INERTIAL PROPERTIES IN GROUPS

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Dedicated to the memory of Silvia Lucido

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ABSTRACT. Let G be a group and φ be an endomorphism of G. A subgroup H of G is called φ-*inert* if $H^\varphi \cap H$ has finite index in the image H^φ. The subgroups that are φ-inert for all inner automorphisms of G are widely known and studied in the literature, under the name *inert* subgroups.

The related notion of *inertial endomorphism*, namely an endomorphism φ such that all subgroups of G are φ-inert, was introduced in [34] and thoroughly studied in [37, 35]. The "dual" notion of *fully inert subgroup*, namely a subgroup that is φ-*inert* for all endomorphisms of an abelian group A, was introduced in [51] and further studied in [25, 54, 76].

The goal of this paper is to give an overview of up-to-date known results, as well as some new ones, and show how some applications of the concept of inert subgroup fit in the same picture even if they arise in different areas of algebra. We survey on classical and recent results on groups whose inner automorphisms are inertial. Moreover, we show how inert subgroups naturally appear in the realm of locally compact topological groups or locally linearly compact topological vector spaces, and can be helpful for the computation of the algebraic entropy of continuous endomorphisms.

1. **Preliminaries**

1.1. **Introduction.** A subgroup H of a group G is *inert* if the index $|H : H^g \cap H|$ is finite for all $g \in G$. Clearly normal and finite subgroups are inert. Many generalizations of the notion of normality have appeared in the literature (see for example [71]). The concept of inert subgroup allows to put many of them in a common framework. It seems to have been introduced in 1993 in papers of Belyaev [3, 4]

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as a tool in the investigation of infinite simple groups. In a survey paper on this subject [5] Belyaev gives credit to Kegel for coining the term *inert subgroup*. The concept of inert subgroup tacitly involves inner automorphisms of a group and has been therefore extended to general automorphisms ([34]) in 2012. Then in [51] the concept has been extended to general endomorphisms of abelian groups. Let us give a basic definition which reveals to be a powerful tool. One may replace the relation of invariance of a subgroup with respect to an endomorphism by a weaker condition which is trivially satisfied by all finite subgroups.

Definition 1.1. *Let φ be an endomorphism and H a subgroup of a group G. Then H is called φ-inert if $H^\varphi \cap H$ has finite index in the image H^φ.*

A subgroup H is *inert* in a group G if and only if H is φ-inert in G for each inner automorphism φ of G. The relevant novelty in the point of view adopted in [51] is imposing the relation from Definition 1.1 between a *single pair* of a subgroup and an endomorphism, allowing one to extend it further to arbitrary pairs of a family of subgroups and a family of endomorphisms. In §1.4 we also recall the possibility to distinguish "uniformly" inert subgroups as in [9].

The concept of inert subgroup has many applications in different areas of algebra. In this paper we try to collect some of them and continue the work done by previous surveys as [5, 71, 74] even if we are conscious that we cover only a part of the subject. Many of these results have been presented at several meetings "Ischia Group Theory", an environment that stimulated many interesting and fruitful discussions. Here we also present some new results, propose problems and topics for further investigations.

This survey is organized as follows. In the rest of this section (§§1.2–1.4) we give an overview of the paper. In Section 2 we will consider groups in which all or many subgroups are inert or have a stronger property. In Section 3 we will consider groups of automorphisms φ, which have the property that each subgroup is φ-inert and φ^{-1}-inert (as in [34, 38]) or some stronger property. In Section 4 we will consider endomorphisms φ of an abelian group A for which each subgroup of A is φ-inert and the ring that they form (as in [37, 35]). In Section 5 we will dually consider fully inert subgroups, that is subgroups of A which are φ-inert with respect to all endomorphisms φ of A. There are satisfactory results for many classes of abelian groups (as in [54, 76]). Moreover, we will describe abelian groups which are inert in their divisible hull (as in [51]). We will also consider some variations of these problems concerning the relation of fully inert subgroups to fully invariant ones (as in [14, 77]).

In Section 6 we recall the algebraic entropy functions ent and h_{alg} for group endomorphisms, as well as the notion of growth of group endomorphism (as in [42, 43, 47, 66, 67]), which naturally extends the classical function of growth studied by Milnor and Gromov in geometric group theory [79]. Following [51], we show how the use of φ-inert subgroup leads to a better behaved intrinsic entropy funtion $\widetilde{\text{ent}}$ which works well also in non-torsion abelian groups. Among others, we discuss an extension of the entropy function $\widetilde{\text{ent}}$ to the case of non-abelian groups. The final part of the section offers various

examples, from the realm of topological groups or topological vector spaces, where inert subgroups appear in a natural way and are helpful, again, for the computation of the entropy.

Notation, terminology and basic facts. We recall here briefly some notation, definitions and basic facts of those used more frequently in the paper. For the sake of brevity, we omit others that can be found in the monographs [62, 83, 92].

The set of positive natural numbers will be denoted by $\mathbb{N}$. We use both multiplicative and additive notation for group operation and correspondingly right- and left-hand notation for endomorphisms. As usual, the *order* of a group G is the cardinality of the underling set and the order of an element g of G is the order of the subgroup $\langle g \rangle$ generated by g. We denote by id_G (or simply id, when no confusion is possible) the identity map of G. If φ is an endomorphism of G, we say that a subgroup H of G is φ-invariant if $\varphi(H) \subseteq H$. If Γ is a group acting on G, then we say that a subgroup H of G is Γ-invariant if $H^\gamma \leq H$ for each $\gamma \in \Gamma$. We denote $H_\Gamma := \cap_{\gamma \in \Gamma} H^\gamma$ (resp. $H^\Gamma := \langle H^\gamma | \gamma \in \Gamma \rangle$) the largest (resp. smallest) Γ-invariant subgroup of G contained in (resp. containing) H. In particular, H_G is the core of H in G and H^G is the normal closure of H in G.

We use A to denote exclusively abelian groups, while $t(A)$, A_π and $D(A)$ denote the torsion subgroup, the π-component and the maximum divisible subgroup of A, respectively. Further, if $n \in \mathbb{N}$, we denote $A[n] := \{a \in A \mid na = 0 \}$. If $nA = 0$, we say that A is *bounded* by n and the least such n is called the exponent of A. We just say that A is bounded or that A has finite exponent, if A is bounded by some n. A Prüfer (or *quasi-cyclic*) p-group is a group isomorphic to the p-component $\mathbb{Z}(p^\infty)$ of the quotient $\mathbb{Q}/\mathbb{Z}$ of the additive group of the rational numbers. If $n \in \mathbb{N}$ then $\pi(n)$, is the set of prime divisors of n. We denote $\mathbb{Q}^n := \mathbb{Q} \oplus \cdots \oplus \mathbb{Q}$ (n times). If π is a set of primes, we denote by π' the set of primes not in π and by $\mathbb{Q}^\pi$ the ring (or just the additive group) of rational numbers whose denominator is divisible by primes in π only.

Recall that the torsion free rank $r_0(A)$ of an abelian group A is the cardinality of any maximal independent subset of elements of infinite order. Then $r_0(A) < \infty$ if and only if $A/t(A)$ is isomorphic to a subgroup of $\mathbb{Q}^n$ for some $n \in \mathbb{N}$. If p is a prime, $r_p(A)$ denotes the cardinality of any maximal independent subset of elements of A with order p. It follows that $r_p(A) < \infty$ if and only if A_p is the direct sum of finitely many cyclic or quasi-cyclic groups. Furthermore, the rank of A is $r(A) := \max\{r_0(A), \sup\{r_p(A) : p \ prime\}\}$.

If $\mathfrak{P}$ and $\mathfrak{P}_1$ are properties for groups, then we say that a group G is *locally* $\mathfrak{P}$ if each finitely generated subgroup of G has $\mathfrak{P}$ and that G is $\mathfrak{P}$-*by*-$\mathfrak{P}_1$ if G has a normal subgroup N such that N has $\mathfrak{P}$ and G/N has $\mathfrak{P}_1$. A group G is said *metabelian* if its derived subgroup G' is abelian. Moreover, a group is called *locally graded* if each of its non-trivial finitely generated subgroups has a non-trivial finite image. A group is *residually* $\mathfrak{P}$ if the trivial subgroup is the intersection of the set of normal subgroups N such that G/N has $\mathfrak{P}$. Recall that the class of locally graded groups contains all locally (soluble-by-finite) groups and all locally residually finite groups.

Recall that a subgroup H of a group G is said *subnormal* in G if there is a finite series $H = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$ where each term is normal in the successive one. A *Chernikov group* is a group which

has a subgroup of finite index which is a direct product of finitely many Prüfer groups (for possibly different primes).

1.2. **Inert subgroups.** Let us introduce some further terminology.

Definition 1.2. *If H, K are subgroups of a group G we say that:*

    (1) *H is* almost contained *in K (and write $H \leq_a K$) if $|H : (K \cap H)| < \infty$;*

    (2) *H is* commensurable *with K (and write $H \sim K$) if $H \leq_a K$ and $K \leq_a H$.*

In these terms, for an endomorphism φ of G, a subgroup H is φ-inert if and only if the image H^φ is almost contained in H. The relation $\leq_a$ is reflexive and transitive; the equivalence relation that it generates is usually called commensurability. Two subgroups H and K are *commensurable* if their intersection $H \cap K$ has finite index in both of them, that is $H \leq_a K$ and $K \leq_a H$. Thus, a subgroup of a group is inert if and only if it is commensurable with each of its conjugates.

Accordingly, in a recent paper [88] in the setting of Model Theory, Placin and Wagner consider the *almost normalizer* $\tilde{N}_G(H)$ as the subgroup formed by the elements $g \in G$ such that H^g is commensurable with H.

Commensurability is compatible with taking the intersection of finitely many subgroups and taking quotients. It is preserved under projectivities, i.e. subgroup lattice isomorphisms, even when not induced by group isomorphisms, as proved by Zacher [110] in 1980. Commensurability appears explicitly mentioned in 1987 by Heineken and Specht [100] who studied groups with only finitely many commensurability classes. Later, in 1997, Heineken and Kurdachenko in [81] considered groups whose *normal* subgroups are partitioned in finitely many commensurability classes.

Not surprisingly, the structure of a proper inert subgroup in a simple group is far from arbitrary. For example, it must be residually finite. We do not go deeper in this direction leading somewhat far from the scope of this survey (see [3, 4, 5, 56, 57] for more details).

Clearly, a subgroup commensurable with an inert subgroup, is inert itself. In particular, subgroups which are commensurable with a normal subgroup are inert. Therefore, a subgroup H of a group G having a normal core H_G of finite index $|H : H_G|$, is inert. Such subgroups are called *normal-by-finite* in [13, 29, 71]. Dually, a subgroup H of a group G having finite index $|H^G : H|$ in its normal closure H^G is inert. Such subgroups are called *nearly-normal* in [71].

Nevertheless, an inert subgroup need not be commensurable with a normal subgroup, as shown by the following example due to Peter Neumann. Let G be the alternating group of an arbitrary infinite set S, i.e. the group of permutations of S which move only finitely many elements and are product of an even number of transposition. Let P be any partition of S into finite subsets, each of cardinality > 2, and H the subgroup of elements of G which carry each member of P into itself and act on each of these as an even permutation. Then H is inert but not commensurable with a normal subgroup of G (as G is simple), while H has both order and index equal to $|S|$. Other non-trivial examples of inert subgroups are permutable subgroups (see [93]). Further examples and basic properties of inert

subgroups can be found in [3, 93]. Notice that all the subgroups of the so-called dihedral group (over $\mathbb{Z}$) $D := \langle x, y \mid x^2 = 1, y^x = y^{-1} \rangle = \langle y \rangle \rtimes \langle x \rangle$ are inert.

1.3. **The inertial correspondence.** The straightforward proof of the following fact is omitted.

Fact 1.3. *In the lattice of subgroups of a group, the relation $\sim$ of commensurability is the equivalence generated by the transitive relation $\leq_a$. Both relations are compatible with the meet operation (i.e. intersection) and endomorphisms.*

If the underlying group A is abelian, then the relations $\leq_a$ and $\sim$ are compatible with the join operation (i.e. sum) as well. Thus, commensurability is a congruence in the lattice $l(A)$ of subgroups of A. $\qquad\square$

Consequently, the intersection of finitely many inert subgroups is inert. However the intersection of infinitely many inert subgroups need not be inert (see Corollary 2.7). Further, the join of a pair of inert subgroups of a group G need not be inert, if G is non-abelian. This can be seen as in [93] by considering the soluble group

$$G := \langle a, b, x, y \mid a^b = a, x^2 = 1, y^x = y^{-1}, a^x = a^{-1}, b^x = b, a^y = ab, b^y = b \rangle = \langle a, b \rangle \rtimes \langle x, y \rangle,$$

where $A := \langle a, b \rangle$ is free-abelian of rank 2 and $D := \langle x, y \rangle$ is a dihedral group. The infinite subgroup D is generated by the finite subgroups $\langle x \rangle$ and $\langle xy \rangle$, but D is not inert in G, as $D \cap D^a = 1$.

