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INVESTIGATION OF STATIC AND DYNAMIC BEHAVIOR OF ANISOTROPIC INHOMOGENEOUS SHALLOW SHELLS BY SPLINE APPROXIMATION METHOD

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Abstract. The present report proposes an efficient approach to solving within the framework of the classic and refined models the stress-strain problems of shallow shells as well as the problems on free vibrations. In accordance with the approach the initial system of partial differential equations is reduced to one-dimensional problems by using approximation of the solution in terms of basic splines in one coordinate. The boundary-value problems obtained and eigenvalue boundary-value problems for systems of ordinary differential high-order equations are solved by the stable numerical method of discrete ortogonalization.

Keywords: shallow shells, varying thickness, refined and classic formulation, stress-strain state, free vibration.

1. Introduction

Shallow shells made of orthotropic materials are widely used for construction of structure elements in modern engineering (Fig. 1). To estimate their strength under possible conditions of service operation, it is necessary to have the information about the stress-strain state (Cowper *et al.* 1970; Gould 1988; Григоренко и др. 1987) and dynamic characteristics (Graff 1991; Lee *et al.* 1984; Liew *et al.* 1997) of the mechanical objects being considered.

Currently the problems of computational mathematics, mathematical physics, and mechanics, spline-functions are widely solved (Fan and Cheung 1983; Завьялов и др. 1980). It is due to advantages of the spline-approximation techniques in comparison with others. As basic advantages, the following can be referred: stability of splines in respect to local disturbances, i.e. behaviour of the spline near a point does not affect the behaviour of the spline as a whole as, for instance, this holds in the case of the polynomial approximation; fast convergence of the spline-interpolation in contrast to polynomial one; simplicity and convenience in realization of algorithms for constructing and calculating splines by personal computers. Use of spline unctions in various variational, projective, and other discrete-continual methods makes it possible to obtain appreciable results in comparison with those the classical apparatus of polynomials would yield, to simplify essentially their numerical realization, and to obtain the desired solution with a high-degree accuracy (Grigorenko and Zakhariichenko 2004; Grigorenko and Yaremchenko 2004).

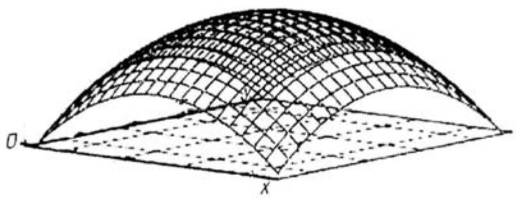

Fig. 1. Shallow rectangular in plan shell

2. Basic relations and constitutive equations

2.1. Free vibrations of shallow shells in classic formulation

According to the Mushtari–Donnell–Vlasov's theory of shallow shells, the natural transverse vibrations of these shells are described by the equations

$$\frac{\partial N_x}{\partial x} + \frac{\partial S}{\partial y} = 0\,;\ \frac{\partial S}{\partial x} + \frac{\partial N_y}{\partial y} = 0\,;$$
$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - k_1 N_x - k_2 N_y = \rho h \frac{\partial^2 w}{\partial t^2}\,; \quad (1)$$
$$\frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y} = Q_x\,;\ \frac{\partial M_y}{\partial y} + \frac{\partial H}{\partial x} = Q_y\,,$$

where x and y are the Cartesian coordinates of a point on the mid-surface ($0 \le x \le a$, $0 \le y \le b$), t is time, w – the shell deflection, and ρ – the density of the material (rotary and in-plane inertia are not included there).

The normal and shear forces N_x, N_y, and S and the bending and twisting moments M_x, M_y, and H satisfy the following relations:

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$$N_x = C_{11}\left(\frac{\partial u}{\partial x} + k_1 w\right) + C_{12}\left(\frac{\partial v}{\partial y} + k_2 w\right);$$

$$S = C_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right); M_x = -\left(D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2}\right);$$

$$N_y = C_{12}\left(\frac{\partial u}{\partial x} + k_1 w\right) + C_{22}\left(\frac{\partial v}{\partial y} + k_2 w\right); \quad (2)$$

$$H = -2D_{66}\frac{\partial^2 w}{\partial x \partial y}; \; M_y = -\left(D_{12}\frac{\partial^2 w}{\partial x^2} + D_{22}\frac{\partial^2 w}{\partial y^2}\right),$$

where

$$C_{ij} = B_{ij} h(x,y), \; D_{ij} = B_{ij}\frac{h^3(x,y)}{12};$$

$$B_{11} = \frac{E_1}{1-\nu_1\nu_2}, \; B_{12} = \frac{\nu_2 E_1}{1-\nu_1\nu_2} = \frac{\nu_1 E_2}{1-\nu_1\nu_2}; \quad (3)$$

$$B_{22} = \frac{E_2}{1-\nu_1\nu_2}, \; B_{66} = G_{12}, \; i,j \in \{1,2,6\}.$$

E_1, E_2, G_{12}, ν_1, ν_2 are the elastic and shear moduli and Poisson's ratios; k_1 and k_2 – the curvatures of mid-surface, u, v, w – components of displacements vector.

