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Variable sampling interval run sum $\bar{X}$ chart with estimated process parameters

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Abstract. The $\bar{X}$ type control chart is often evaluated by assuming the process parameters are known. However, the exact values of process parameters are hardly known and thus Phase-I dataset is needed to estimate them. In this paper, the performance of the variable sampling interval run sum $\bar{X}$ chart with estimated process parameters is evaluated by using the performance measure of the average of the average time to signal (AATS) and the optimal design of the proposed chart in minimizing the out-of-control AATS is developed. The performance measure of the standard deviation of the average time to signal (SDATS) is then used to identify the number of Phase-I samples (w) needed to have an in-control AATS performance close to its known process parameter case. Results show that large w is needed to minimize the performance gap between known and unknown process parameters cases of the VSI RS $\bar{X}$ chart.

1 Introduction

The $\bar{X}$ chart is the first control chart invented by Shewhart in 1924 for process monitoring [1]. However, the $\bar{X}$ chart is not sensitive in signaling an out-of-control, when small and moderate mean shifts are detected. Thus, the more advancement control chart such as run sum (RS-$\bar{X}$) chart was proposed.

RS-$\bar{X}$ chart was developed by [2]. It is also well known as a zone chart, where the control chart is segregated into many regions according to the preference of the practitioners. The run sum chart is shown to have better performance than the Shewhart charts with and with-out run-rules [3]. In addition, the performance of the run sum chart is comparable with the performance of the exponentially weighted moving average (EWMA) and the cumulative sum (CUSUM) charts when the number of regions of the chart is increased [4]. To enhance the performance as well as to simplify the design of the run sum chart, [5] integrated the fast initial response (FIR) feature into the run sum chart with one parameter. They pointed out that the proposed chart is competitive with the CUSUM, FIR CUSUM and Shewhart FIR CUSUM charts. To have a quick detection of the changes of the process mean, [6] employed the variable sampling interval (VSI) schemes on the RS-$\bar{X}$ control charts. They showed that their proposed charts outperform the RS-$\bar{X}$ chart for all of the shift sizes under steady state and zero state modes, respectively. However, all these mentioned run sum-type charts assumed the process parameters are known in monitoring process. However, without such assumption, an in-control Phase-I dataset is needed by the practitioners to estimate the process parameters. Thus, numerous research studied the impact of the estimated process parameters on the performance of control charts.

Some of the contemporary works of estimated parameters based control charts are discussed. [7] evaluated the impact of estimated control limits on the adaptive $\bar{X}$ chart, where the results showed that the run length performance are significantly impacted by the parameter estimation. [8] investigated the process parameter estimation on the double sampling S^2 chart. To address the variability in run length performance and practitioner to practitioner variation in estimating process parameters, [9] employed the performance measures of AARL and SDARL to evaluate the RS-$\bar{X}$ chart with estimated process parameters.

To the best of our knowledge, the study on the impact of estimated process parameters on the variable sampling interval run sum $\bar{X}$ chart (VSI RS-$\bar{X}$) chart has not been proposed in the literature. Therefore, this paper proposes the VSI RS-$\bar{X}$ chart with estimated process parameters-based on the criteria of average of the average time to signal (AATS) and standard deviation of the average time to signal (SDATS). The rest of the paper is organized as follows: Section 2 demonstrates the optimal design of the VSI RS-$\bar{X}$ chart with estimated process parameters in minimizing the out-of-control average of the average time to signal (AATS_1). The optimal charting parameters and the performance of the chart under known and unknown process parameters are studied in Section 3. Lastly, conclusions and recommendations are summarized in Section 4.

