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ESTIMATION AND PREDICTION FOR TYPE-II HYBRID CENSORED DATA FOLLOW FLEXIBLE WEIBULL DISTRIBUTION

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1. Introduction

In parametric estimation problems, we often assume that a set of randomly selected samples of pre-specified number of units, say n, is available. It is further assumed that random observations follow a specified distribution. But in practice, observations on all the units may not be available because observations corresponding to some units are either not recorded or lost during intermediate transmission. The data thus obtained are termed as censored sample. Estimation procedures for censored sample cases exist in existing literature. It is worthwhile to mention here that the censoring may not be the most efficient way to conduct experiments from theoretical point of view because it reduces the number of observations. But due to cost and time consideration, practical and administrative constraints, the availability of censored observations is unavoidable. In the other words, the censoring schemes compensate the loss of efficiency of an estimator by providing the administrative convenience at lower costs.

In statistical literature, a number of censoring schemes e.g. time censoring (Type I censoring), item censoring (Type II censoring), Type-I and Type-II hybrid censoring and progressive censoring etc., have been discussed. It may be noted that termination of the experiment at a pre-specified time (say, T) guarantees the duration of the test but efficiency may vary as the number of observations in the censored sample becomes random variable. On other hand, if the test is terminated at pre-specified number of observations, the duration of test becomes random variable although this censoring guarantees a specified efficiency. The hybrid censoring (a mixture of Type-I and Type-II censoring schemes) provides a more flexible and administratively convenient life-testing procedure. In Type-I hybrid censoring, the test is terminated whenever a pre-specified time T of the duration of experiment has reached or a pre-specified number (R) of the observations

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have been obtained i.e. if $X_{R:n}$ denotes the Rth ordered failure time, the termination of the experiment take place at $\min(T, X_{R:n})$. Thus, it guarantees that the duration of the test would be less than T but number of observation will vary and will be less than R. The Type-I hybrid censoring scheme was originally proposed by Epstein (1954) and used in reliability acceptance test MIL-STD-781C (1977). Many authors discussed the estimation of the unknown parameters for various probability distributions in case of Type-I hybrid censored data (see, Dubey *et al.* (2011); Chen and Bhattacharyya (1988); Kundu (2007); Park and Balakirshnan (2012); Gupta and Singh (2012); Rastogi and Tripathi (2012) and Yadav *et al.* (2016)).

In some areas like medical, military and aeronautics etc, level of efficiency is of greater concern than the duration of experiment. Thus, one can propose a censoring scheme where test is terminated whenever a pre-specified number of observations has been obtained and pre-specified time for the duration of experiment has reached i.e. the termination of the experiment take place at $\max(T, X_{R:n})$. Such a censoring scheme called Type-II hybrid censoring scheme. It may be noted here that in this censoring scheme the total number of observation is random but it will not be less than R. Thus, it guarantees a minimum efficiency but the duration of the test will be vary and may be more than T. Balakrishnan and Kundu (2013) provides a detailed review on hybrid censoring schemes with its generalisations and application in competing risks and step-stress modelling. In addition to it, Childs *et al.* (2003); Ganguly *et al.* (2012); Banerjee and Kundu (2008); Kohansal *et al.* (2015); Al-Zahrani and Gindwan (2014) can also be referred for the estimation of the parameters under Type-II hybrid censoring schemes.

Prediction of future order statistics comes up quite naturally in several real life situations. Here, we consider the estimation of future order statistics under Bayesian paradigm. Dansmore (1974); Lawless (1971) are the first two articles that address the prediction problems and introduced the concept of predictive posterior. Since then, prediction problems have been extended to various censoring schemes. Ebrahmini (1992) discussed two sample prediction problems for exponential distribution under Type-I hybrid censoring. Shafay and Balakrishnan (2012) and Singh *et al.* (2016) discussed the one- and two sample prediction problems based on Type-I hybrid censored sample for general class of distribution and generalized Lindley distribution respectively. Balakrishnan and Shafay (2012) developed estimation procedure for one- and two sample prediction problems based on Type-II hybrid censored sample for general class of distribution.

An extension of the exponential distribution, called as flexible Weibull by Bebbington *et al.* (2007), is considered to be the lifetime model. The density function of the flexible Weibull (FW) distribution is given by

$$f(x;\alpha,\beta) = \left(\alpha + \frac{\beta}{x^2}\right)\exp\left(\alpha x - \frac{\beta}{x}\right)\exp\left(-e^{\left(\alpha x - \frac{\beta}{x}\right)}\right), x > 0, \beta > 0, \alpha > 0. \quad (1)$$

The cumulative density function (CDF) and hazard rate function (HRF), at epoch

t, corresponding to the pdf (1) are given by

$$F(t;\alpha,\beta)=1-\exp\left(-e^{\left(\alpha t-\frac{\beta}{t}\right)}\right) \tag{2}$$

$$h(t;\alpha,\beta)=\left(\alpha+\frac{\beta}{t^2}\right)\exp\left(\alpha t-\frac{\beta}{t}\right). \tag{3}$$

The mean time to failure (MTF) is given by

$$\text{MTF}=E(X)=\int_0^\infty \exp\left(-e^{\left(\alpha x-\frac{\beta}{x}\right)}\right)dx. \tag{4}$$

This distribution is capable to accommodate the various forms of the hazard rate including IFR, IFRA, and MBT (modified bathtub), and in particular, allows considerable flexibility in modelling the "pre-useful" (i.e., infancy) period. Figure 1 shows the pdf and hazard function of FW distribution with various combinations of the parameters. Bebbington *et al.* (2007) have verified the goodness-of-fit of this distribution in comparison to the Weibull, modified Weibull, reduced additive Weibull and extended Weibull distributions for data set of failure times of secondary reactor pumps and recommended this distribution as a more suitable alternative to the various generalizations of the Weibull distribution. Motivating from the applicability of the FW distribution, Singh *et al.* (2013, 2015) discussed the use of FW distribution under censoring mechanism and developed classical and Bayes estimators for the FW parameters under Type-II and progressive Type-II censoring schemes respectively. They also derived One- and Two-sample predictive posteriors for future order statistics on the basis of censored data. Selim and Salem (2014) provided recurrence relations for moments of k-th upper record values from flexible Weibull Distribution. Afify (2016) also discussed the flexible Weibull distribution under progressive Type-I interval censoring schemes respectively.

Although a vast literature is available on the estimation under hybrid censoring, limited attention has been paid to the prediction of future ordered statistics based on hybrid censored sample. On the other hand, the estimation and prediction for flexible Weibull distribution is considered only for conventional Type-II (Singh *et al.*, 2013), progressive Type-II (Singh *et al.*, 2015) and progressive Type-I interval censored data sets. Good applicability notwithstanding, the Type-II hybrid censoring is overlooked in literature. Realizing the pertinency of the Type-II hybrid censoring and flexible Weibull distribution, we, in this article, aim to develop estimation procedures for estimating parameters and future observations based on Type-II hybrid censored data that follow the flexible Weibull distribution.

The rest of the paper is organized as follows. In Section 2, the estimation of the parameters, reliability, hazard rate and MTF is approached by using both classical and Bayesian methods. We constructed point and interval estimates of the unknown parameters. Bayesian one sample and two sample prediction problems are discussed in Section

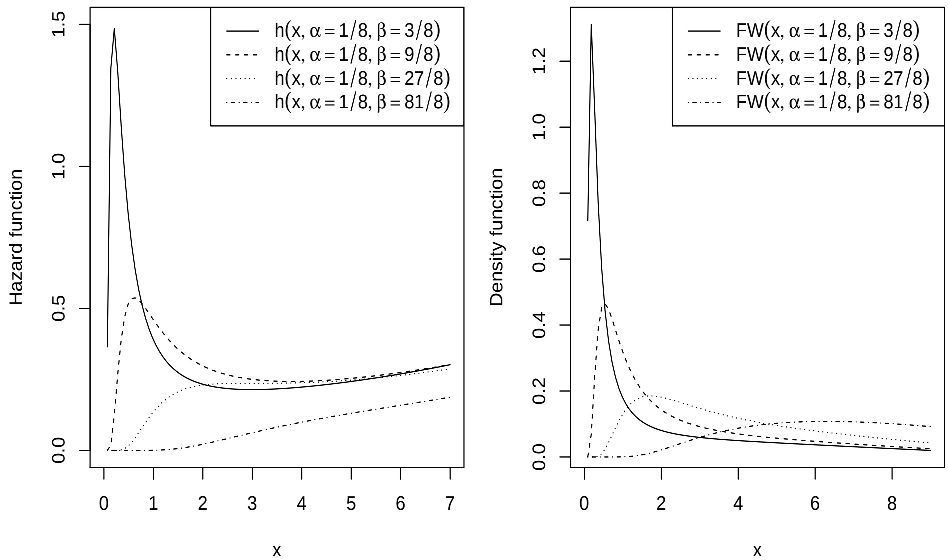

Figure 1 – The pdf and hazard function of flexible Weibull distribution (Bebbington *et al.*, 2007).

3. The predictive bounds of future samples are also constructed in respective section. To compare the performance of the proposed estimators, Monte Carlo simulation results are summarised in Section 4. A real data set is analysed in Section 5 for illustration. At the end of this paper, conclusions are given in Section 6.

2. Estimation of the parameters, reliability, hazard rate and MTF under Type-II hybrid censored sample

2.1. *Estimation based on likelihood*

Suppose that n identical units are placed on a life-test and their lifetimes follow the FW distribution defined by the pdf (1). The life-test is terminated at pre-specified time (say, T) and at pre-determined number of failure (say, $R \leq n$) whichever occurs later i.e. the test is terminated at $\max(X_{R:n}, T)$, where, $X_{R:n}$ denotes the Rth order statistic. Under such type of censoring scheme (known as Type-II hybrid censoring scheme), the random observations would be obtained under the following cases:
Case I: $\{X_{1:n}, X_{2:n}, \ldots, X_{R:n}\}$ if $T < X_{R:n}$.
Case II:$\{X_{1:n}, X_{2:n}, ..., X_{k:n}\}$ if $X_{R:n} \leq T$, $R \leq k < n$.
Case III:$\{X_{1:n}, X_{2:n}, ..., X_{n:n}\}$ if $X_{n:n} \leq T$.

