TERM-ORDERING FREE INVOLUTIVE BASES

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ABSTRACT. In this paper, we consider a monomial ideal $J \lhd P := A[x_1, \dots, x_n]$, over a commutative ring A, and we face the problem of the characterization for the family $\mathcal{M}f(J)$ of all homogeneous ideals $I \lhd P$ such that the A-module P/I is free with basis given by the set of terms in the Gröbner escalier $\mathsf{N}(J)$ of J. This family is in general wider than that of the ideals having J as initial ideal w.r.t. any term-ordering, hence more suited to a computational approach to the study of Hilbert schemes.
For this purpose, we exploit and enhance the concepts of multiplicative variables, complete sets and involutive bases introduced by Janet in [19, 20, 21] and we generalize the construction of J-marked bases and term-ordering free reduction process introduced and deeply studied in [1, 6] for the special case of a strongly stable monomial ideal J.
Here, we introduce and characterize for every monomial ideal J a particular complete set of generators $\mathcal{F}(J)$, called stably complete, that allows an explicit description of the family $\mathcal{M}f(J)$. We obtain stronger results if J is quasi stable, proving that $\mathcal{F}(J)$ is a Pommaret basis and $\mathcal{M}f(J)$ has a natural structure of affine scheme.

The final section presents a detailed analysis of the origin and the historical evolution of the main notions we refer to.

1. INTRODUCTION.

Let $P := A[x_1, ..., x_n]$ be the polynomial ring in n variables over a commutative ring A. The problem we address is the following.

Problem: *Given any monomial ideal $J \lhd P$ find a characterization for the family $\mathcal{M}f(J)$ of all homogeneous ideals $I \lhd P$ such that the A-module P/I is free with basis given by the set of terms in the Gröbner escalier $\mathsf{N}(J)$ of J.*

The most relevant examples of ideals that belong to this family are those such that J is their initial ideal w.r.t. some term-ordering, but in general they form a proper subset of $\mathcal{M}f(J)$. Therefore, we must overcome the Gröbner framework.
A computational description of the whole family $\mathcal{M}f(J)$ is obtained in [1, 6] for J strongly stable. These families are optimal for many applications, for instance for an effective study of Hilbert schemes (see [2]). However, the strong stability of the monomial ideal J is a rather limiting condition.

In the present paper we give an overall view on what can be said about the above question for an *arbitrary* monomial ideal J, enhancing some ideas introduced by Janet in [19, 20, 21].
The ideas we mainly deal with are those of *multiplicative variable* and *complete system*, leading to the so called *Janet decomposition* for terms. These concepts date back to the late nineteenth century and the first decades of the twentieth. In a historical note at the

2010 *Mathematics Subject Classification.* 14C05, 14Q20, 13P10.

The third author was supported by the framework of PRIN 2010-11 *Geometria delle varietà algebriche*, cofinanced by MIUR.

end of the paper we present a detailed overview of their appearances, evolution and applications.

In Janet's theory the ideals I are generated by those we call now *involutive bases*, a set which contains as a subset those that are also Gröbner bases. Indeed, Janet develops his ideas assuming to be in *generic coordinates*. Hence the homogeneous ideals I and J he considers satisfy many good properties that always hold after having performed a *generic linear change of coordinates*. In particular, J is the generic initial ideal of I w.r.t. the (deg)-revlex ordering.

From a computational point of view, a general change of coordinates is remarkably heavy. For this reason, we consider interesting enhancing the theory and obtaining analogous results not assuming this hypothesis. Indeed, Janet's ideas permit to go beyond this context and to recover results and techniques of both Gröbner basis theory and J-marked basis theory. In fact we do not need to impose a term-ordering on the given polynomial ring.

We identify two essential features that are key points for most computations in both the above frameworks:

I) I is generated by a set of polynomials, marked on the terms of a suitable generating set of the monomial ideal J;
II) there is a reduction process w.r.t. these marked polynomials, that is used to rewrite each element of P/I as an element of the free A-module $\langle \mathsf{N}(J)\rangle$

Janet's notions of multiplicative variable and complete system allow to construct such marked set of generators for I and to define an efficient reduction process.

First of all, we examine and compare two different definitions of multiplicative variable that Janet presents in [19, 20] and in [21], that are equivalent in general coordinates. We underline similarities and differences and introduce the notion of *stably complete* set of terms, when both conditions hold. We show that every monomial ideal J has only one stably complete set of generators (possibly made of infinitely many terms) that we call *star set* and denote by $\mathcal{F}(J)$.

Furthermore, we define a reduction procedure with respect to a homogeneous set of polynomials marked on a stably complete system $\mathcal{F}(J)$ and prove its noetherianity. As a consequence we are able to give a first, general answer to Problem 1 .

Of course, the most interesting cases are those of ideals J such that their generating stably complete set M is finite. We prove that they are the *quasi stable* ideals and that $\mathcal{F}(J)$ is their Pommaret basis. Among them, those such that $\mathcal{F}(J)$ coincides with the monomial basis are exactly the *stable* ones.
For the class of quasi stable ideals J we give a more complete and effective answer to Problem 1. Indeed, we prove that our description of $\mathcal{M}f(J)$ is natural, in the sense that it defines a representable functor from the category of $\mathbb{Z}$-algebras to the category of sets. Finally we give an effective procedure computing equations for the scheme that represents this functor.

After introducing all the notation (section 2), we introduce the Janet decomposition for semigroup ideals and order ideals (section 3).
More precisely, we recall the notion of multiplicative variables ([19, 20, 21]) and complete system, pointing out that in the cited papers Janet uses two *non equivalent* definitions of multiplicative variables and completeness.

For our purpose, we then introduce the notion of *stable completeness* as a junction between the two notions. Given a complete system of terms M, we define a decomposition for terms in $J = (M)$, called *star decomposition*, in analogy with the star product introduced in [1, 6] for strongly stable ideals.
In section 4 we define a very special stably complete set, i.e. the *star set* $\mathcal{F}(J)$, introducing the *stable ideals*, for which $\mathcal{F}(J)$ is the minimal generating set of J and the *quasi stable ideals* for which the finiteness condition is respected.
In section 5, we define M-marked polynomials, bases and families, for a complete set M, again generalizing the definitions given for the generating set of a strongly stable ideal. Then, we define a noetherian reduction process for homogeneous polynomial w.r.t. the elements of a M-marked set. In section 6 we associate to marked families a representable functor and give a procedure computing equations for the scheme that defines it.

Finally, section 7 contains the historical note.

2. NOTATION.

Consider the polynomial ring $P := A[x_1, ..., x_n] = \bigoplus_{d\in\mathbb{N}} P_d$ in n variables and coefficients in the base ring A.
When an order on the variables comes into play, we consider $x_1 < x_2 < ... < x_n$.
In case of $n = 2, 3$ we will usually set $P = A[x, y]$, $x < y$ and $P = A[x, y, z]$, $x < y < z$.
The symbol $<_{Lex}$ will denote the lexicographic term-ordering according to this order on the set of variables.
The *set of terms* of P is

$$\mathcal{T} := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},\ (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n\}$$

and we define also

$$\mathcal{T}[1, m] := \mathcal{T} \cap k[x_1, ..., x_m] = \{x_1^{\alpha_1} \cdots x_m^{\alpha_m} / (\alpha_1, ..., \alpha_m) \in \mathbb{N}^m\}.$$

For every polynomial $f \in P$, $deg(f)$ is its usual degree and $deg_i(f)$ is its degree with respect to the variable x_i.
For each $p \in \mathbb{N}$, and for all $W \subseteq P$,

$$W_p := \{f \in W\ :\ f \text{ homogeneous and } deg(f) = p\};$$

in particular:

$$\mathcal{T}_p := \{\tau \in \mathcal{T}\ :\ deg(\tau) = p\},\ \ |\mathcal{T}_p| = dim_k(P_p) = \binom{p+n-1}{n-1}.$$

If $f \in P$, we denote by $\mathrm{Supp}(f)$ the *support* of f, i.e. the set of all the terms in $\mathcal{T}$, appearing in f with non-zero coefficient.

Fixed a polynomial $f \in \mathcal{P}$ and a term-ordering $<$, we call *leading term* of f the maximal element in $\mathrm{Supp}(f)$ w.r.t. $<$ and we denote it $\mathsf{T}(f)$. Its coefficient is the *leading coefficient* of f. Given a term $\tau = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathcal{T}$, we set

$$\max(\tau) = \max\{x_i\ :\ x_i \mid \tau\} \quad , \quad \min(\tau) = \min\{x_i\ :\ x_i \mid \tau\}$$

the maximal and the minimal variable appearing in τ with non-zero exponent.

Definition 2.1. Given a term $\tau \in \mathcal{T}$ and a variable $x_j \mid \tau$, the term $\frac{\tau}{x_j}$ is the j-th *predecessor* of τ.

Definition 2.2. Let $F = \{\tau_1, ..., \tau_s\} \subseteq \mathcal{T}$ be an ordered subset of terms, generating an ideal $J = (F)$. The module

$$Syz(F) = \{(g_1, ..., g_s) \in P^s, \sum_{i=1}^{s} g_i \tau_i = 0\}$$

is the *syzygy module* of F.
We denote an element in $Syz(F)$ as $(g_1, ..., g_s)$ and we call it *syzygy among* F.

Definition 2.3. A set $\mathsf{N} \subset \mathcal{T}$ is called *order ideal* if

$$\forall s \in \mathcal{T},\ t \in \mathsf{N}: \quad s|t \Rightarrow s \in \mathsf{N}.$$

Observe that N is an order ideal if and only if the complementary set $I := \mathcal{T} \setminus \mathsf{N}$ is a *semigroup ideal*, i.e. $\forall t \in \mathcal{T}$, $\tau \in I \Rightarrow t\tau \in I$.
If I is either a monomial ideal or a semigroup ideal, we will denote by $\mathsf{N}(I)$ the order ideal $\mathsf{N} := \mathcal{T} \setminus I$ and by $\mathsf{G}(I)$ its *monomial basis*, namely the minimal set of terms generating I.

