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ORIGINAL ARTICLE

On the Hausdorff distance between the shifted Heaviside step function and the transmuted Stannard growth function

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Dedicated to Professor Roumen Anguelov on the occasion of his 60th anniversary

***Abstract*—In this paper we study the one-sided Hausdorff distance between the shifted Heaviside step–function and the transmuted Stannard growth function. Precise upper and lower bounds for the Hausdorff distance have been obtained. We present a software module (intellectual property) within the programming environment *CAS Mathematica* for the analysis of the growth curves. Numerical examples, illustrating our results are given, too.**

***Keywords*-Transmuted Stannard growth function; Heaviside step function; Hausdorff distance; Upper and lower bounds.**

I. Introduction and Preliminaries

The Stannard function finds numerous applications in many scientific fields, including population dynamics, bacterial growth, population ecology, plant biology, chemistry, agriculture, demography, financial mathematics, statistics and fuzzy set theory [1]–[5].

Definition 1. For $\gamma \in \mathbb{R}$ define the shifted Heaviside step function as [12]:

$$h_\gamma(t) = \begin{cases} 0, & \text{if} \quad t < 0, \\ [0,1], & \text{if} \quad t = \gamma, \\ 1, & \text{if} \quad t > \gamma. \end{cases} \tag{1}$$

Definition 2. Define the shifted Stannard growth function $S(t)$ as [1]–[5]:

$$S(t) = \frac{1}{\left(1 + e^{\frac{-(\beta + k(t-\gamma))}{m}}\right)^m}, \tag{2}$$

where β, k and $m \in \mathbb{R}$ are the growth parameters. We note that the slope of (2) at $t = \gamma$ is equal to: $\frac{ke^{-\frac{\beta}{m}}}{\left(1+e^{-\frac{\beta}{m}}\right)^{m+1}}$.

Definition 3. A random variable T is said to have a transmuted distribution if its cumulative distribution function (*cdf*) is given by [6], [7]:

$$G_1(t) = (1+\lambda)F_1(t) - \lambda F_1^2(t), \quad |\lambda| \le 1, \tag{3}$$

where $F_1(t)$ is the *cdf* of the base distribution.

"λ - transmuting" of (*cdf*) is a familiar technique from the field of probability distributions with application to insurance mathematics.

Definition 4. The Hausdorff distance $\rho(f,g)$ between two interval functions f,g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$ [8], [9], [12].

More precisely, we have

$$\rho(f,g) = \max\{ \sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \qquad (4)$$

$$\sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\},$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A=(t_A,x_A)$, $B=(t_B,x_B)$ in $\mathbb{R}^2$ is $||A-B|| = \max(|t_A-t_B|,|x_A-x_B|)$.

Sigmoidal growth curves typically have three parts (phases, time intervals): lag, log and stationary parts. It is a challenging question to characterize mathematically these phases. The lag time (interval) is practically important in many medical and biotechnological applications as this time is responsible for the acceleration or inhibition of the process and the possibility of controlling the lag time depends on the understanding of the hidden mechanisms of the corresponding process [10], [11].

Usually the lag time is defined by means of the uniform distance between the sigmoidal function and the induced cut function. We propose a new definition for the lag time by means of the Hausdorff distance between the sigmoidal function and the induced step function.

In this work we prove estimates for the one–sided Hausdorff approximation of the shifted Heaviside step–function by transmuted Stannard growth function.

Let us point out that the Hausdorff distance is a natural measuring criteria for the approximation of bounded discontinuous functions [12], [13].