The likelihood function under such a Type-II hybrid censoring scheme can be written as follows

$$\ell(\alpha,\beta|\underline{\mathbf{x}}) = \frac{n!}{(n-r_0)!}\prod_{i=1}^{r_0}\left(\alpha+\beta/x_{(i)}^2\right)\phi(x_{(i)})\exp\left(-\sum_{i=1}^{r_0}\phi(x_{(i)})-(n-r_0)\phi(t_0)\right), \quad (5)$$

where $r_0 = \begin{cases} R & \text{for case-I} \\ k & \text{for case-II} \\ n & \text{for case-III} \end{cases}$, $t_0 = \begin{cases} X_{R:n} & \text{for case-I} \\ T & \text{for case-II} \\ X_{n:n} & \text{for case-III} \end{cases}$ and $\phi(x) = \exp(\alpha x - \beta/x)$.

The log-likelihood function is given by

$$\log(\ell) = \ln\left(\frac{n!}{(n-r_0)!}\right)+\sum_{i=1}^{r_0}\ln(\alpha+\beta/x_{(i)}^2)+\sum_{i=1}^{r_0}\ln\phi(x_{(i)})-\sum_{i=1}^{r_0}\phi(x_{(i)})-(n-r_0)\phi(t_0). \quad (6)$$

The MLEs $\hat{\alpha}$ and $\hat{\beta}$ can be obtained as the simultaneous solution of the following log-likelihood equations

$$\sum_{i=1}^{r_0}\frac{1}{\left(\alpha+\beta/x_{(i)}^2\right)}+\sum_{i=1}^{r_0}x_{(i)}-\sum_{i=1}^{r_0}x_{(i)}\phi(x_{(i)})-(n-r)t_0\phi(t_0)=0 \quad (7)$$

$$\sum_{i=1}^{r_0}\frac{1/x_{(i)}^2}{\left(\alpha+\beta/x_{(i)}^2\right)}-\sum_{i=1}^{r_0}\frac{1}{x_{(i)}}+\sum_{i=1}^{r_0}\frac{\phi(x_{(i)})}{x_{(i)}}+(n-r_0)\frac{\phi(t_0)}{t_0}=0. \quad (8)$$

It can be seen that above equations cannot be solved explicitly and one needs iterative method to solve the above equations numerically. Here, we propose the use of fixed point iteration method which can be routinely applied as follows: from Equations (7) and (8), we can write the fixed point iteration equations as

$$\alpha = \phi_1(\alpha,\beta) = \frac{\sum_{i=1}^{r_0}\left(1+\frac{\beta}{\alpha x_{(i)}^2}\right)^{-1}}{h_1(\alpha,\beta)} \quad (9)$$

$$\beta = \phi_2(\alpha,\beta) = \frac{\sum_{i=1}^{r_0}\frac{1}{x_{(i)}^2}\left(\frac{\alpha}{\beta}+\frac{1}{x_{(i)}^2}\right)^{-1}}{h_2(\alpha,\beta)}, \quad (10)$$

where $h_1(\alpha,\beta) = \sum_{i=1}^{r_0}x_{(i)}\phi(x_{(i)})+(n-r_0)t_0\phi(t_0)-\sum_{i=1}^{r_0}x_{(i)}$ and $h_2(\alpha,\beta) = \sum_{i=1}^{r_0}\frac{1}{x_{(i)}} - \sum_{i=1}^{r_0}\frac{\phi(x_{(i)})}{x_{(i)}}-(n-r_0)\frac{\phi(t_0)}{t_0}$. By using the guess values α^{j-1} and β^{j-1} for α and β, one can get the next values of α and β as α^j and β^j respectively from Equations (9) and (10). For some pre-assigned tolerance limit $\epsilon > 0$, (α^j,β^j) will be the desired solution of Equations (7) and (8) if these satisfy the following inequalities $|\alpha^{j-1} - \alpha^{j}| \leq \epsilon$ and $|\beta^{j-1} - \beta^{j}| \leq \epsilon$, j=1,2,3,....

The exact distribution of MLEs cannot be obtained explicitly. However, the asymptotic properties of MLEs can be used to construct the confidence intervals for the parameters. Under some regularity conditions, the MLEs $(\hat{\alpha}, \hat{\beta})$ follow approximately bivariate normal distribution with mean $\mu^{'} = (\alpha, \beta)$ and variance matrix $\Sigma = I^{-1}(\hat{\alpha}, \hat{\beta})$, where $I(\hat{\alpha}, \hat{\beta})$ is the observed Fisher's information matrix defined by

$$I\left(\hat{\alpha}, \hat{\beta}\right) = \begin{bmatrix} -\frac{d}{d\alpha^2}\log(\ell) & -\frac{d}{d\beta d\alpha}\log(\ell) \\ -\frac{d}{d\alpha d\beta}\log(\ell) & -\frac{d}{d\beta^2}\log(\ell) \end{bmatrix}_{\left(\hat{\alpha}, \hat{\beta}\right)},$$

where

$$\frac{d}{d\alpha^2}\log(\ell) = -\sum_{i=1}^{r_0} \frac{1}{\left(\alpha + \beta/x_{(i)}^2\right)^2} - \sum_{i=1}^{r_0} x_{(i)}^2 \phi(x_{(i)}) - (n - r_0) t_0^2 \phi(t_0) \tag{11}$$

$$\frac{d}{d\beta^2}\log(\ell) = -\sum_{i=1}^{r_0} \frac{\left(1/x_{(i)}^2\right)^2}{\left(\alpha + \beta/x_{(i)}^2\right)^2} + \sum_{i=1}^{r_0} \frac{\phi(x_{(i)})}{x_{(i)}^2} + (n - r_0)\frac{\phi(t_0)}{t_0^2} \tag{12}$$

$$\frac{d}{d\alpha d\beta}\log(\ell) = \frac{d}{d\beta d\alpha}\log(\ell) = -\sum_{i=1}^{r_0} \frac{\left(1/x_{(i)}^2\right)}{\left(\alpha + \beta/x_{(i)}^2\right)^2} + \sum_{i=1}^{r_0} \phi(x_{(i)}) + (n - r_0)\phi(t_0). \tag{13}$$

The diagonal elements of $I^{-1}(\hat{\alpha}, \hat{\beta})$ provides the asymptotic variances for the parameters α and β respectively. A two-sided $100(1-\gamma)\%$ normal approximation confidence interval of α can be obtained as

$$\left\{\hat{\alpha} \mp Z_{\gamma/2}\sqrt{\text{var}(\hat{\alpha})}\right\}.$$

Similarly, a two-sided $100(1-\gamma)\%$ normal approximation confidence interval of β can be obtained as

$$\left\{\hat{\beta} \mp Z_{\gamma/2}\sqrt{\text{var}(\hat{\beta})}\right\},$$

where Z is the standard normal variate (SNV).

2.2. *Bayes estimation*

In this section, we proposed Bayes estimators of the unknown parameters of FW distribution based on Type-II censored sample. For Bayesian estimation, we need to specify prior distributions for the parameters. Berger (1985) have discussed various approaches for prior specifications. The easiest way of prior specifications is to assume the functional form for the prior density and then elicit its parameter(s) (called hyper- parameter(s)). The gamma distribution is quite flexible to accommodate various shapes of the density and provides conjugacy and mathematical ease in some cases, and has closed form expressions for moments. In Bayesian estimation, the gamma distribution has been widely exploited as a prior distribution for the parameters of various lifetime distributions. In this study, we also assume that α and β are independent gamma random variables. The prior distribution of α and β are given by

$$g(\alpha) \propto \alpha^{b-1} e^{-\alpha a}; a > 0, b > 0, \alpha > 0 \tag{14}$$

$$g(\beta) \propto \beta^{d-1} e^{-\beta c}; c > 0, d > 0, \beta > 0, \tag{15}$$

where the hyper-parameters a, b, c and d are assumed to be known, and chosen to reflect the prior belief about the unknown parameters. These can be obtained, if we can guess the expected values of the parameter as prior mean (M) and confidence in the guessed value as prior variance (V). Thus, a, b, c and d can be elicited by solving the following prior moments equations: $\frac{b}{a} = M$, $\frac{b}{a^2} = V$, $\frac{d}{c} = M$ and $\frac{d}{c^2} = V$. Note that when $a = b = c = d = 0$, the prior densities reduce to non-informative improper prior distributions as

$$g(\alpha) \propto \frac{1}{\alpha}; \alpha > 0$$

$$g(\beta) \propto \frac{1}{\beta}; \beta > 0.$$

In some cases, the above prior distribution coincides with scale invariant Jeffrey prior (see, Berger (1985); Jeffrey (1961)). The joint posterior density of α and β can be obtained as

$$\begin{aligned} \pi(\alpha, \beta \mid \underline{\mathbf{x}}) = \frac{1}{C} \alpha^{b-1} \beta^{d-1} \exp\Bigg(& \sum_{i=1}^{r_0} \ln\left(\alpha + \beta / x_{(i)}^2\right) + \sum_{i=1}^{r_0} \ln \phi(x_{(i)}) \\ & - \sum_{i=1}^{r_0} \phi(x_{(i)}) - (n - r_0)\phi(t_0) - \alpha a - \beta c \Bigg), \end{aligned} \tag{16}$$

where C is the normalizing constant. It is well established that Bayes estimator of any parametric function, $\Psi(\alpha, \beta)$ under squared error loss is the expected value of that function with respect to the posterior distribution, i.e. $E_\pi[\Psi(\alpha, \beta)]$. Therefore, Bayes estimates of α and β are the means of their posteriors. Similarly, the Bayes estimates of the reliability and hazard rate under squared error loss can be readily obtained as

$$\hat{R}(t)=\frac{1}{C}\int_{\alpha}\int_{\beta}\alpha^{b-1}\beta^{d-1}\exp\left(\sum_{i=1}^{r_0}\ln\left(\alpha+\beta/x_{(i)}^2\right)+\sum_{i=1}^{r_0}\ln\phi(x_{(i)})\right.$$
$$\left.-\sum_{i=1}^{r_0}\phi(x_{(i)})-(n-r_0)\phi(t_0)-\phi(t)-\alpha a-\beta c\right)d\beta d\alpha \tag{17}$$

$$\hat{h}(t)=\frac{1}{C}\int_{\alpha}\int_{\beta}\left(\alpha+\beta/t^2\right)\alpha^{b-1}\beta^{d-1}\exp\left(\sum_{i=1}^{r_0}\ln\left(\alpha+\beta/x_{(i)}^2\right)+\sum_{i=1}^{r_0}\ln\phi(x_{(i)})\right.$$
$$\left.-\sum_{i=1}^{r_0}\phi(x_{(i)})-(n-r_0)\phi(t_0)+\ln\phi(t)-\alpha a-\beta c\right)d\beta d\alpha. \tag{18}$$