Let us now introduce the dynamical aspect of the inertial relation given by Definition 1.1. Note that, when the underlying group is abelian (in additive notation), a subgroup H is φ-inert if and only if the factor group $(H + \varphi(H))/H$ is finite.

Fact 1.4. *Let A be an abelian group, Φ a subset of the ring $End(A)$ of endomorphisms of A and $\mathcal{H}$ a subset of the lattice $l(A)$ of subgroups of A. Then:*

(1) *the set $\mathcal{I}_\Phi(A)$ of subgroups which are φ-inert for each $\varphi \in \Phi$ is a sublattice of $l(A)$ closed under commensurability;*

(2) *the set $\mathcal{I}_\mathcal{H} End(A)$ of endomorphisms φ for which each $H \in \mathcal{H}$ is φ-inert is a subring of $End(A)$.* $\qquad\square$

Proof. The proof is rather straightforward and follows from [37, 51]. Here we only recall an argument that shows that $\mathcal{I}_\Phi(A)$ is closed under commensurability. If A is additively written and $H_1 \sim H \in \mathcal{I}_\Phi(A)$, then for every endomorphism $\varphi \in \Phi$ each of the factors in the following series is finite

$$H \cap H_1 \leq H \leq H + \varphi(H) \leq H + H_1 + \varphi(H) \leq H + H_1 + \varphi(H) + \varphi(H_1).$$

Hence, the same holds for the intermediate factor in the series

$$H \cap H_1 \leq H_1 \leq H_1 + \varphi(H_1) \leq H + H_1 + \varphi(H) + \varphi(H_1).$$

Therefore, $H_1 \in \mathcal{I}_\Phi(A)$. $\qquad\square$

In the sequel we simply write $\mathcal{I}_\varphi(A)$, when $\Phi = \{\varphi\}$ is a singleton, and $\mathcal{I} End(A)$ for $\mathcal{I}_{l(A)} End(A)$. This case, introduced in [37], deserves a special attention and will be largely discussed in following sections. Note that in the next definition the underlying group G is not necessarily abelian.

Definition 1.5. [37] *An endomorphism φ of a group G such that H is φ-inert for each subgroup H of G is called an* inertial endomorphism.

In other words, an endomorphism φ of a group G is inertial if the image of H is almost contained in H for each subgroup H of G. According to Fact 1.4(2), the inertial endomorphisms of any abelian group A form a subring, say $\mathcal{I}\mathrm{End}(A)$, of the whole ring $\mathrm{End}(A)$.

The subring $\mathcal{I}\mathrm{End}(A)$ contains the ideal $F(A)$ formed by the endomorphisms of A with finite image. We come back to this ring in §4.2, while in §3.4 we focus our attention on the group $\mathcal{I}\mathrm{Aut}(A)$ generated by the automorphisms γ for which each subgroup is γ-inert. This group is somewhat larger than the group of invertibles of $\mathcal{I}\mathrm{End}(A)$, that we will call *bi-inertial automorphisms.* Note that the automorphism $\varphi : x \mapsto 2x$ of $\mathbb{Q}^{\mathbb{N}}$ is inertial but not bi-inertial, as $\mathbb{Z}^{\mathbb{N}}$ is not φ^{-1}-inert).

The counterpart of Definition 1.5 is the following concept proposed in [51]:

Definition 1.6. [51] *A subgroup H of an abelian group G is called* fully inert *if H is φ-inert for each endomorphism φ of G.*

Clearly, a subgroup which is commensurable with a fully invariant subgroup is fully inert. On the other hand, $\mathbb{Z}$ is a fully inert subgroup of $\mathbb{Q}$, but it is not commensurable to any fully invariant subgroup of $\mathbb{Q}$. In Section 5 we will investigate the difference between these two properties and will discuss results on fully inert subgroups of (divisible) abelian groups.

1.4. **Uniformly inert subgroups.** The following definition is motivated by a nice result by Bergman and Lenstra [9] in 1989 (see Theorem 1.8).

Definition 1.7. *A subgroup H of a group G is called* uniformly inert *with respect to a subgroup K of G if there is $n \in \mathbb{N}$ such that $|H : H^g \cap H| \leq n$ for each $g \in K$.*
If it is necessary to emphasize n, we may also say n-uniformly inert.
In case $K = G$, we simply say H is uniformly inert *(resp., n-uniformly inert).*

Note that, since K is a subgroup, in the above definition we could equivalently write that both indeces $|H : H^g \cap H|$ and $|H^g : H^g \cap H|$ are bounded by some natural number for each $g \in K$, since $|H : H^\gamma \cap H| = |H^{-1} : H^{-1} \cap H|$ for any automorphism γ of G.

Roughly speaking, the next theorem states that a subgroup of a group is commensurable with a normal subgroup if and only if it is uniformly inert.

Theorem 1.8. [9] *If H and K are subgroups of a group G then the following conditions are equivalent:*

  (1) *H is uniformly inert with respect to K;*
  (2) *H is commensurable with a subgroup N of G which is normalized by K.*

Moreover, in case H is n-uniformly inert w.r.t. K, then there is $N \leq G$ such that N is normalized by K, $|N : H \cap N| \leq n$, $|H : H \cap N| \leq n^{n^n}$ and there is also $M \leq G$ such that $N \leq M$, M is normalized by K, M/N is finite and $|H : H \cap M| \leq n$.

Recall that the bound n^{n^n} is due to P. M. Neumann and the particular case when $G = K$ had been previously considered by Schlichting [97] in 1980 when generalizing a result by Baer [2]. We state a corollary in additive notation.

Corollary 1.9. *For a subgroup H of an abelian group A the following are equivalent:*

(1) *there is n such that $|H + \gamma(H)/H| \leq n$ for each $\gamma \in Aut(A)$;*

(2) *H is commensurable with a characteristic subgroup K of A.*

Moreover, there is a characteristic subgroup K of A such that $|(H + K)/(H \cap K)| \leq n^{n^n+1}$.

Proof. It is clear that (2) implies (1). Assume (1) and let $G = A \rtimes Aut(A)$ be the holomorph group of A and note that for each $g \in G$ one has

$$|H/(H^g \cap H)| = |H^{g^{-1}}/(H \cap H^{g^{-1}})| = |\langle H, H^{g^{-1}} \rangle/H| \leq n.$$

Thus one may apply Theorem 1.8 with $K := G$. So H is commensurable with a normal subgroup N of G. Then $N \cap A$ is a characteristic subgroup of A commensurable with H . $\square$

Theorem 1.8 and Corollary 1.9 suggest the following definitions. For every abelian group A we denote by $\mathcal{I}nv(A)$ the sublattice of $l(A)$ consisting of all fully invariant subgroups of A, and by $\mathcal{I}(A) = \mathcal{I}_{End(A)}(A)$ the family of all fully inert subgroups of A. According to Fact 1.4, $\mathcal{I}(A)$ is a sublattice of $l(A)$, stable under $\sim$, i.e., a subgroup which is commensurable with some fully inert subgroup of A is also fully inert. Easy examples show that in general $\mathcal{I}nv(A)$ is not stable under $\sim$, in other words, the family $\mathcal{I}nv^{\sim}(A)$ of all subgroups of A commensurable with some fully inert subgroup of A properly contains the lattice $\mathcal{I}nv(A)$, and it is a lattice on its own, according to Fact 1.3 and the fact that $\mathcal{I}nv(A)$ itself is a (complete) lattice.

Definition 1.10. *A subgroup H of an abelian group A is* uniformly fully inert, *if there is n such that $|(H + \varphi(H))/H| \leq n$ for each $\varphi \in End(A)$.*

By an argument similar to that used in Fact 1.4 one proves the following:

Proposition 1.11. *If A is an abelian group, then the set $\mathcal{I}_u(A)$ of uniformly fully inert subgroups of A is a sublattice of $l(A)$ which is closed under $\sim$ and*

$$\mathcal{I}nv(A) \subseteq \mathcal{I}nv^{\sim}(A) \subseteq \mathcal{I}_u(A) \subseteq \mathcal{I}(A). \tag{$*$}$$

Section 5 is mainly dedicated to the following:

Problem 1.12. *Study the interrelations between the above lattices.*

2. **Groups with many inert subgroups**

It is well known that Dedekind groups, that is groups G in which all subgroups are normal, have a restricted structure (see [93]). In particular, if G is a non-abelian Dedekind group, then the groups $G/Z(G)$ and G' have respectively order 4 and 2.

2.1. **Groups in which each subgroup is inert.** A group is called *inertial* (or totally inert, or TIN) if all its subgroups are inert. Clearly, all finite groups and Dedekind groups are inertial. Moreover, if a group G has a finite normal subgroup F such that G/F is an inertial group, then G is inertial.

The first result we recall on inertial groups is a theorem by Beidelman and Heineken [11] in 2000 on groups which are inertial in a rather trivial way.

Theorem 2.1. [11] *Let G be an infinite group. Then all infinite subgroups of G are normal if and only if one of the following holds:*

(1) *G is a Dedekind group;*
(2) *G is an extension of a Prüfer p-group by a finite Dedekind group;*
(3) *G is an extension of a non-abelian infinite group M by a Dedekind group and all infinite subgroups of G contain M.*

We state now a result by Robinson [93] in 2006 which guarantees that the class of inertial groups is rather large. Recall that a group G is said to be an *FC-group* if each element of G has a finite number of conjugates. Note that the number of conjugates of an element a of G equals the index of the centralizer $C_G(a) := \{g \in G \mid ag = ga\}$.

Theorem 2.2. [93] *FC-groups are inertial.*

The class of inertial groups can be very complex. For example all *quasi-finite* groups (that is groups in which all proper subgroups are finite) are inertial. In particular Tarski p-groups (that is, infinite simple groups in which all proper non-trivial subgroup have order p) are inertial. On the other hand, Belyaev, Kuzucuoglu and Seckin [6] in 1999 have proved:

Theorem 2.3. [6] *No infinite locally finite simple group can be inertial.*

Dixon, Evans and Tortora [58] in 2010 have extended this result to simple locally graded groups.

Theorem 2.4. [58] *Let G be an inertial infinite simple group. Then G is a finitely generated periodic group without involutions such that $C_G(x)$ is finite for all $1 \neq x \in G$ and in which every proper subgroup is residually finite. In particular, G is not locally graded.*

Corollary 2.5. [58] *An inertial simple group G of finite exponent is quasi-finite.*

It seems to be unknown whether any inertial simple group is necessarily quasi-finite. By restricting the investigation to the realm of generalized soluble groups, Robinson [93] in 2006 has proved the following theorem. Recall that a hyper-(abelian-by-finite) group is a group which has an ascending (i.e. well ordered) series of normal subgroups whose factors are either abelian or finite. We already noticed that a dihedral group over $\mathbb{Z}$ is inertial.

Theorem 2.6. [93] *Let G be a finitely generated hyper-(abelian-by-finite) group. Then G is inertial if and only if G has a torsion-free abelian normal subgroup A of finite index in which elements of G induce by conjugation either the identity or the inversion map.*

From [93] we recall that in a polycyclic group every subgroup is the intersection of subgroups of finite index, but not every subgroup of a polycyclic group is necessarily inert. According to Theorem 2.6, this is witnessed by the free nilpotent group of class and rank 2. Thus, we have:

Corollary 2.7. *The intersection of an infinite family of inert subgroups need not to be inert.*

It is easy to show that if a group G has normal subgroups F and A, with $F \leq A$, such that $|F|$ and $|G/A|$ are finite, A/F is abelian and each subgroup of A/F is normalized by G, then the group G is inertial. In such a case, one says that G is *inertial of elementary type.*

Concerning non finitely-generated inertial groups, from [83] we recall that a soluble group G has *finite abelian total rank*, or is a soluble FATR-group, if G has a finite series in which each factor is abelian with finite rank and is either torsion-free or a p-group. In [93] D. Robinson has described soluble inertial FATR-groups. We remark that also groups of non-elementary type occur, but here we do not report the classification as it requires technical details. We just deduce the following.

Theorem 2.8. [93] *Let G be a soluble inertial FATR-group. Then G is finite-by-metabelian-by-finite.*

2.2. **Groups in which each subgroup is strongly inert.** Another way to avoid dealing with quasi-finite groups is strengthening the condition of being inert. Thus, De Falco, de Giovanni, Musella, Trabelsi [39] in 2013 have considered the following definition.