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2 VSI RS-$\bar{X}$ chart with estimated process parameters

When a chart consists of the unknown process parameters such as the mean (μ_0) and standard deviation (σ_0), an in-control Phase-I dataset is needed to estimate them. By assuming a Phase-I dataset consists of w samples, where each of the sample consists of size n such as $\{X_{i,1}, X_{i,2}, ..., X_{i,n}\}$ for $i = 1, 2, ..., w$. By further considering the Phase-I data are normally distributed with μ_0 and σ_0, i.e. $X_{i,j} \sim N\left(\mu_0, \sigma_0^2\right)$ and there is independence within and between samples. The commonly used estimator for μ_0 and σ_0 are defined as ([10])

$$\hat{\mu}_0 = \frac{1}{wn}\sum_{i=1}^{w}\sum_{j=1}^{n} X_{i,j} \tag{1}$$

and

$$\hat{\sigma}_0 = \sqrt{\frac{1}{w(n-1)}\sum_{i=1}^{w}\sum_{j=1}^{n}\left(X_{i,j} - \bar{X}_i\right)^2}, \tag{2}$$

respectively, where $\bar{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_{i,j}$ is the sample mean of i^{th} sample.

In the Phase-II process, let $Y_{i,1}, Y_{i,2}, ..., Y_{i,n}$ be the n observations of sample i and we further assume that $Y_{i,j}$ follows a normal $N\left(\mu_0 + \delta\sigma_0, \sigma_0^2\right)$ distribution, where $j = 1, 2, ..., n$. Note that, $\delta = 0$ and $\delta \neq 0$ indicate the process is statistically in-control and out-of-control, respectively.

Fig. 1 depicts the M regions above the centre line (CL), i.e. $\hat{R}_1, \hat{R}_2, ..., \hat{R}_{(M-1)}, \hat{R}_M$ and M regions below the CL, i.e. $\hat{R}_{1b}, \hat{R}_{2b}, ..., \hat{R}_{(Mb-1)}, \hat{R}_{Mb}$ of the VSI RS-$\bar{X}$ chart with estimated process parameters. The M regions above the CL are allocated with positive integer scores, $0 \le +S_1 \le +S_2 \le ... \le +S_K$, while the M regions below the CL are allocated with negative integer scores, $-S_M \le ... \le -S_2 \le -S_1 \le 0$. The interval of the upper regions ($\hat{R}_H$) and lower regions ($\hat{R}_{Hb}$) of the CL for the chart are shown as

$$\hat{R}_H = [\hat{\mu}_0 + A_{H-1}\hat{\sigma}_0, \hat{\mu}_0 + A_H\hat{\sigma}_0) \tag{3a}$$

and

$$\hat{R}_{Hb} = [\hat{\mu}_0 - A_H\hat{\sigma}_0, \hat{\mu}_0 - A_{H-1}\hat{\sigma}_0) \tag{3b}$$

where, $A_0 = 0$, $A_M = \infty$ and $A_H = \frac{k}{\sqrt{n}}\left[\frac{3H}{q-1}\right]$ for H = 1, 2,.., $M-1$, and k is the constant chosen to set the value of in-control average of the average time to signal (AATS_0).

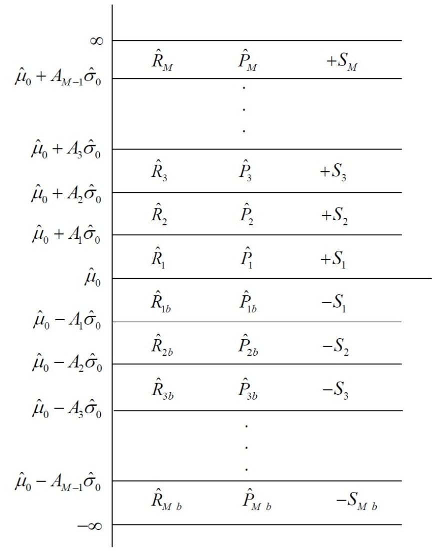

Fig. 1. The regions (M) with the corresponding scores and probabilities of the VSI RS-$\bar{X}$ chart with estimated process parameters.

The estimated process parameter-based VSI RS-$\bar{X}$ chart reacts based on the upper cumulative scores (U_i) and lower cumulative scores (L_i), defined as

$$U_i = \begin{cases} 0 & \text{if } \bar{Y}_i < CL \\ U_{i-1} + S\left(\bar{Y}_i\right) & \text{if } \bar{Y}_i \ge CL \end{cases}, \tag{4a}$$

$$L_i = \begin{cases} 0 & \text{if } \bar{Y}_i \ge CL \\ L_{i-1} + S\left(\bar{Y}_i\right) & \text{if } \bar{Y}_i < CL \end{cases}, \tag{4b}$$

where $i = 1, 2, ...$ and $H = 1, 2, ...M$, while

$$S(\bar{Y}_i) = \begin{cases} S_H & \text{if } \bar{Y}_i \in \hat{R}_H \\ -S_H & \text{if } \bar{Y}_i \in \hat{R}_{Hb} \end{cases} \tag{5}$$

The chart signals an out-of-control signal if $U_i \ge +S_M$ or $L_i \le -S_M$, where, $+S_M$ and $-S_M$ act as the triggering scores for the M regions above CL and M regions below CL, respectively.