In the same way, the Bayes estimate of the MTF can be given by

$$\widehat{\mathrm{MTF}}(t)=\frac{1}{C}\int_{\alpha}\int_{\beta}\int_{t=0}^{\infty}\alpha^{b-1}\beta^{d-1}\exp\left(\sum_{i=1}^{r_0}\ln\left(\alpha+\beta/x_{(i)}^2\right)+\sum_{i=1}^{r_0}\ln\phi(x_{(i)})\right.$$
$$\left.-\sum_{i=1}^{r_0}\phi(x_{(i)})-(n-r_0)\phi(t_0)-\phi(t)-\alpha a-\beta c\right)dt\,d\beta d\alpha \tag{19}$$

It can be seen that the above expressions (17-19) cannot be solved in nice closed form. Therefore, we proposed to use Markov chain Monte Carlo (MCMC) methods namely Gibbs sampler (Smith and Roberts, 1993) and Metropolis Hastings algorithm (Hastings, 1970; Brooks, 1998), to draw the random sample from the joint posterior so that sample based inference can be drawn. For implementing the Gibbs algorithm, the full conditional posterior densities of α and β are given by

$$\pi_1(\alpha\mid\beta,\underline{\mathbf{x}})\propto\alpha^{b-1}e^{-\alpha\left(\sum_{i=1}^{r}x_{(i)}+a\right)}$$
$$\exp\left(\sum_{i=1}^{r_0}\ln\left(\alpha+\beta/x_{(i)}^2\right)-\sum_{i=1}^{r_0}\phi(x_{(i)})-(n-r_0)\phi(t_0)\right) \tag{20}$$

$$\pi_2(\beta\mid\alpha,\underline{\mathbf{x}})\propto\beta^{d-1}e^{-\beta\left(\sum_{i=1}^{r}\frac{1}{x_{(i)}}+c\right)}$$
$$\exp\left(\sum_{i=1}^{r_0}\ln\left(\alpha+\beta/x_{(i)}^2\right)-\sum_{i=1}^{r_0}\phi(x_{(i)})-(n-r_0)\phi(t_0)\right). \tag{21}$$

The simulation algorithm consists of the following steps

Step 1. Initialize the values of $\{\alpha^{(0)},\beta^{(0)}\}$.

Step 2. At stage j, by using previous points, generate candidate points $\{\alpha_c^{(j)}, \beta_c^{(j)}\}$ from proposal densities $\{q_1\left(\alpha^{(j)}|\alpha^{(j-1)}\right), q_2\left(\beta^{(j)}|\beta^{(j-1)}\right)\}$.

Step 3. Generate $u \sim U(0,1)$ and set

$$\alpha^{(j)} = \begin{cases} \alpha_c^{(j)} & \text{if } u \leq \frac{\pi_1\left(\alpha_c^{(j)}|\beta^{(j-1)}, \underline{\mathbf{x}}\right) q_1\left(\alpha^{(j-1)}|\alpha_c^{(j)}\right)}{\pi_1\left(\alpha^{(j-1)}|\beta^{(j-1)}, \underline{\mathbf{x}}\right) q_1\left(\alpha_c^{(j)}|\alpha^{(j-1)}\right)} \\ \alpha^{(j-1)} & \text{otherwise} \end{cases}$$

and then

$$\beta^{(j)} = \begin{cases} \beta_c^{(j)} & \text{if } u \leq \frac{\pi_2\left(\beta_c^{(j)}|\alpha^{(j)}, \underline{\mathbf{x}}\right) q_2\left(\beta^{(j-1)}|\beta_c^{(j)}\right)}{\pi_2\left(\beta^{(j-1)}|\alpha^{(j)}, \underline{\mathbf{x}}\right) q_2\left(\beta_c^{(j)}|\beta^{(j-1)}\right)} \\ \beta^{(j-1)} & \text{otherwise.} \end{cases}$$

Step 4. Repeat steps 2-3 for all values of $j = 1, 2, \ldots, M$.

Step 5. Using simulated posterior samples, obtain Bayes estimates of the parameters by using following formulae: $\hat{\alpha} = E_\pi(\alpha \mid \underline{\mathbf{x}}) \approx \frac{1}{M-M_0} \sum_{k=1}^{M-M_0} \alpha_k$, $\hat{\beta} = E_\pi(\beta \mid \underline{\mathbf{x}}) \approx \frac{1}{M-M_0} \sum_{k=1}^{M-M_0} \beta_k$, where, M_0 is the burn-in-period of Markov Chain.

Step 6. Similarly, obtain the Bayes estimates of reliability, hazard rate and MTF by using following formulae

$$\hat{R}(x) \approx \frac{1}{M-M_0} \sum_{k=1}^{M-M_0} \exp\left(-e^{\alpha_k x - \beta_k/x}\right)$$

$$\hat{h}(x) \approx \frac{1}{M-M_0} \sum_{k=1}^{M-M_0} \left(\alpha_k + \beta_k/x^2\right) \exp\left(\alpha_k x - \beta_k/x\right)$$

$$\widehat{\text{MTF}} \approx \frac{1}{M-M_0} \sum_{k=1}^{M-M_0} \left\{ \int_0^\infty \exp\left(-e^{\alpha_k x - \beta_k/x}\right) dx \right\}.$$

Step 7. Construct HPD credible intervals for α and β by using the results given in Chen and Shao (1998). Let $\{\Theta_{(i)} = \left\{\alpha_{(i)}, \beta_{(i)}\right\}; i = 1, 2, ..., M\}$ be an ergodic MCMC sample from $\pi(\Theta|\underline{\mathbf{x}})$ and let $R_{j*}(M) = \left(\Theta_{(j^*)}, \Theta_{(j^*+[(1-\psi)M])}\right)$. Then, the $100(1-\psi)\%$ HPD credible intervals for Θ is $R_{j*}(M)$, where j^* is chosen so that

$$\Theta_{(j^*+[(1-\psi)M])} - \Theta_{(j^*)} = \min_{1 \leq j \leq M-[(1-\psi)M]} \left(\Theta_{(j+[(1-\psi)M])} - \Theta_{(j)}\right).$$

That is, $R_{j*}(n)$ has the smallest interval width among all $R_j(n)$.

3. Predictive posterior analysis

In this section, we have derived the density of future order statistics from the FW distribution using Type-II hybrid censored sample. The pdf of the future sample is referred to as the predictive density. The prediction problems has been classified in two ways: (i) One sample prediction and (2) Two sample prediction, which are discussed in the next consecutive sub-sections.

3.1. One sample prediction

In Type-II hybrid censoring scheme, the data in hand consist of only few observed items out of all units/items put on test. In such situations, the experimenter may be interested to know the life times of the removed surviving units on the basis of the sample in hand. Let $r_0 < n$ and we wish to predict $X_{s:n}(r_0 < s \leq n)$. The conditional distribution of sth order statistics given Type-II hybrid censored sample $\underline{\mathbf{x}}$ shall be as follows

For Case-I:

$$f_1(x_{s:n}|\underline{\mathbf{x}}\,,\alpha,\beta) = f(x_{s:n}|x_{R:n},\alpha,\beta) = \frac{f_{Nu}(X_{s:n}=x_{s:n},X_{R:n}=x_{R:n}|\alpha,\beta)}{f_{De}(X_{R:n}=x_{R:n}|\alpha,\beta)}. \tag{22}$$

Therefore

$$f_1(x_{s:n}|\underline{\mathbf{x}}\,,\alpha,\beta) = \frac{(n-R)!}{(n-s)!(s-R-1)!}\frac{[F(x_{s:n})-F(x_{R:n})]^{s-R-1}[1-F(x_{s:n})]^{n-s}f(x_{s:n})}{[1-F(x_{R:n})]^{n-R}}, \tag{23}$$

where $\underline{\mathbf{x}} = \{x_{1:n}, x_{2:n}, ..., x_{R:n}\}, x_{s:n} > x_{R:n}$ and $R < s \leq n$. Putting (1) and (2) in (23), we get

$$\begin{aligned} f_1(x_{s:n}|\underline{\mathbf{x}}\,,\alpha,\beta) = {} & (s-R-1)\binom{n-R}{n-s}\left(\alpha+\beta/x_{s:n}^2\right)\phi(x_{s:n})\sum_{l=0}^{s-R-1}(-1)^l \\ & \binom{s-R-1}{l}\exp[-(n-s+l+1)(\phi(x_{s:n})-\phi(x_{R:n}))]. \end{aligned} \tag{24}$$

For Case-II:

$$\begin{aligned} f_2(x_{s:n}|\underline{\mathbf{x}}\,,K=k,\alpha,\beta) = {} & \frac{1}{\Pr(R \leq K \leq s-1)} \\ & \sum_{k=R}^{s-1}\frac{f_{Nu}(X_{s:n}=x_{s:n},X_{k:n}=x_{k:n}|K=k,\alpha,\beta)}{f_{De}(X_{k:n}=x_{k:n}|\alpha,\beta)}\Pr(K=k), \end{aligned} \tag{25}$$

where

$$f_{Nu}(\ldots) = \frac{n![F(T)]^{k-1}[F(x_{s:n})-F(T)]^{s-k-1}[1-F(x_{s:n})]^{n-s}f(T)f(x_{s:n})}{(k-1)!(s-k-1)!(n-s)!}$$

and

$$f_{De}(X_{k:n}=x_{k:n}|\alpha,\beta)=\frac{n!}{(k-1)!(n-k)!}[F(T)]^{k-1}[1-F(T)]^{n-k}f(T).$$