Definition 2.4 ([26])**.** A *(monic) marked polynomial* is a polynomial $f \in P$ together with a fixed term τ of $\mathrm{Supp}(f)$, called *head term of* f and denoted by $\mathrm{Ht}(f)$ and such that its coefficient is equal to 1_A.

We can extend to marked polynomials the notion of S-polynomial.

3. Janet decomposition.

In this section we loosely base on the paper [19], where Janet first defines the notion of *multiplicative variable* for a term τ with respect to a given set $M \subseteq \mathcal{T}$.

For completeness' sake, we recall Janet's decomposition for terms in the semigroup ideal generated by M into disjoint classes.

Each of them contains:

(1) a term $\tau \in M$;
(2) the set of monomials obtained multiplying τ by products of multiplicative variables, that we call *offspring of* τ and denote $\mathrm{off}_M(\tau)$.

The main difference with respect to Janet's papers is that we remove the finiteness condition on M, showing that it is not necessary for our purposes.

Definition 3.1. [19, ppg.75-9] Let $M \subset \mathcal{T}$ be a set of terms and $\tau = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be an element of M. A variable x_j is called *multiplicative* for τ with respect to M if there is no term in M of the form $\tau' = x_1^{\beta_1} \cdots x_j^{\beta_j} x_{j+1}^{\alpha_{j+1}} \cdots x_n^{\alpha_n}$ with $\beta_j > \alpha_j$. We will denote by $\mathrm{mult}_M(\tau)$ the set of multiplicative variables for τ with respect to M.

Definition 3.2. With the previous notation, the *offspring* of τ with respect to M is the set

$$\mathrm{off}_M(\tau) := \{\tau x_1^{\lambda_1} \cdots x_n^{\lambda_n} \mid \text{where } \lambda_j \neq 0 \text{ only if } x_j \text{ is multiplicative for } \tau \text{ w.r.t. } M\}.$$

Example 3.3. Consider the set $M = \{x_1^3, x_2^3, x_1^4 x_2 x_3, x_3^2\} \subseteq k[x_1, x_2, x_3]$.
Let $\tau = x_1^3$, so $\alpha_1 = 3$, $\alpha_2 = \alpha_3 = 0$. The variable x_1 is multiplicative for τ w.r.t M since there are no terms $\tau' = x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} \in M$ satisfying both conditions:

- $\beta_1 > 3$;
- $\beta_2 = \beta_3 = 0$.

On the other hand, x_2 is not multiplicative for τ since $\tau'' = x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3} = x_2^3 \in M$: $\gamma_2 = 3 > 0 = \alpha_2$, $\gamma_3 = \alpha_3 = 0$.
Similarly, x_3 is not multiplicative since $x_3^2 \in M$.
In conclusion, we have $\text{mult}_M(\tau) = \{x_1\}$.

Remark 3.4. Observe that, by definition of multiplicative variable, the only element in $\text{off}_M(\tau) \cap M$ is τ itself.
Indeed, if $\tau \in M$ and also $\tau\sigma \in M$ for a non constant term σ, then $max(\sigma)$ cannot be multiplicative for τ, hence $\tau\sigma \notin \text{off}_M(\tau)$.

In paper [19], Janet defines multiplicative variables as in Definition 3.1 and he provides both a decomposition for the semigroup ideal $\mathsf{T}(M)$ generated by a finite set of terms M and a decomposition for the complementary set $\mathsf{N}(M)$.
On the other hand, in [20, 21], he defines multiplicative variables in the following way.

3.5. *A variable x_j is* multiplicative *for $\tau \in \mathcal{T}$ if and only if $x_j \leq \min(\tau)$.*

These two definitions of multiplicative variables appear to be very different.
First of all, in the first formulation, the set of multiplicative variables for a term in M depends on the whole set M, while in the second it is completely independent on the set M. Indeed, the two notions are not equivalent for a general set M, as shown by the following examples.

Example 3.6. In $k[x_1, x_2, x_3]$ consider the ideal $I = (x_1^2 x_2, x_1 x_2^2)$ and let M be its monomial basis. Then, $\text{mult}_M(x_1^2 x_2) = \{x_1, x_3\}$ and $\text{mult}_M(x_1 x_2^2) = \{x_1, x_2, x_3\}$ while only x_1 can be multiplicative according to the other notion of multiplicative variable.

Example 3.7. Taken the set $M = \{x_1^2 x_2, x_1 x_2^2\} \subseteq k[x_1, x_2]$, we get $\text{mult}_M(x_1 x_2^2) = \{x_1, x_2\}$, while of course $x_1 \leq \min(x_1 x_2^2)$ but $x_2 > \min(x_1 x_2^2)$.

However, they are equivalent in Janet setting, that is if M is the generating set of the generic initial ideal of homogeneous ideals I.
More generally, we will see that they turn out to be equivalent also if M is the monomial basis $\mathsf{G}(J)$ of a strongly stable ideal J and if M is the special set of generators of any monomial ideal J that we will introduce in section 4 and denote by $\mathcal{F}(J)$.
We will see that stronger results can be proved when a set M is such that *the two definitions* of multiplicative variables *coincide*.
The following definition will be a key point in this paper.

Definition 3.8. [19, ppg.75-9] A set of terms $M \subset \mathcal{T}$ is called *complete* if for every $\tau \in M$ and $x_j \notin mult_M(\tau)$, there exists $\tau' \in M$ such that $x_j\tau \in \text{off}_M(\tau')$.
Moreover, M is *stably complete* if it is complete and for every $\tau \in M$ it holds $\text{mult}_M(\tau) = \{x_i \mid x_i \leq \min(\tau)\}$.
If a set M is stably complete and finite, then it is the *Pommaret basis* of $J = (M)$ and we denote it by $\mathcal{H}(J)$.

Remark 3.9. If $M = \{\tau\} \subseteq \mathcal{P}$ is a singleton, it is complete, with $\text{mult}(\tau) = \{x_1, ..., x_n\}$.

Let us examine some examples.

Example 3.10. In $k[x_1, x_2, x_3]$ consider the ideal $I = (x_1^2, x_1x_2, x_3)$.
Both $M_0 = \{x_1^2, x_1x_2, x_3\}$ and each generating set of I with the shape
$M_i = \{x_1^2, x_1x_2, x_3, x_2x_3, ..., x_2^i x_3\}$ are complete systems of terms. In fact, for M_0:

- $\text{mult}_{M_0}(x_1^2) = \{x_1\}$, $x_1^2x_2 \in \text{off}_{M_0}(x_1x_2)$, $x_1^2x_3 \in \text{off}_{M_0}(x_3)$;
- $\text{mult}_{M_0}(x_1x_2) = \{x_1, x_2\}$, $x_1x_2x_3 \in \text{off}_{M_0}(x_3)$;
- $\text{mult}_{M_0}(x_3) = \{x_1, x_2, x_3\}$.

For M_i, $i \geq 1$:

- $\text{mult}_{M_i}(x_1^2) = \{x_1\}$, $x_1^2x_2 \in \text{off}_{M_i}(x_1x_2)$, $x_1^2x_3 \in \text{off}_{M_i}(x_3)$;
- $\text{mult}_{M_i}(x_1x_2) = \{x_1, x_2\}$, $x_1x_2x_3 \in \text{off}_{M_i}(x_2x_3)$;
- $\text{mult}_{M_i}(x_3) = \{x_1, x_3\}$, $x_2x_3 \in \text{off}_{M_i}(x_2x_3)$;
- $\text{mult}_{M_i}(x_2^j x_3) = \{x_1, x_3\}$, $x_2^{j+1}x_3 \in \text{off}_{M_i}(x_2^{j+1}x_3)$, $0 \leq j < i$;
- $\text{mult}_{M_i}(x_2^i x_3) = \{x_1, x_2, x_3\}$.

Example 3.11. Consider the ideal $J = (xy) \triangleleft k[x, y]$.
The monomial basis $M_0 = \mathsf{G}(J) = \{xy\}$ is a complete system with $\text{mult}_{M_0}(xy) = \{x, y\}$. Also the set $M = \{x^h y \mid h \geq 1\} \subseteq k[x, y]$, $x < y$, is a complete system, again according to the first definition. It generates the same ideal (xy), but has infinitely many elements. Anyway, it is not stably complete. In fact, for each $x^h y \in M$, $\text{mult}_M(x^h y) = \{y\}$, since no terms of the form $x^l y^e$ with $e > 1$ belong to M; on the other hand $x \notin \text{mult}_M(x^h y)$ since $x^{h+1}y \in M$.

Example 3.12. Let M be the set of terms $\{x, y^2\}$ in $k[x, y]$, with $x < y$.
The multiplicative variables for every term in M are those lower than or equal to its minimal one:
$\text{mult}(x) = \{x\}$ $\text{mult}(y^2) = \{x, y\}$.
However, M is not complete since yx does not belong to the offspring of any term in M.

The following example shows that a complete generating set of terms can loose completeness when the ideal is enlarged.

Example 3.13. Let $M = \{x^2, xy\} \subset k[x, y]$ and $J = (M)$. It is a complete system, but it is not stably complete, since y is multiplicative for xy, although $\min(xy) = x$.
Adding to M a term in $\mathsf{N}(J)$, we get a new set M_0 and $J_0 = (M_0)$, whose Janet decomposition clearly changes. For example, if $M_0 = \{x^2, xy, y^2\}$ we get a stably complete system.
On the other hand, if $M_0 = \{x^2, xy, y^3\}$ the system is not complete anymore, since xy^2 does not belong to the offspring of any term in the set.

