domatic number

If $K_{p,p}$ is the complete bipartite graph with $p \geq 3$, then we have shown in [12] that $d_{dR}(K_{p,p}) = p$. Using the next theorem, we prove a more general result.

**Theorem 2.** Let $G$ be a graph of order $n$. If $G$ contains $p \geq 2$ vertices of degree less or equal $n-2$, then $d_{dR}(G) \leq n - \lceil \frac{p}{2} \rceil$.

*Proof.* Let $\{f_1, f_2, \ldots, f_d\}$ be a DRD family on $G$ with $d = d_{dR}(G)$. According to [4], we can assume, without loss of generality, that $f_i(x) \in \{0, 2, 3\}$ for each $x \in V(G)$ and $1 \leq i \leq d$. Let $A_i$ be the set of vertices such that $f_i(x) \geq 2$ for $x \in A_i$ and $1 \leq i \leq d$. Since $\{f_1, f_2, \ldots, f_d\}$ is a DRD family on $G$, we note that $A_j \cap A_k = \emptyset$ for $1 \leq j \neq k \leq d$. The hypothesis that $G$ has $p \geq 2$ vertices of degree less or equal $n-2$ shows that there are at most $n-p$ vertex sets $A_i$ with $|A_i| = 1$ and all other such vertex sets are of cardinality at least two. This leads to

$$d_{dR}(G) \leq n - p + \left\lfloor \frac{p}{2} \right\rfloor = n - \left\lceil \frac{p}{2} \right\rceil.$$

$\square$

**Example 1.** *Let $M$ be a matching of the complete graph $K_n$ such that $|M| = k$ and $2k \leq n$. Let $H = K_n - M$, and let $u_1, u_2, \ldots, u_{n-2k}$ be the vertices of degree $n-1$ in $H$. If*

$$M = \{x_{n-2k+1}y_{n-2k+1}, x_{n-2k+2}y_{n-2k+2}, \ldots, x_{n-k}y_{n-k}\},$$

*then define the functions $f_i(u_i) = 3$ and $f_i(x) = 0$ for $x \in V(H) \setminus \{u_i\}$ for $1 \leq i \leq n-2k$ and $f_i(x_i) = f_i(y_i) = 2$ and $f_i(x) = 0$ for $x \in V(H) \setminus \{x_i, y_i\}$ for $n-2k+1 \leq i \leq n-k$. Then $\{f_1, f_2, \ldots, f_{n-k}\}$ is a DRD family on $H$ and therefore $d_{dR}(H) \geq n-k$. Applying*

*Theorem 2, we deduce that $d_{dR}(H) = n - k$. This example shows that Theorem 2 is sharp for p even.*
*For odd p, let $M$ be a matching and $T$ be the edges of a triangle of $K_n$ such that the edges of $M$ and $T$ are not adjacent. Now $K_n - (M \cup T)$ shows that Theorem 2 is also sharp for p odd.*

**Theorem 3.** Let $G = K_{n_1,n_2,\ldots,n_r}$ be the complete $r$-partite graph with $r \geq 2$ and $n_1 = n_2 = \ldots = n_r = q \geq 2$. Then $d_{dR}(G) = \lfloor \frac{rq}{2} \rfloor$.

*Proof.* Applying Theorem 2, we obtain $d_{dR}(G) \leq \lfloor \frac{rq}{2} \rfloor$. Let $X_1, X_2, \ldots, X_r$ be the partite sets of $G$, and let $v_1, v_2, \ldots, v_{rq}$ be the vertex set of $G$ such that $v_{jr+i} \in X_i$ for $0 \leq j \leq q-1$ and $1 \leq i \leq r$. Now define the function $f_i$ by $f_i(v_{2i-1}) = f_i(v_{2i}) = 3$ and $f_i(x) = 0$ for $x \neq v_{2i-1}, v_{2i}$ for $1 \leq i \leq \lfloor \frac{rq}{2} \rfloor$. Then $f_i$ is a DRD function on $G$ for $1 \leq i \leq \lfloor \frac{rq}{2} \rfloor$ such that

$$f_1(x) + f_2(x) + \ldots + f_{\lfloor \frac{rq}{2} \rfloor}(x) \leq 3$$

for each vertex $x \in V(G)$. Therefore $\{f_1, f_2, \ldots, f_{\lfloor \frac{rq}{2} \rfloor}\}$ is a double Roman dominating family on $G$ and thus $d_{dR}(G) \geq \lfloor \frac{rq}{2} \rfloor$. This yields to $d_{dR}(G) = \lfloor \frac{rq}{2} \rfloor$. □

In [12], we have proved the following two results.