Therefore

$$f_2(x_{s:n}|\underline{\mathbf{x}},\alpha,\beta)=\sum_{k=R}^{s-1}\frac{(n-k)!\Pr(K=k)}{(n-s)!(s-k-1)!}\frac{(F(x_{s:n})-F(T))^{s-k-1}(1-F(x_{s:n}))^{n-s}f(x_{s:n})}{\Pr(R\le K\le s-1)(1-F(T))^{n-k}}, \quad (26)$$

where $\underline{\mathbf{x}}=\{x_{1:n},x_{2:n},...,x_{k:n}\},k<s\le n,x_{s:n}>T$, and K is a random variable, represents the number of $x_i's$ that are at most T. Then

$$P(K=k)=\binom{n}{k}F(T)^k[1-F(T)]^{n-k};\quad k=0,\ 1,\ ...,\ n.$$

Putting (1) and (2) in (26), then equation (26) becomes

$$\begin{aligned}f_2(x_{s:n}|\underline{\mathbf{x}},\alpha,\beta)=&\frac{1}{\Pr(R\le K\le s-1)}\left(\alpha+\beta/x_{s:n}^2\right)\phi(x_{s:n})\sum_{k=R}^{s-1}\sum_{l=0}^{s-k-1}(-1)^l\\&(s-k-1)\binom{n-k}{n-s}\binom{s-k-1}{l}\Pr(K=k)\\&\exp[-(n-s+l+1)(\phi(x_{s:n})-\phi(T))].\end{aligned}\quad (27)$$

The One-sample predictive posterior density of future observables is defined by

For Case-I:

$$f_1^\star(x_{s:n}\,|\,\underline{\mathbf{x}})=\int_0^\infty\int_0^\infty f_1(x_{s:n}\,|\,\alpha,\beta,\underline{\mathbf{x}})\,\pi(\alpha,\beta\,|\,\underline{\mathbf{x}})\,d\alpha d\beta. \quad (28)$$

For Case-II:

$$f_2^\star(x_{s:n}\,|\,\underline{\mathbf{x}})=\int_0^\infty\int_0^\infty f_2(x_{s:n}\,|\,\alpha,\beta,\underline{\mathbf{x}})\,\pi(\alpha,\beta\,|\,\underline{\mathbf{x}})\,d\alpha d\beta. \quad (29)$$

The above predictive posteriors can not be reduced to any standard distribution and hence numerical methods are required to explore the properties of the posteriors. Here, we can used the M-H algorithm to draw the sample from the predictive posteriors and obtain the estimates and predictive intervals for future observations using the approach as discussed in the previous section. The survival function of future sample can be simply defined as

For Case-I:

$$S_y(T_0)=1-\int_{y=x_{(r)}}^{T_0}\int_0^\infty\int_0^\infty f(x_{s:n}\,|\,\alpha,\beta,\underline{\mathbf{x}})\,\pi(\alpha,\beta\,|\,\underline{\mathbf{x}})\,d\alpha d\beta dy. \quad (30)$$

For Case-II:

$$S_y(T_0) = 1 - \int_{y=T}^{T_0} \int_0^{\infty} \int_0^{\infty} f(x_{s:n} \mid \alpha, \beta, \underline{\mathbf{x}}) \, \pi(\alpha, \beta \mid \underline{\mathbf{x}}) \, d\alpha \, d\beta \, dy. \tag{31}$$

We can also obtain the two sided $100(1-\alpha)\%$ prediction intervals (L_s, U_s) for $y_{(s)}$ by solving the following equations

$$\Pr\left(y_{(s)} > U_x \mid \underline{\mathbf{x}}\right) = \frac{\alpha}{2} \tag{32}$$

$$\Pr\left(y_{(s)} > L_x \mid \underline{\mathbf{x}}\right) = 1 - \frac{\alpha}{2}. \tag{33}$$

Confidence intervals can be obtained by using any suitable iterative procedure as above equations cannot be solved directly.

3.2. Two sample prediction

In some other situation, we may be interested in the failure time of the kth ordered observation from future sample of size N from the same lifetime distribution which is independent of observed sample i.e. $p\left(y_{(s)} \mid \alpha, \beta, \underline{\mathbf{x}}\right) = p\left(y_{(s)} \mid \alpha, \beta\right)$. This leads to the two sample prediction problems. The pdf of sth order statistic is given by

$$p\left(y_{(s:N)} \mid \alpha, \beta\right) = \frac{N!}{(s-1)!(N-s)!} \left[F\left(y_{(s:N)}\right)\right]^{k-1} \left[R\left(y_{(s:N)}\right)\right]^{N-k} f\left(y_{(s:N)}\right). \tag{34}$$

Putting (1) and (2) in (34), we get

$$p\left(y_{(s:N)} \mid \alpha, \beta\right) = \frac{N! \, \phi(y_{s:N})}{(s-1)!(N-s)!} \left(\alpha + \beta / y^2_{(s:N)}\right) \sum_{j=0}^{s-1} \binom{s-1}{j} (-1)^j \, e^{-\phi(y_{s:N})(N-s-j-1)}. \tag{35}$$

The Two-sample predictive posterior density of sth future observation is defined by

$$p\left(y_{(s:N)} \mid \alpha, \beta\right) = \int_0^{\infty} \int_0^{\infty} p\left(y_{(s)} \mid \alpha, \beta, \underline{\mathbf{x}}\right) \pi(\alpha, \beta \mid \underline{\mathbf{x}}) \, d\alpha \, d\beta. \tag{36}$$

The above predictive posterior density is same for both the cases (Case-I and Case-II) since the density in (34) does not dependent on Type-II censoring and we can obtain the predictive density for Case-I and Case-II by substituting the posterior $\pi(\alpha, \beta \mid \underline{\mathbf{x}})$ of the respective case. The above posterior density can be numerically explored by using MCMC methods as discussed in the previous section. The survival function of future sample can be simply defined as

$$S_y(T_0) = 1 - \int_{y=0}^{T_0} \int_0^{\infty} \int_0^{\infty} p\left(y_{(s)} \mid \alpha, \beta, \underline{\mathbf{x}}\right) \pi(\alpha, \beta \mid \underline{\mathbf{x}}) \, d\alpha \, d\beta \, dy. \tag{37}$$

We can also obtain the two sided $100(1-\alpha)\%$ prediction intervals (L_k, U_k) for $y_{(k)}$ by solving the following two equations

$$P\left(y_{(s)} > U_x \mid \underline{\mathbf{x}}\right) = \frac{\alpha}{2} \tag{38}$$

$$P\left(y_{(s)} > L_x \mid \underline{\mathbf{x}}\right) = 1 - \frac{\alpha}{2}. \tag{39}$$

Confidence intervals can be obtained by using any suitable iterative procedure as above equations cannot be solved directly.

4. Simulation Study

In this section, we conduct simulation study to compare the performance of the classical and Bayesian estimation procedures under different Type-II hybrid censoring schemes. The comparison between the MLEs and Bayes estimators of the parameters have been made in terms of their mean squared errors(MSEs) and average width of confidence/HPD intervals. For this purpose, we generated random sample of size n (= 20 small, 30 medium and 50 large) from the pdf (1) by using probability integral transformation for given α and β. Here, we took arbitrarily $\alpha = 2$ and $\beta = 2$. The random sample can be drawn by using the following formula

$$x = \frac{1}{2\alpha}\left[\log(-\log(1-u)) + \sqrt{(\log(-\log(1-u)))^2 + 4\alpha\beta}\right],$$

where u is the standard uniform variate. From each generated sample, we obtained the MLEs and Bayes estimates for the parameters along with their confidence/HPD intervals. The MLEs and Bayes estimates for the reliability characteristics have also been obtained. Under gamma prior distribution, the hyper-parameters are chosen such that the prior mean equals to the true value of the parameter and with specified prior variance equals to 1. From the prior moments equations (given in Section 2.2), we obtained $a = c = 2$, $b = d = 4$. It is denoted by Gamma-1. The prior with hyper-parameters identical to zero ($a = c = b = d = 0$) is referred to as Gamma-0. We repeated the process 1000 times and average estimates, the MSEs, and the average confidence/HPD intervals are recorded. The simulation results are summarized in Tables 2-5.

4.1. Specification of censoring parameters

In this subsection of simulation study, we consider the problem of specification of the censoring parameters R and T, i.e. what pair of values of R and T should be considered for studying the performance of the estimates. In such case, one can first set the value of (R) the number of failures and take T by analysing the distribution of R^{th} order statistic form the sample of size n. For this purpose, two approaches, predictive inference and bootstrap procedures have been utilized.

4.1.1. Bootstrap procedure

Bootstrapping is a method of estimating statistical parameters from the sample by means of resampling with replacement. Like other non-parametric approaches, bootstrapping does not make any assumptions about the distribution of the sample. The major assumption behind bootstrapping is that the sample distribution is a good approximation to the population distribution and samples are independent and identically distributed. The bootstrap procedure is due to the pinion effort of Efron and Tibshirani (1986) that can be routinely applied using the following steps.

Step 1. Generate sample $\{x_1, x_2, ..., x_n\}$ of size n form the pdf (1) by using inversion method. Then estimated distribution function is given by $\hat{F}\left(x, \hat{\Theta}\right)$, where, $\Theta = \{\alpha, \beta; \alpha > 0, \beta > 0\}$.

Step 2. Generate a bootstrap sample $\{x_1^*, x_2^*, ..., x_n^*\}$ of size n from $\hat{F}\left(x, \hat{\Theta}\right)$. Calculate bootstrap estimates of statistics $t(\underline{\mathbf{x}})$ of interest using bootstrap sample. Here, $t(\underline{\mathbf{x}}) = X_{R:n}^*$.

Step 3. Repeat step 2, BOOT-times.