Definition 2.9. [39] *A subgroup H of a group G is said to be* strongly inert *if H has finite index in $\langle H, H^g \rangle$ for each $g \in G$.*

A group in which all subgroups are strongly inert is called strongly inertial.

Clearly, every nearly-normal subgroup is strongly inert. Moreover, if the commutator subgroup G' of G is finite of size n, then the index of H in $\langle H, H^g \rangle$ is at most n for each $g \in G$ and H is strongly inert. Then we recall a celebrated theorem by Neumann [87] in 1955 which deals in fact with strongly inertial groups.

Theorem 2.10. [87] *In a group G each subgroup is nearly-normal if and only if the group has finite derived subgroup (i.e. G is finite-by-abelian).*

Clearly permutable subgroups are strongly inert (see [39]). Other examples of strongly inert subgroups are inert subgroups which are subnormal (see [36]). On the other hand, finite subgroups in a dihedral group (over $\mathbb{Z}$) are not strongly inert (although trivially inert). We also recall that in an FC-group all subgroups are strongly inert (see [39]).

A finitely generated group is strongly inertial if and only it it is an FC-group, as we see in next theorem.

Theorem 2.11. [39] *Every finitely generated strongly inertial group is centre-by-finite.*

Corollary 2.12. *Let G be a group whose finitely generated subgroups are strongly inert. Then the commutator subgroup G' of G is locally finite, i.e. G is (locally finite)-by-abelian.*

Thus, a strongly inertial simple group must be finite, since otherwise $G' = G$ is locally finite, contradicting Theorem 2.3.

Recall that a group G is called *minimax* if it has a series of finite length whose factors satisfy either the minimal or the maximal condition on subgroups. Recall also that if G is any soluble-by-finite minimax group, then its finite residual R (i.e. the intersection of all subgroups of finite index) is the direct product of finitely many Prüfer subgroups. If this subgroup R is actually a p-group we say that G is *p-primary*. Then in [39] a full classification of soluble-by-finite p-primary minimax strongly inertial groups is obtained. We omit details, but we recall a relevant consequence.

Corollary 2.13. [39] *A soluble-by-finite minimax group is strongly inertial if and only if it is inertial and has no infinite dihedral sections.*

2.3. **Groups in which each subgroup is normal-by-finite.** We are going to introduce a condition dual to nearly-normality. Recall that a subgroup of a group G is called *normal-by-finite* (or sometimes *almost-normal*) if H/H_G is finite. A normal-by-finite subgroup is inert.

Definition 2.14. *Let G be a group.*
  (1) *G is a CF-group if $\forall H \leq G$ $|H/H_G| < \infty$;*
  (2) *G is a BCF-group if $\exists n \in \mathbb{N}$ $\forall H \leq G$ $|H/H_G| \leq n$.*

Since normal-by-finite subgroups are also called "core-finite" by some authors, here CF stands for "core-finite" and BCF for "boundedly core-finite". The class of CF-groups seems to be difficult to handle since all quasi-finite groups are CF. In fact normal-by-finite (and even finite) subgroups are not necessarily strongly inert. However, the problem is tractable in a restricted class of groups. It has been treated by Buckley, Cutolo, Lennox, Neumann, Rinauro, Smith and Wiegold around 1995 in [13, 29, 99].

Theorem 2.15. [13, 99] *A CF-group G whose periodic quotients are locally finite is abelian-by-finite. Moreover, if G is periodic, then G is BCF.*

Note that every group which has an ascending normal series whose factors are either locally nilpotent or finite has the property that periodic quotients are locally finite.

Problem 2.16. *Is every locally graded CF-group abelian-by-finite?*

Let us see how periodic abelian-by-finite CF-groups may be described in a satisfactory way using also results from [34, 61]. Recall that an automorphism γ of a group G is said a *power automorphism* if $H^\gamma = H$ for each subgroup $H \leq G$. It is well-known (see [92]) that, if G is an abelian p-group, then there exists a p-adic integer α such that $a^\gamma = a^\alpha$ for all $a \in G$. Here a^α stands for a^n, where n is any integer congruent to α modulo the order of a.

Theorem 2.17. *A locally finite group G is CF if and only if it has a normal abelian subgroup $A = C \times (D \times E)$ of finite index and such that:*
  (1) *$\pi(D) = \pi(E)$ is finite and $\pi(C) \cap \pi(DE) = \emptyset$;*

(2) *D is divisible of finite rank r and E has finite exponent e;*

(3) *G induces by conjugation power automorphisms on C, D, E.*

For such a group G one has $|H/H_G| \leq e^r \cdot |G/A|$ for every $H \leq G$. Therefore, G is a BCF-group.

Concerning the non-periodic case, recall that a power automorphism of a non-periodic abelian group is either the identity or the inversion map, say for short that it is *the power* ± 1 respectively. Consider the group $G := A \rtimes \langle \gamma \rangle$ where $A = C_\infty \times C_{p^\infty}$ (here C_∞ and C_{p^∞} are an infinite cyclic group and a Prüfer *p*-group) and γ is the automorphism of *A* acting as the identity (resp., the inversion) map on C_∞ (resp. on C_{p^∞}). Then *G* is CF, but not BCF.

Theorem 2.18. [18] *A non-periodic abelian-by-finite group G is CF if and only if there is a normal abelian subgroup A of finite index such that either G acts by conjugation on A as the power ± 1, or there is a G-series $1 \leq V \leq A$ such that:*

(1) *G/V is periodic and CF,*

(2) *V is finitely generated free-abelian and G acts by conjugation on V as the power ± 1.*

Moreover, G is BCF if and only if G is CF and

(3) *there is $B \leq A$ such that G/B has finite exponent and G acts by conjugation on B as the power ± 1.*

2.4. **Groups in which each subgroup is commensurable with a normal subgroup.** In order to put the above results in a common framework, Casolo, Dardano and Rinauro [18] have recently considered the class of *CN-groups*, that is, groups in which each subgroup is commensurable with a normal subgroup. Clearly, a group is inertial if and only if (IN) below holds. Moreover, in view of Theorem 1.8, a group is a CN-group if and only if (CN) below holds.

Definition 2.19. *Let G be a group. Consider the follwing properties for G:*

(IN) $\forall H \leq G \ \forall g \in G \ \exists n \in \mathbb{N} \ \ |H : (H \cap H^g)| \leq n;$

(CN) $\forall H \leq G \ \exists n \in \mathbb{N} \ \forall g \in G \ \ |H : (H \cap H^g)| \leq n;$

(BCN) $\exists n \in \mathbb{N} \ \forall H \leq G \ \forall g \in G \ \ |H : (H \cap H^g)| \leq n.$

By Theorem 1.8, *BCN-groups* (boundedly CN) are groups *G* for which there is $n \in \mathbb{N}$ such that for each $H \leq G$ there is $N \lhd G$ such that $|HN/H \cap N| \leq n$.

It is easy to show that the class of CN-groups is contained in the class of so-called *sbyf-groups*, that is, groups in which each subgroup *H* contains a subnormal subgroup *S* of *G* such that the index $|H : S|$ is finite, say *H* is *subnormal-by-finite*. Sbyf-groups have been studied by Heineken [80] in 1996 and Casolo [17] in 2016.

Theorem 2.20. *Let G a locally finite sbyf-group. Then:*

(1) [80] *G is nilpotent-by-Chernikov;*

(2) [17] *G is (locally nilpotent)-by-finite.*

Problem 2.21. *Study sbyf-groups whose periodic quotients are locally finite.*

The next result can be found in the forthcoming paper [18].

Theorem 2.22. [18] *Let G be a CN-group. If every periodic quotient of G is locally finite, then G is finite-by-abelian-by-finite.*

It is easy to see that for an abelian-by-finite group the properties CN and CF are equivalent. However, for every prime p there is a nilpotent p-group with the property CN which is neither finite-by-abelian nor abelian-by-finite ([18]).

Theorem 2.23. [18] *Let G be a finite-by-abelian-by-finite group.*
- (1) *G is* CN *if and only if it is finite-by-*CF.
- (2) *G is* BCN *if and only if it is finite-by-*BCF.

It follows that if the group G is periodic and finite-by-abelian-by-finite, then G is BCN if and only if it is CN. The more restrictive property BCN remains treatable when we consider the wider class of locally graded groups.

Theorem 2.24. [18] *A locally graded BCN-group is finite-by-abelian-by-finite.*

The proof of the above results will appear in [18]. Here we just notice that in the proof of Theorem 2.22 one reduces first to the case when G is locally nilpotent and soluble, by virtue of Theorem 2.20. Then one reduces to the case when G is nilpotent, by using Theorem 2.26 below and some technical lemmas. Then, finally, one uses techniques introduced by Möhres [86] for nilpotent groups.

Definition 2.25. *Let A be an abelian group and $\Gamma \leq Aut(A)$ a group of automorphisms of A. Consider the following properties for the pair (Γ, A):*

- (P) *$\forall H \leq A \quad H = H^\Gamma$;*
- (AP) *$\forall H \leq A \quad |H/H_\Gamma| < \infty$;*
- (BP) *$\forall H \leq A \quad |H^\Gamma/H| < \infty$;*
- (CP) *$\forall H \leq A \quad \exists K = K^\Gamma \leq A$ such that $H \sim K$, i.e. H, K are commensurable;*
- (P̃) *Γ has (P) on the factors of a series $1 \leq V \leq D \leq A$ whose members are Γ-invariant and*
    - (1) *V is free abelian of finite rank and A/V is periodic,*
    - (2) *D/V is divisible of finite rank and $\pi(D/V)$ is finite,*
    - (3) *the p-component of A/D is bounded for each $p \in \pi(D/V)$.*

According to [13], A is said to be *Γ-hamiltonian*, when (P) holds. Obviously both (AP) and (BP) imply (CP). Moreover, (AP), (BP) and (CP) *are equivalent, provided that A is abelian and Γ is finitely generated, while they are in fact different in general even when both A and Γ are elementary abelian p-groups* (see [34]).

Theorem 2.26. *Let Γ be a group acting on an abelian group A. Then:*

(1) [61] Γ *has* (AP) *on* A *if and only if there is a* Γ-*invariant subgroup* A_1 *such that* A/A_1 *is finite and* Γ *has either* (P) *or* ($\tilde{\text{P}}$) *on* A_1;

(2) [16] Γ *has* (BP) *on* A *if and only if there is a* Γ-*invariant subgroup* A_0 *such that* A_0 *is finite and* Γ *has either* (P) *or* ($\tilde{\text{P}}$) *on* A/A_0;

(3) [18] Γ *has* (CP) *on* A *if and only if there are* Γ-*invariant subgroups* $A_0 \leq A_1 \leq A$ *such that* A_0 *and* A/A_1 *are finite and* Γ *has either* (P) *or* ($\tilde{\text{P}}$) *on* A_1/A_0.

Corollary 2.27. *For a group* Γ *acting on an abelian group* A, *the following are equivalent:*

(1) Γ *has* (AP) *on* A/A_0 *for a finite* Γ-*invariant subgroup* A_0 *of* A;

(2) Γ *has* (BP) *on a* Γ-*invariant subgroup* A_1 *of* A *of finite index;*

(3) Γ *has* (CP) *on* A.

In closing this subsection we observe that the definition of inert subgroup suggests considering also another natural subclass of the class of inertial groups.

Problem 2.28. *Describe (soluble) groups which have the property:*

$(BIN) \quad \forall g \in G \ \exists n \in \mathbb{N} \ \forall H \leq G \ \ |H : (H \cap H^g)| \leq n.$

2.5. **Groups in which each subnormal subgroup is inert.** The study of groups in which subnormal subgroups are subject to some restrictions is a standard problem in the theory of groups. The class of soluble groups in which subnormal subgroups are normal (usually called T-groups) and its generalizations has been studied by many authors. Gaschütz [63] treated the finite case in 1957. Then Robinson [91] treated the general case in 1964. Of course one has to consider classes of groups in which there are sufficiently many subnormal subgroups.

Theorem 2.29. [91] *Let* G *be a soluble group in which subnormal subgroups are normal. Then* G *is metabelian. If* G *is finitely generated, then* G *is finite or abelian.*

In 1985 Franciosi and de Giovanni considered weaker conditions.

Theorem 2.30. [59] *Let* G *be a soluble group in which subnormal subgroups are either normal or finite. Then the derived subgroup* G' *of* G *is nilpotent and* G'' *is cyclic of prime power order.*

An application of this result can be found in [31]. On the other hand, in 1989 Casolo [16] studied the class of groups whose subnormal subgroups are nearly-normal. These are exactly those groups in which the relation of being nearly-normal is transitive. They have been called T^*-groups.