**Theorem 4.** If $G$ is a graph, then $d_{dR}(G) \leq \delta(G) + 1$.

**Theorem 5.** Let $G$ be a graph of order $n$. If $G \neq K_n$ and $\overline{G} \neq K_n$, then

$$d_{dR}(G) + d_{dR}(\overline{G}) \leq n.$$

For a great family of graphs, we can improve the Nordhaus-Gaddum bound of Theorem 5.

**Theorem 6.** Let $G$ be a graph of order $n$ such that $\delta(G), \delta(\overline{G}) \geq 1$. If $n$ is odd or if $n$ is even and $\delta(G) \leq \frac{n}{2} - 2$ or $\delta(\overline{G}) \leq \frac{n}{2} - 2$, then

$$d_{dR}(G) + d_{dR}(\overline{G}) \leq n - 1.$$

*Proof.* Since $\delta(G), \delta(\overline{G}) \geq 1$, we observe that $\Delta(G), \Delta(\overline{G}) \leq n - 2$.
If $n$ is odd, then it follows from Theorem 2 that

$$d_{dR}(G) + d_{dR}(\overline{G}) \leq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor = n - 1.$$

If $n$ is even, then assume, without loss of generality, that $\delta(G) \leq \frac{n}{2} - 2$. Applying Theorems 2 and 4, we obtain

$$d_{dR}(G) + d_{dR}(\overline{G}) \leq \left(\frac{n}{2} - 2\right) + 1 + \frac{n}{2} = n - 1,$$

and the proof is complete. □

If $G = K_{p,p}$ for $p \geq 2$, then we have $d_{dR}(G) + d_{dR}(\overline{G}) = 2p = n(G)$. This example demonstrates that Theorem 6 is not valid for $n$ even and $\delta(\overline{G}) = \frac{n}{2} - 1$ in general. For odd $n$ we will improve Theorem 6.

**Theorem 7.** Let $G$ be a graph of odd order $n$. If $G, \overline{G} \neq K_n, K_n - e$, where $e$ is an arbitray edge of $K_n$, then

$$d_{dR}(G) + d_{dR}(\overline{G}) \leq n - 1.$$

*Proof.* If $\delta(G), \delta(\overline{G}) \geq 1$, then the result follows from Theorem 6. Assume now, without loss of generality, that $\delta(G) = 0$. Then it follows that $d_{dR}(G) = 1$. Since $\overline{G} \neq K_n, K_n - e$, there are at least two edges $e_1, e_2 \in E(G)$. Hence $\overline{G}$ contains at least three vertices of degree less or equal $n - 2$. We deduce from Theorem 2 that $d_{dR}(\overline{G}) \leq n - 2$, and we obtain $d_{dR}(G) + d_{dR}(\overline{G}) \leq 1 + n - 2 = n - 1$. □

Note that if $G = K_n$, then $d_{dR}(G) + d_{dR}(\overline{G}) = n + 1$, and if $G = K_n - e$, then $d_{dR}(G) + d_{dR}(\overline{G}) = (n - 1) + 1 = n$.
For some regular graphs we will improve the upper bound of Theorem 4.

**Theorem 8.** Let $G$ be a $\delta$-regular graph ($\delta \geq 2$) of order $n = p(\delta + 1) + r$ with integers $p \geq 1$ and $1 \leq r \leq \delta$. If $\frac{3r}{\delta+1}$ is not an integer, then $d_{dR}(G) \leq \delta$.

*Proof.* Let $\{f_1, f_2, \ldots, f_d\}$ be a DRD family on $G$ such that $d = d_{dR}(G)$. It follows that

$$\sum_{i=1}^{d} \omega(f_i) = \sum_{i=1}^{d} \sum_{v \in V(G)} f_i(v) = \sum_{v \in V(G)} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V(G)} 3 = 3n. \tag{1}$$

Since $\frac{3r}{\delta+1}$ is not an integer, Theorem 1 yields to

$$\gamma_{dR}(G) \geq \left\lceil \frac{3n}{\delta+1} \right\rceil = \left\lceil \frac{3p(\delta+1) + 3r}{\delta+1} \right\rceil = 3p + \left\lceil \frac{3r}{\delta+1} \right\rceil > 3p + \frac{3r}{\delta+1}. \tag{2}$$

Suppose to the contrary that $d = \delta + 1$. Then we deduce from the inequality chains (1) and (2) that

$$3n \geq \sum_{i=1}^{d} \omega(f_i) \geq \sum_{i=1}^{d} \gamma_{dR}(G) > (\delta+1)\left(3p + \frac{3r}{\delta+1}\right) = 3p(\delta+1) + 3r = 3n.$$

This is a contradiction and thus $d_{dR}(G) \leq \delta$. □

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