Step 4. Let $\{t_{(1)}, t_{(2)}, ..., t_{(\mathrm{BOOT})}\}$ be the ordered value of $\{t_1, t_2, \ldots, t_{\mathrm{BOOT}}\}$. The empirical distribution function of $\{t_1, t_2, ..., t_B\}$ is defined as $\hat{G}(x) = \left\{\frac{\text{Number of } (t_b < x)}{\mathrm{BOOT}}\right\}$. Then, mean and the two sided $100(1-\gamma)\%$ boot-p confidence intervals for $t(\underline{\mathbf{x}})$ can be obtained as

$$\bar{t}(\underline{\mathbf{x}}) = \frac{1}{\mathrm{BOOT}} \sum_{i=1}^{\mathrm{BOOT}} t_i(\underline{\mathbf{x}})$$
$$t(\underline{\mathbf{x}}) \in \{t_{(\mathrm{BOOT}\times\gamma/2)}, t_{(\mathrm{BOOT}\times(1-\gamma/2))}\}$$

Step 5. Repeat steps 1-4, 1000 of times and calculate average of the estimates and confidence intervals for the statistics of interest.

From Table 1, it can be easily seen that for $n = 30$ and $R = 18$, the censoring time T may take values between 0.85 and 1.07. We have considered the following combination of censoring parameters.

- $n = 20, R(= 12, 16), T(= 0.90, 1.00, 1.25, 1.50)$
- $n = 30, R(= 18, 24), T(= 0.90, 1.00, 1.25, 1.50)$
- $n = 50, R(= 35, 40), T(= 0.90, 1.00, 1.25, 1.50)$

TABLE 1
Average Bootstrap and predictive estimates and 95% confidence intervals of Rth order statistics for fixed values of $\alpha = 2$ and $\beta = 2$.

n	R	Bootstrap procedure			Prediction approach		
		Mean	Lower	Upper	Mean	Lower	Upper
20	12	0.96165	0.82585	1.09620	0.95718	0.81814	1.09655
20	16	1.09773	0.95659	1.23930	1.09982	0.95617	1.24496
30	18	0.96749	0.85484	1.07882	0.96496	0.84377	1.08602
30	24	1.10712	0.98998	1.22489	1.10937	0.98317	1.23701
50	35	1.03952	0.95008	1.12855	1.03968	0.93885	1.14124
50	40	1.11514	1.02220	1.20792	1.11723	1.01287	1.22323

4.2. *Candidate to the independent MH algorithm*

In Metropolis-Hastings (MH) algorithm, the most conman task is to choose appropriate candidate density for the posterior and it is very tricky since there are infinite number of choice to the proposal density. There is a vast literature available on MCMC methods. There have been various suggestion given for the selection of proposal density. Here, we proposed the use of normal approximation of the posterior. The means of the approximating normal proposal density of α and β are calculated as the solution of the following normal equations

$$\sum_{i=1}^{r_0} \frac{1}{\left(\alpha+\beta/x_{(i)}^2\right)} + \sum_{i=1}^{r_0} x_{(i)} - a - \sum_{i=1}^{r_0} x_{(i)}\phi(x_{(i)}) - (n-r)t_0\phi(t_0) + \frac{b-1}{\alpha} = 0$$

$$\sum_{i=1}^{r_0} \frac{1/x_{(i)}^2}{\left(\alpha+\beta/x_{(i)}^2\right)} - \sum_{i=1}^{r_0} \frac{1}{x_{(i)}} - c + \sum_{i=1}^{r_0} \frac{\phi(x_{(i)})}{x_{(i)}} + (n-r_0)\frac{\phi(t_0)}{t_0} + \frac{d-1}{\beta} = 0$$

and variances as the diagonal elements of the inverse of estimated Hessian matrix. We have generated posterior sample using MH algorithm with normal proposal distribution and checked the convergence of the sequences of α and β for their stationary distributions through different starting values. It was observed that all the Markov chains reached to the stationary condition very quickly. First thousand MCMC iterations (Burn-in period) have been discarded from the generated sequence.

In the predictive case, the approximate properties of the predictive posterior density can not be possible to obtain analytically as the density contains many integrals and sums. The normal proposal is not a suitable choice for the predictive posterior density since predictive posterior density is not symmetric. So, the method discussed above can not be applied here. In such situation, one can take any asymmetric shaped distributions as the proposal density similar to the target density because the shape of the candidate relative to the target density affects the convergence of the algorithm. We found that the three parameter Weibull distribution provides enough flexibility to cover the target density with different values of its parameters. The density of the three parameter Weibull distribution is given by

$$g(x) = \frac{w_2}{w_3}\left(\frac{x - w_1}{w_3}\right)^{w_2 - 1} e^{-\left(\frac{x-w_1}{w_2}\right)^{w_3}}, x > w_1 > 0, w_2 > 0, w_3 > 0,$$

where the location parameter w_1 has been taken as

$$\text{for one sample prediction, } w_1 = \begin{cases} X_{R:n} & \text{for case-I} \\ T & \text{for case-II,} \end{cases}$$

$$\text{for two sample prediction, } w_1 = 0.$$

The parameters w_2 and w_3 can be set by the experimenter such that the chain converges to its stationary distribution with satisfactory acceptance rate.

4.3. *Performance of the estimators*

On the basis of simulation results, we observe the following

- The MSEs of all the estimators decreases as sample increases for fixed values of α and β.
- In all the considered cases, it was noticed that the Bayes estimators show smaller MSEs than the MLEs.
- The Bayes estimators obtained under non-informative prior (Gamma-0) behave more or less similar to the MLEs. Although, the Bayes estimates obtained under Gamma-1 is more efficient than those obtained under Gamma-0 prior. This indicates that the Bayesian procedure with appropriate prior information provides more accurate estimates of the parameters.
- The width of the HPD credible intervals is smaller than the width of the asymptotic confidence intervals in all the cases.
  However, the width of the confidence/HPD intervals decreases as sample size increases.
- The Bayes estimators and MLEs of the reliability and MTF show more or less equal MSEs. However, the MSEs of the Bayes estimators of hazard rate are smaller than that of MLEs.

- For fixed values of n and R, the MSEs of all the estimators decrease as T increases.
- Generally, for fixed values of n and T, the MSEs of all the estimators decrease as R increases.

TABLE 2

Average estimates of the parameters, reliability, hazard and MTSF and corresponding MSEs (in brackets) with varying R and T for fixed $n=20, \alpha=2, \beta=2$, $R(1)=0.368$, $h(1)=4$ *and* $MTF=0.910$.

R	Set-up			T=0.9	T=1.0	T=1.25	T=1.50
R=12	Classical		α	2.252(0.557)	2.137(0.370)	2.079(0.178)	2.122(0.159)
			β	2.202(0.430)	2.139(0.347)	2.098(0.219)	2.129(0.214)
			R(t)	0.351(0.012)	0.369(0.009)	0.375(0.008)	0.370(0.007)
			h(t)	5.044(8.195)	4.448(3.361)	4.192(1.666)	4.297(1.545)
			MTF	0.910(0.004)	0.916(0.003)	0.916(0.003)	0.913(0.003)
	Bayes	Gamma-0	α	1.898(0.401)	1.841(0.328)	1.934(0.165)	2.001(0.128)
			β	1.957(0.302)	1.923(0.264)	1.969(0.186)	2.016(0.177)
			R(t)	0.389(0.009)	0.396(0.009)	0.380(0.007)	0.373(0.006)
			h(t)	4.011(4.191)	3.727(2.510)	3.905(1.439)	4.072(1.218)
			MTF	0.935(0.005)	0.938(0.005)	0.917(0.003)	0.911(0.003)
		Gamma-1	α	1.908(0.137)	1.876(0.134)	1.941(0.099)	1.992(0.080)
			β	1.954(0.114)	1.939(0.114)	1.970(0.111)	2.004(0.109)
			R(t)	0.384(0.008)	0.389(0.007)	0.378(0.006)	0.371(0.006)
			h(t)	3.919(1.703)	3.779(1.471)	3.934(1.151)	4.068(1.009)
			MTF	0.923(0.004)	0.926(0.004)	0.914(0.003)	0.911(0.003)
R=16	Classical		α	2.175(0.296)	2.161(0.280)	2.082(0.177)	2.122(0.159)
			β	2.160(0.295)	2.155(0.292)	2.099(0.220)	2.129(0.215)
			R(t)	0.363(0.009)	0.364(0.008)	0.374(0.007)	0.370(0.007)
			h(t)	4.575(3.675)	4.507(2.924)	4.195(1.655)	4.297(1.545)
			MTF	0.911(0.003)	0.912(0.003)	0.916(0.003)	0.913(0.003)
	Bayes	Gamma-0	α	1.973(0.222)	1.964(0.213)	1.937(0.163)	2.002(0.128)
			β	1.996(0.225)	1.991(0.224)	1.972(0.186)	2.016(0.177)
			R(t)	0.376(0.007)	0.377(0.007)	0.380(0.007)	0.373(0.006)
			h(t)	4.077(2.378)	4.026(1.956)	3.908(1.426)	4.072(1.218)
			MTF	0.916(0.003)	0.916(0.003)	0.9166(0.003)	0.911(0.002)
		Gamma-1	α	1.962(0.109)	1.956(0.105)	1.942(0.098)	1.992(0.080)
			β	1.982(0.114)	1.979(0.114)	1.971(0.112)	2.004(0.109)
			R(t)	0.375(0.007)	0.376(0.006)	0.378(0.006)	0.372(0.006)
			h(t)	4.038(1.496)	4.005(1.289)	3.937(1.141)	4.068(1.010)
			MTF	0.915(0.003)	0.915(0.003)	0.915(0.003)	0.911(0.002)

TABLE 3
Average estimates of the parameters, reliability, hazard and MTF and corresponding MSEs (in brackets) with varying R and T for fixed $n = 30, \alpha = 2, \beta = 2$, $R(1) = 0.3678$, $h(1) = 4$ and $MTF = 0.9104$.