Theorem 2.31. [16] *Let* G *be a soluble* T^*-*group. Then* G *is finite-by-metabelian. If* G *is finitely generated, then* G *is abelian-by-finite.*

A stronger condition was considered by Franciosi and de Giovanni in 1993.

Theorem 2.32. [60] *Let* G *be a soluble infinite group in which subnormal subgroups are either normal or have finite index. Then* G *is metabelian.*

In 1995, Franciosi, de Giovanni and Newell considered a dual condition and studied T_*-groups, that is, groups whose subnormal subgroups are normal-by-finite. These are exactly those groups in which the relation of being normal-by-finite is transitive.

Theorem 2.33. [61] *Let G be a soluble T_*-group. Then G is metabelian-by-finite. Moreover, the derived subgroup G' is abelian-by-finite. If G is finitely generated, then G is abelian-by-finite.*

In order to put these results in the same framework, we consider the class of soluble group in which subnormal subgroups are (strongly) inert and report a result for finitely generated groups. Say that a group G is *semidihedral* on a torsion-free abelian subgroup A if G acts on A by means of inertial automorphisms and $C_G(A) = A$. This results to be equivalent to the fact that $G = A \rtimes K$ where A is torsion-free abelian and K acts faithfully on A by means of inertial automorphisms (see [36]). It is easy to verify that inertial automorphisms of a torsion-free abelian group are multiplications, according to Definition 4.1 and Theorems 4.2 and 4.5 below. Therefore, $K = K_0 \times T$ where K_0 is free-abelian and T has order at most 2 (for details see [36]). The reader must be warned that the word *semidihedral* has been used also with a different meaning in other areas of group theory.

Theorem 2.34. [36] *Let G be finitely generated soluble-by-finite group. Then each subnormal subgroup of G is inert if and only if G has a finite normal subgroup F such that G/F is a semidihedral group.*

Let us deduce a corollary generalizing the above stated fact that finitely generated soluble T^*- and T_*-groups are both abelian-by-finite.

Corollary 2.35. *Let G be a finitely generated soluble group. Then the following are equivalent:*

1) *each subnormal subgroup of G is commensurable with a normal subgroup;*
2) *G has a finitely generated torsion-free abelian subgroup A_0 of finite index such that G acts on A_0 by either the identity or the inversion map;*
3) *G is CF, i.e., each subgroup is normal-by-finite.*

Proof.   Assume (1). Apply Theorem 2.34 and let F be a finite normal subgroup such that G/F is semidihedral, thus $G/F = A/F \rtimes K/F$ where A/F is torsion-free abelian and K/F acts faithfully on A/F by means of inertial automorphisms (hence multiplications). Since each subgroup of A/F is commensurable with a subgroup which is normalized by G, these multiplications are either the identity or the inversion map. Since K/F acts faithfully on A/F, we can conclude that K/F and G/A have order 2. Then A is finitely generated. Moreover, A is finite-by-abelian, hence FC (see [92]). Then $Z = Z(A)$ is a finitely generated abelian subgroup with finite index in G. Now we can apply Theorem 4.5 below (or Theorem 4 in [34]) and deduce the existence of $A_0 \leq Z$ like in statement (2). Finally, it is clear that (2) implies (3) and (3) implies (1). $\square$

The next proposition shows that we are extending both T^* and T_* in a natural way.

Proposition 2.36. *For a group G the following conditions are equivalent:*

(1) *each subnormal subgroup of G is commensurable with a normal subgroup;*

(2) *the relation of being commensurable with a normal subgroup is transitive.*

Proof. Assume (1) and let H and K be subgroups with the property that there are subgroups H_1 and K_1 such that $H \sim H_1 \lhd K \sim K_1 \lhd G$. We wish to prove that there is H_2 such that $H \sim H_2 \lhd G$. The subgroup $K_2 := (K \cap K_1)_{KK_1}$ has finite index in KK_1 and so $H_1 \cap K_2$ has finite index in H_1 as $H_1 \leq KK_1$. Hence, $H \sim H_1 \sim H_1 \cap K_2$, according to Fact 1.3. On the other hand $H_1 \cap K_2 \lhd K_2 \lhd K_1 \lhd G$, hence $H_1 \cap K_2$ is in turn commensurable with a normal subgroup. Thus, (2) holds. The converse is obvious. $\qquad\square$

Problem 2.37. *Describe the structure of soluble groups G in which every subnormal subgroup is commensurable with a normal subgroup.*

3. **Groups of inertial automorphisms of a group**

3.1. **The group of finitary automorphisms of a group.** We start this section by recalling a basic instance of inertial automorphisms. Let us extend the notation for the centralizer subgroup to denote the subgroup of fixed-points of a set of endomorphisms of a group.

Definition 3.1. *An endomorphism γ of a group G is called* finitary *if the subgroup $C_G(\gamma) := \{x \in G \mid x^\gamma = x\}$ has finite index in G.*

Clearly, a group is an FC-group if and only if all its inner automorphisms are finitary. In any case, finitary automorphism form a subgroup, say FAut(G), of the group of all automorphisms of G. The group FAut(G) seems to have been studied for the first time in 1975 in [111] by A. E. Zalesskii (see Theorem 3.5(2) below) who called *bounded automorphisms* what we here call finitary automorphisms (following [104]) .

The case when the underlying group is abelian, has been considered independently by Wehrfritz in 2002.

Theorem 3.2. [104] *If A is an abelian group, then the group* FAut(A) *of all finitary automorphisms of A is locally finite.*

Imposing some additional restraints on the underlying abelian group A, one can describe the group FAut(A). For example, observe that if A is torsion-free (or divisible), then FAut$(A) = 1$, since for each $\gamma \in$ FAut(A) the subgroup $A/ker(\gamma - id) \simeq \text{im}(\gamma - id)$ is simultaneously finite and torsion-free, hence trivial.

Now we report two results which suggest a construction for FAut(A) for some abelian groups A. If G is any group and N a normal subgroup of G, denote by St(G, N) the stability group in Aut(G) of the series $G \geq N \geq 1$, that is, the set of automorphisms $\gamma \in$ Aut(G) such that N is a $\langle\gamma\rangle$-invariant normal subgroup of G and γ acts as the identity map on both N and G/N. Then it is well-known that St(G, N) is abelian and, if G is abelian, we have St$(G, N) \simeq$ Hom$(G/N, N)$.

Theorem 3.3. [38] *Let A be an abelian p-group such that $D := D(A)$ has finite rank and A/D is bounded. Then, $\Sigma := \mathrm{St}(A, D)$ is a bounded abelian p-group and there is a subgroup $\Phi \simeq \mathrm{FAut}(A/D)$ of $\mathrm{FAut}(A)$ such that $\mathrm{FAut}(A) = \Sigma \rtimes \Phi$ where the automorphisms induced by Φ via conjugation on Σ are finitary and this action is faithful.*

Theorem 3.4. [38] *Let A be an abelian group and $T = t(A)$. If A/T is finitely generated (resp., T is bounded), then $\Sigma := \mathrm{St}(A, T)$ is a periodic (resp., bounded) abelian group and there is a subgroup $\Phi_1 \simeq \mathrm{FAut}(T)$ of $\mathrm{FAut}(A)$ such that $\mathrm{FAut}(A) = \Sigma \rtimes \Phi_1$, where Φ_1 induces finitary automorphisms by conjugation on Σ .*

   In the case when $A/T \neq 0$ is finitely generated, the above action by conjugation is faithful.

Concerning the general case in which the underlying group G need not be abelian the following holds.

Theorem 3.5. [7, 8, 85, 111] *Let G be any group and $\mathrm{FAut}(G)$ be the group of all finitary automorphisms of G. Then:*

   (1) $\mathrm{FAut}(G)$ *is abelian-by-(locally finite);*
   (2) $\mathrm{FAut}(G)$ *is locally (centre-by-finite), hence (locally finite)-by-abelian;*
   (3) $\mathrm{FAut}(G)$ *is locally finite if and only if $\mathrm{FAut}(G) \cap Inn(G)$ is locally finite.*

Statement (1) has been shown in 2009 by Belyaev and Shved in [7, 8]. Part (2) follows from Zalesskii's investigation in [111] and implies that the elements of $\mathrm{FAut}(G)$ of finite order form a locally finite subgroup. Finally, part (3) follows from results in [85] by Menegazzo and Robinson in 1987.

Recently, Shved has obtained the following result. Recall that a finitary linear group is a group of automorphisms of a vector space which fix poitwise a subspace of finite codimension.

Theorem 3.6. [98] *Let G be any group, then $\mathrm{FAut}(G)$ has a normal series*

$$1 \leq \Gamma_1 \leq \Gamma_2 \leq \mathrm{FAut}(G)$$

such that:

   (1) Γ_1 *is nilpotent of class al most 4;*
   (2) Γ_2/Γ_1 *is periodic and has an ascending central series of type at most ω;*
   (3) $\mathrm{FAut}(G)/\Gamma_2$ *is embeddable into a direct product of finitary linear groups over prime fields.*

In [98] more details are given in the case when G is a p-group (finitary automorphisms are called *virtually trivial* in that paper).

Let us consider now a particular type of finitary automorphism that we will use in §3.5.

Definition 3.7. *An automorphism γ of a group G is called* strongly finitary *if there is a finite $\langle \gamma \rangle$-invariant normal subgroup N of G such that γ acts trivially on G/N.*

In the above notation, since the correspondence $x \mapsto [x, \gamma]$ gives an injection of a right transversal of $C_G(\gamma)$ in G into N, it results $|G : C_G(\gamma)| \leq |N|$. Thus, the strongly finitary automorphisms of G form a subset $s\mathrm{FAut}(G)$ of $\mathrm{FAut}(G)$. Moreover, if G is abelian we have $s\mathrm{FAut}(G) = \mathrm{FAut}(G)$. In fact, if $\gamma \in \mathrm{FAut}(G)$ then $A/ker(\gamma - id) \simeq \mathrm{im}(\gamma - id) =: N$ is finite and γ acts on G/N as the identity map. On the other hand, in the dihedral group $\langle x, y \mid x^2 = 1, y^x = y^{-1} \rangle$ the inner automorphism induced by y is finitary but not strongly finitary. Thus we can have $s\mathrm{FAut}(G) \neq \mathrm{FAut}(G)$.

Then let us state an easy extension of Theorem 3.2.

Theorem 3.8. *Let G be a group. Then the group $s\mathrm{FAut}(G)$ of all strongly finitary automorphisms of G is locally finite and coincides with the subgroup of $\mathrm{FAut}(G)$ formed by the elements of finite order.*

Proof. Let $\gamma_1, \ldots, \gamma_n \in s\mathrm{FAut}(G)$. If γ_i acts trivially on G/N_i, then the subgroup $N := N_1 \cdots N_n$ is finite, normal in G and $\langle \gamma_i \rangle$-invariant for each i. Then each γ_i acts trivially on G/N and $s\mathrm{FAut}(G)$ is a subgroup of $\mathrm{FAut}(G)$. Moreover, if $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$, then the subgroup $C := C_\Gamma(N)$ has finite index in Γ and stabilizes the series $1 \leq N \leq G$. Thus C is abelian and finitely generated. With $m = |N|$, for each γ_i and $a \in G$ we have $[\gamma_i^m, a] = [\gamma_i, a]^m = 1$. Then C and Γ are finite. Hence $s\mathrm{FAut}(G)$ is locally finite.

On the other hand, if $\gamma \in \mathrm{FAut}(G)$ and $\Gamma_1 := \langle \gamma \rangle$ is finite, consider $G \rtimes \Gamma_1$ as a subgroup of the holomorph group of G. By Dicman's Lemma (see [92]), Γ_1^G is finite. Let $N := [G, \Gamma_1] \leq \Gamma_1^G$, then N is a finite normal subgroup of G and Γ_1 acts trivially on G/N. Hence $\gamma \in s\mathrm{FAut}(G)$. $\qquad\square$

3.2. **The group of automorphisms of a group fixing all but finitely many subgroups.** Power automorphisms are trivially inertial and have been studied by many authors, mainly by C.D.H. Cooper [26] in 1968.

Theorem 3.9. [26] *The set $\mathrm{PAut}G$ of all power automorphisms of a group G is a normal, abelian, residually finite subgroup of the full automorphism group $\mathrm{Aut}(G)$ of G.*

The subgroup $N(G)$ of all elements of a group G which induce by conjugation a power automorphism is called the *norm*, it coincides with the intersection of all normalizers of subgroups of G. A group G is a Dedekind group if and only if $G = N(G)$. Clearly we have $Z(G) \leq N(G)$. Furthermore, by a classical results by Wielandt, we have $N(G) \leq Z_2(G)$ where $Z_2(G)/Z(G) := Z(G/Z(G))$ is the so-called second centre of G. This follows also from the fact that power automorphisms are central, that is, they act trivially on the central factor group $G/Z(G)$ (see [26]). Equivalently, an automorphism of G is central if and only if it centralizes the group $Inn(G)$ of inner automorphisms (in the full automorphism group $\mathrm{Aut}G$ of G) .