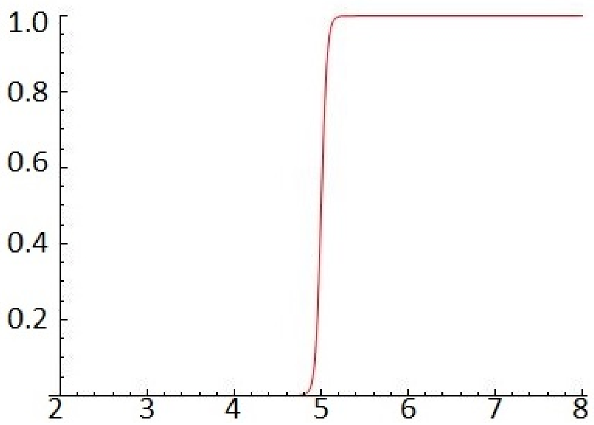

Fig. 1. Approximation of the shifted Heaviside step function by transmuted Stannard growth function for the following data: $k = 16$, $m = 0.52$, $\beta = 0.01$, $t_r = 5$; Hausdorff distance $d = 0.0801797$.

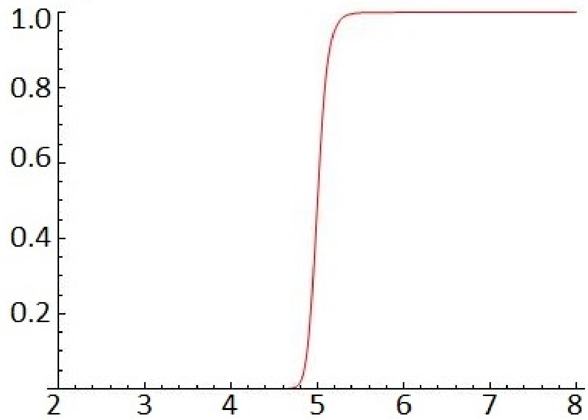

Fig. 2. Approximation of the shifted Heaviside step function by transmuted Stannard growth function for the following data: $k = 26$, $m = 2.1$, $\beta = 1$, $t_r = 5$; Hausdorff distance $d = 0.112237$.

II. Main Results

For $\gamma,\beta,m \in \mathbb{R}$ consider the following transmuted Stannard function

$$S^*(t) = \frac{1+\lambda}{\left(1+e^{\frac{-(\beta+k(t-\gamma))}{m}}\right)^m} - \qquad (5)$$

$$\frac{\lambda}{\left(1+e^{\frac{-(\beta+k(t-\gamma))}{m}}\right)^{2m}}, \quad |\lambda| \le 1.$$

Function $S^*(t)$ from (5) satisfies:

$$S^*(\gamma) = \frac{1+\lambda}{\left(1+e^{-\frac{\beta}{m}}\right)^m} - \frac{\lambda}{\left(1+e^{-\frac{\beta}{m}}\right)^{2m}} = \frac{1}{2} \tag{6}$$

hence

$$\lambda = \frac{0.5(1+z)^{2m} - (1+z)^m}{(1+z)^m - 1}; \quad z = e^{-\frac{\beta}{m}}. \tag{7}$$

We study the Hausdorff approximation d of the Heaviside step function $h_\gamma(t)$ by the transmuted Stannard function (5)–(7) and look for an expression for the error of the best one–sided approximation.

Let

$$\begin{aligned} A &= (1+\lambda)\left(1+e^{-\frac{\beta}{m}}\right)^{-m} - \lambda\left(1+e^{-\frac{\beta}{m}}\right)^{-2m} \\ B &= 1 - 2e^{-\frac{\beta}{m}}\left(1+e^{-\frac{\beta}{m}}\right)^{-1-2m} k\lambda \\ &\quad + e^{-\frac{\beta}{m}}\left(1+e^{-\frac{\beta}{m}}\right)^{-1-m} k(1+\lambda), \quad k \in \mathbb{R}. \end{aligned} \tag{8}$$

The following Theorem gives upper and lower bounds for d.