The system of equations (1–2) yields 3 equivalent differential equations for the 3 displacements u, v, and w of the mid-surface:

$$C_{11}\frac{\partial^2 u}{\partial x^2} + \frac{\partial C_{11}}{\partial x}\frac{\partial u}{\partial x} + C_{66}\frac{\partial^2 u}{\partial y^2} + \frac{\partial C_{66}}{\partial y}\frac{\partial u}{\partial y} +$$

$$(C_{12}+C_{66})\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial C_{66}}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial C_{12}}{\partial x}\frac{\partial v}{\partial y} +$$

$$(C_{11}k_1 + C_{12}k_2)\frac{\partial w}{\partial x} + \frac{\partial (C_{11}k_1 + C_{12}k_2)}{\partial x} w = 0;$$

$$C_{66}\frac{\partial^2 v}{\partial x^2} + \frac{\partial C_{66}}{\partial x}\frac{\partial v}{\partial x} + C_{22}\frac{\partial^2 v}{\partial y^2} + \frac{\partial C_{22}}{\partial y}\frac{\partial v}{\partial y} +$$

$$(C_{12}+C_{66})\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial C_{12}}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial C_{66}}{\partial x}\frac{\partial u}{\partial y} +$$

$$(C_{12}k_1 + C_{22}k_2)\frac{\partial w}{\partial y} + \frac{\partial (C_{12}k_1 + C_{22}k_2)}{\partial y} w = 0; \quad (4)$$

$$D_{11}\frac{\partial^4 w}{\partial x^4} + D_{22}\frac{\partial^4 w}{\partial y^4} + 2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} +$$

$$2\frac{\partial D_{11}}{\partial x}\frac{\partial^3 w}{\partial x^3} + 2\frac{\partial D_{22}}{\partial y}\frac{\partial^3 w}{\partial y^3} + 2\frac{\partial}{\partial y}\left(D_{12}+2D_{66}\right)\frac{\partial^3 w}{\partial x^2 \partial y} +$$

$$2\frac{\partial}{\partial x}\left(D_{12}+2D_{66}\right)\frac{\partial^3 w}{\partial x \partial y^2} + \left(\frac{\partial^2 D_{11}}{\partial x^2} + \frac{\partial^2 D_{12}}{\partial y^2}\right)\frac{\partial^2 w}{\partial x^2} +$$

$$\left(\frac{\partial^2 D_{12}}{\partial x^2} + \frac{\partial^2 D_{22}}{\partial y^2}\right)\frac{\partial^2 w}{\partial y^2} + 4\frac{\partial^2 D_{66}}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} +$$

$$(C_{11}k_1^2 + 2C_{12}k_1k_2 + C_{22}k_2^2)w +$$

$$(C_{11}k_1 + C_{12}k_2)\frac{\partial u}{\partial x} + (C_{12}k_1 + C_{22}k_2)\frac{\partial v}{\partial y} + \rho h\frac{\partial^2 w}{\partial t^2} = 0.$$

It is assumed that all points of the plate vibrate harmonically with a frequency ω, i.e. $u(x,y,t) = \tilde{u}(x,y)e^{i\omega t}$, $v(x,y,t) = \tilde{v}(x,y)e^{i\omega t}$, $w(x,y,t) = \tilde{w}(x,y)e^{i\omega t}$ (the symbol "~" is omitted hereafter).

Finally we obtain

$$\frac{\partial^2 u}{\partial x^2} = a_1\frac{\partial u}{\partial x} + a_2\frac{\partial^2 u}{\partial y^2} + a_3\frac{\partial u}{\partial y} + a_4\frac{\partial^2 v}{\partial x \partial y} +$$

$$a_5\frac{\partial v}{\partial x} + a_6\frac{\partial v}{\partial y} + a_7\frac{\partial w}{\partial x} + a_8 w;$$

$$\frac{\partial^2 v}{\partial x^2} = b_1\frac{\partial v}{\partial x} + b_2\frac{\partial^2 v}{\partial y^2} + b_3\frac{\partial v}{\partial y} + b_4\frac{\partial^2 u}{\partial x \partial y} +$$

$$b_5\frac{\partial u}{\partial x} + b_6\frac{\partial u}{\partial y} + b_7\frac{\partial w}{\partial y} + b_8 w; \quad (5)$$

$$\frac{\partial^4 w}{\partial x^4} = c_1\frac{\partial^3 w}{\partial x^3} + c_2\frac{\partial^4 w}{\partial x^2 \partial y^2} + c_3\frac{\partial^3 w}{\partial x^2 \partial y} + c_4\frac{\partial^2 w}{\partial x^2} +$$