In 1993 Cutolo [28] studied the group $\mathrm{QAut}(G)$ of the so-called *quasi-power* automorphisms of a group G, that is, automorphisms which fix all but finitely many subgroups of G.

Theorem 3.10. [28] *Let G be an infinite group. Then $\mathrm{QAut}(G)$ is abelian and residually finite.*
If $\mathrm{QAut}(G) \neq \mathrm{PAut}(G)$, then the subgroups of G which are not fixed under the action of $\mathrm{QAut}G$ generate a finite characteristic subgroup F such that G/F is a finite extension of a p-subgroup. In

particular, G is periodic. Moreover, every infinite subgroup of G which is locally finite is Prüfer-by-finite.

3.3. **The group of automorphisms of a group fixing all infinite subgroups.** If $IN(G)$ is the subgroup of the group G defined as the intersection of all normalizers of infinite subgroups of G we have clearly $Z(G) \leq N(G) \leq IN(G)$. In 2000, Beidelman and Heineken in [11] give many results about this subgroup. We report just a couple of facts.

Theorem 3.11. [11] *If G is a non-periodic group, then $IN(G)$ is abelian. Moreover, if $IN(G)$ is non-periodic, then $IN(G) = Z(G)$.*

In 1990 Curzio, Franciosi and de Giovanni [27] had considered the group I-Aut(G) of all automorphisms γ of G such that $H^\gamma = H$ for each infinite subgroup H of G. These automorphisms are clearly inertial and I-Aut(G) is a normal subgroup of Aut(G) containing the groups PAut(G) and QAut(G), see [28]. We summarize their results as follows.

Theorem 3.12. [27] *Let G be a group.*

(1) *If G is non-periodic, then* I-Aut(G) *is abelian.*

(2) *There are infinite hypercentral p-groups G for which* I-Aut(G) *is not abelian.*

(3) *If G is locally finite with no infinite section which are a simple groups and* I-Aut(G) *is not abelian, then G is Chernikov.*

(4) *If G is (locally radical)-by-(finite) and* I-Aut$(G) \neq$ PAut(G), *then G is Chernikov.*

(5) *If G is either FC or nilpotent and* I-Aut$(G) \neq$ PAut(G), *then G is Prüfer-by-finite.*

Recall that a *radical* group is a group which has an ascending series with locally nilpotent factors.

In [11] Beidelman and Heineken have continued the study of the group I-Aut(G) and have given further information. For example, they show that in most cases the equality I-Aut$(G) =$ PAut(G) actually holds and give a complete characterization of Chernikov groups satisfying I-Aut$(G) \neq$ PAut(G). Their results are rather technical. For the sake of brevity, we recall only the following one and we refer the reader to [11] for the rest.

Theorem 3.13. [11] *If G is an infinite locally finite group, then* I-Aut(G) *is metabelian.*

Recently, De Falco, de Giovanni, Musella and Sysak in [40] have introduced the concept of *weakly power automorphism*, that is an automorphism γ of a group G such that $H^\gamma = H$ for each non-periodic subgroup H of G. Denote by WAut(G) the group of all such automorphisms. Clearly WAut(G) contains I-Aut(G).

Theorem 3.14. [40] *Let G be a non-periodic group.*

(1) *The group* WAut(G) *is abelian.*

(2) *If G is generated by non-periodic elements, then* WAut$(G) =$ PAut(G).

(3) *If G' is metabelian-by-finite, then* WAut$(G) =$ PAut(G).

(4) *There exists a soluble non-periodic group (with derived length 4) such that* WAut$(G) \neq$ PAut(G).

3.4. **The group of inertial automorphisms of an abelian group.** As we observed above, the inverse of an inertial automorphism need not to be inertial. On the other hand, if G is any group and $\gamma \in Aut(G)$, then both γ and γ^{-1} are inertial if and only if $H \sim H^\gamma$ for each $H \leq G$. Therefore, we introduce a couple of definitions.

Definition 3.15. *If γ is an automorphism of a group G we say that:*

(1) *γ is* bi-inertial *if both γ and γ^{-1} are inertial;*

(2) *γ is* almost-power *if every subgroup H of G contains a subgroup K of finite index such that $K^\gamma = K$.*

According to the terminology introduced in Section 2.3, an automorphism γ is almost-power if and only if the group $\langle \gamma \rangle$ has (AP). Clearly an almost-power automorphism is bi-inertial. Moreover, a periodic inertial automorphism is almost-power. The definition of almost-power automorphism was introduced in [61] where it was shown that the set of almost-power automorphisms of an abelian group A is a subgroup of $\mathrm{Aut}(A)$. By Fact 1.3, it follows that *the set of all bi-inertial automorphisms of any group G is a subgroup of* $\mathrm{Aut}(G)$ which contains all almost-power automorphisms and seems to be an object of interest. However, in the general case, this group may contain even a Tarski group, hence it can be too complicated. Also a centreless FC-group may be regarded as a group of bi-inertial automorphisms. On the other hand, if the underlying group is abelian the picture is clearer, as next theorem shows. As we announced, we will focus attention on the group $\mathcal{I}\mathrm{Aut}(A)$ *generated by the inertial automorphisms* of an abelian group A and report results from [38]. We warn that in [34] by inertial one means what here is called bi-inertial. The terminology of the present paper is consistent with [37, 38].

Theorem 3.16. [34, 37] *Let γ, γ_1 be automorphisms of an abelian group A.*

(1) *If γ, γ_1 are both inertial, then the commutator $[\gamma, \gamma_1]$ is finitary.*

(2) *The group $\mathcal{I}\mathrm{Aut}(A)$ coincides with the set of the products $\gamma\gamma_1^{-1}$ where both γ, γ_1 are inertial.*

(3) *If $r_0(A) < \infty$, then γ is inertial if and only if γ is bi-inertial.*

(4) *If $r_0(A) = \infty$, then γ is bi-inertial if and only if γ is almost-power.*

Recall that $\mathcal{I}\mathrm{Aut}(A)$ contains the group $\mathrm{FAut}(A)$, which is locally finite, by Theorem 3.2. In the case A is periodic, we can give many details on the group $\mathcal{I}\mathrm{Aut}(A)$ in next theorem. If π is a set of primes, we regard $\mathrm{FAut}(A_\pi)$ as naturally embedded in $\mathrm{FAut}(A)$.

Theorem 3.17. [38] *Let A be a periodic abelian group, then $\mathcal{I}\mathrm{Aut}(A) = \mathrm{PAut}(A) \cdot \Delta \cdot \mathrm{FAut}(A)$ where Δ is a direct product of cyclic groups (with $\Delta = 1$, if A is reduced). Moreover, all inertial automorphisms of A are almost-power.*

Therefore, $\mathcal{I}\mathrm{Aut}(A)$ is central-by-(locally finite), hence (locally finite)-by-abelian.

Furthermore, there is a set π of primes such that $\Delta \cdot \mathrm{FAut}(A) = \mathrm{FAut}(A_\pi) \times (\Sigma \rtimes \mathcal{I})$, where Σ is an abelian π'-group with bounded primary components, $\Delta \leq \mathcal{I}$ and $\mathcal{I}$ acts faithfully on Σ by inertial automorphisms.

To treat the general case, we list results from [38] in a unique theorem. Let $\mathcal{I}\mathrm{Aut}_1(A)$ be the group of bi-inertial automorphisms centralizing $A/t(A)$.

Theorem 3.18. [38] *Let A be an abelian group.*

(1) *If A is torsion-free, then $\mathcal{I}\mathrm{Aut}(A)$ is abelian.*

(2) $\mathrm{FAut}(A) \leq \mathcal{I}\mathrm{Aut}_1(A)$ *and equality holds if $r_0(A) = \infty$.*

(3) $\mathcal{I}\mathrm{Aut}(A) = \mathcal{I}\mathrm{Aut}_1(A) \times Q(A)$ *where $Q(A)$ is isomorphic to a multiplicative group of rational numbers.*

(4) $\mathcal{I}\mathrm{Aut}_1(A) \times \{\pm 1\}$ *coincides with the group of almost-power automorphisms of A.*

(5) $\mathcal{I}\mathrm{Aut}(A)$ *is (locally finite)-by-abelian.*

(6) *There is a normal subgroup $\Gamma \leq \mathcal{I}\mathrm{Aut}_1(A)$ such that:*

   i) $\mathcal{I}\mathrm{Aut}_1(A)/\Gamma$ is locally finite;

   ii) Γ acts by means of power automorphisms on its derived subgroup Γ', which is a periodic abelian group.

(7) $\mathcal{I}\mathrm{Aut}(A)$ *is metabelian-by-(locally finite).*

(8) *If $A = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}$, then $\mathcal{I}\mathrm{Aut}(A)$ is not (locally nilpotent)-by-(locally finite).*

Recall that a metabelian group G in which each subgroup of the derived subgroup G' is normal in G is said to be a *KI-group*. Such groups have been studied by Subbotin in a series of papers (see [101] and the references therein). A further instance of KI-group is given in the following theorem. By $Fit(G)$ we denote the subgroup generated by all normal nilpotent subgroups of G

Theorem 3.19. [32] *Let G be a group and $\mathrm{Aut}_{sn}(G)$ be the group of all automorphisms γ of G such that $H^\gamma = H$ for each subnormal subgroup H of G. If G is soluble and either G is nilpotent-by-(finitely generated) or $\mathrm{Fit}(G)$ is non-periodic, then $\mathrm{Aut}_{sn}(G)$ is a KI-group.*

Let us recall that, to show that a group of automorphisms is a KI-group one can use arguments like the following lemma which is inspired by the celebrated argument introduced by P. Hall to prove that the stability group of a finite series must be nilpotent (see [19, 33]).

Lemma 3.20. [33] *Let A be an abelian group, $\sigma, \gamma \in \mathrm{Aut}(A)$ and $m_1, m_2 \in \mathbb{Z}$. If σ stabilizes a series $0 \leq A_1 \leq A$ and $a_1^\gamma = a_1^{m_1}$ for each $a_1 \in A_1$, $[a]^{\gamma^{-1}} = [a]^{m_2}$ for each $[a] \in A/A_1$, then $\sigma^\gamma = \sigma^{m_1 m_2}$.*

One can give a detailed description of $\mathcal{I}\mathrm{Aut}(A)$ in some cases.

Theorem 3.21. [38] *Let A be an abelian group with $r_0(A) < \infty$ and $T := t(A)$ be bounded. Then there are an abelian subgroup $\Sigma := \mathrm{St}(A, T)$ of finite exponent and a subgroup Γ_1 of $\mathcal{I}\mathrm{Aut}_1(A)$ such that $\mathcal{I}\mathrm{Aut}_1(A) = \Sigma \rtimes \Gamma_1$, where $\Gamma_1 \simeq \mathcal{I}\mathrm{Aut}(T)$ induces via conjugation on Σ inertial automorphisms.*

Theorem 3.22. [38] *Let A be a non periodic abelian group and $T := t(A)$ with A/T finitely generated. Then there are a periodic abelian group $\Sigma := \mathrm{St}(A, T)$ and a subgroup Γ_1 of $\mathcal{I}\mathrm{Aut}_1(A)$ such that $\mathcal{I}\mathrm{Aut}_1(A) = \Sigma \rtimes \Gamma_1$, where $\Gamma_1 \simeq \mathcal{I}\mathrm{Aut}(T)$ induces via conjugation on Σ inertial automorphisms and the action is faithful.*

If in addition T is not bounded, then $\mathcal{I}\mathrm{Aut}_1(A)$ is not nilpotent-by-(locally finite). Further, if $A_{2'}$ is unbounded, then $\mathcal{I}\mathrm{Aut}_1(A)$ is not even (locally nilpotent)-by-(locally finite).

3.5. **The group of strongly inertial automorphisms.** As observed above, groups of inertial automorphisms can have a complicated structure. This suggests to consider a stronger condition.

Definition 3.23. *An automorphism γ of a group G is called* strongly inertial *if and only if:*

$$\forall H \leq G \quad |\langle H, H^\gamma \rangle : (H \cap H^\gamma)| < \infty.$$

This is equivalent to the fact that both H and H^γ have finite index in their join $\langle H, H^\gamma \rangle$. Clearly a strongly inertial automorphism is inertial. Moreover, we have $|\langle H, H^\gamma \rangle : (H \cap H^\gamma)| = |\langle H^{\gamma^{-1}}, H \rangle : (H^{\gamma^{-1}} \cap H)|$. Therefore, if γ is strongly inertial, then γ^{-1} is strongly inertial as well. Then a strongly inertial automorphism is bi-inertial. Moreover, when the underlying group is abelian, an inertial automorphism is strongly inertial and the two notions coincide.