The following technical lemma will be very useful throughout the paper. As a first application, we will prove that a system of terms M (possibly infinite) is *complete* if and only if the offsprings of the elements in M form a partition of the semigroup ideal generated by M.

Lemma 3.14. [22, pg.23] *Let τ, τ' be elements of a set of terms M and x_j be a variable such that $x_j \notin mult_M(\tau)$ and $x_j\tau \in off_M(\tau')$. Then $\tau <_{Lex} \tau'$. If, moreover, $x_j \leq min(\tau)$, then $\tau x_j = \tau' \in M$.*

Proof. First of all, we observe that $\tau \neq \tau'$, since $x_j \notin mult_M(\tau)$. By definition of offspring, we have that $\tau x_j = \tau'\sigma'$, where σ' is a product of multiplicative variables for

τ'. Let us assume by contradiction that $\tau >_{Lex} \tau'$ and let x_i be the maximal variable such that $deg_i(\tau) > deg_i(\tau')$. Then, $x_i|\sigma'$, hence $x_i \in mult_M(\tau')$, but this is impossible by definition of multiplicative variable, since also τ is in M.

Now let us assume that $x_j \leq min(\tau)$ and $\sigma' \neq 1$. If $x_j|\sigma'$, then $\tau = \frac{\sigma'}{x_j}\tau' \in M \cap \mathrm{off}_M(\tau')$, which is not possible by Remark 3.4. If, on the contrary, $x_j \not|\sigma'$ we get a contradiction with the previous assertion, since in this case $\tau' \leq_{Lex} \frac{\tau'\sigma'}{\max(\sigma')} <_{Lex} \frac{\tau'\sigma'}{x_j} = \tau$. $\square$

Theorem 3.15. *Let M be a set of terms (possibly infinite).*
If $\tau, \tau' \in M$ and $\tau \neq \tau'$, then $\mathit{off}_M(\tau) \cap \mathit{off}_M(\tau') = \emptyset$.
If, moreover, M is complete and $\mathsf{T}(M)$ is the semigroup ideal it generates, then $\forall \gamma \in \mathsf{T}(M)$, $\exists \tau \in M$ such that $\gamma \in \mathit{off}_M(\tau)$. Hence, the offsprings of the elements in M give a partition of $\mathsf{T}(M)$.

Proof. To prove the first assertion, let us assume by contradiction that $\tau\sigma = \tau'\sigma' \in \mathrm{off}_M(\tau) \cap \mathrm{off}_M(\tau') \neq \emptyset$ and let $\tau >_{lex} \tau'$. If x_i is the maximal variable such that $deg_i(\tau) > deg_i(\tau')$, then $x_i|\sigma'$. By definition of offspring, $x_i \in mult_M(\tau')$, but this is impossible by definition of multiplicative variable, since also τ is in M.

Now we assume that M is complete and prove the second fact. We argue by contradiction. Suppose $\mathsf{T}(M) \supsetneq O := \bigcup_{\sigma \in M} \mathrm{off}_M(\sigma)$ and take any term γ in $\mathsf{T}(M) \setminus O$. As M generates $\mathsf{T}(M)$, there are terms in M that divide γ: let τ be the one which is maximal with respect to $<_{lex}$. If $\gamma = \tau\sigma$, the term σ contains at least a variable x_i which is not multiplicative for τ, since $\tau\sigma \notin \mathrm{off}_M(\tau)$. Then $\gamma = \tau x_i \eta$ and $\tau x_i \notin \mathrm{off}_M(\tau)$.

By the completeness of M, we have $\tau x_i \in O$, namely there is a term $\tau' \in M$ such that $\tau x_i = \tau'\sigma' \in \mathrm{off}_M(\tau')$. By Lemma 3.14 *i)*, $\tau' >_{Lex} \tau$, and this is not possible since $\tau'|\gamma = \tau x_i \eta = \tau'\sigma'\eta$. $\square$

Thanks to the previous result, if M is a complete system, each term in $\mathsf{T}(M)$ can be written in a unique way as a product of

(1) an element $\tau \in M$;
(2) a term $x^\eta = x_i^{\eta_i} \cdots x_j^{\eta_j}$, with $x_i, ..., x_j \in \mathrm{mult}_M(\tau)$.

This fact suggests the following

Definition 3.16. Let M be a complete system of terms. The *star decomposition* of every term $\gamma \in (M)$ with respect to M, is the unique couple of terms (τ, η), with $\tau \in M$, such that $\gamma = \tau\eta$ and $\gamma \in \mathrm{off}_M(\tau)$. If (τ, η) is the star decomposition of γ with respect to M, we will write $\gamma = \tau *_M \eta$.

Remark 3.17. From the results stated above, we obtain the following explicit formula for the Hilbert function of $P/(M)$:

$$H(P/(M))(k) = \binom{k+n}{n} - \sum_{\tau \in M \; deg(\tau) \leq k} \binom{k - deg(\tau) + s_\tau - 1}{s_\tau - 1},$$

where s_τ is the number of multiplicative variables for τ w.r.t M and we set equal to 0 every binomial with a negative numerator or a negative denominator. Thus, this formula makes sense even if $|M| = \infty$, since for every k there are only finitely many non-zero summands.
If M is a finite set of terms and r is the maximal degree of its elements, this formula gives the value of the Hilbert polynomial for every $k \geq r$.

The following lemma will be very useful for the reduction process we will define in section 5.

Lemma 3.18. *Let M be a stably complete system of terms and let γ be a term such that $\gamma = \tau *_M \eta$ and also $\gamma = \sigma\eta'$ with $\sigma \notin \mathsf{T}(M)$.*
Then $\eta' >_{Lex} \eta$.

Proof. By definition of stable completeness, $\min(\tau) \geq \max(\eta)$. If $\eta' <_{Lex} \eta$, then $\eta' | \eta$ and $\tau | \sigma$. This is not possible since $\tau \in \mathsf{T}(M)$ and $\sigma \notin \mathsf{T}(M)$. □

4. STAR SET AND QUASI STABLE IDEALS

We introduce here a special set of terms. We will prove that it is a complete system with many interesting properties in common with the minimal monomial basis of strongly stable ideals.

Definition 4.1. Given a monomial ideal $J \triangleleft P$ we define the *star set* as

$$\mathcal{F}(J) := \{x^\alpha \in \mathcal{T} \setminus \mathsf{N}(J) \mid \frac{x^\alpha}{\min(x^\alpha)} \in \mathsf{N}(J)\}.$$

Theorem 4.2. *For every monomial ideal J, the star set $\mathcal{F}(J)$ is the unique stably complete system of generators of J. Hence, if M is stably complete, $M = \mathcal{F}((M))$.*

Proof. Let $\tau := x_k^{\alpha_k} \cdots x_n^{\alpha_n}$ be any monomial in $\mathcal{F}(J)$.

Assume x_i is not multiplicative, so that $x_i\tau \in J$, $x_i\tau = \tau'\sigma', \tau' \in M$. Then Lemma 3.14 implies $\tau <_{Lex} \tau'$ whence $x_i > \min(\tau)$.

Let $x_i > x_k := min(\tau)$ and set $\sigma_0 := \tau x_i$, $\sigma_r := \frac{\sigma_{r-1}}{min(\sigma_{r-1})}$ for $r = 1 \ldots, \alpha_k + \cdots + \alpha_{i-1}$ and note that $x_i^{\alpha_i} \cdots x_n^{\alpha_n} \notin J$, since it divides $\frac{\tau}{min(\tau)}$, while $\sigma := \sigma_0 \in J$, since it is a multiple of τ. Then, in the sequence of terms $\sigma_i, 0 \leq i \leq \alpha_k + \cdots + \alpha_{i-1}$, we find an element σ_j that belongs to J, while the following one does not.

Then $\sigma_j \in \mathcal{F}(J)$, so that $x_i\tau \in \mathrm{off}_{\mathcal{F}(I)}(\sigma_j)$ and x_i is not multiplicative for τ w.r.t. $\mathcal{F}(I)$.

Take $\tau = x_k^{\alpha_k} \cdots x_n^{\alpha_n} \in \mathcal{F}(J)$, and a variable $x_i \notin \mathrm{mult}_{\mathcal{F}(J)}(\tau)$. By the previous result $x_i > x_k = \min(\tau)$. By definition of non-multiplicative variable, there is a term $\sigma' = x_i^t x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n} \in \mathcal{F}(J)$, for some integer $t > \alpha_i$. Let us consider the minimum one.

If $t = \alpha_i + 1$, then $x_i\tau = x_k^{\alpha_k} \cdots x_i^t \cdots x_n^{\alpha_n} \in \mathrm{off}_{\mathcal{F}(J)}(\sigma')$.

If, on the contrary, $t > \alpha_i + 1$, then $\sigma'' = x_i^{\alpha_i+1} \cdots x_n^{\alpha_n} \in \mathsf{N}(J)$ by definition. Let us consider, as in the previous proof, the sequence of terms $\sigma_0 := \tau x_i \in J$, $\sigma_r := \frac{\sigma_{r-1}}{min(\sigma_{r-1})}$ for $r = 1 \ldots, \sum_{j=k}^{k-1} \alpha_j$. Since the last one is σ'', we can find in this sequence a suitable $\sigma_j \in I$ such that $\sigma_{j+1} \in \mathsf{N}(J)$, that is $\sigma_j \in \mathcal{F}(J)$ and $x_i\tau \in \mathrm{off}_{\mathcal{F}(J)}(\sigma_j)$.