$$c_5\frac{\partial^3 w}{\partial x \partial y^2} + c_6\frac{\partial^2 w}{\partial x \partial y} + c_7\frac{\partial^4 w}{\partial y^4} + c_8\frac{\partial^3 w}{\partial y^3} +$$

$$c_9\frac{\partial^2 w}{\partial y^2} + c_{10}w + c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y},$$

where

$a_m = a_m(x,y)$, $b_m = b_m(x,y)$,

$m = 1,\ldots,8$; $c_n = c_n(x,y)$, $n = 1, \ldots, 9, 11, 12$,

$c_{10} = c_{10}(x,y,\omega)$.

Boundary conditions for displacements are specified on the boundaries $x = 0, a$ and $y = 0, b$.
Clamped boundary at $y = const$:

$$u = v = w = \frac{\partial w}{\partial y} = 0 \text{ at } y = 0, y = b; \quad (6)$$

hinged boundary:

$$u = \frac{\partial v}{\partial y} = w = \frac{\partial^2 w}{\partial y^2} = 0 \text{ at } y = 0, y = b; \quad (7)$$

one boundary hinged and the other clamped:

$$u = \frac{\partial v}{\partial y} = w = \frac{\partial^2 w}{\partial y^2} = 0 \text{ at } y = 0,$$

$$u = v = w = \frac{\partial w}{\partial y} = 0 \text{ at } y = b. \quad (8)$$

Similar conditions can also be prescribed on the boundaries $x = const$ (replacing y by x and v by u in Eqs (6–8)).

2.2. Stress-strain state of shallow shells in refined formulation

The equilibrium equations of refined Timoshenko-Mindlin type shell theory (Григоренко и др. 1987) are

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0; \; \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0;$$

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}-k_1N_x-k_2N_y+q=0; \quad (9)$$

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{yx}}{\partial y}-Q_x=0,\quad \frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x}-Q_y=0,$$

$$N_{xy}-k_2M_{yx}-N_{yx}+k_1M_{xy}=0,$$

where N_x, N_y, N_{xy}, and N_{yx} are the tangential forces; Q_x and Q_y are the shearing forces; M_x, M_y, M_{xy}, and M_{yx} are the bending and twisting moments.

The elastic relations for orthotropic shells symmetric across the thickness about the chosen coordinate surface are

$$N_x=C_{11}\varepsilon_x+C_{12}\varepsilon_y;\quad N_y=C_{12}\varepsilon_x+C_{22}\varepsilon_y;$$

$$N_{xy}=C_{66}\varepsilon_{xy}+2k_2D_{66}\kappa_{xy};\quad N_{yx}=C_{66}\varepsilon_{xy}+2k_1D_{66}\kappa_{xy};$$

$$M_x=D_{11}\kappa_x+D_{12}\kappa_y;\quad M_y=D_{12}\kappa_x+D_{22}\kappa_y; \quad (10)$$

$$M_{yx}=M_{xy}=2D_{66}\kappa_{xy};\quad Q_x=K_1\gamma;\quad Q_y=K_2\gamma_y,$$

where

$$\varepsilon_x=\frac{\partial u}{\partial x}+k_1w;\quad \varepsilon_y=\frac{\partial v}{\partial y}+k_2w;\quad \varepsilon_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x};$$

$$\kappa_x=\frac{\partial \psi_x}{\partial x}-k_1^2w;\quad \kappa_y=\frac{\partial \psi_y}{\partial y}-k_2^2w;$$

$$2\kappa_{xy}=\frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x};$$

$$\gamma_x=\psi_x-\vartheta_x;\quad \gamma_y=\psi_y-\vartheta_y; \quad (11)$$

$$\vartheta_x=-\frac{\partial w}{\partial x}+k_1u;\quad \vartheta_y=-\frac{\partial w}{\partial y}+k_2v;$$

$$K_1=\frac{5}{6}hG_{xz};\quad K_2=\frac{5}{6}hG_{yz},$$

where $\varepsilon_x,\varepsilon_y,\varepsilon_{xy}$ are the tangential strains of the coordinate surface; $\kappa_x,\kappa_y,\kappa_{xy}$ – the flexural strains of the coordinate surface; ϑ_x, ϑ_y – the angles of rotation of the normal regardless of transverse shear; γ_x,γ_y are – angles of rotation of the normal due to transverse shear; ψ_x, ψ_y – the complete angles of rotation of the rectilinear element.