Proposition 3.24. *Strongly inertial automorphisms of any group G form a subgroup $s\mathcal{I}\mathrm{Aut}(G)$ of $\mathrm{Aut}(G)$ containing the subgroup $s\mathrm{FAut}(G)$ of strongly finitary automorphisms of G.*

Proof. If γ_1 and γ_2 are strongly inertial, then the subgroup $H \cap H^{\gamma_1 \gamma_2}$ has finite index in H for each $H \leq G$, since $\gamma_1 \gamma_2$ is inertial, as we said above. Moreover, H has finite index in $H_1 := \langle H, H^{\gamma_1} \rangle$ which in turn has finite index in $\langle H_1, H_1^{\gamma_2} \rangle$ which contains $\langle H, H^{\gamma_1 \gamma_2} \rangle$. For what concerns $s\mathrm{FAut}(G)$ see Theorem 3.8 and note that, if $\gamma \in s\mathrm{FAut}(G)$ acts trivially on G/N, then $\langle H, H^\gamma \rangle \leq HN$. $\qquad\qquad\square$

In the dihedral group $\langle x, y \mid x^2 = 1, y^x = y^{-1} \rangle$ the inner automorphism induced by y is finitary hence inertial, but not strongly inertial as $\langle x, x^y \rangle$ is infinite.

Clearly, *a group is strongly inertial if and only if each inner automorphism is strongly inertial.*

Problem 3.25. *Study the group $s\mathcal{I}\mathrm{Aut}(G)$, when G is any group.*

4. **Endomorphisms of an abelian group**

4.1. **Inertial endomorphisms of an abelian group.** From §1.3 recall that an endomorphism φ of an abelian group A such that $H + \varphi(H)/H$ is finite for each subgroup H of A is called an inertial endomorphism. Recall also that an endomorphism φ of A is called finitary if the subgroup $\{x \in A \mid \varphi(x) = x\}$ has finite index in A.

Note that finitary endomorphisms are trivially inertial. More generally, if an endomorphism φ of an abelian group A acts as an inertial endomorphism modulo a finite subgroup of A (or on a subgroup of finite index), then φ is inertial on the whole A. To consider other instances of inertial endomorphisms we need to introduce some terminology.

Definition 4.1. *An endomorphism φ of an abelian group A is called a* multiplication *if one of the following holds:*

(1) *A is periodic and $\varphi(H) \subseteq H$ for each subgroup H of A;*

(2) *A is not periodic and there are $m, n \in \mathbb{Z}$ such that $A = nA$, $A_{\pi(n)} = 0$ and $\varphi(nx) = mx$ for all $x \in A$*

If (1) holds, as we noticed in §2.1, for a power automorphism γ, there exist a p-adic integer α such that we have $\gamma(a) = \alpha(a)$ for all $a \in A_p$ (here we are in additive notation for A and $\alpha(a)$ stands for na, where n is any integer congruent to α modulo the order of a). We can say that φ acts as the multiplication by α on A_p. In case (2) the abelian group A has a natural structure of $\mathbb{Q}^{\pi(n)}$-module and we say that φ acts as the multiplication by $\frac{m}{n} \in \mathbb{Q}$. By abuse of notation, we write $\varphi = \frac{m}{n}$. Note that we use the word "multiplication" in a way different from [62]. Ours are in fact "scalar" multiplications.

Theorem 4.2. [37, 35] *Let A be an abelian group.*

(1) *The multiplications of A form a ring $M(A)$ and commute with any endomorphism.*

(2) *If A is torsion-free, then all inertial endomorphisms of A are multiplications.*

(3) *If $r_0(A) < \infty$, then all multiplications are inertial.*

In order to study inertial endomorphisms of torsion abelian groups, note first that a standard argument shows that *an endomorphism φ of an abelian torsion group A is inertial if and only if φ is inertial on all primary components A_p and multiplication on all but finitely many of them.* So that we are reduced to consider inertial endomorphisms of abelian p-groups.

Theorem 4.3. [37] *Let φ be an endomorphism of an abelian p-group A. Then φ is inertial if and only if there is a φ-invariant subgroup A_0 of A of finite index such that either (1) or (2) holds:*

(1) *φ acts as a multiplication on A_0;*

(2) *$A_0 = B \oplus D$ where B is bounded, D is divisible of finite rank, B and D are φ-invariant and φ acts as a multiplication (by possibly different p-adics) on both B and D.*

If φ is inertial, then there is $m \in \mathbb{N}$ such that for each $X \leq A$ there are φ-invariant subgroups X_*, X^* of A such that $X_* \leq X \leq X^*$ and $|X^*/X_*| \leq m$.

For the case of non-periodic groups we have the following.

Theorem 4.4. [37] *Let φ be an endomorphism of a non-periodic abelian group A. Then φ is inertial if and only if either (1) or (2) holds:*

(1) *there is a φ-invariant subgroup A_0 of A of finite index such that $\varphi = m \in \mathbb{Z}$ on A_0;*

(2) *there are $\frac{m}{n} \in \mathbb{Q}$, $r \in \mathbb{N}$ and a φ-invariant subgroup $V \simeq \mathbb{Q}^{\pi(n)} \oplus \cdots \oplus \mathbb{Q}^{\pi(n)}$ (r times) of A such that*

　　i) *φ acts as the multiplication by $\frac{m}{n}$ on V;*

　　ii) *the factor group A/V is torsion and φ induces an inertial endomorphism on A/V;*

　　iii) *$A_{\pi(n)}$ is bounded.*

We give also a more detailed (and technical) characterization for the action of finitely many inertial endomorphisms.

Theorem 4.5. [37]. *Let $\varphi_1, \ldots, \varphi_l$ be finitely many endomorphisms of an abelian group A. Then each φ_i is inertial if and only if there is a subgroup A_0 of A of finite index which is φ_i-invariant for each i and such that either (1) or (2) holds:*

(1) *φ_i acts as multiplication on A_0 by some $m_i \in \mathbb{Z}$ for each i;*

(2) *$A_0 = B \oplus D \oplus C$ where B, C, D are φ_i-invariant for each i and there exist finite sets of primes π, π_1 such that $\pi \subseteq \pi_1$ and*

    i) *$B \oplus D$ is the π_1-component of A_0, where B is bounded and D is a divisible π'-group of finite rank;*

    ii) *C is a π-divisible subgroup with $C_\pi = \{0\}$ and there is a subgroup $V \simeq \mathbb{Q}^\pi \oplus \cdots \oplus \mathbb{Q}^\pi$ (finitely many times) which is φ_i-invariant for each i and such that C/V is a π_1-divisible π'-group;*

    iii) *each φ_i acts by multiplication on each $B, D, V, C/V$ where φ_i acts as $m_i/n_i \in \mathbb{Q}$ on V and on all p-components of D such that the p-component of C/V is infinite and $\pi = \pi(n_1 \cdots n_l)$.*

4.2. The ring of inertial endomorphisms of an abelian group.
We have already seen that inertial endomorphisms of any abelian group A form a ring, denoted by $\mathcal{I}\mathrm{End}(A)$, containing the ideal $F(A)$ of endomorphisms with finite image.

In [35] one finds a rather satisfactory description of the whole ring $\mathcal{I}\mathrm{End}(A)$. To report it in the general case we would need to introduce many tools which are too technical for the present paper. Here we recall the perhaps most important results and refer the reader to [35].

Theorem 4.6. [37] *The ring $\mathcal{I}\mathrm{End}(A)$ of inertial endomorphisms of any abelian group A is commutative modulo its ideal $F(A)$ formed by the endomorphisms of A with finite image.*

In the case $r_0(A) < \infty$, one may wonder if the ring $\mathcal{I}\mathrm{End}(A)$ is generated by the subrings $M(A)$ and $F(A)$. Actually, this is not the case, even for p-groups. However for p-groups we have the following theorem. Say that an abelian p-group is *critical* when the maximum divisible subgroup $D(A)$ has positive finite rank and $A/D(A)$ is bounded but infinite.

Theorem 4.7. [35] *Let A be an abelian p-group.*

(1) *If A is non-critical then $\mathcal{I}\mathrm{End}(A) = F(A) + M(A)$.*

(2) *If A is critical, then $\mathcal{I}\mathrm{End}(A) = M(A) + F(A) + N$, where $N \simeq \mathbb{Z}(p^e)$ and p^e is the exponent of $A/D(A)$.*

4.3. Inertiality for linear mappings.
Recall that in 1988 Hall and Phillips in [78, 90] considered finitary linear maps (i.e. endomorphisms) of the vector space V to be those (invertible) linear maps β such that the subspace $(\beta - id)V$ is finite-dimensional. This is clearly equivalent to the fact that β acts as the identity map on a subspace of finite codimension. The study of groups of finitary linear maps has received much attention in the literature (see [82, 84, 105], and the bibliography therein for example). Along the same lines, Dardano and Rinauro [35] have considered the corresponding inertial properties for linear maps.

Theorem 4.8. [35] *Let β be an endomorphism of an infinite dimensional vector space V.*

(1) *Each subspace H of V has finite codimension in $H + \beta(H)$ if and only if β acts as a scalar multiplication on a finite codimension subspace of V.*

(2) *For each subspace H of V the codimension of $H \cap \beta(H)$ in H is finite if and only if β acts as multiplication by a non-zero scalar on a finite codimension subspace.*

Corollary 4.9. *The ring of endomorphisms β of a vector space V such that each subspace H has finite codimension in $H + \varphi(H)$ is the sum of the subring of scalar multiplications and the subring of finitary endomorphisms.*

5. **Fully inert vs fully invariant subgroups of abelian groups**

5.1. **The main problem and a conjecture.** The main object of this section is the study of solutions to the general Problem 1.12 on the interrelations between the following sublattices (of the lattice of subgroups of an abelian group A) that we introduced at the end of §1.4, see Proposition 1.4:

$$\mathcal{I}nv(A) \ \subseteq \ \mathcal{I}nv^\sim(A) \ \subseteq \ \mathcal{I}_u(A) \ \subseteq \ \mathcal{I}(A). \tag{*}$$

Two natural more specific problems arise right away:

Problem 5.1.

(1) *Describe $\mathcal{I}nv^\sim(A)$ for an abelian group A, i.e., determine when is a fully inert subgroup of A commensurable with a fully invariant subgroup of A.*

(2) *Characterize the class $\mathfrak{I}$ of abelian groups A with $\mathcal{I}(A) = \mathcal{I}nv^\sim(A)$, i.e., such that every fully inert subgroup of A is commensurable with some fully invariant subgroup of A.*

Let us involve now also the lattice $\mathcal{I}_u(A)$ in the chain (*). This gives rise to other two classes: the class $\mathfrak{I}_*$ (resp., $\mathfrak{I}_u$) of all groups A where the equality $\mathcal{I}nv^\sim(A) = \mathcal{I}_u(A)$ (resp., $\mathcal{I}_u(A) = \mathcal{I}(A)$) holds. Clearly, $\mathfrak{I} = \mathfrak{I}_* \cap \mathfrak{I}_u$, as $\mathfrak{I}$ is the class of all abelian groups A such that $\mathcal{I}nv^\sim(A) = \mathcal{I}(A)$.

Conjecture 5.2. *The class $\mathfrak{I}_*$ coincides with the class of all abelian groups, i.e., a subgroup H of an abelian group A is uniformly fully inert if and only if H is commensurable with a fully invariant subgroup of A.*

Let us note that the positive answer to the above conjecture implies the equality $\mathfrak{I} = \mathfrak{I}_u$, or equivalently, the inclusion $\mathfrak{I}_u \subseteq \mathfrak{I}_*$.

We are not aware even if the following weaker conjecture holds true:

Question 5.3. *Is the equality $\mathfrak{I} = \mathfrak{I}_u$ (or, equivalently, the inclusion $\mathfrak{I}_u \subseteq \mathfrak{I}_*$) true?*

We shall see in the sequel that the known results support our stronger Conjecture 5.2.

Now we state a useful proposition which shows that the study of (uniformly) fully inert subgroups of finite direct sums can be reduced to box-like subgroups. It was proved in [51, Lemma 4.1] for fully inert

subgroups of *divisible* abelian groups, but the proof remains valid for (uniformly) fully inert subgroups of *arbitrary* abelian groups.