In order to prove that every stably complete set of terms M, with $J = (M)$ is exactly $\mathcal{F}(J)$, we first notice that clearly $\mathsf{G}(J) \subseteq M$ and $\mathsf{G}(J) \subseteq \mathcal{F}(J)$.
Moreover, it is sufficient to prove that $\mathcal{F}(J) \subseteq M$. Let $\sigma \in \mathcal{F}(J)$, i.e. $\frac{\sigma}{\min(\sigma)} = \omega \in \mathsf{N}(J)$. Then, there exists $\tau \in M$ such that $\sigma \in \mathrm{off}(\tau)$ and so $\sigma = \tau\eta$, with either $\eta = 1$ or $\max(\eta) \leq \min(\tau)$.
This implies that either $\tau = \sigma$ or $\tau \mid \omega$, but the second alternative is impossible since both $\tau \in M$ and $\omega \in \mathsf{N}(J)$.

□

Remark 4.3. i. For an arbitrary monomial ideal J the set $\mathcal{F}(J)$ can be infinite. For example, if $J = (x) \triangleleft k[x, y]$, $x < y$, then $\mathcal{F}(J) = \{xy^n \mid n \in \mathbb{N}\}$.

ii. Not all the complete systems turn out to be of the form of a star set.
For example, the complete system $M = \{x^h y,\ h \geq 1\} \subseteq k[x, y]$ of Example 3.11 is not the star set of the ideal $J := (M)$.
Indeed, $\mathsf{N}(J) = \{x^m,\ m \geq 0\} \cup \{y^l,\ l > 0\}$ and all the terms of the form xy^k, $k > 1$, do not belong to M, even if $\frac{xy^k}{\min(xy^k)} = y^k \in \mathsf{N}(M)$.
Moreover, for $h > 1$, $\frac{x^h y}{x} = x^{h-1}y \in M$, so $x^h y \notin \mathcal{F}(J)$.

Better results hold if the monomial ideal J satisfies one of the following conditions, weaker then the strongly stable property (see section 6).

Definition 4.4. A monomial ideal J is called *stable* if it holds

$$\tau \in J,\ x_j > \min(\tau) \Longrightarrow \frac{x_j \tau}{\min(\tau)} \in J$$

A monomial ideal J is called *quasi stable* if it holds

$$\tau \in J,\ x_j > \min(\tau) \Longrightarrow \exists t \geq 0 :\ \frac{x_j^t \tau}{\min(\tau)} \in J.$$

We will show that this notion of quasi stable ideal coincides with the one in [37], by proving that J actually has a Pommaret basis.

Remark 4.5.
- Obviously, a stably complete system M is also stable, and a stable set is also quasi stable.
- In order to verify whether the conditions above are satisfied for a given ideal J it is sufficient to check the terms in the basis $\mathsf{G}(J)$.

Proposition 4.6. *Let J be a monomial ideal. Then TFAE:*

i) *J is stable*
ii) $\mathcal{F}(J) = \mathsf{G}(J)$

Proof. *i) ⇒ ii)* The inclusion $\mathsf{G}(J) \subseteq \mathcal{F}(J)$ is true for every monomial ideal by definition of star set. We prove now that $\gamma \notin \mathcal{F}(J)$ for every term $\gamma \in J \setminus \mathsf{G}(J)$.
By hypothesis, $\exists \tau \in \mathsf{G}(J)$, such that $\gamma = \tau\sigma$ and $\sigma \neq 1$.
Let $x_k := \min(\gamma)$. If $x_k | \sigma$, then $\frac{\gamma}{\min(\gamma)} = \tau \frac{\sigma}{x_k} \in J$, so that $\gamma \notin \mathcal{F}(J)$.

If, on the other hand, $x_k \not| \sigma$ and x_j is any variable dividing σ, then $x_j > x_k$ and $x_k = \min(\tau)$. By the stability of J we have $\frac{x_j \tau}{x_k} \in J$, hence $\frac{\gamma}{x_k} = \frac{\tau\sigma}{x_j}\frac{x_j}{x_k} \in J$, hence again $\gamma \notin \mathcal{F}(J)$.

ii) ⇒ i) If *ii)* holds, then $\mathsf{G}(J)$ is the only stably complete system generating J. By remark 4.5, we can check the stability on the terms $x^\alpha \in \mathsf{G}(J)$. Let $x_j > x_k := min(x^\alpha)$. By hypothesis there exists $x^\beta \in \mathsf{G}(J)$ such that $x_j x^\alpha \in \mathrm{off}_{\mathsf{G}(J)}(x^\beta)$, and, since $x^\alpha \in \mathsf{G}(J)$, of course $x^\alpha x_j \notin \mathsf{G}(J)$. Hence $x^\beta | \frac{x_j x^\alpha}{x_k}$ and so $\frac{x^\alpha x_j}{x_k} \in J$. □

Proposition 4.7. *Let J be a monomial ideal. Then TFAE:*

i) *J is quasi stable*
ii) $|\mathcal{F}(J)| < \infty$
iii) *$\mathcal{F}(J) = \mathcal{H}(J)$ is the Pommaret basis of J.*

Proof. i) ⇒ ii) Let a be the maximum of the degrees of elements in $\mathsf{G}(J)$ and let t be such that $\frac{x_j^t x^\alpha}{\min(x^\alpha)} \in J$ for every $x^\alpha \in \mathsf{G}(J)$ and $x_j > min(x^\alpha)$. We prove that $\mathcal{F}(J)$ is contained in $P_{<d}$ where $d := a + tn$. Let $x^\alpha x^\eta \in J_{\geq d}$ with $x^\alpha \in \mathsf{G}(J)$ and x_k be $\min(x^\alpha x^\eta)$. If $x_k | x^\eta$, then obviously $\frac{x^\alpha x^\eta}{x_k} = x^\alpha \frac{x^\eta}{x_k} \in J$, so $x^\alpha x^\eta \notin \mathcal{F}(J)$. If, on the other hand, $x_k \not| x^\eta$, then $x_k = \min(x^\alpha)$. Moreover, every variable dividing x^η is higher than x_k and at least one of them , let us call it x_j, appears in x^η with exponent $\geq t$, as $\deg(x^\eta) \geq nt$. Then $\frac{x_j^t x^\alpha}{x_k} \in J$, hence $\frac{x^\alpha x^\eta}{x_k} = \frac{x_j^t x^\alpha}{x_k} \cdot \frac{x^\eta}{x_j^t} \in J$ and $x^\alpha x^\eta \notin \mathcal{F}(J)$.

ii) ⇒ iii) By *ii)* $\mathcal{F}(J)$ is finite, and by 4.2 is stably complete, so it is clearly the Pommaret basis of J.

iii) ⇒ i) By remark 4.5, we check the quasi stability on the terms $x^\alpha \in \mathsf{G}(J)$. Let $x_j > x_k := min(x^\alpha)$. By the hypothesis on the finiteness of $\mathcal{F}(J)$, there exists $m \gg 0$ such that $x^\alpha x_j^m \notin \mathcal{F}(J)$. Moreover, being $\mathcal{F}(J)$ a stably complete system, there exists $x^\beta \in \mathcal{F}(J)$ such that $x_j^m x^\alpha \in \text{off}_{\mathcal{F}(J)}(x^\beta)$ and $x^\beta | \frac{x_j^m x^\alpha}{x_k}$. Therefore, $\frac{x_j^m x^\alpha}{x_k} \in J$, namely J is quasi stable. □

Example 4.8. In $k[x, y, z]$ with $x < y < z$:

- considered $J = (z, y^2)$, we get $M = \mathcal{F}(J) = \mathsf{G}(J) = \{z, y^2\}$, since J is stable;
- taken the ideal $J' = (z^2, y)$, we get $M = \mathcal{F}(J) = \{z^2, yz, y\} \supset \mathsf{G}(J)$. In fact, J is quasi stable, but it is not stable;
- given $J = (y)$, the star set is $M = \mathcal{F}(J) = \{z^k y \mid k \geq 0\}$, and it holds $|\mathcal{F}(J)| = \infty$, since J is not stable.

5. M-MARKED SETS AND REDUCTION PROCESS.

In this section, we generalize the notions of J-marked polynomial, J-marked basis and J-marked family given in [1, 6] for J strongly stable.
In those papers, the involved polynomials are marked on the monomial basis of the given monomial ideal J. Here, we give the analogous definitions for any monomial ideal, provided that the involved polynomials are marked on a complete generating system in the sense of definition 3.8.
After determining the setting, we extend to it the reduction process of the quoted papers.
At the end, we will see that such a generalized procedure does not need to be noetherian for every complete system of terms. We will need to add some hypotheses on the given complete system in order to overcome this problem.
We point out that, as in [1, 6], we do not introduce any term-ordering and this represents an important difference w.r.t. Janet's papers.
Moreover, we consider polynomials with coefficients in a ring, not necessarily in a field.

Definition 5.1. Let M be a complete system of terms and J be the ideal it generates.

- A *M-marked set* is a finite set $\mathcal{G}$ of homogeneous (monic) marked polynomials $f_\alpha = x^\alpha - \sum c_{\alpha\gamma} x^\gamma$, with $\text{Ht}(f_\alpha) = x^\alpha \in M$ and $\text{Supp}(f_\alpha - x^\alpha) \subset \mathsf{N}(J)$, so that $|\text{Supp}(f) \cap J| = 1$.

- A *M-marked basis* $\mathcal{G}$ is a M-marked set such that $\mathsf{N}(J)$ is a basis of $P/(\mathcal{G})$ as A-module, i.e. $P = (\mathcal{G}) \oplus \langle \mathsf{N}(J)\rangle$ as an A-module.
- The *M-marked family* $\mathcal{M}f(M)$ is the set of all homogeneous ideals I that are generated by a M-marked basis.