From (9) – (11) we obtain

$$\frac{\partial^2 u}{\partial x^2}=a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}+a_{13}\frac{\partial^2 u}{\partial y^2}+a_{14}\frac{\partial v}{\partial x}+$$

$$a_{15}\frac{\partial v}{\partial y}+a_{16}\frac{\partial^2 v}{\partial x\partial y}+a_{17}w+a_{18}\frac{\partial w}{\partial x}+$$

$$a_{19}\frac{\partial \psi_x}{\partial y}+a_{1,10}\frac{\partial^2 \psi_x}{\partial y^2}+a_{1,11}\frac{\partial \psi_y}{\partial x}+a_{1,12}\frac{\partial^2 \psi_y}{\partial x\partial y},$$

$$\frac{\partial^2 v}{\partial x^2}=a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}+a_{23}\frac{\partial^2 u}{\partial x\partial y}+a_{24}v+a_{25}\frac{\partial v}{\partial x}+$$

$$a_{26}\frac{\partial v}{\partial y}+a_{27}\frac{\partial^2 v}{\partial y^2}+a_{28}w+a_{29}\frac{\partial w}{\partial y}+a_{2,10}\frac{\partial \psi_x}{\partial x}+ \quad (12)$$

$$a_{2,11}\frac{\partial^2 \psi_x}{\partial x\partial y}+a_{2,12}\psi_y+a_{2,13}\frac{\partial \psi_y}{\partial y}+a_{2,14}\frac{\partial^2 \psi_y}{\partial y^2},$$

$$\frac{\partial^2 w}{\partial x^2}=a_{31}u+a_{32}\frac{\partial u}{\partial x}+a_{33}v+a_{34}\frac{\partial v}{\partial y}+a_{35}w+$$

$$a_{37}\frac{\partial w}{\partial y}+a_{36}\frac{\partial w}{\partial x}+a_{38}\frac{\partial^2 w}{\partial y^2}+a_{39}\psi_x+$$

$$a_{3,10}\frac{\partial \psi_x}{\partial x}+a_{3,11}\psi_y+a_{3,12}\frac{\partial \psi_y}{\partial y}+a_{3,13}q,$$

$$\frac{\partial^2 \psi_x}{\partial x^2}=a_{41}u+a_{42}w+a_{43}\frac{\partial w}{\partial x}+a_{44}\psi_x+a_{45}\frac{\partial \psi_x}{\partial x}+$$

$$a_{46}\frac{\partial \psi_x}{\partial y}+a_{47}\frac{\partial^2 \psi_x}{\partial y^2}+a_{48}\frac{\partial \psi_y}{\partial x}+a_{49}\frac{\partial \psi_y}{\partial y}+a_{4,10}\frac{\partial^2 \psi_y}{\partial x\partial y},$$

$$\frac{\partial^2 \psi_y}{\partial x^2}=a_{51}v+a_{52}w+a_{53}\frac{\partial w}{\partial y}+a_{54}\frac{\partial \psi_x}{\partial x}+a_{55}\frac{\partial \psi_x}{\partial y}+$$

$$a_{56}\frac{\partial^2 \psi_x}{\partial x\partial y}+a_{57}\psi_y+a_{58}\frac{\partial \psi_y}{\partial x}+a_{59}\frac{\partial \psi_y}{\partial y}+a_{5,10}\frac{\partial^2 \psi_y}{\partial y^2}.$$

Clamped boundary at $y = const$:

$$u=v=w=0,\quad \psi_x=\psi_y=0;\ \text{at}\ y=0,\quad y=b; \quad (13)$$

hinged boundary:

$$u=w=0,\quad \frac{\partial v}{\partial y}=0,\quad \psi_x=0,\quad \frac{\partial \psi_y}{\partial y}=0;$$

$$\text{at}\ y=0,\quad y=b. \quad (14)$$

3. Method of solution

3.1. Free vibrations of shells

The solution of the system of equations (5) is sought in the form

$$u=\sum_{i=0}^{N}u_i(x)\varphi_i(y),\quad v=\sum_{i=0}^{N}v_i(x)\chi_i(y),$$

$$w=\sum_{i=0}^{N}w_i(x)\psi_i(y), \quad (15)$$

where $u_i(x)$, $v_i(x)$, and $w_i(x)(i = 0,..., N)$ are the unknown functions; $\varphi_i(y)$, $\chi_i(y)$ – functions constructed using cubic B-splines and $\psi_i(y)$ – functions constructed using quintic B-splines (Завьялов и др. 1980) and they are selected so as to satisfy the boundary conditions at y = const using linear combinations of cubic and quintic B-splines (Grigorenko and Kryukov 1995).