Proposition 5.4. [51] *Let $A = A_1 \oplus \cdots \oplus A_n$ be an abelian group and $H \leq A$. For $i = 1, \ldots, n$, let $H_i := H \cap A_i$ and $H_* := H_1 \oplus \cdots \oplus H_n$. Then the following conditions are equivalent:*

(a) *$H \in \mathcal{I}(A)$ (resp., $H \in \mathcal{I}_u(A)$);*

(b) *$[H : H_*] < \infty$ and $H_* \in \mathcal{I}(A)$ (resp., $H_* \in \mathcal{I}_u(A)$).*

In case these conditions are fulfilled, then $H_i \in \mathcal{I}(A_i)$ (resp., $H_i \in \mathcal{I}_u(A_i)$) for all i.

The next theorem from [51] gives some necessary conditions satisfied by a fully inert subgroup of a divisible abelian group.

Theorem 5.5. [51, Theorem 3.8] *A proper fully inert subgroup H of a divisible abelian group D has finite torsion-free rank and $r_0(H) = r_0(D)$. Moreover, if H has some non-trivial divisible p-quotient, then H contains the p-primary component of D.*

The equality $r_0(H) = r_0(D)$ is not given in [51], but it easily follows, as the existence of a non-torsion element $x \in D$ such that $\langle x \rangle \cap H = \{0\}$, will produce an endomorphism $\varphi \in \mathrm{End}(D)$ such that $x \in \varphi(H)$, so $r_0(H + \varphi(H))/H) > 0$, witnessing $H \notin \mathcal{I}_\varphi(G)$, a contradiction.

The above theorem and the next proposition provide easy examples showing that neither $\mathfrak{I}$, nor $\mathfrak{I}_u$ coincide with the class of all abelian groups.

Proposition 5.6.

(1) *Every finitely generated full rank subgroup of a torsion-free abelian group A of finite rank is fully inert.*

(2) *$\{\{0\}, \mathbb{Q}^n\} = Inv(\mathbb{Q}^n) = Inv^\sim(\mathbb{Q}^n) = \mathcal{I}_u(\mathbb{Q}^n)$.*

(3) *$H \in \mathcal{I}(\mathbb{Q}^n)$ if and only if $H = H_1 \oplus \cdots \oplus H_n$ where the H_i are all isomorphic to a fixed subgroup of $\mathbb{Q}$.*

Hence $\mathbb{Q}^n \notin \mathfrak{I}_u$.

Proof. (1) has been proved in [52, Lemma 2.4] (see also [51, Lemma 2.3]).

To prove (2) let $\{0\} \neq H \in \mathcal{I}_u(\mathbb{Q}^n)$. If H is divisible, we have $r_0(H) = n$ and consequently $H = \mathbb{Q}^n$, by Theorem 5.5. If H is not divisible, then there is a prime p such that $H \neq pH$. The multiplication $\varphi = \frac{1}{p}$ is an automorphism of A with $H < \varphi(H)$. Therefore, $|\varphi(H)/H| \geq p$ and -by induction- $|\varphi^n(H)/H| \geq p^n$ contradicting $H \in \mathcal{I}_u(A)$. This proves the equality $\mathcal{I}_u(\mathbb{Q}^n) = \{\{0\}, \mathbb{Q}^n\}$ and, in view of (*), also (2).

To prove (3), pick $H \in \mathcal{I}(\mathbb{Q}^n)$ and note that $r_0(H) = n$, by Theorem 5.5. Now the assertion follows from Proposition 5.4 and [51, Theorem 4.9].   □

5.2. **Partial solutions of Problem 5.1.** Dikranjan, Salce and Zanardo [54] have shown that the class $\mathfrak{I}$ contains all free abelian groups:

Theorem 5.7. [54] *A fully inert subgroup of a free abelian group A is commensurable with a fully invariant subgroup of A.*

Recall that a fully invariant subgroups of a free abelian group A must be of the form nA for some $n \in \mathbb{N}$.

In the case of periodic abelian groups, the problem is more complicated reflecting the fact that fully invariant subgroups of p-groups have a more complicated structure. On one hand, Goldsmith, Salce and Zanardo [76] have shown that p-groups with some further restriction belong to $\mathfrak{I}$:

Theorem 5.8. [76, Theorem 3.10] *A fully inert subgroup of a direct sum of cyclic p-groups A is commensurable with a fully invariant subgroup of A.*

On the other hand, one has the following example of a p-group that does not belong to $\mathfrak{I}$. It is built using a construction of Pierce of abelian p-groups A whose endomorphism ring $End(A)$ is not that large, as $\text{End}(A) = J_p id_A + E_s(A)$, where $J_p id_A = M(A)$ is the ring of multiplication by p-adics and $E_s(A)$ is the ideal of *small* endomorphisms φ (i.e., such that for each $k \in \mathbb{N}$ there is an $m \in \mathbb{N}$ with $\varphi((p^m A)[p^k]) = 0$).

Example 5.9. [76] *There exist p-groups A that contain fully inert subgroups which are not commensurable with any fully invariant subgroup of A, i.e., A does not belong to $\mathfrak{I}$.*

Recently also a result of Chekhlov [24] about completely decomposable finite rank torsion-free abelian group has appeared. Here we state an extension of that result.

Theorem 5.10. [24] *For a completely decomposable finite rank torsion-free abelian group $A = A_1 \oplus \cdots \oplus A_n$, where $r_0(A_i) = 1$ for each $i = 1, \ldots, n$, the following are equivalent:*

   (1) *$A \in \mathfrak{I}$, i.e., every fully inert subgroup is commensurable with a fully invariant subgroup;*
   (2) *$pA_i \neq A_i$ and the types of A_i and A_j are either equal or incomparable, for each i, j and each prime p.*
   (3) *$A \in \mathfrak{I}_u$ i.e., every fully inert subgroup is uniformly fully inert.*

Proof. The equivalence $(1) \leftrightarrow (2)$ is proved in [24]. On the other hand, the implication $(1) \to (3)$ is trivial.

To prove $(3) \to (2)$, assume by contradiction that $pA_1 = A_1$ for some prime p. Then the multiplication φ_1 by $\frac{1}{p}$ is an automorphism of A_1. Pick for every $i = 1, \ldots, n$ a cyclic subgroup C_i of A_i and let $H = C_1 \oplus \cdots \oplus C_n$. This subgroup H is fully inert, by Proposition 5.6, while the subgroup $C_1 \cong \mathbb{Z}$ of A_1 is not uniformly fully inert since for each m we have $|C_1 + \varphi_1^m(C_1)/C_1| = p^m$. Moreover, φ_1 may be extended to an endomorphism φ of A such that $\varphi \restriction_{A_i} = 0$ for each $i \neq 1$. Then $|H + \varphi^m(H)/H| \geq p^m$ for each m and H cannot be uniformly fully inert in A either.

To prove the second part of the statement, argue by contradiction and assume (without loss of generality) that the type of A_1 is strictly smaller that the type of A_2. Pick $x_i \in A_i$ such that for the p-heights we have $h_{p_m}(x_1) < h_{p_m}(x_2)$ for infinitely many primes p_m with $m \in \mathbb{N}$. Then, for every $m \in \mathbb{N}$ there is a homomorphism $\psi_m : A_1 \to A_2$ such that $\psi_m(x_1) = \frac{1}{p_1 \cdots p_m} x_2$. Let $C_i := \langle x_i \rangle$. Then

$C_2 \leq \psi_m(C_1)$ and $|\psi_m(C_1) : C_2| \geq m$. We can extend ψ_m to an endomorphism ψ of the whole A such that $\psi \restriction_{A_i} = 0$ for each $i \neq 1$. Then we can show, as above, that ψ_m witnesses the failure of $H = C_1 \oplus \cdots \oplus C_n$ to be uniformly fully inert, although H is fully inert, a contradiction. $\qquad\square$

The characterization of the completely decomposable torsion-free abelian groups A such that every fully inert subgroup of A is commensurable with a fully invariant subgroup was extended in a more recent paper of Chekhlov [25, Theorem 2] to the case when A has finitely many homogeneous components.

5.3. **Self-inert groups and fully inert subgroups of divisible groups.** Although the case of divisible groups was the first one to be treated in the pioneering paper [51], we are facing it only here as it is notoriously more complicated. We call an abelian group G *self-inert*, if G is fully inert in its divisible hull. This concept was introduced in [51] under the term *inert*, that we avoid here for obvious reasons.

We collect in the next theorem the essential facts necessary for the description of the self-inert groups.

Theorem 5.11. [51, Theorem 4.10] *If A is a self-inert group, then $A = t(A) \oplus B$ splits and the subgroups $t(A)$ and B are both self-inert.*

   (1) [51, Theorem 4.9] *A torsion-free abelian group A is self-inert if and only if it is either divisible or completely decomposable homogeneous of finite rank.*
   (2) [51, Theorem 5.2] *An abelian p-group A is self-inert if and only if it is either divisible or bounded with at most one infinite Ulm-Kaplansky invariant.*
   (3) [51, Theorem 5.3] *A torsion abelian group A is self-inert if and only if each p-primary component of A is a self-inert p-group and all but finitely many of them are either divisible or have a single nonzero Ulm-Kaplansky invariant.*

The conditions provided by the first assertion of above theorem are only *necessary*. Conditions that are simultaneously *necessary and sufficient* for a mixed abelian group to be self-inert can be found in [51, Theorem 5.5]. We skip them as they are rather technical and will not be used in the sequel in this survey.

The following simple observation can be helpful in the case for fully invariant subgroups:

Remark 5.12. *Let φ an endomorphism of an abelian group A.*
   (1) *If H is a subgroup of an abelian group A containing a fully invariant subgroup N of A and $\bar{\varphi} : A/N \to A/N$ is the naturally induced endomorphism of A/N, then $H \in \mathcal{I}_\varphi(A)$ if and only if $HN \in \mathcal{I}_{\bar{\varphi}}(A/N)$.*
   (2) *Combining item (1) with Proposition 5.6 , we deduce that if $r_0(A) < \infty$ and H is a subgroup of A containing $t(A)$ with $r_0(H) = r_0(A)$, then $H \in \mathcal{I}(A)$ if $H/t(A)$ is finitely generated.*

Now we resolve Problem 5.1 for divisible groups:

Theorem 5.13. *For a divisible abelian group D the following are equivalent:*

(1) $D \in \mathfrak{I}$; i.e., every fully inert subgroup of D is commensurable with a fully invariant subgroup of D;

(2) either $r_0(D) = 0$ or $r_0(D)$ is infinite;

(3) $D \in \mathfrak{I}_u$, i.e., every fully inert subgroup of D is uniformly inert.

Proof. The implication (1) $\to$ (3) is trivial. In order to prove the implication (3) $\to$ (2) in a counter-positive form we need to check first that a non-torsion divisible group D of finite torsion-free rank does not belong to $\mathcal{I}_u$. Indeed, if $0 < r(D) = n < \infty$, then we can write, up to isomorphism, $D = \mathbb{Q}^n \oplus t(D)$. Now the subgroup $G = \mathbb{Z}^n \oplus t(D)$ of D is fully inert by Remark 5.12. On the other hand, H is not uniformly fully inert in view of the last assertion of Proposition 5.4 and Lemma 5.6.

To prove the remaining implication (2) $\to$ (1) we note first that according to Theorem 5.5, a divisible group D of infinite torsion-free rank admits no proper fully inert subgroups, hence $D \in \mathcal{I}$ vacuously. Therefore, we are only left with the proof of the fact that all torsion divisible abelian groups D belong to $\mathcal{I}$.

Assume that $G \in \mathcal{I}(D)$, we have to prove that $G \sim H$ for some fully invariant subgroup H of D. Let $D = \bigoplus_p D_p$ and $G = \bigoplus_p G_p$ be the respective primary decompositions of D and G. Clearly, $G \in \mathcal{I}(D)$ if and only if $G_p \in \mathcal{I}(D_p)$ for every p and G_p is a fully invariant subgroup of D_p for all but finitely many p. Let

$$\Pi = \{ p \in \mathbb{P} : G_p \text{ is a fully invariant subgroup of } D_p \} \quad \text{and} \quad \pi = \mathbb{P} \setminus \Pi.$$

Then $G_1 = \bigoplus_{p \in \Pi} G_p$ is a fully invariant subgroup of D and $G = G_1 \oplus G_\pi$, where $G_\pi = \bigoplus_{p \in \pi} G_p$. Our next aim (see Claim) will be to prove that for every $p \in \pi$ there exists a fully invariant subgroup H_p of D_p such that $G_p \sim H_p$ for every $p \in \pi$. Then $H_\pi := \bigoplus_{p \in \pi} H_p$ is a fully invariant subgroup of $\bigoplus_{p \in \pi} D_p$ and $G_\pi \sim H_\pi$, so $G \sim G_1 \oplus H_\pi$, and the latter is a fully invariant subgroup of D.

Claim. For every $p \in \pi$ there exists a fully invariant subgroup H_p of D_p such that $G_p \sim H_p$.

Proof. If $G_p = D_p$, there is noting to prove. Hence, assume that $G_p \neq D_p$ in the sequel.