Remark 5.2. Observe that the above definition of marked family $\mathcal{M}f(M)$ is consistent with that given in the Introduction of $\mathcal{M}f(J)$ for a monomial ideal J. Indeed, if $I \in \mathcal{M}f(M)$, then $I \in \mathcal{M}f(J)$ with $J = (M)$. On the other hand, for every given J there are complete systems M that generate it, for instance $M = \mathcal{F}(J)$ and $\mathcal{M}f(J) = \mathcal{M}f(M)$. In fact, if $I \in \mathcal{M}f(J)$, every polynomial h can be uniquely written as a sum $f + g$ with $f \in I$ and $g \in \langle \mathsf{N}(J)\rangle$; especially for every $x^\alpha \in M$, we have

$$x^\alpha = f_\alpha + g_\alpha,\ f_\alpha \in I \text{ and } g_\alpha \in \langle \mathsf{N}(J)\rangle. \tag{1}$$

Then I contains the M-marked basis

$$\mathcal{G} = \{f_\alpha = x^\alpha - g_\alpha\ ,\ x^\alpha \in \mathcal{M}\}.$$

Furthermore $\mathcal{G}$ is a M-marked basis since $(\mathcal{G}) \subseteq I$ and $P = (\mathcal{G}) + \langle \mathsf{N}(J)\rangle = I \oplus \langle \mathsf{N}(J)\rangle$.

The only difference between the two notations $\mathcal{M}f(J)$ and $\mathcal{M}f(M)$ with M a complete system generating J, is that using the second one we present every ideal of the family by means of a special set of generators depending on M. Note that, by the definition itself of $\mathcal{M}f(J)$, we can assert that for every ideal $I \in \mathcal{M}f(J)$ the M-marked basis generating it is *unique*.

We define now a reduction procedure for terms and polynomials, with respect to an homogeneous set $\mathcal{G}$ of polynomials, marked on a complete system of terms M.
The usual reduction process with respect to $\mathcal{G}$ consists of substituting each term $x^\alpha x^\eta$, multiple of an head term $x^\alpha = \mathrm{Ht}(f_\alpha)$, with the polynomial $(x^\alpha - f_\alpha)x^\eta = g_\alpha x^\eta$.
We add an extra condition to the standard procedure, namely that this substitution can be performed only in the case $x^\alpha x^\eta = x^\alpha *_M x^\eta$.

Definition 5.3. Let M be a complete system and $\mathcal{G}$ a M-marked set. We will denote by $\xrightarrow{\mathcal{G}}$ the transitive closure of the relation $h \xrightarrow{\mathcal{G}} h - cf_\alpha x^\eta$, where $x^\alpha x^\eta = x^\alpha *_M x^\eta$ is a term that appears in h with a non-zero coefficient c. We will say that $\xrightarrow{\mathcal{G}}$ is noetherian if the length r of any sequence $h = h_0 \xrightarrow{\mathcal{G}} h_1 \xrightarrow{\mathcal{G}} \dots \xrightarrow{\mathcal{G}} h_r$ is bounded by an integer number $m = m(h)$. This is equivalent to say that if we continue rewriting terms in this way we always obtain, after a finite number of reductions, a polynomial whose support is contained in $\mathsf{N}(J)$.

We will write $h \xrightarrow{\mathcal{G}}_* g$ if $h \xrightarrow{\mathcal{G}} g$ and $\mathrm{Supp}(g) \subset \mathsf{N}(J)$.

In general, the relation $\xrightarrow{\mathcal{G}}$ is not noetherian, namely there are sequences of reduction of infinite length.

Example 5.4. Let $M := \{xz, yz, y^2\}$ a set of terms in $k[x, y, z]$ with $x < y < z$. We find the following sets of multiplicative variables:

- $\mathrm{mult}_M(xz) = \{x, z\}$
- $\mathrm{mult}_M(y^2) = \{x, y\}$
- $\mathrm{mult}_M(yz) = \{x, y, z\}$

and check that M is complete.
Let $\mathcal{G}$ the M-marked set $\{f_{xz} = xz - xy,\ f_{yz} = yz - z^2,\ f_{y^2} = y^2\}$.
Then we have the infinite sequence of reductions:

$$xz^2 = xz *_M z \xrightarrow{\mathcal{G}} xz^2 - f_{xz}z = xyz = yz *_M x \xrightarrow{\mathcal{G}} xyz - f_{yz}x = xz^2$$

However, the reduction $\xrightarrow{\mathcal{G}}$ is always noetherian if $\mathcal{G}$ is marked on a stably complete system. In order to prove this fact we will use the following special subset of the ideal $(\mathcal{G})$.

Definition 5.5. Let $\mathcal{G}$ be a M-marked set on a complete system of terms M and let $J := (M)$. For each degree s, we will denote by $\mathcal{G}^{(s)}$ the set of homogeneous polynomial

$$\mathcal{G}^{(s)} := \{f_\alpha x^\eta \mid x^\alpha *_M x^\eta \in (M)_s\}$$

marked on the terms of J_s in the natural way $\mathrm{Ht}(f_\alpha x^\eta) = x^\alpha x^\eta$.

Remark 5.6. Observe that if $\mathcal{G}$ is a M-marked set on a stably complete system of terms M, for every homogeneous polynomial g of degree s, $g \xrightarrow{\mathcal{G}} h$ implies that $g - h = \sum_{i=1}^m c_i f_{\alpha_i} x^{\eta_i} \in \langle \mathcal{G}^{(s)} \rangle$.

It is worth noticing as a direct consequence of Lemma 3.18 that if $f_\alpha x^\eta \in \mathcal{G}$, then every term in Supp$(x^\alpha x^\eta - f_\alpha x^\eta)$ either belongs to $\mathsf{N}((M))$ or is of the type $x^{\alpha'} *_M x^{\eta'}$ with $x^{\eta'} <_{Lex} x^\eta$.

Lemma 5.7. *Let $\mathcal{G}$ be a M-marked set on the stably complete system of terms $M = \mathcal{F}(J)$.*

1. *Every term in Supp$(x^\beta x^\epsilon - f_\beta x^\epsilon)$ either belongs to $\mathsf{N}((M))$ or is of the type $x^\alpha *_M x^\eta$ with $x^\eta <_{Lex} x^\epsilon$.*
2. *If $f_\beta \in \mathcal{F}(J)$, then all the polynomials $f_{\alpha_i} x^{\eta_i} \in \mathcal{G}^{(s)}$ used in the reduction of $x^\beta x^\epsilon$ (except $f_\beta x^\epsilon$ if it belongs to $\mathcal{G}^{(s)}$) are such that $x^\epsilon >_{Lex} x^{\eta_i}$.*
3. *If $g = \sum_{i=1}^m c_i f_{\alpha_i} x^{\eta_i}$, with $c_i \in k - \{0\}$ and $f_{\alpha_i} x^{\eta_i} \in \mathcal{G}^{(s)}$ pairwise different, then $g \neq 0$ and its support contains some term of the ideal J.*

Proof. (1) is a direct consequence of Lemma 3.18.

(2) Assume that the statement holds for every term $x^{\beta'} x^{\epsilon'}$, with $x^{\epsilon'} <_{Lex} x^\epsilon$. At a first step of reduction of $x^\beta x^\epsilon$ we use the polynomial $f_\alpha x^\eta$ where $x^\beta x^\epsilon = x^\alpha *_M x^\eta$, so that $x^\eta \leq_{Lex} x^\epsilon$; moreover every term in the support of the obtained polynomial either belongs to $\mathsf{N}((M))$ or is of the type $x^{\alpha'} *_M x^{\eta'}$ with $x^{\eta'} <_{Lex} x^\eta$ (Remark 5.6). Then we conclude since we assumed the property holds for all those terms.

(3) We assume that the summands in g are ordered so that $x^{\eta_1} \geq_{Lex} x^{\eta_i}$ for every $i = 1, \dots, m$ and show that $x^{\eta_1 + \alpha_1}$ belongs to the support of g.

The term $x^{\alpha_1 + \eta_1}$ cannot appear as the head of $f_{\alpha_i} x^{\eta_i}$ for some $i \neq 1$ because the star decomposition of a term is unique. Moreover it cannot appear in $f_{\alpha_i} x^{\eta_i} - x^{\alpha_i + \eta_i}$ since $x^{\alpha_1 + \eta_1} = x^\beta x^{\eta_i}$, with $x^\beta \in \mathsf{N}(J)$ would imply $x^{\eta_i} >_{Lex} x^{\eta_1}$ (see Lemma 3.18), against the assumption. □

Theorem 5.8. *Let $\mathcal{G}$ be a M-marked set on a stably complete system of terms M and let J be the ideal generated by M.*

Then the reduction process $\xrightarrow{\mathcal{G}}$ is noetherian and, for every integer s, $P_s = \langle \mathcal{G}^{(s)} \rangle \oplus \langle \mathsf{N}(J)_s \rangle$. Indeed, for every $h \in P_s$

$$h = f + g \text{ with } f \in \langle \mathcal{G}^{(s)} \rangle \text{ and } g \in \langle \mathsf{N}(J)_s \rangle \iff h \xrightarrow{\mathcal{G}}_* g \text{ and } f = h - g$$

Proof. Let $\mathcal{G} = \{f_\alpha \mid x^\alpha \in M\}$.
We observe that we have $\langle \mathcal{G}^{(s)} \rangle \cap \langle \mathsf{N}(J)_s \rangle = \{0\}$ by Lemma 5.7.

In order to prove that the module $\langle \mathcal{G}^{(s)} \rangle + \langle \mathsf{N}(J)_s \rangle$ coincides with P_s it is sufficient to show that it contains all the terms in $J_s \setminus M$, being obvious for those in M, for which $x^\alpha = f_\alpha + g_\alpha$ (see 1).

Let τ be a term in J_s.
If $\tau = x^\alpha *_M x^\eta$, we may assume of having already proved the statement for all the terms $\tau' = x^{\alpha'} *_M x^{\eta'}$ with $x^{\eta'} <_{Lex} x^\eta$.