Substituting (15) into Eqs (5), we require that they be satisfied at prescribed collocation points $\xi_k\in[0,b]$, $k=0,...,N$. If the mesh has an even number of nodes ($N=2n+1$) and the collocation points are such that $\xi_{2i}\in[y_{2i},y_{2i+1}]$, $\xi_{2i+1}\in[y_{2i},y_{2i+1}]$, $(i=\overline{0..n})$, then the interval $[y_{2i}, y_{2i+1}]$ has 2 collocation points, and the adjacent intervals $[y_{2i+1}, y_{2i+2}]$ do not have such points. Within each of the intervals $[y_{2i}, y_{2i+1}]$, collocation points are selected as follows:

$\xi_{2i} = x_{2i} + z_1 h_y, \quad \xi_{2i+1} = y_{2i} + z_2 h_y \quad (i = 0,\ 1,\ 2,\ ...,\ n),$

where $z_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}$; $z_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}$ are the roots of a quadratic Legendre polynomial on the interval [0, 1]. Such collocation points are optimal and substantially increase the accuracy of approximation. As a result, we obtain a system of $3(N + 1)$ linear differential equations for u_i, v_i, and w_i. With the notation

$\mathbf{\Phi}_l = [\varphi_i^{(l)}(\xi_k)]$, $\mathbf{X}_l = [\chi_i^{(l)}(\xi_k)]$, $\mathbf{\Psi}_m = [\psi_i^{(m)}(\xi_k)]$,

$i,k = 0,...,N$, $l = 0,...,2$, $m = 0,...,4$;

$\bar{u}^T = \{u_0,...,u_N\}$, $\bar{v}^T = \{v_0,...,v_N\}$, $\bar{w}^T = \{w_0,...,w_N\}$;

$\bar{a}_r^T = \{a_r(x,\xi_0),...,a_r(x,\xi_N)\}$,

$\bar{b}_r^T = \{b_r(x,\xi_0),...,b_r(x,\xi_N)\}$, $r = 1,...,8$;

$\bar{c}_p^T = \{c_p(x,\xi_0),...,c_p(x,\xi_N)\}$, $p = 1,...,9,11,12$,

$\bar{c}_{10}^T = \{c_{10}(x,\xi_0,\omega),...,c_{10}(x,\xi_N,\omega)\}$,

and the notation $\bar{c}\cdot\mathbf{A}$ of the matrix $[c_i a_{ij}]$, where $\mathbf{A} = [a_{ij}]$, ($i, j = 0,...,N$) is a matrix and $\bar{c}^T = \{c_0,...,c_N\}$ is a vector, the system of differential equations becomes

$$\bar{u}'' = \mathbf{\Phi}_0^{-1}\{(\bar{a}_1\cdot\mathbf{\Phi}_0)\bar{u}' + (\bar{a}_2\cdot\mathbf{\Phi}_2 + \bar{a}_3\cdot\mathbf{\Phi}_1)\bar{u} + (\bar{a}_4\cdot\mathbf{X}_1 + \bar{a}_5\cdot\mathbf{X}_0)\bar{v}' + +(\bar{a}_6\cdot\mathbf{X}_1)\bar{v} + (\bar{a}_7\cdot\mathbf{\Psi}_0)\bar{w}' + (\bar{a}_8\cdot\mathbf{\Psi}_0)\bar{w}\},$$

$$\bar{v}'' = \mathbf{X}_0^{-1}\{(\bar{b}_1\cdot\mathbf{X}_0)\bar{v}' + (\bar{b}_2\cdot\mathbf{X}_2 + \bar{b}_3\cdot\mathbf{X}_1)\bar{v} + (\bar{b}_4\cdot\mathbf{\Phi}_1 + \bar{b}_5\cdot\mathbf{\Phi}_0)\bar{u}' + (\bar{b}_6\cdot\mathbf{\Phi}_1)\bar{u} + (\bar{b}_7\cdot\mathbf{\Psi}_1)\bar{w} + (\bar{b}_8\cdot\mathbf{\Psi}_0)\bar{w}\}. \quad (16)$$

$$\bar{w}^{IV} = \mathbf{\Psi}_0^{-1}\{(\bar{c}_1\cdot\mathbf{\Psi}_0)\bar{w}''' + (\bar{c}_2\cdot\mathbf{\Psi}_2 + \bar{c}_3\cdot\mathbf{\Psi}_1 + \bar{c}_4\cdot\mathbf{\Psi}_0)\bar{w}'' + (\bar{c}_5\cdot\mathbf{\Psi}_2 + \bar{c}_6\cdot\mathbf{\Psi}_1)\bar{w}' + (\bar{c}_7\cdot\mathbf{\Psi}_4 + \bar{c}_8\cdot\mathbf{\Psi}_3 + \bar{c}_9\cdot\mathbf{\Psi}_2 + \bar{c}_{10}\cdot\mathbf{\Psi}_0) + (\bar{c}_{11}\cdot\mathbf{\Phi}_0)\bar{u}' + (\bar{c}_{12}\cdot\mathbf{X}_1)\bar{v}.$$

This system can be normalized:

$$\frac{d\bar{Y}}{dx} = \mathbf{A}(x,\omega)\bar{Y} \quad (0 \le x \le a) \quad (17)$$