R	Set-up			T=0.90	T=1.00	T=1.25	T=1.50
R=18	Classical		α	2.155(0.294)	2.068(0.209)	2.048(0.111)	2.079(0.095)
			β	2.123(0.220)	2.078(0.191)	2.063(0.135)	2.085(0.129)
			R(t)	0.357(0.008)	0.371(0.006)	0.373(0.005)	0.369(0.004)
			h(t)	4.638(4.030)	4.227(1.908)	4.111(1.010)	4.191(0.914)
			MTF	0.910(0.003)	0.916(0.002)	0.915(0.002)	0.913(0.001)
	Bayes	Gamma-0	α	1.931(0.236)	1.876(0.202)	1.954(0.106)	2.002(0.082)
			β	1.970(0.175)	1.939(0.162)	1.979(0.121)	2.013(0.112)
			R(t)	0.382(0.007)	0.390(0.006)	0.377(0.005)	0.372(0.004)
			h(t)	4.000(2.543)	3.755(1.630)	3.926(0.927)	4.048(0.784)
			MTF	0.925(0.003)	0.928(0.003)	0.915(0.002)	0.911(0.001)
		Gamma-1	α	1.935(0.116)	1.897(0.109)	1.957(0.076)	1.998(0.059)
			β	1.967(0.092)	1.951(0.091)	1.980(0.084)	2.007(0.081)
			R(t)	0.379(0.006)	0.387(0.005)	0.376(0.004)	0.371(0.003)
			h(t)	3.957(1.423)	3.791(1.126)	3.939(0.803)	4.048(0.699)
			MTF	0.919(0.003)	0.923(0.003)	0.915(0.002)	0.911(0.001)
R=24	Classical		α	2.092(0.157)	2.091(0.155)	2.049(0.111)	2.079(0.094)
			β	2.092(0.165)	2.091(0.165)	2.063(0.135)	2.085(0.127)
			R(t)	0.368(0.005)	0.368(0.005)	0.373(0.005)	0.369(0.004)
			h(t)	4.276(1.603)	4.266(1.522)	4.112(1.008)	4.191(0.914)
			MTF	0.912(0.002)	0.913(0.002)	0.915(0.002)	0.913(0.001)
	Bayes	Gamma-0	α	1.964(0.133)	1.963(0.132)	1.954(0.106)	2.003(0.082)
			β	1.988(0.139)	1.987(0.139)	1.979(0.121)	2.013(0.112)
			R(t)	0.376(0.005)	0.376(0.004)	0.375(0.004)	0.372 (0.004)
			h(t)	3.977(1.236)	3.969(1.185)	3.926(0.924)	4.048(0.784)
			MTF	0.915(0.002)	0.915(0.002)	0.915(0.002)	0.911(0.001)
		Gamma-1	α	1.962(0.083)	1.961(0.083)	1.957(0.075)	1.997(0.059)
			β	1.983(0.089)	1.982(0.089)	1.979(0.085)	2.006(0.081)
			R(t)	0.375(0.004)	0.375(0.004)	0.377(0.004)	0.371(0.004)
			h(t)	3.971(0.946)	3.965(0.913)	3.941(0.798)	4.048(0.699)
			MTF	0.915(0.002)	0.915(0.002)	0.914(0.002)	0.911(0.001)

TABLE 4
Average estimates of the parameters, reliability, hazard and MTF and corresponding MSEs (in brackets) with varying R and T for fixed $n = 50, \alpha = 2, \beta = 2$, $R(1) = 0.3678$, $h(1) = 4$ and MTF = 0.9104.

R	Set-up			T=0.90	T=1.00	T=1.25	T=1.50
R=35	Classical		α	2.089(0.137)	2.078(0.128)	2.029(0.070)	2.045(0.055)
			β	2.078(0.122)	2.072(0.119)	2.041(0.086)	2.052(0.077)
			R(t)	0.364(0.004)	0.365(0.003)	0.37214(0.002)	0.370(0.002)
			h(t)	4.293(1.361)	4.240(1.126)	4.054(0.544)	4.093(0.453)
			MTF	0.911(0.0014)	0.912(0.0013)	0.914(0.0012)	0.912(0.0011)
	Bayes	Gamma-0	α	1.991(0.119)	1.981(0.113)	1.974(0.068)	2.001(0.051)
			β	2.004(0.106)	1.999(0.105)	1.993(0.081)	2.010(0.072)
			R(t)	0.373(0.003)	0.374(0.003)	0.374(0.003)	0.371(0.002)
			h(t)	4.044(1.089)	4.000(0.936)	3.943(0.522)	4.013(0.417)
			MTF	0.914(0.001)	0.9151(0.001)	0.913(0.001)	0.911(0.001)
		Gamma-1	α	1.986(0.082)	1.978(0.079)	1.974(0.055)	1.998(0.042)
			β	1.999(0.075)	1.995(0.075)	1.991(0.064)	2.007(0.058)
			R(t)	0.372(0.003)	0.373(0.003)	0.374(0.002)	0.371(0.002)
			h(t)	4.024(0.836)	3.981(0.733)	3.949(0.476)	4.013(0.393)
			MTF	0.914(0.001)	0.914(0.001)	0.913(0.001)	0.911(0.001)
R=40	Classical		α	2.065(0.101)	2.065(0.101)	2.029(0.069)	2.045(0.055)
			β	2.064(0.104)	2.064(0.104)	2.041(0.086)	2.051(0.077)
			R(t)	0.367(0.003)	0.367(0.003)	0.372(0.002)	0.370(0.002)
			h(t)	4.182(0.888)	4.183(0.901)	4.053(0.543)	4.094(0.453)
			MTF	0.911(0.001)	0.911(0.001)	0.913(0.001)	0.912(0.001)
	Bayes	Gamma-0	α	1.990(0.090)	1.990(0.090)	1.974(0.068)	2.000(0.051)
			β	2.004(0.093)	2.004(0.093)	1.993(0.080)	2.010(0.071)
			R(t)	0.373(0.003)	0.372(0.003)	0.375(0.002)	0.371(0.002)
			h(t)	4.008(0.760)	4.010(0.772)	3.943(0.521)	4.013(0.417)
			MTF	0.913(0.001)	0.913(0.001)	0.913(0.001)	0.911(0.001)
		Gamma-1	α	1.986(0.068)	1.986(0.068)	1.974(0.054)	1.998(0.042)
			β	1.999(0.071)	1.999(0.070)	1.992(0.064)	2.007(0.058)
			R(t)	0.372(0.003)	0.372(0.002)	0.374(0.002)	0.371(0.002)
			h(t)	4.002(0.637)	4.003(0.647)	3.949(0.475)	4.013(0.393)
			MTF	0.913(0.001)	0.913(0.001)	0.913(0.001)	0.911(0.001)

TABLE 5
Average confidence/HPD intervals of α and β from different hybrid Type-II censoring scheme for fixed values of $\alpha = 2, \beta = 2$.

n, R	Set-up			T=0.90	T=1.0	T=1.25	T=1.50
n=20, R=12	Classical		α	0.943, 3.561	0.979, 3.295	1.301, 2.858	1.415, 2.828
			β	1.113, 3.291	1.115, 3.162	1.239, 2.958	1.298, 2.959
	Bayes	Non	α	1.322, 2.476	1.327, 2.353	1.585, 2.283	1.683, 2.318
			β	1.348, 2.592	1.335, 2.535	1.428, 2.541	1.484, 2.576
		G	α	1.468, 2.349	1.469, 2.284	1.634, 2.246	1.709, 2.275
			β	1.413, 2.532	1.406, 2.503	1.471, 2.503	1.512, 2.527
n=20, R=16	Classical		α	1.221, 3.129	1.217, 3.113	1.304, 2.859	1.415 ,2.828
			β	1.213, 3.107	1.211, 3.098	1.240, 2.959	1.298, 2.960
	Bayes	Non	α	1.551, 2.396	1.545, 1.603	1.587, 2.285	1.683, 2.318
			β	1.423, 2.595	1.420, 2.589	1.429, 2.542	1.484, 2.576
		G	α	1.606, 2.317	2.384, 2.310	1.636, 2.247	1.709, 2.275
			β	1.464, 2.531	1.462, 2.527	1.472, 2.504	1.512, 2.527
n=30, R=18	Classical		α	1.122, 3.188	1.138, 2.999	1.422, 2.674	1.516, 2.643
			β	1.266, 2.981	1.262, 2.893	1.372, 2.754	1.421, 2.750
	Bayes	Non	α	1.474, 2.391	1.463, 2.291	1.674, 2.235	1.748, 2.256
			β	1.460, 2.496	1.448, 2.451	1.534, 2.442	1.580, 2.467
		G	α	1.553, 2.317	1.544,2.249	1.699, 2.214	1.762, 2.256
			β	1.505, 2.456	1.498, 2.429	1.560, 2.422	1.595, 2.441
n=30, R=24	Classical		α	1.342, 2.843	1.341, 2.840	1.423, 2.675	1.516, 2.643
			β	1.343, 2.841	1.343, 2.839	1.372, 2.755	1.421, 2.750
	Bayes	Non	α	1.630, 2.299	1.629, 2.297	1.674, 2.235	1.748, 2.256
			β	1.521, 2.474	1.521, 2.473	1.534, 2.443	1.580, 2.467
		G	α	1.662, 2.259	1.661, 2.259	1.699, 2.214	1.762, 2.233
			β	1.547, 2.441	1.547, 2.440	1.560, 2.421	1.595, 2.441
n=50, R=35	Classical		α	1.420, 2.758	1.416, 2.740	1.549, 2.509	1.618, 2.472
			β	1.468, 2.688	1.465, 2.679	1.512, 2.571	1.546, 2.557
	Bayes	Non	α	1.693, 2.291	1.686, 2.279	1.758, 2.190	1.807, 2.193
			β	1.624, 2.397	1.620, 2.392	1.645, 2.351	1.674, 2.359
		G	α	1.711, 2.260	1.706, 2.250	1.769, 2.179	1.814, 2.183
			β	1.636, 2.374	1.633, 2.369	1.656, 2.339	1.681, 2.346
n=50, R=40	Classical		α	1.492, 2.638	1.492, 2.638	1.549, 2.510	1.618, 2.472
			β	1.493, 2.636	1.493, 2.636	1.512, 2.571	1.546, 2.557
	Bayes	Non	α	1.733, 2.247	1.733, 2.247	1.758, 2.191	1.807, 2.193
			β	1.639, 2.378	1.639, 2.378	1.769, 2.179	1.674, 2.359
		G	α	1.747, 2.227	1.747, 2.227	1.646, 2.351	1.814, 2.183
			β	1.649, 2.361	1.649, 2.361	1.656, 2.339	1.681, 2.346

5. Real data analysis

In this section, a real data set has been analysed to illustrate our discussed methodologies. The data represents the times between failures of secondary reactor pumps. This data was first reported by Suprawhardana *et al.* (1999), and modelled by Bebbington *et al.* (2007) using the FW distribution. Singh *et al.* (2013) have discussed the classical and Bayesian estimation methods under the Type-II censoring schemes of the data set. The times between failures of 23 secondary reactor pumps are as follows

2.160,	0.746,	0.402,	0.954,	0.491,	6.560,
4.992,	0.347,	0.150,	0.358,	0.101,	1.359,
3.465,	1.060,	0.614,	1.921,	4.082,	0.199,
0.605,	0.273,	0.070,	0.062,	5.320.	