Theorem 2.1 *For the Hausdorff distance d between the function $h_\gamma(t)$ and the transmuted Stannard function (5)–(7) the following inequalities hold for $|\lambda| \leq 1$ and $B > 4$:*

$$d_l = \frac{A}{2B} < d < A\frac{\ln(2B)}{2B} = d_r. \tag{9}$$

Proof. We need to express d in terms of k, β and m. The Hausdorff distance d satisfies the relation

$$F(d) := S^*(\gamma - d) = \frac{1+\lambda}{\left(1+e^{-\frac{\beta-kd}{m}}\right)^m} - \frac{\lambda}{\left(1+e^{-\frac{\beta-kd}{m}}\right)^{2m}} - d = 0. \tag{10}$$

Consider the function

$$G(d) = A - Bd.$$

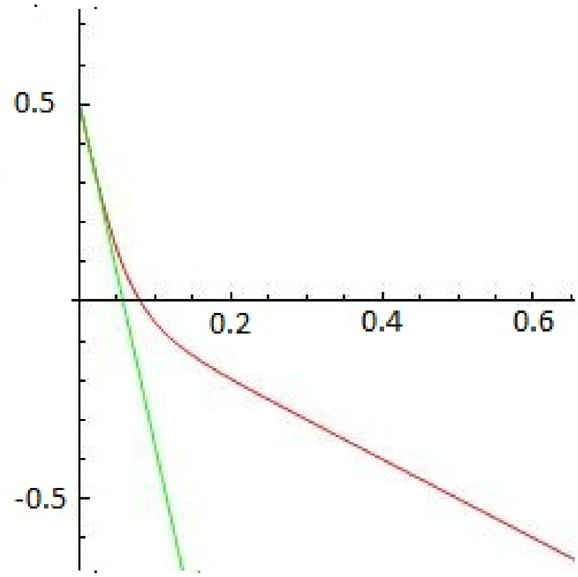

Fig. 3. The functions $F(d)$ and $G(d)$ for $k = 16$, $m = 0.52$, $\beta = 0.01$, $t_r = 5$.

By means of Taylor expansion we obtain

$$G(d) - F(d) = O(d^2).$$

Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 3). Further, for $|\lambda| \leq 1$ and $B > 4$ we have

$$\begin{aligned} G(d_l) &= \frac{A}{2} > 0, \\ G(d_r) &= A\left(1 - 0.5\ln(2B)\right) < 0. \end{aligned}$$

This completes the proof of the theorem.

Some computational examples using relations (9) are presented in Table 1. The last column of Table 1 contains the values of d computed by solving the nonlinear equation (10).

β	k	m	λ	γ	d_l	d_r	d from(10)
1	26	2.1	0.594719	5	0.0480564	0.112559	0.112237
0.1	24	1.3	0.300329	5	0.0407241	0.102127	0.10138
0.01	16	0.52	−0.957987	5	0.0287694	0.0821452	0.0801797
0.63	21	0.875	−0.997848	5	0.0317872	0.0875911	0.0793406
0.1	28	0.7	−0.63833	5	0.0228017	0.0704065	0.0692928

TABLE I
BOUNDS FOR d COMPUTED BY EQUATION (9) FOR VARIOUS β, k, m.

```
Clear[β];
Clear[k];
Clear[m];
Clear[γ];
k = Input["  k"]; (*16 *)
Print["  k = ", k];
m = Input["  m"]; (*0.52 *)
Print["  m = ", m];
β = Input["  β"]; (*0.01 *)
Print["  β = ", β];
γ = Input[" γ "]; (*5 *)
Print[" γ = ", γ];
λ = (0.5 * (1 + Exp[-β / m]) ^ {2 * m} - (1 + Exp[-β / m]) ^ m) / ((1 + Exp[-β / m]) ^ {m} - 1)
Print["  λ = ", λ];
```

$$A = -\left(1 + e^{-\frac{\beta}{m}}\right)^{-2m} \lambda + \left(1 + e^{-\frac{\beta}{m}}\right)^{-m} (1 + \lambda);$$