$\bar{Y}^T = \{u_0,u_1,...,u_N,u_0',u_1',...,u_N',\ v_0,v_1,...,v_N,\ v_0',v_1',...,v_N',\ w_0,w_1,...,w_N,\ w_0',w_1',...,w_N',\ w_0'',w_1'',...,w_N'',w_0''',w_1''',...,w_N'''\}$;

$A(x,\omega)$ is a square matrix of order $8(N+1)\times 8(N+1)$. The boundary conditions for this system can be expressed as

$$\mathbf{B}_1\bar{Y}(0) = \bar{0}, \quad \mathbf{B}_2\bar{Y}(a) = \bar{0}. \quad (18)$$

To solve the eigenvalue problem for the system of ordinary differential equations (17) with the boundary conditions (18), we will combine discrete orthogonalization with incremental search (Григоренко и др. 1986).

3.2. Stress-strain state of shells

The solution of boundary-value problem (12)–(14) can be represented as

$$u = \sum_{i=o}^{N} u_i(x)\varphi_{1i}(y),\quad v = \sum_{i=o}^{N} v_i(x)\varphi_{2i}(y),$$
$$w = \sum_{i=o}^{N} w_i(x)\varphi_{3i}(y),\quad \psi_x = \sum_{i=o}^{N} \psi_{xi}(x)\varphi_{4i}(y), \quad (19)$$
$$\psi_y(x,y) = \sum_{i=o}^{N} \psi_{yi}(x)\varphi_{5i}(y),$$

where $u_i, v_i, w_i, \psi_{xi}, \psi_{yi}$ are the searched functions of the variable x $\varphi_{ji}(y)$ ($j = \overline{1,5}$; $i = 0,1,...,N$) are the linear combinations of B-splines third power.

If a resolving function is equal to zero, then

$\varphi_{j0}(y) = -4B_3^{-1}(y) + B_3^0(y)$;

$\varphi_{j1}(y) = B_3^{-1}(y) - \frac{1}{2}B_3^0(y) + B_3^1(y)$;

$\varphi_{ji}(y) = B_3^i(y)$ $(i = 2,\ 3,...,N-2)$.

If the derivative of a resolving function with respect to s is equal to zero, then

$\varphi_{j0}(y) = B_3^0(y)$; $\varphi_{j1}(y) = B_3^{-1}(y) - \frac{1}{2}B_3^0(y) + B_3^1(y)$;

$\varphi_{ji}(y) = B_3^i(y)$ $(i = 2,\ 3,...,N-2)$.

The functions $\varphi_{j,N-1}(y)$ and $\varphi_{j,N}(y)$ can be represented similarly.

Substituting (19) into Eq. (12) and boundary conditions (14), we require that they be satisfied at prescribed collocation points. We obtain a dimensional boundary problem that can be solved by the discrete orthogonalization method. The full solving technique is described in 3.1.

The results of calculation are presented for square in plane isotropic shell displacements wE/q at all hinged boundaries in cross-section $y = a/2$ (Table). The parameters of shell are a = 10, h = 0.4, k_1 = 0.05, k_2 =0, ν = 0.3, q = const.

$\frac{x}{a}$	Spline-approximation method			Fourier series solution
	$N = 9$	$N = 17$	$N = 21$	
0.1	1098.8	1117.3	1119.3	1121.0
0.2	2016.7	2052.9	2056.6	2060.0
0.3	2679	2730	2735.4	2740.0
0.4	3070.9	3132.3	3138.6	3144.3
0.5	3199.8	3264.9	3271.7	3277.7

The results were obtained by spline-approximation and Fourier series methods. As follows from Table, the solution approximate to the exact one with increase in quantity of collocation points. It can be reliability criterion of the technique proposed.

4. Numerical results

4.1. Studying the natural vibrations of shells basing on the Mushtari–Donnell–Vlasov's theory

We will use the proposed approach to study the spectrum of natural vibrations of a square shallow shell with varying thickness and different boundary conditions. The thickness of the plate varies by the formula

$$h(x) = h_0\left[\alpha(6\frac{x^2}{a^2} - 6\frac{x}{a} + 1) + 1\right]. \quad (20)$$

The material of the shell is orthotropic (Лехницкий 1957) with Young's moduli $E_1 = 4.76 \cdot 10^4$ MPa, $E_2 = 2.07 \cdot 10^4$ MPa, shear moduli $G_{12} = 0.531 \cdot 10^4$ MPa, $G_{13} = 0.501 \cdot 10^4$ MPa, $G_{23} = 0.434 \cdot 10^4$ MPa and Poisson's ratios $\nu_1 = 0.149$, $\nu_2 = 0.0647$, $1/k_1 = 1/k_2$ = 12.5; 3.125; 1.5625 (there $1/k_1$ and $1/k_2$ are dimensionless radiuses of curvatures)

The following boundary conditions were used:

– the entire boundary is clamped (A);

– two adjacent sides are clamped and the other sides are hinged (B).