We have $x^\alpha x^\eta = f_\alpha x^\eta + (x^\alpha - f_\alpha)x^\eta$ where $\mathrm{Supp}(x^\alpha - f_\alpha) \subset \mathsf{N}(J)$. If x^β is any term in this support, then either $x^{\beta+\eta} \in \mathsf{N}(J)$ or $x^{\beta+\eta} = x^{\alpha'} *_M x^{\eta'}$ with $x^{\eta'} <_{Lex} x^\eta$ by Lemma 3.18. This allows us to conclude $P_s = \langle \mathcal{G}^{(s)} \rangle + \langle \mathsf{N}(J)_s \rangle$.

Finally, in order to prove that $\xrightarrow{\mathcal{G}}$ is noetherian it is sufficient to observe that every step of reduction substitutes a term of J of the type $x^\alpha *_M x^\eta$ with $x^\alpha x^\eta - f_\alpha x^\eta$. Indeed, by remark 5.6, each $\tau \in \mathrm{Supp}(x^\alpha x^\eta - f_\alpha x^\eta) \setminus \mathsf{N}((M))$ has the form $x^{\alpha'} *_M x^{\eta'}$, $x^{\eta'} <_{Lex} x^\eta$ and this permits to conclude by induction. $\square$

As a straightforward consequence of the previous result, we obtain the following

Corollary 5.9. *If M is a stably complete system and $\mathcal{G}$ is a M-marked set, the following are equivalent:*

- *$\mathcal{G}$ is a M-marked basis*
- *for every s: $\langle \mathcal{G}^{(s)} \rangle = (\mathcal{G})_s$*
- *for every $h \in (\mathcal{G})$: $h \xrightarrow{\mathcal{G}}_* 0$*
- *if $h - g \in (\mathcal{G})$ and $\mathrm{Supp}(g) \subset \mathsf{N}(J)$, then $h \xrightarrow{\mathcal{G}}_* g$.*

Remark 5.10. We point out that if $\mathcal{G}$ is a M-marked set, but not a M-marked basis, then there are polynomials in the ideal $(\mathcal{G})$ whose support is contained in $\mathsf{N}((M))$. Hence, we do not have a "normal form" of a polynomial h modulo $(\mathcal{G})$, since, in general, there are several polynomials g' such that $\mathrm{Supp}(g') \subset \mathsf{N}(J)$ and $h - g' \in (\mathcal{G})$. However, the reduction process $h \xrightarrow{\mathcal{G}}_* g$ with respect to a $\mathcal{F}(J)$-marked set $\mathcal{G}$ gives a unique reduced polynomial g for every polynomial h.

Using the reduction process, we can now answer Problem 1 and characterize the ideals I that belong to the marked family $\mathcal{M}f(J)$.

Theorem 5.11. *Let $\mathcal{G}$ be a $\mathcal{F}(J)$-marked set. Then:*

$$(\mathcal{G}) \in \mathcal{M}f(J) \Longleftrightarrow \forall f_\beta \in \mathcal{G},\ \forall x_i > \min(x^\beta):\ f_\beta x_i \xrightarrow{\mathcal{G}}_* 0$$

Proof. Since "$\Rightarrow$" is a straightforward consequence of Corollary 5.9, we only prove "$\Leftarrow$". More precisely, we prove that $(\mathcal{G})_m = (\mathcal{G}^{(m)})$, showing that if $f_\beta \in \mathcal{G}$ and $\deg(x^{\beta+\epsilon}) = m$, then $f_\beta x^\epsilon$ is either an element of $\mathcal{G}^{(m)}$ itself or a linear combination of polynomials in $\mathcal{G}^{(m)}$.
If this were not true, we can choose an element $f_\beta x^\epsilon \notin \langle \mathcal{G}^{(m)} \rangle$ with x^ϵ minimal with respect to $<_{Lex}$. As $f_\beta x^\epsilon \notin \mathcal{G}^{(m)}$, at least one variable x_i appearing in x^ϵ with nonzero exponent is non-multiplicative for x^β. Let $x^\epsilon = x_i x^{\epsilon'}$. By hypothesis $f_\beta x_i \xrightarrow{\mathcal{G}}_* 0$, so that $f_\beta x_i$ is a linear combination $\sum c_i f_{\alpha_i} x^{\eta_i}$ of polynomials in $\mathcal{G}^{(|\beta|+1)}$. By Lemma 5.7 we have $x^{\eta_i} <_{Lex} x_i$.

Now $f_\beta x^\epsilon = (f_\beta x_i)x^{\epsilon'} = (\sum c_i f_{\alpha_i} x^{\eta_i})x^{\epsilon'} = \sum c_i f_{\alpha_i} x^{\eta_i+\epsilon'}$, where $x^{\eta_i+\epsilon'} <_{Lex} x_i x^{\epsilon'} = x^\epsilon$. Now we get a contradiction, since $f_{\alpha_i} x^{\eta_i+\epsilon'} \in \langle \mathcal{G}^{(m)} \rangle$ by the minimality of x^ϵ. □

Example 5.12. Let J be the monomial ideal (x^3, xy, y^3) in $k[x,y]$ with $x < y$. Its star set is $\mathcal{F}(J) = \{x^3, xy, xy^2, y^3\}$. Using the criterion given in Theorem 5.11, we can easily check that the $\mathcal{F}(J)$-marked set $\mathcal{G} := \{f_1 := \mathbf{x^3},\ f_2 := \mathbf{xy} - x^2 - y^2,\ f_3 := \mathbf{xy^2},\ f_4 = \mathbf{y^3}\}$ (in bold the head terms) is a $\mathcal{F}(J)$-market basis:

- $yf_1 = xf_1 + x^2 f_2 + xf_3 \xrightarrow{\mathcal{G}}_* 0$,
- $yf_2 = f_1 - xf_2 - f_4 \xrightarrow{\mathcal{G}}_* 0$
- $yf_3 = xf_4 \xrightarrow{\mathcal{G}}_* 0$.

This is a simple example of a marked basis which is not a Gröbner basis. In fact, it is obvious that $\mathrm{Ht}(f_2) = xy$ cannot be the leading term of f_2 with respect to any term-ordering and, more generally, that J cannot be the initial ideal of the ideal $(\mathcal{G})$, even though $(\mathcal{G}) \oplus \mathsf{N}(J) = k[x,y]$.

A wider family of ideals of this type are presented in [6, Example 3.18 and Appendix].

Remark 5.13. Observe that we can perform the first step of reduction of the polynomial $f_\beta x_i$ rewriting the head $x^\beta x_i$ throughout $f_\alpha x^\eta$ with $x^\beta x_i = x^\alpha x^\eta \in \mathrm{off}(x^\alpha)$. In this way we obtain $f_\beta x_i \xrightarrow{\mathcal{G}} f_\beta x_i - f_\alpha x^\eta$, namely the *$S$-polynomial*

$$S(f_\beta, f_\alpha) := \frac{lcm(x^\beta, x^\alpha)}{x^\beta} f_\beta - \frac{lcm(x^\beta, x^\alpha)}{x^\alpha} f_\alpha.$$

Therefore we could reformulate the criterion given by Theorem 5.11 as follows:

$$(\mathcal{G}) \in \mathcal{M}f(J) \Longleftrightarrow \forall f_\alpha, f_\beta \in \mathcal{G} :\ S(f_\alpha, f_\beta) \xrightarrow{\mathcal{G}}_* 0.$$

However Theorem 5.11 shows that it is sufficient to check a special subset of the S-polynomials that corresponds to the basis for the first syzygies of the terms in $\mathcal{F}(J)$. If J is quasi stable, this basis is the one considered in [36]. It is obvious that the maximal degree of these special S-polynomials cannot exceed $1 + \max\{\deg(x^\alpha) \mid x^\alpha \in \mathcal{F}(J)\}$. Indeed, if J is quasi stable, $reg(J) = \max\{deg(\tau),\ \tau \in \mathcal{F}(J)\}$ as proved in [19, 37, 16].

Remark 5.14. If J is a quasi stable monomial ideal and $\mathcal{G}$ is a $\mathcal{F}(J)$-marked set, then there are only a finite number of reduction to perform in order to decide if a $\mathcal{F}(J)$-marked set $\mathcal{G}$ is a basis. We will use this algorithm in order to endow the marked family $\mathcal{M}f(J)$ of a structure of affine scheme

If the considered monomial ideal is not quasi stable, then the (unique) stably complete generating set is *infinite*. Actually this does not necessarily exclude we can exploit it even from a computational point of view.

6. Marked families, schemes and functors

In this section we follow [1, 6] and show how it is possible to associate a scheme to each marked family $\mathcal{M}f(J)$. Due to the naturality of this construction, we can mimic that of [23], and define marked families as functors.

Our results are very similar, but more general, than those of [1, 6, 23]; in fact in those papers the ideal J is assumed to be strongly stable. Recall that a monomial ideal J is called *strongly stable* if for every term $\tau \in J$ and pair of variables $x_i,\ x_j$ such that $x_i|\tau$ and $x_i < x_j$, then also $\frac{\tau x_j}{x_i}$ belongs to J.

Obviously, a strongly stable ideal is also stable, so that $\mathcal{F}(J) = \mathsf{G}(J)$. If J is strongly stable, the notions of $\mathsf{G}(J)$-marked sets, $\mathsf{G}(J)$-marked bases and $\mathsf{G}(J)$-marked family introduced in the previous sections exactly correspond to those of J-marked sets, J-marked bases, J-marked family considered in [1, 6] and the reduction procedure $\xrightarrow{\mathcal{G}}$ with respect to a $\mathsf{G}(J)$-marked set $\mathcal{G}$ introduced in definition 5.3 coincides with the one used in those papers.

Moreover, for such an ideal J, the scheme structure that we will define is the same obtained in [1, 6] and used in [2, 23] for a local study of Hilbert schemes. Indeed, for every monomial ideal J, if $I \in \mathcal{M}f(J)$, then the ideals I and J share the same Hilbert polynomial (and also the same Hilbert function), so that they correspond to points in the same Hilbert scheme.