For simplicity, we consider the initial value $U_0 = L_0 = 0$ for the estimated parameters-based VSI RS-$\bar{X}$ chart. Let t_i be the sampling interval of selecting the next sample of the estimated VSI RS-$\bar{X}$ chart based on the cumulative scores values, (i.e. U_i and L_i), then t_i is defined as:

$$t_i = \begin{cases} d_1 & \text{if } S_M / D \le U_i < S_M \text{ or} -S_M < L_i \le -S_M / D \\ d_2 & \text{if } 0 \le U_i < S_M / D \text{ or} -S_M / D < L_i \le 0 \end{cases}, \tag{6}$$

where D is the positive integer decided by user to control the threshold limit, while d_1 and d_2 denote the short and long sampling intervals, respectively, Note that $d_2 > d_1 > 0$. The probability of $\bar{Y}_i$ falls in the region $\hat{R}_H$ is calculated as

$$\hat{P}_H = \Pr\left(\bar{Y}_i \in \hat{R}_H\right)$$
$$= \Phi\left(\left(\hat{\mu}_0 - \mu_0\right)\frac{\sqrt{n}}{\sigma_0} + A_H \frac{\hat{\sigma}_0\sqrt{n}}{\sigma_0} - \delta\sqrt{n}\right) - \Phi\left(\left(\hat{\mu}_0 - \mu_0\right)\frac{\sqrt{n}}{\sigma_0} + A_{H-1} \frac{\hat{\sigma}_0\sqrt{n}}{\sigma_0} - \delta\sqrt{n}\right) \quad (7)$$

By letting $U = \left(\hat{\mu}_0 - \mu_0\right)\left(\sqrt{n}/\sigma_0\right)$ and $V = \left(\hat{\sigma}_0\sqrt{n}/\sigma_0\right)$, $\hat{P}_H$ is simplified as

$$\hat{P}_H = \Phi\left(U + A_H V - \delta\sqrt{n}\right) - \Phi\left(U + A_{H-1}V - \delta\sqrt{n}\right) \quad (7a)$$

Similarly, the probability of $\bar{Y}_i$ falls in the region $\hat{R}_{Hb}$ can be computed as

$$\hat{P}_{Hb} = \Phi\left(U - A_{H-1}V - \delta\sqrt{n}\right) - \Phi\left(U - A_H V - \delta\sqrt{n}\right) \quad (7b)$$

Here, the probability density functions (pdf) of U and V can be obtained from [9].

The performance measures of AATS and SDATS are used to evaluate the performance of the estimated process parameter-based VSI RS-$\bar{X}$ chart. The AATS of the chart is computed as ([11])

$$\text{AATS}(\delta) = \int_{-\infty}^{+\infty}\int_{0}^{+\infty} \widehat{\text{ATS}} f_{(U)}\left(u \mid w\right) f_V\left(v \mid w, n\right) dv du, \quad (8)$$

where $\widehat{\text{ATS}}$ denotes the average time to signal of the estimated case, and it is defined as ([6])

$$\widehat{\text{ATS}} = \boldsymbol{q}^T\left(\boldsymbol{I} - \hat{\boldsymbol{Q}}\right)^{-1}\boldsymbol{t} - \boldsymbol{q}^T\boldsymbol{t} \quad (9)$$

Note that $\hat{\boldsymbol{Q}}$ is a $p \times p$ transition probability matrix of estimated process parameters-based VSI RS-$\bar{X}$ chart. where, the value of p depends on the scores combinations and [9] described the procedure of finding $\hat{\boldsymbol{Q}}$. The column vector $\boldsymbol{t}$ consists of the elements t_i, for $i = 1, 2, ..p$, determined by Eqs (6), while, $\boldsymbol{q}^T = \left(1, 0, 0, ..., 0\right)$ is a initially probability vector.