Figs 2–4 show the dimensionless natural frequencies of the shell $\bar{\omega}_i = \omega_i a^2 \sqrt{\rho h_0 / D_{11}}$ as a function of the parameter α for A (solid line) and B (dashed line) boundary conditions.

From Figs 2–3 follows, that the first frequencies of orthotropic shells of a variable thickness at the big radiuses of curvature increase, and the second frequencies decrease practically linearly at increasing α. Under boundary conditions B, first two frequencies increase with increasing α. At the further reduction of the main radiuses of curvature the first frequencies decrease, and for the second both increasing and decreasing under certain boundary conditions is possible with increasing α (Fig. 4). The higher frequencies, basically, increase nonlinearly, though their decreasing is possible also since some value α. Such behaviour of frequencies is caused by simultaneous influence both of variable thickness and the orthotropy of material.

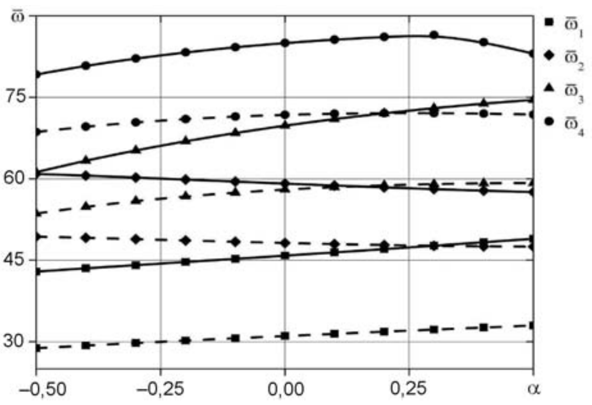

Fig. 2. Dimensionless frequencies $\bar{\omega}$ of vibrations of shallow shells with different boundary conditions as the functions of the parameter α ($1/k_1 = 1/k_2 = 12.5$)

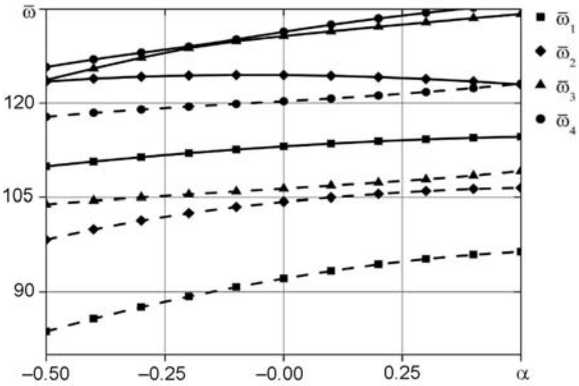

Fig. 3. Dimensionless frequencies $\bar{\omega}$ of vibrations of shallow shells with different boundary conditions as the functions of the parameter α ($1/k_1=1/k_2 = 3.125$)

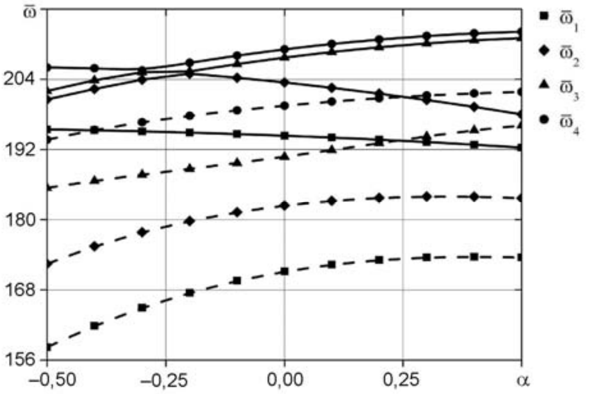

Fig. 4. Dimensionless frequencies $\bar{\omega}$ of vibrations of shallow shells with different boundary conditions as the functions of the parameter α ($1/k_1=1/k_2 = 1.5625$)

4.2. Stressed state of non-thin shallow shells basing on the Timoshenko-Mindlin's theory

Let us analyze, as an example, the stress–strain state a doubly curved isotropic shallow shell with square planform and varying thickness under uniform normal pressure $q = q_0$ = const. The thickness of the shell (Fig. 1) varies by (20). The input data: $a = b = 10$, $k_1 = 1/10$, $k_2 = 0$, $h_0 = 1$, $\alpha = -0.4, -0.2, 0, 0.2, 0.4$, Figs 5–7 show the thickness dependence of the displacements and stresses in the section $y = a/2$ on the lateral surfaces of the shell clamped at 3 edges and hinged at one edge. It can be seen that w, σ_x^+, and σ_x^- are distributed asymmetrically.