The scheme we associate to $\mathcal{M}f(J)$ only depends on the monomial ideal J, but the way we use in order to define it needs a set of generators M complete, finite and such that for every M-marked set $\mathcal{G}$ the reduction procedure $\xrightarrow{\mathcal{G}}$ is noetherian.

Then, in the following J will be a quasi stable monomial ideal and M will be its finite star-set $\mathcal{F}(J)$, that is its Pommaret basis $\mathcal{H}(J)$.

Let $\{x_1^{\alpha}, ..., x_s^{\alpha}\}$ be the monomials in M and consider the polynomial ring $B := A[C]$, where C is a compact notation for the set of variables $C_{i,\beta}$ $i = 1, \ldots, s$ and $x^{\beta} \in \mathsf{N}(J)_{|\alpha_i|}$. We also define the M-marked set in $B[x_1, ..., x_n]$

$$\mathcal{G} := \{f_{\alpha_i} := x^{\alpha_i} + \sum C_{i,\beta} x^{\beta} \mid x^{\beta} \in \mathsf{N}(J)_{|\alpha_i|}, \mathsf{Ht}(f_{\alpha_i}) = x^{\alpha_i}\}.$$

Clearly, every M-marked set can be obtained specializing $\mathcal{G}$, namely as $\phi(\mathcal{G})$ for a suitable morphism of A-algebras $\phi : A[C] \to A$. Moreover, by the uniqueness of the M-marked basis generating each ideal in $\mathcal{M}f(J)$, we can assert that for every ideal $I \in \mathcal{M}f(J)$ there exists a unique specialization ϕ such that $(\phi(\mathcal{G})) = I$.

We use Theorem 5.11 in order to construct a set of polynomials $\mathcal{R}$ that will define the scheme we associate to M. If g is a polynomial in $B[x_1, ..., x_n]$, we denote with $\mathrm{coeff}_x(g)$ the set of coefficients of g with respect to the only set of variables $x_1, \ldots, x_n$; hence $\mathrm{coeff}_x(g) \subset B = A[C]$ is a set of polynomials in the variables C. For every $x^{\alpha_i} \in M$ and $x_j > \min(x^{\alpha_i})$, let $g_{\alpha_i,j} \in B[x_1, \ldots, x_n]$ be such that $f_{\alpha_i} x_j \xrightarrow{\mathcal{G}}_* g_{\alpha_i,j}$.

Definition 6.1. Let M be a stably complete system in $\mathcal{T}$, A be any ring, and $\mathcal{R}$ be the union of $\mathrm{coeff}_x(g_{\alpha_i,j})$ for every $x^{\alpha_i} \in M$ and $x_j > \min(x^{\alpha_i})$.

We will call *M-marked scheme* over the ring A, and denote with $\mathbf{Mf}_M(A)$ the affine scheme $\mathrm{Spec}(A[C]/(\mathcal{R}))$.

Remark 6.2. Every M-marked set in $A[x_1, \ldots, x_n]$ is a M-marked basis if and only if the coefficients of the terms in the tails satisfy the conditions given by $\mathcal{R}$.

In particular, if $A = k$ is an algebraically closed field, then the closed points of $\mathbf{Mf}_M(A)$ correspond to the ideals in the marked family $\mathcal{M}f(J)$ where J is the ideal in $k[x_1, \ldots, x_n]$ generated by M.

Remark 6.3. The above construction of $\mathcal{R}$ is in fact *independent* from the fixed commutative ring A, in the sense that it is preserved by extension of scalars. We can first choose $\mathbb{Z}$ as the coefficient ring and then apply the standard map $\mathbb{Z} \to A$.

More formally, for every stably complete set of terms M we can define a functor between the category of $\mathbb{Z}$-algebras to the category of sets

$$\underline{\mathbf{Mf}}_M : \underline{\mathbb{Z}\text{-Alg}} \to \underline{\text{Set}}$$

that associates to any $\mathbb{Z}$-algebra A the set $\underline{\mathbf{Mf}}_M(A) := \mathcal{M}f(MA[x_1, \dots, x_n])$ and to any morphism $\phi : A \to B$ the map

$$\begin{array}{rccc} \underline{\mathbf{Mf}}_J(\phi) : & \underline{\mathbf{Mf}}_M(A) & \longrightarrow & \underline{\mathbf{Mf}}_M(B) \\ & \text{I} & \longmapsto & I \otimes_A B. \end{array}$$

Moreover, again following [23], it is possible to prove that $\underline{\mathbf{Mf}}_M$ is a representable functor represented by the scheme $\mathbf{Mf}_M(\mathbb{Z}) = \operatorname{Spec}(\mathbb{Z}[C]/(\mathcal{R}))$.

7. Historical notes.

Through the trivial interepretation of derivatives

$$\frac{1}{\alpha_1! \cdots \alpha_n!} \frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}},$$

in terms of the corresponding term $\tau = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \in \mathcal{T}$, Riquier [27, 28, 29] was able to algebraically transform the problem of solving differential partial equations in terms of ideal membership.

After introducing the concept (but not the notion) of S-polynomials he proved that if the normal form (in terms of Gauss-Buchberger reduction) of each S-polynomial among the elements of the basis $\mathcal{G}$ goes to zero then

- the given basis $\mathcal{G}$ generates the related ideal;
- the generic solution of the PDE can be given (and computed) as series in terms of initial conditions which can be described and formulated in terms of a Hironaka-Galligo-like decomposition [18, 10] (but more general) of the related *escalier* N;

if not all normal forms are 0, then, exactly as in Buchberger Algorithm, the non-zero normal forms are included in the basis and the procedure is repeated.

For instance, the system [29, pp.188-9]

$$\frac{\partial^3 u}{\partial y^3} = A(x,y,z), \quad \frac{\partial^2 u}{\partial x \partial z} = B(x,y,z), \quad \frac{\partial^3 u}{\partial x^2 \partial y} = C(x,y,z),$$

must satisfy the integrability conditions

$$\frac{\partial^2 A}{\partial x \partial z} = \frac{\partial^3 B}{\partial y^3}, \quad \frac{\partial^2 A}{\partial x^2} = \frac{\partial^2 C}{\partial y^2}, \quad \frac{\partial^2 B}{\partial x \partial y} = \frac{\partial C}{\partial z};$$

in which case the initial conditions have the shape

$$\begin{cases} \left.\begin{array}{rcl} u & = & \phi_0(z) \\ \frac{\partial u}{\partial y} & = & \phi_1(z) \\ \frac{\partial^2 u}{\partial y^2} & = & \phi_2(z) \end{array}\right\} & x - x_0 = y - y_0 = 0, \\ \\ \left.\begin{array}{rcl} \frac{\partial u}{\partial x} & = & \alpha_0 \\ \frac{\partial^2 u}{\partial x \partial y} & = & \alpha_1 \\ \frac{\partial^3 u}{\partial x \partial y^2} & = & \alpha_2 \end{array}\right\} & x - x_0 = y - y_0 = z - z_0 = 0, \\ \\ \frac{\partial^2 u}{\partial x^2} = \psi(x) & y - y_0 = z - z_0 = 0. \end{cases}$$

In his theory, Riquier was assuming that the set $\mathcal{T}$ of the terms was ordered by a term-ordering; he was mainly using [29, p.67] the deglex ordering induced by $x_1 > x_2 > \cdots > x_n$, but he gave a large class of term-orderings to which his theory was applicable; actually (but he never stated that) his characterization is the classical one of all term-orderings [9, 30]. He was however forced to restrict himself to degree-compatible term-orderings in order to be granted convergency.

In his gaussian reduction, Riquier, as Buchberger, considered as head term of each "marked" polynomial its maximal term.

In his considerations on generic initial ideal, Delassus [7], followed by Robinson [31] used (deg)-rev-lex induced by $x_1 < x_2 \cdots < x_n$ and the minimal term as head term of each "marked" polynomial.

In order to "harmonize" the two notations, Janet in [19, 22] applied deglex induced by $x_1 < x_2 < \cdots < x_n$ and chose the maximal term as head term, but expressed all terms as (!) $x_n^{\alpha_n} x_{n-1}^{\alpha_{n-1}} \dots x_1^{\alpha_1}$, while in [20] went back to use deglex induced by $x_1 > x_2 > \cdots > x_n$.

What is worst, in [21] Janet not only applied deglex induced by $x_1 < x_2, \cdots < x_n$ but presented all results within his notation; so, in his presentation of Delassus's result, the head term is again, *à la* Buchberher, the maximal one.

This is not helpful, as regards his reformulation of the previous results on generic initial ideals and stability; thus while, for Robinson [31, 32] and Gunther [14, 15] a generic initial ideal $\epsilon(I)$ satisfies

$$\mu \in \epsilon(I),\ x_h \mid \mu, i < h \implies x_i \frac{\mu}{x_h} \in \epsilon(I),$$

according [21] the formula is

$$\mu \in \epsilon(I), x_h \mid \mu, i > h \implies x_i \frac{\mu}{x_h} \in \epsilon(I).$$

Under the suggestion of Hadamard [24], Janet dedicated his doctorial thesis [19] to a reformulation of Riquier's results in terms of Hilbert's results [17].

In particular, given a finite set of monomials M, he associates to each term $\tau \in M$, as functions of its relation with the other elements of M, a set of variables which he labels *multiplicative* (Definition 3.1) and a subset of terms in (M) which he called his *class* and which we labelled as its *offspring* and considered M *complete* (Definition 3.8) when the disjoint offsprings of M cover (M).

He then gave [19, p.80] a *procédé régulier pour obtenir un système complet base d'un module donné* which *ne pourra se prolonger indéfiniment*; it simply consisted to enlarge M with the elements $xt \notin \cup_{\tau\in M}\mathrm{off}_M(\tau), t \in M$, x non-multiplicative for t.