Three hybrid censoring schemes have been considered for analysis of this real data, namely

Scheme 1: R=20, T=3.0
Scheme 2: R=18, T=4.5
Scheme 3: R=18, T=2.0

Using these schemes, the censored samples are obtained from the complete data set and the MLEs and Bayes estimates along with the corresponding asymptotic confidence intervals (CIs) and HPD credible intervals are computed. In order to guess the starting values, the log-likelihood profile is studied. Figure 2 shows the log-likelihood curve with respect to the given sequence of the parameters. From the Figure 2, we can see that the likelihood achieves its maximum at neighborhood of $(\alpha,\beta) = (0.2, 0.2)$ which we used as initial guess in the fixed point iteration method. Table 6 provides the MLEs and Bayes estimates for Type-II hybrid censoring schemes based on real data set. Table 7 consists the CIs and HPD intervals for Type-II hybrid censoring schemes based on real data set. The estimates, obtained under three censoring schemes given above, of MTF are summarised in Table 6 and estimated reliability and hazard functions for real data are plotted in Figure 4.

Since, in real study, there is no prior information available regarding the true values of the parameters, the hyper-parameters are chosen such that the prior mean (M) equals to the MLE of the parameter and prior variance (V) is 1. Thus hyper-parameters are obtained by using the following relations: $\frac{b}{a} = \hat{\alpha}_{mle}$, $\frac{b}{a^2} = 1$, $\frac{d}{c} = \hat{\beta}_{mle}$ and $\frac{d}{c^2} = 1$. We call these priors as Gamma-1. In Table 6, Gamma-0 denotes the especial form of the gamma prior with hyper-parameters identical to zero.

By using MCMC algorithm, ten thousands sample points have been generated from the posterior. The properties of the simulated samples have been studied. The posterior plots and convergence plots are plotted in Figure 3. Clearly, the MCMC chains (posterior samples) are well mixed and converged to their stationary distributions, and proximately normally distributed. The scattered diagram shown in Figure 3 indicates that correlation between the chains of α and β is negligible.

One- and two-sample predictive densities are also explored through the use of MCMC methods. The summary statistics of one- and two-sample predictive densities are provided in Tables 8-12. It is to be observed that the standard error (SD) as well as the width of confidence intervals increases as the values of s increases i.e. the predictive densities become more and more wider for larger order statistics. The survival functions corresponding to one- and two-sample predictive densities for different values of s are obtained by using the Monte Carlo technique and plotted in Figures 5 and 6 respectively.

TABLE 6
MLEs and Bayes estimates for real data set.

Scheme		MLE	Gamma-1	Gamma-0
Scheme 1	$\hat{\alpha}$	0.1908	0.1721	0.1717
	$\hat{\beta}$	0.2606	0.2512	0.2502
	$\widehat{\text{MTF}}$	1.5581	1.8000	1.8119
Scheme 2	$\hat{\alpha}$	0.1744	0.1557	0.1555
	$\hat{\beta}$	0.2624	0.2531	0.2521
	$\widehat{\text{MTF}}$	1.6768	1.9713	1.9777
Scheme 3	$\hat{\alpha}$	0.2751	0.2323	0.2281
	$\hat{\beta}$	0.2557	0.2458	0.2466
	$\widehat{\text{MTF}}$	1.1682	1.8334	2.2050

TABLE 7
The 95% asymptotic confidence intervals (CI) and HPD credible intervals for real data set.

Scheme		CI		HPD(Gamma-1)		HPD(Gamma-0)	
		Lower	Upper	Lower	Upper	Lower	Upper
Scheme 1	α	0.0723	0.3094	0.0850	0.2581	0.0854	0.2587
	β	0.1315	0.3897	0.1681	0.3395	0.1635	0.3404
Scheme 2	α	0.0624	0.2864	0.0721	0.2346	0.0711	0.2369
	β	0.1333	0.3914	0.1643	0.3421	0.1667	0.3421
Scheme 3	α	0.0704	0.4799	0.0794	0.3916	0.0552	0.3971
	β	0.1254	0.3860	0.1598	0.3375	0.1579	0.3398

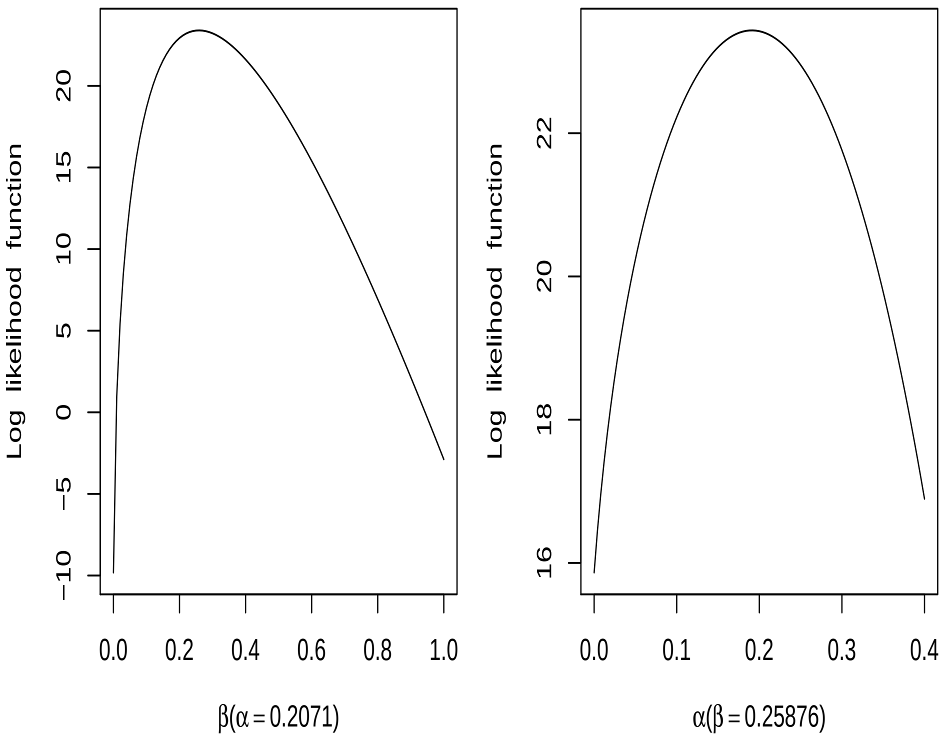

Figure 2 – Log likelihood function based on scheme 1.

TABLE 8
Sample based summary for one sample predictive posterior based on Scheme 1 for real data set.

s	Q1	Median	Mean	Q3	SD	95% CI	
						Lower	Upper
21	4.352	4.710	5.039	5.335	0.9974	4.0823	7.1579
22	5.013	5.732	6.179	6.765	1.6705	4.1613	9.7376
23	6.226	7.613	8.106	9.293	2.5828	4.2734	13.167

TABLE 9
Sample based summary for one-sample predictive posterior based on Scheme 2 for real data set.

s	Q1	Median	Mean	Q3	SD	96% CI	
						Lower	Upper
21	5.003	5.663	6.091	6.628	1.5953	4.5002	9.0190
22	5.434	6.407	6.933	7.842	2.1252	4.5012	10.9323
23	6.505	7.993	8.770	10.110	3.3039	4.5178	15.0635

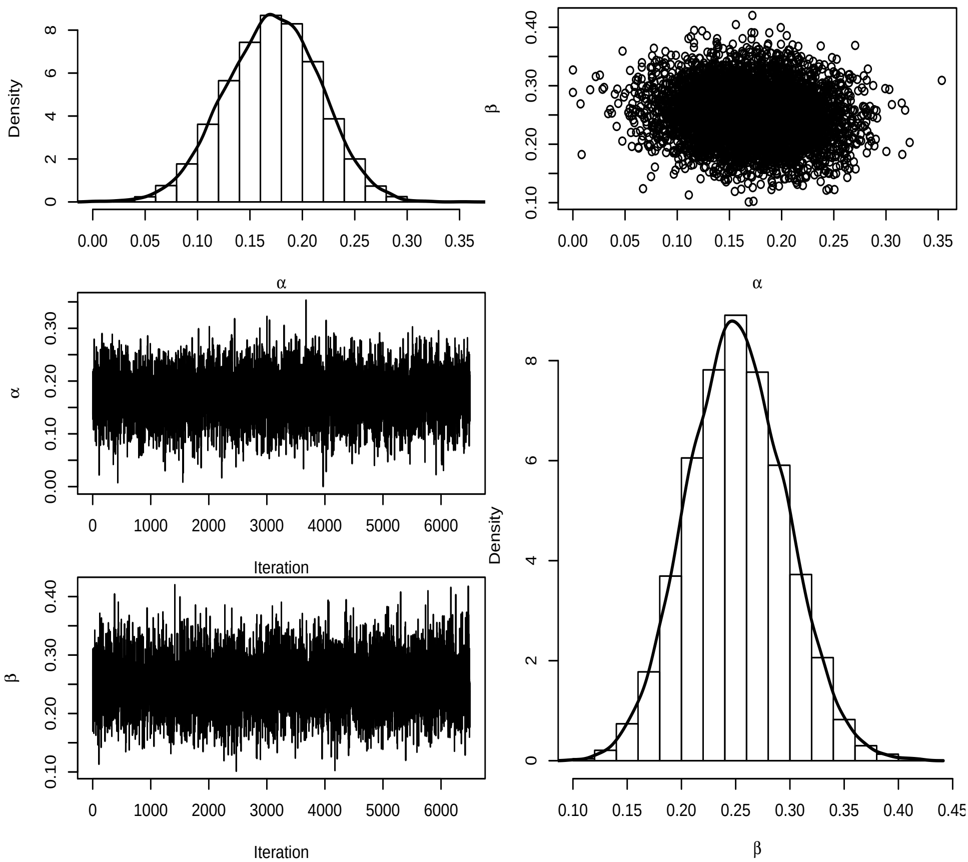

Figure 3 – Histograms, trace plots and scattered diagram of the posterior (Gamma-0) samples of α and β based on scheme 1.