Fig. 5 demonstrates that the maximum displacement is slightly shifted from the point of the rise toward the hinged edge, the maximum increasing with α. As the thickness increases in this zone, the deflection decreases insignificantly. Fig. 6 shows how the stress on the outside surface depends on the thickness. It can be seen that the maximum of σ_x^+ is shifted from the point of therise toward the hinged edge and increases with α.

Fig. 7 shows the stress distribution on the inside surface. The stress patterns on the inside and outside surfaces of the shell are qualitatively close and differ by sign. Quantitatively, the maximum stresses σ_x^- are almost twice as great as σ_x^+.

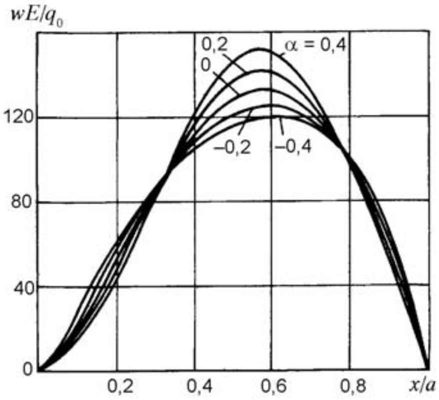

Fig. 5. Distribution of displacements depend on parameter α

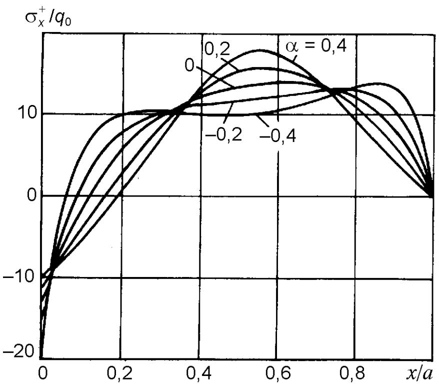

Fig. 6. Distribution of stresses σ_x^+ depends on parameter α

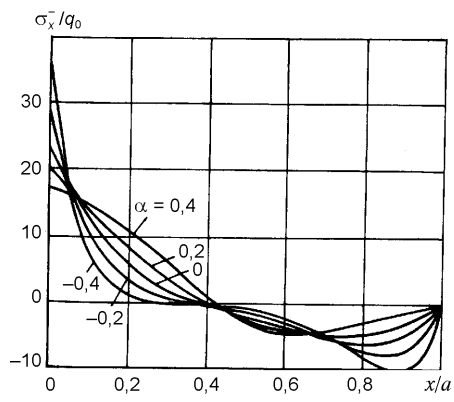

Fig. 7. Distribution of stresses σ_x^- depends on parameter α

5. Conclusions

1. The paper proposes a numerical-analytical approach to investigation of the stress-stain state and natural vibrations of orthotropic varying thickness plates and shells. The approach includes 2 stages. At the first stage an initial eigen-value or boundary problem for the systems of partial differential equations is reduced to the eigen-value (boundary) problem for the system of high-order ordinary differential equations by representing the desired solution in the form of segment of series in spline-collocations and choosing collocation points in the domain under consideration. The obtained one-dimensional eigen-value (boundary) problems are solved by the stable numerical method of discrete-ortogonalization in combination with the step-by-step search method what provides highly accurate solution.

2. The applied problems for natural vibrations (Mushtari–Donnell–Vlasov's theory) and stress-strain state (Timoshenko-Mindlin's theory.) of shallow shells with varying thickness under different boundary conditions are solved.

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LĖKŠTŲJŲ ANIZOTROPINIŲ NEHOMOGENINIŲ KEVALŲ STATINĖS IR DINAMINĖS ELGSENOS TYRIMAI,TAIKANT SPLAINŲ APROKSIMACIJOS METODĄ

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Santrauka

Nagrinėjamas efektyvus lėkštųjų kevalų įtempių, deformacijų ir laisvųjų svyravimų nustatymo algoritmas, pagrįstas klasikiniais ir tobulesniais skaičiavimo modeliais. Algoritme dalinių diferencialinių lygčių sistema yra transformuojama į vienmatį uždavinį, sprendžiamą naudojant pagrindinių splainų aproksimaciją į vieną koordinatę. Gaunamas kraštinis uždavinys, kuris sprendžiamas kaip kraštinis savųjų reikšmių nustatymo uždavinys. Uždavinio sąlygas atitinka įprastų aukštesnės eilės diferencialinių lygčių sistema, kuriai spręsti taikomi patikimi diskrečiosios ortogonalizacijos skaitiniai metodai.

Reikšminiai žodžiai: lėkštieji kevalai, kintamasis storis, modifikuotoji ir klasikinė formuluotės, įtempių ir deformacijų būvis, laisvieji svyravimai.

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