The SDATS is defined as ([11])

$$\text{SDATS}(\delta) = \sqrt{E\left(\widehat{\text{ATS}}^2\right) - \left[E\left(\widehat{\text{ATS}}\right)\right]^2}, \quad (10)$$

Where

$$E[\widehat{\text{ATS}}^2] = \int_{-\infty}^{+\infty}\int_{0}^{+\infty} \widehat{\text{ATS}}^2 f_{(U)}\left(u \mid w\right) f_V\left(v \mid w, n\right) dv du.$$

Consequently, to minimize the out-of-control AATS $\left(\text{AATS}_1\right)$ of the estimated process parameter-based VSI RS-$\bar{X}$ chart, the associated optimal algorithm is mathematically expressed as

$$\underset{(k, d_2, S_1, S_2, ..., S_M)}{\text{Minimize}} \ \text{AATS}_1(\delta) \quad (11)$$

subject to the constraints of in-control $\text{AATS} = \text{AATS}_0$ and in-control average sampling interval $\text{ASI}_0 = t_0$.

3 Numerical analysis

In this section, the methodology presented in Section 2 is used to evaluate the performance of the VSI RS-$\bar{X}$ chart when the process parameters are estimated from Phase-I samples. Table 1 shows the optimal charting parameters $\{k, d_2(S_1, S_2, ..., S_M)\}$ in minimizing the AATS_1 with respect to the constraints of $\text{AATS}_0 = 370$ and $\text{ASI}_0 = 1$, by considering the combinations of $n = 5$, $d_1 = 0.01$, $M = 4$, $D = 4$, $\delta \in \{0.2, 0.4, 0.6, 0.8, 1, 1.5, 2\}$ and $w \in \{20, 40, 80\}$.

Table 2 provides the optimal AATS_1 and the associated out-of-control SDATS (SDATS_1) values of the estimated VSI RS-$\bar{X}$ chart for $w \in \{20, 40, 80\}$ corresponding to the optimal parameters given in Table 1 and the out-of-control ATS (ATS_1) values for known parameters case (i.e. $w = \infty$). The values of the AATS_1 and SDATS_1 are computed using Eqs. (8) and (10). For the chart performance to be at acceptable level, [12] suggested that the SDATS value should not more than 10% of the ATS value. The details can refer to [9]. From Table 2, we observe most of the SDATS_1 values are greater than its corresponding 10% of the ATS_1 values. Furthermore, the chart's performance based on the estimated process parameters is significantly different compared to its known parameters counterpart, especially when the values of δ and w are small.

Table 1. Values of optimal parameters of the 4 regions estimated process parameter-based VSI RS-$\bar{X}$ chart when $\text{AATS}_0 = 370$, $d_1 = 0.01$ and $D = 4$.

		$w = 20$						$w = 40$						$w = 80$					
n	δ	k	d_2	S_1	S_2	S_3	S_4	k	d_2	S_1	S_2	S_3	S_4	k	d_2	S_1	S_2	S_3	S_4
5	0.2	1.2336	1.5089	0	2	5	8	1.2196	1.5496	0	2	3	7	1.2103	1.5695	0	2	3	7
	0.4	1.2336	1.5089	0	2	4	7	1.2256	1.5418	0	2	4	7	1.217	1.5605	0	2	4	7
	0.6	1.2336	1.5089	0	2	5	8	1.2256	1.5418	0	2	5	8	1.217	1.5605	0	2	4	7
	0.8	1.2336	1.5089	0	2	4	7	1.2256	1.5418	0	2	5	8	1.217	1.5605	0	2	4	7
	1.0	1.1665	1.6008	0	2	3	8	1.154	1.6385	0	2	3	8	1.1432	1.6616	0	2	3	8
	1.5	1.1665	1.6008	0	2	3	8	1.154	1.6385	0	2	3	8	1.1432	1.6616	0	2	3	8
	2.0	1.1593	1.6131	0	2	2	7	1.1458	1.6524	0	2	2	7	1.1344	1.6763	0	2	2	7