Janet can now formulate [22, p.75] Riquier's procedure; we can assume to have a finite basis $\mathcal{G} \subset \mathcal{P}$; denoting $M = \{\mathsf{T}(f) : f \in \mathcal{G}\}$,

- we enlarge M in order to made it complete and at the same time
- we similarly enlarge $\mathcal{G}$, adding xg to $\mathcal{G}$ when we add $x\mathsf{T}(g) \notin \cup_{\tau\in M}\mathrm{off}_M(\tau)$;
- we then perform Riquier's test, which, for a complete systems, consists in computing the normal form of each element $xg, g \in \mathcal{G}$, x non-multiplicative for $\mathsf{T}(g)$.

Janet [19, p.112-3] further remarks (in connection with Hilbert's syzygy theory) that the reduction-to-zero of all such elements give a basis S of the syzygy module of $\mathcal{G}$. Actually he repeatedly applied the same procedure to S, thus computing a resolution of $\mathcal{G}$ and anticipating Schreyer's Algorithm [33].

Next, in 1924, Janet [20] moved his interest in extending the study to the homogeneous case, adapting his approach on one side to the solution of partial differential equation given by E. Cartan [3, 4, 5] via his characters and test and on the other side to the introduction by Delassus [7] of the concept of generic initial ideal and the precise description of it given by Robinson [31, 32] and Gunther [14, 15]; he thus discussed the notion of *système de forms (de même ordere) en involution*. The notion, as he explains, is independent from the variable chosen and allows to assign to the system a series of values $\sigma_i^{(p)}, 1 \le i \le n, p \in \mathbb{N}$ which [22, p.87] *sont évidemment invariables lorsq'on fait un changement linéaire et homogène des variables indépendantes* which, under the assumption of generality, allow to describe the structure of the *generic escalier* of the considered ideal.
The procedure, given a finite set $\mathcal{G}$ of forms, repeatedly produces *à la* Macaulay a linear basis $\mathcal{B}_p$ of $(\mathcal{G})_p$ by performing linear algebra on the set $\{x_i g : g \in \mathcal{B}_{p-1}, 1 \le i \le n\}$; termination is granted when the formula (2) below is satisfied.

Given a homogeneous ideal $I \subset k[x_1, x_2, \dots, x_n]$, where the variables are assumed to be generic, so that $\mathsf{N}(I)$ is stable, Janet defined [20, pp.30-2],[21, p.30],[22, pp.90-1],[24, p.93, p.99] multiplicative variables according 3.5, introduced values $\sigma_i^{(p)}(I)$ (or $\sigma_i^{(p)}$ for short when no confusion is possible) for every $1 \le i \le n$, and $p \in \mathbb{N}$, which can be described as

$$\sigma_i^{(p)} := \#\{\tau \in \mathsf{N}(I), \deg(\tau) = p, \min(\tau) = i\}$$

and, fixing a value p and denoting $\sigma_i := \sigma_i^{(p)}$, and $\sigma'_i := \sigma_i^{(p+1)}$ proved

Proposition 7.1 (Janet). *It holds,*

1. $\sigma'_1 + \sigma'_2 + \ldots + \sigma'_n \le \sigma_1 + 2\sigma_2 + \ldots + n\sigma_n$;
2. $\sum_{i=1}^n {\sigma'}_i = \sum_{i=1}^n i\sigma_i \implies {\sigma'}_j = \sum_{i=j}^n \sigma_i$ *for each* j.
3. $\sum_{i=1}^n {\sigma'}_i = \sum_{i=1}^n i\sigma_i \implies \sum_{i=1}^n \sigma_i^{(P+1)} = \sum_{i=1}^n i\sigma_i^{(P)}$ *for each* $P > p$.

He can then state

Definition 7.2 (Janet). [22, pp.90-1] A finite set $E \subset \mathcal{P}$ of forms of degree at most p generating the ideal $I \subset P$, is said to be *involutive*[1] if, with the present notation, it

[1] *en involution.*

satisfes the formula

$$\sum_{i=1}^{n} \sigma_i^{(p+1)} = \sum_{i=1}^{n} i\sigma_i^{(p)}. \tag{2}$$

□

Thus, once the iterated Macaualy-like procedure satisfies (2) at degree $\bar{p}$ then it successfuly terminates and the *finite* bases produced by it is involutive; Janet is therefore able to present the ideal $\{\tau \in \mathsf{T}(I), \deg(\tau) \geq \bar{p}\}$ by explicitly producing[22] the decomposition

$$\{\tau \in \mathsf{T}(I), \deg(\tau) \geq \bar{p}\} = \sqcup_{\tau \in M} \mathrm{off}_M(\tau)$$

where M is the stably complete set $M = \{\tau \in \mathsf{T}(I), \deg(\tau) \geq \bar{p}\}$ and to express its Hilbert polynomial as

$$^hH_I(t) = \sum_{h=1}^{n-1} \binom{t-p+h-1}{h-1} \sigma_h^{(p)}(I).$$

In our context, the characterization of $\sigma_i^{(p)}$ and definition 7.2 lead to the following

Proposition 7.3. *With the previous notation, if J is a quasi stable monomial ideal, then*

$$\sum_{i=1}^{n} \sigma_i^{(p+1)}(J) = \sum_{i=1}^{n} i\sigma_i^{(p)}(J).$$

The same equality holds if I is a homogeneous ideal generated by a J-marked basis $\mathcal{G}$ with J quasi stable.

Therefore $\mathcal{G}$ is an involutive basis.

Proof. For the first statement we observe that if $p \geq \overline{p}$ every term $\tau \in J_{p+1}$ can be written in a unique way as a product $\tau = \theta x_i$, with $\theta \in J_p$ and x_i a multiplicative variable for θ, i.e. $x_i \leq \min(\theta)$.

If I is the homogeneous ideal generated by a J-marked set $\mathcal{G}$, then for the corresponding $f_\tau \in \mathcal{G}^{(p+1)}$ we have $f_\tau = f_\theta x_i$ with f_θ in $\mathcal{G}^{(p)}$ and of course $x_i \leq \min(\theta)$.

If $\mathcal{G}$ is a J-marked basis, then we get the equality since $(\mathcal{G})_t = (\mathcal{G}^{(t)})$ for every t (Corollary 5.9). □

Note that for an ideal I generated by a J-marked set $\mathcal{G}$ which is not a marked basis, only the inequality $\sum_{i=1}^{n} \sigma_i^{(p+1)} \leq \sum_{i=1}^{n} i\sigma_i^{(p)}$ holds true, since $(\mathcal{G})_t \supseteq (\mathcal{G}^{(t)})$.

The iterated Macaualy-like procedure gives also a fine decomposition of $\mathsf{N}(I)_{\geq \bar{p}-1}$ as follows:

- he partitions the set $\mathsf{N}(I)_{\overline{p}-1}$ as $\mathsf{N}_{\overline{p}-1} = \sqcup_{i=0}^{n-1} N_i$ associating to
  - N_0 the monomials $\tau \in \mathsf{N}_{\overline{p}-1}(I)$ for which $x_1\tau \in \mathsf{T}(I)$;
  - while each of the σ_1 elements $\tau = \frac{\upsilon}{x_1} \in \mathsf{N}(I)_{\overline{p}-1} \setminus N_0, \mathrm{class}(\upsilon) = 1$, is inserted in N_i if it is one of the σ_i elements which can be expressed as $\tau = \frac{\upsilon_i}{x_i}, \mathrm{class}(\upsilon_i) = i$ but is not one of the σ_{i+1} elements which can be expressed as $\tau = \frac{\upsilon_{i+1}}{x_{i+1}}, \mathrm{class}(\upsilon_{i+1}) = i+1$.
- he then associate to each $\tau \in N_i$ $\mathrm{mult}(\tau) = \{x_j, 1 \leq j \leq i\}$ as multiplicative variables and $\mathrm{off}(\tau) := \{\tau\omega, \omega \in \mathcal{T}[1,i]\}$ as its offspring

- and states

$$\{\tau \in \mathsf{N}(I), \deg(\tau) \geq \bar{p} - 1\} = \sqcup_{i=0}^{n-1} \sqcup_{\tau \in N_i} \mathrm{off}(\tau).$$

Riquier's and Janet's results were introduced to the Computational Algebra commutative at the MEGA-90 Symposium in 1990 by a survey by Pommaret [25] of his theory and, two years later, through a paper by F. Schwarz [34] where he remarked:

> The concept of a Gröbner base and algorithmic methods for constructing it for a given system of multivariate polynomials has been established as an extremaly important tool in commutative algebra. It seems to be less well known that similar ideas have been applied for investigating partial differential equations (pde's) around the turn of the century in the pioneering work of the French mathematicians Riquier and Janet. [...] [T]heir theory [...] is basically a critical-pair/completion procedure. All basic concepts like a term-ordering, reductions and formation of critical pairs are already there.

This prompted V. Gerdt to suggest his coworkers Zharkov and Blinkov to investigate whether the results by Janet and Pommaret were translatable from pde's to polynomial rings in order to produce an effective alternative approach to Buchberger's Algorithm; the conclusion of this investigation [39, 40] was successful — the proposed algorithm was able to give a solution with a speed-up of 20 w.r.t. degrevlex Buchberger's algorithm on classical test-suites and caused sensation in the community.

Unfortunately, among the two constructions proposed by Janet, they hitted the involutive one, which is *not* a Buchberger-like procedure and presented it as such, remarking that in general does not terminate and that the basis is not necessarily finite unless the ideal is 0-dimensional. What is worst, they attributed to Pommaret their mistakes, thus introducing in literature a "bad" fictional Pommaret division compared with the "good" Janet division (related to Janet completion [19] procedure).

An algorithm based on Janet's notion [19] of completeness is reported in [12, 13, 11]

Involutiveness is the argument of the *Habilitation* thesis (2002) of Seiler [35, 36, 37]; an improved version has recently appeared as [38]. Finiteness is a required condition for the notion of Pommaret bases [16].

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