TABLE 10
Sample based summary for one-sample predictive posterior based on Scheme 3 for real data set.

s	Q1	Median	Mean	Q3	SD	95% CI	
						Lower	Upper
19	2.296	2.492	2.750	2.860	0.9614	2.1601	4.0382
20	2.631	3.020	3.309	3.622	1.0404	2.1777	5.3281
21	3.047	3.617	4.030	4.513	1.4944	2.2722	6.9468
22	3.611	4.369	4.857	5.501	1.8494	2.4379	8.6736
23	4.471	5.559	6.150	7.135	2.4599	2.7360	11.082

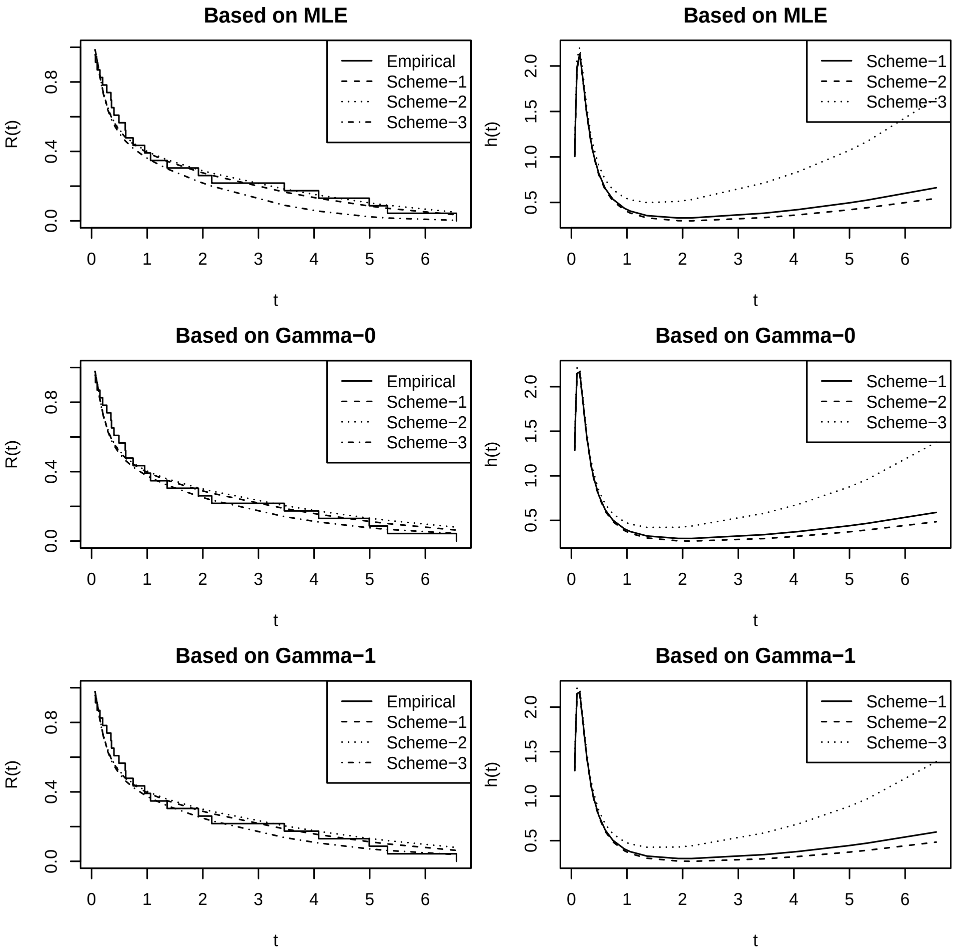

Figure 4 – Estimated reliability and hazard functions for real data set.

TABLE 11
Sample based two-sample predictive posterior characteristics for the complete real data set.

s	Q1	Median	Mean	Q3	SD	95% CI	
						Lower	Upper
5	0.1253	0.1586	0.1719	0.2041	0.0671	0.0693	0.2996
10	0.2607	0.3522	0.4067	0.4864	0.2217	0.1286	0.8456
15	0.6914	1.0300	1.2610	1.5430	0.9101	0.2153	2.8233
20	2.1780	2.9780	3.3190	3.9790	1.6896	0.7028	6.7756
23	4.2640	5.5890	6.1810	7.4200	2.7001	1.8915	11.917

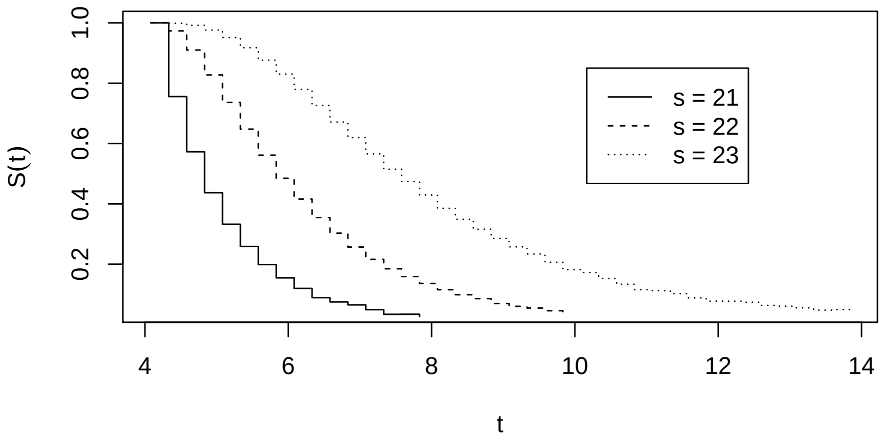

Figure 5 – One Sample Survival functions for different values of *s* for real data set.

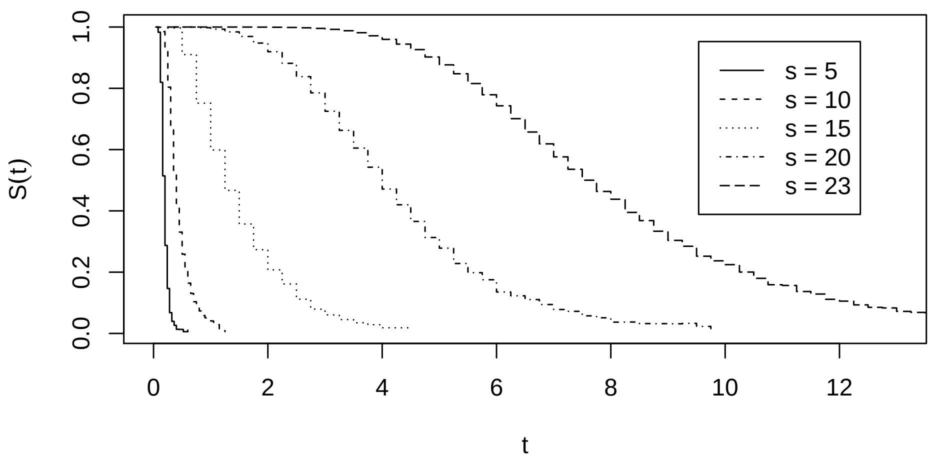

Figure 6 – Two Sample Survival functions for different values of *s* for real data set.

TABLE 12
Sample based two-sample predictive posterior characteristics from scheme 3 for the real data set.

s	Q1	Median	Mean	Q3	SD	95% CI	
						Lower	Upper
5	0.1253	0.1584	0.1704	0.2009	0.0651	0.0704	0.3006
10	0.2659	0.3474	0.4028	0.4748	0.2145	0.1231	0.8306
15	0.6784	0.9935	1.2520	1.4910	0.9865	0.2153	2.8014
20	2.1720	2.9650	3.3780	4.0410	1.8166	0.7920	7.0615
23	4.2990	5.5520	6.1700	7.3550	2.7144	2.1728	12.009

6. Conclusion

The maximum likelihood estimation and Bayesian estimation procedures for flexible Weibull distribution are developed under the hybrid Type-II censoring schemes. The estimation of the parameters as well as reliability characteristics ($R(t)$, $h(t)$ and MTF) are considered under both the paradigms. The long-run performances of the proposed estimators are compared on the bases of simulated samples generated from the distribution for various hybrid Type-II censoring schemes. Bayes estimators are constructed under the SELF using independent gamma priors of the parameters. In fact, Bayes estimators provide the precise estimates of the parameters as they show smaller mean squared errors than that of MLEs. Furthermore, We also discussed the problem of estimating future observables and one- and two-sample predictive posteriors of the future order statistics are derived. The posteriors become quite complicated and can not be reduced to any explicit forms. Metropolis-Hastings technique is utilised to draw the samples from the posteriors and sample based summary of predictive posteriors is presented. Finally, we believe that the methodologies discussed in this paper will be very useful for researchers, reliability practitioners and scientists in physics and medicine where the analysis of UBT data under censoring mechanism needs to be performed.

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SUMMARY

In this paper, we proposed Bayes estimators for estimating the parameters, reliability, hazard rate, mean time to failure from flexible Weibull distribution using Type-II hybrid censored sample. Bayes estimators have been obtained under squared error loss function assuming independent gamma prior distributions for the parameters. The maximum likelihood estimators along with asymptotic distributions have also been discussed. The performances of the estimators have been compared with respect to the various Type-II hybrid censoring schemes. For approximating the posteriors, we proposed the use of Markov chain Monte Carlo techniques such as Gibbs sampler and Metropolis-Hastings algorithm. Further, Bayesian One- and Two-sample prediction problems have also been considered. A real data set has been analysed for illustration purposes.

Keywords: Flexible Weibull distribution; Type-II hybrid censored sample; Maximum likelihood estimation; Bayesian estimation; Bayesian prediction.