$$B = 1 - 2 e^{-\frac{\beta}{m}} \left(1 + e^{-\frac{\beta}{m}}\right)^{-1-2m} k \lambda + e^{-\frac{\beta}{m}} \left(1 + e^{-\frac{\beta}{m}}\right)^{-1-m} k (1 + \lambda);$$

```
dl = A / (2 * B)
Print["  The lower bound dl = ", dl]
dr = A * Log[2 * B] / (2 * B)
Print["  The upper bound dr = ", dr]
Print["The following nonlinear equation is used to determination of the one-sided Hausdorff distance between
shifted Heaviside step function and transmuted Stannard growth curve - d: "];
Print["(1+λ)*1/(1+Exp[-(β-k*d)/m])^m-λ*(1/(1+Exp[-(β-k*d)/m])^m)^2 -d=0"];
FindRoot[(1 + λ) * 1 / (1 + Exp[-(β - k * d) / m]) ^ m - λ * (1 / (1 + Exp[-(β - k * d) / m]) ^ m) ^ 2 - d, {d, 0}]
g1 = Plot[(1 + λ) * 1 / (1 + Exp[-(β - k * d) / m]) ^ m - λ * (1 / (1 + Exp[-(β - k * d) / m]) ^ m) ^ 2 - d, {d, 0, 1}, PlotRange → {-1, 1},
   PlotStyle → {Red}, AspectRatio → 1];
g2 =
```

$$\mathrm{Plot}\Bigg[\frac{\left(1 + e^{-\frac{\beta}{m}}\right)^{-m} \left(-0.5` \left(1 + e^{-\frac{\beta}{m}}\right)^{m} + 0.5` \left(1 + e^{-\frac{\beta}{m}}\right)^{2m}\right)}{-1.` + \left(1 + e^{-\frac{\beta}{m}}\right)^{m}} +$$

$$\frac{\left(1 + e^{-\frac{\beta}{m}}\right)^{-m} \left(\left(1 + e^{-\frac{\beta}{m}}\right)^{m} + e^{\frac{\beta}{m}} \left(1 + e^{-\frac{\beta}{m}}\right)^{m} - 1.` \left(1 + e^{-\frac{\beta}{m}}\right)^{2m} - 1.` e^{\frac{\beta}{m}} \left(1 + e^{-\frac{\beta}{m}}\right)^{2m} - 1.` k + 1.` \left(1 + e^{-\frac{\beta}{m}}\right)^{m} k - 0.5` \left(1 + e^{-\frac{\beta}{m}}\right)^{2m} k\right) d}{\left(1.` + e^{\frac{\beta}{m}}\right) \left(-1.` + \left(1 + e^{-\frac{\beta}{m}}\right)^{m}\right)},$$

```
   {d, 0, 1}, PlotRange → {-1, 1}, AspectRatio → 0.75, PlotStyle → {Green}];
Show[g1, g2]
Plot[(1 + λ) * 1 / (1 + Exp[-(β + k * (t - γ)) / m]) ^ m - λ * (1 / (1 + Exp[-(β + k * (t - γ)) / m]) ^ m) ^ 2, {t, 2, 8}, PlotRange → {0, 1},
 PlotStyle → {Red}, AspectRatio → 0.7]
```

Fig. 4. A simple module implemented in CAS Mathematica for the computation and visualization of the Hausdorff distance between the Heaviside step function and the transmuted Stannard growth function.

III. Conclusion remarks

New estimates for the Hausdorff distance between an interval Heviside step function and its best approximating Stannard function are obtained.

On Fig. 1 and Fig. 2 appropriate illustrations of some approximations of the shifted Heaviside step function by transmuted Stannard growth function are given.

We propose a software module within the programming environment *CAS Mathematica* for the analysis of the considered growth curves (see Fig. 4). The module offers the following possibilities:

i) generation of the shifted Stannard curve under user-defined values for k, m, β;

ii) automatic check of the condition $|\lambda| \leq 1$ that guarantees the existence of sigmoidality of the transmuted Stannard curve;

iii) software tools for animation and visualization.

The Hausdorff approximation of the interval step function by the logistic and other sigmoidal functions is discussed from various approximation, computational and modelling aspects in [14]–[27].

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