Table 2. $\mathrm{AATS_1}$ and $\mathrm{SDATS_1}$ values of the 4 regions estimated process parameter-based VSI RS-$\bar{X}$ chart when $\mathrm{AATS_0}=370$, $d_1=0.01$ and $D=4$.

n	δ	$w=20$		$w=40$		$w=80$		$w=\infty$
		$\mathrm{AATS_1}$	$\mathrm{SDATS_1}$	$\mathrm{AATS_1}$	$\mathrm{SDATS_1}$	$\mathrm{AATS_1}$	$\mathrm{SDATS_1}$	$\mathrm{ATS_1}$
5	0.2	132.705	222.402	95.851	104.934	76.323	54.059	58.350
	0.4	16.157	30.004	11.871	9.729	10.313	4.954	8.949
	0.6	3.131	2.882	2.7395	1.370	2.565	0.829	2.387
	0.8	0.100	0.598	0.934	0.357	0.898	0.234	0.856
	1.0	0.384	0.204	0.362	0.130	0.350	0.087	0.335
	1.5	0.037	0.021	0.034	0.014	0.033	0.009	0.031
	2.0	0.003	0.002	0.003	0.001	0.003	0.001	0.002

Table 3 presents the values of $\mathrm{AATS_0}$ and in-control SDATS $(\mathrm{SDATS_0})$ for different number of w with the optimal combinations $\{k, d_2, (S_1, S_2, S_3, S_4)\}$ ={1.1121, 1.7095, (0, 1, 2, 4)} and {1.1932, 1.5964, (0, 2, 3, 7)} for $\mathrm{ATS_0}$=200 and 370, respectively. The purpose of this analysis is to identify the number of w needed to generate the $\mathrm{AATS_0}$ values that close to its known process parameters case $(w=\infty)$, where the corresponded $\mathrm{SDATS_0}$ should not exceed 10% of the $\mathrm{ATS_0}$. For example, at least $w=1000$ Phase-I samples are needed to attain $\mathrm{SDATS_0}$ within 10% of the associated $\mathrm{ATS_0}$, when $\mathrm{AATS_0}$ performance of the known process parameter-based VSI RS-$\bar{X}$ is 200 (see $w=\infty$). The bolded values in Table 3 are the values of $\mathrm{SDATS_0}$ that fall within 10% of the associated $\mathrm{ATS_0}$ values, and the corresponding w are suggested, so that the run length performance between the known and unknown process parameters cases is acceptable.

Table 3. $\mathrm{AATS_0}$ and $\mathrm{SDATS_0}$ values of the 4 regions estimated process parameter-based VSI RS-$\bar{X}$ chart for different number of Phase-I samples w, $\mathrm{ATS_0}\in\{200,370\}$, $d_1=0.01$, $D=4$ and $n=5$.

	$\mathrm{ATS_0}=200$		$\mathrm{ATS_0}=370$	
w	$\mathrm{AATS_0}$	$\mathrm{SDATS_0}$	$\mathrm{AATS_0}$	$\mathrm{SDATS_0}$
100	185.080	62.532	336.256	124.893
300	193.913	35.690	355.630	71.301
500	196.186	27.481	360.825	54.831
700	197.233	23.145	363.249	46.133
900	197.835	20.366	364.655	40.564
1000	198.050	**19.305**	365.157	38.438
1100	198.227	18.393	365.572	**36.614**
1150	198.304	17.983	365.754	35.793
1200	198.375	17.600	365.921	35.025
$w=\infty$	200	0	370	0

4 Conclusions

The impacts of the process parameters estimation on the AATS and SDATS of the VSI RS-$\bar{X}$ chart are investigated in this paper. As shown in numerical analysis, the $\mathrm{AATS_0}$ and $\mathrm{AATS_1}$ performances of the estimated process parameter-based VSI RS-$\bar{X}$ chart tend to similar to its known process parameters case when large w is considered. Thus, based on the performance of SDATS, the appropriate number of Phase-I samples required for the parameter estimation is recommended. While, this work focuses on univariate VSI RS chart, in future, the multivariate charts can be investigated under the assumption of estimated process parameters.

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