It will be convenient to fix the divisible hull $\bar{G}_p$ of G_p in D_p as a minimal divisible subgroup of D_p containing G_p. Then $D_p = \bar{G}_p \oplus D^*$ splits. This yields that $G_p \in \mathcal{I}(\bar{G}_p)$, i.e., G_p is self-inert. By item (2) of Theorem 5.11, G_p is bounded and has at most one infinite Ulm-Kaplansky invariant. Since G_p is an essential subgroup of $\bar{G}_p$, this means that $G_p \sim H := \bar{G}_p[p^m]$ for some $m \in \mathbb{N}$. If $r_p(G_p) < \infty$, then G_p and H are finite, so $G_p \sim \{0\}$, and we are done. This proves the claim.

Assume that $r_p(G_p)$ is infinite. Then $r_p(D^*) < \infty$, since otherwise one can easily build an endomorphism of D_p witnessing the failure of G_p to be fully inert in G_p (by simply sending the p-socle $G_p[p]$ to an infinite subgroup of the p-socle of $D^*[p]$ and then extending this to an endomorphism of the divisible group D_p). By $r_p(D^*) < \infty$, we deduce that $D^*[p^m]$ is finite, so $H \sim H \oplus D^*[p^m] = D[p^m]$. This proves the claim (and the theorem), as $G_p \sim H$. $\qquad\square$

The next corollary confirms Conjecture 5.2 in the case of divisible groups:

Corollary 5.14. *The class $\mathfrak{I}_*$ contains all divisible abelian groups, i.e., every uniformly fully inert subgroup of a divisible abelian group D is commensurable with a fully invariant subgroup of D. In particular, $Inv^{\sim}(t(D)) \cup \{D\} = Inv^{\sim}(D)$ and*

    (1) *if $0 < r_0(D) < \infty$, then $\mathcal{I}(D) \neq \mathcal{I}_u(D) = Inv^{\sim}(D)$, hence $D \in \mathfrak{I}_*$;*

    (2) *otherwise, $\mathcal{I}(D) = \mathcal{I}_u(D) = Inv^{\sim}(D)$, hence $D \in \mathfrak{I}$.*

Proof. Let D be a divisible group. The equality $Inv^{\sim}(t(D)) \cup \{D\} = Inv^{\sim}(D)$ is obvious. If $D \in \mathfrak{I}$, then $\mathcal{I}(D) = Inv^{\sim}(D)$ and there is nothing to prove. Assume that $D \notin \mathfrak{I}$. Then $D = \mathbb{Q}^n \oplus t(D)$ for some positive $n \in \mathbb{N}$, by the above theorem. Let $G \in \mathcal{I}_u(D)$. We have to prove that G is commensurable with a fully invariant subgroup. If $G \leq t(G)$, this follows by the above theorem. Let us see that this is the only case that may occur, in case $G \neq D$. Assume that $G \neq t(G)$. By Lemma 5.4, $G \sim G_1 \oplus G_2$, where $G_1 \in \mathcal{I}_u(\mathbb{Q}^n)$ and $G_2 \in \mathcal{I}_u(t(D))$. Our assumption gives $G_1 \neq 0$. By Proposition 5.6, from $G_1 \in \mathcal{I}_u(\mathbb{Q}^n)$ we can deduce that $G_1 = \mathbb{Q}^n$. From Theorem 5.5 we deduce that $G_2 = t(D)$, i.e., $G = D$. $\blacksquare$

5.4. **Generalizations.** To overcame the problem that an abelian group may have many fully invariant subgroups, recently, Calugareanu [14] has introduced *strongly invariant* subgroups, that is subgroups H of a (possibly non-abelian) group G with the property that $f(H) \leq H$ for each homomorphism $f : H \to G$. For example, fully invariant direct factors are clearly strongly invariant. On the other hand if $\mathbb{Q}_p = \{m/n \in \mathbb{Q} \mid gcd(n; p) = 1\}$, then the subgroup $p\mathbb{Q}_p$ of $\mathbb{Q}_p$ is fully invariant but not a strongly invariant subgroup of $\mathbb{Q}_p$ (consider multiplication by $1/p$).

    Notice that unfortunately the usage of word "strong" in [14] is different from that in [39].

Theorem 5.15. [14] *The only strongly invariant subgroups of a reduced abelian p-group A are the subgroups $A[p^n]$ for natural numbers n.*

    Following Braez and Calugareanu [12], in the next statement we call a subgroup H of an abelian group A *strongly inert*, if the factor group $(H + f(H))/H$ is finite for every homomorphism $f : H \to A$. In the vain of Theorem 5.15 we have:

Theorem 5.16. [12] *Let H a strongly inert subgroup (in the terminology of [12]) of an abelian group A. Then:*

    (1) *if A is free abelian, then A/H is finite.*

    (2) *if A is torsion, then H is commensurable with a strongly invariant subgroup of A.*

    Inertiality properties for modules have been considered, along the same lines as for abelian groups, by Goldsmith, Salce and Zanardo [77] in the case of a J_p-module A, where J_p denotes the ring of p-adic integers. A submodule H of A is called fully inert if $(H + \varphi(H))/H$ is finite for all $\varphi \in \mathrm{End}_{J_p}(A)$. Two submodules K, L of A are called commensurable if they are commensurable as subgroups, i.e., if $(K + L)/(K \cap L)$ is finite. We denote by $\mathfrak{I}_p$ the class of all J_p-modules A such that every fully inert submodule of N is commensurable with a fully invariant submodule of A.

Example 5.17. [77, Proposition 2.3] *If A is a free J_p-module of finite rank n, then one can easily see that a non-zero fully invariant submodule of A has the form $p^m A$ for some non-negative $m \in \mathbb{Z}$ and $p^m A \sim A$. Furthermore, a non-zero submodule H of A is fully inert if and only if H has rank n. Clearly, this occurs precisely when H has finite index (equivalently, when A/H is torsion). In such a case, H is commensurable with A. Therefore, $A \in \mathfrak{I}_p$.*

These authors showed that the essential part of this property remains true for all free J_p-modules:

Theorem 5.18. [77] *Let A be a free J_p-module of infinite rank. Then for a submodule H of A the following are equivalent:*

(1) *H is fully inert;*

(2) *H is commensurable with some fully invariant submodule;*

(3) *A/H is a bounded p-group with at most one infinite Ulm-Kaplansky invariant and $(H+pA)/pA$ is finite whenever A/H is infinite.*

This theorem unifies Theorems 2.4 and 2.5 from [77]. The more technical condition in item (3) describes intrinsically the fully inert submodules. The assumption $|(H + pA)/pA| < \infty$ cannot be avoided in general, as shown in [77, Example 2.6].

The equivalence of (1) and (2) in Theorem 5.18 and the next theorem show that free J_p-module of infinite rank as well as complete torsion-free J_p-modules belong to the class $\mathfrak{I}_p$.

Theorem 5.19. [77, Theorem 3.6] *A submodule of a complete torsion-free J_p-module A is fully inert if and only if it is commensurable with a fully invariant submodule (that is with some $p^n A$ for $n \in \mathbb{N}$).*

Finally, an example of a torsion-free J_p-module A with a fully inert submodule which is not commensurable with any of its submodules $p^n A$ is obtained using deep realization theorems of commutative rings as endomorphism rings of J_p-modules ([77, Theorem 4.1])

6. Applications to algebraic entropy of group endomorphisms

We see in this section that the concept of φ-inert subgroup is a very helpful tool also for the study of the dynamical properties of a given endomorphism.

6.1. **Algebraic entropy.** Roughly speaking, the algebraic entropy is a non-negative real number or ∞ assigned to an endomorphism φ of a group G measuring how chaotically φ acts on certain families of (most often, finite) subgroups or subsets of G.

Since the entropy is defined as a limit, one needs a tool providing the existence of limits.

Remark 6.1. *A sequence $\{a_n\}_{n \in \mathbb{N}}$ of non negative real numbers is called* subadditive *if $a_{n+m} \leq a_n + a_m$ for every $n, m \in \mathbb{N}$. According to a folklore fact, known as* Fekete Lemma*, for a subadditive sequence $\{a_n\}_{n \in \mathbb{N}}$, the sequence $\{\frac{a_n}{n}\}_{n \in \mathbb{N}}$ converges and $\lim_{n \to \infty} \frac{a_n}{n} = \inf_{n \in \mathbb{N}} \frac{a_n}{n}$.*

6.1.1. **The entropies** ent **and** h_{alg} **and their basic properties.** For a subset F of an abelian group $(A, +)$, $\varphi \in End(A)$ and $n \in \mathbb{N}$, the *n-th trajectory* of F under the action of φ is the set

$$T_n = T_n(\varphi, F) := F + \varphi(F) + \cdots + \varphi^{n-1}(F).$$

If F is a subgroup, then each $T_n := T_n(\varphi, F)$ is a subgroup and the *trajectory* $T(\varphi, F) := \bigcup_n T_n$ of F under the action of φ is the smallest φ-invariant subgroup of A containing F. The entropy measures how rapidly grows the chain of n-th trajectories $T_n(\varphi, F)$ when approximating the trajectory $T(\varphi, F)$.

When F is finite, the limit

$$\mathcal{H}_{alg}(\varphi, F) := \lim_{n \to \infty} \frac{\log |T_n(\varphi, F)|}{n}$$

exists, according to Remark 6.1, as the sequence $\log |T_n(\varphi, F)|$ is subadditive. This idea was originally used by Adler, Konheim and McAndrew [1] in the case when F is a finite *subgroup*, so they defined the *algebraic entropy* ent of φ by

$$\text{ent}\,(\varphi) := \sup\{\mathcal{H}_{alg}(\varphi, F) : F \text{ finite subgroup of } A\}.$$

Example 6.2 (Bernoulli shift)**.** *The leading example is the (right) Bernoulli shift defined as follows. Let F be a finite abelian group and let $\beta_F : \bigoplus_{n=1}^{\infty} F \to \bigoplus_{n=1}^{\infty} F$ be the endomorphsm defined by*

$$\beta_F(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots).$$

Then ent $(\beta_F) = \log |F|$ [53]. *The Bernoulli shift β_F can be defined for an infinite F as well, but then* ent $(\beta_F) = \infty$.

The entropy function ent is not appropriate in studying endomorphisms of torsion-free groups. To this end one can define an appropriate modification as follows. The *algebraic entropy* $h_{alg}(\varphi)$ of φ is defined as

$$h_{alg}(\varphi) := \sup\{\mathcal{H}_{alg}(\varphi, F) : F \text{ finite subset of } A\}. \tag{1}$$

The use of the same term for two distinct entropy functions may look somewhat embarrassing, but we are following the traditional terminology. Anyway, if will always be clear from the context which is the precise entropy in question.

The subtle question of whether the sup in (1) is attained (i.e., sup is a max) in case when $h_{alg}(\varphi) < \infty$ is still open and will be discussed in Remark 6.8. Clearly, ent $(\varphi) = h_{alg}(\varphi \restriction_{t(A)})$ for any abelian group A and $\varphi \in End(A)$, so ent $(\varphi) \le h_{alg}(\varphi)$ and these entropies coincide when A is torsion. On the other hand, if A is not torsion, then there exists $\varphi \in End(A)$ with ent $(\varphi) < h_{alg}(\varphi)$ (any multiplication by an integer $m > 1$ will do, see Example 6.3).

One can prove that $h_{alg}(\varphi^{-1}) = h_{alg}(\varphi)$ for an automorphism φ, and more generally

$$h_{alg}(\varphi^n) = |n| h_{alg}(\varphi) \text{ for every integer } n. \tag{LL}$$

This rule for computation of the entropy of powers is known as Logarithmic Law. It remains true also for endomorphisms as far as $n > 0$. Note that (LL) implies $h_{alg}(id) = 0$, although this can be directly obtained from the definition. We shall see below that this need not be true in the non-abelian case.

The above definition was proposed in [49], inspired by a similar notion of entropy introduced by Peters [89] in 1979 for automorphisms φ and using the "negative" n-th trajectories (i.e., $T_n(\varphi^{-1}, F)$). Since the limit obviously gives $h_{alg}(\varphi^{-1})$, the values of these two notions coincide for automorphisms, by virtue of (LL). Another leading example of computation of the algebraic entropy is given below (Example 6.3).

In order to describe the algebraic entropy of endomorphisms of $\mathbb{Q}^n$ we need to recall the definition of Mahler measure. Let n be a positive integer, let $f(t) = st^n + a_{n-1}t^{n-1} + \cdots + a_0 \in \mathbb{Z}[t]$ be a primitive polynomial and let $\{\lambda_i : i = 1, \ldots, n\} \subseteq \mathbb{C}$ be the set of all roots of $f(t)$ taken with their multiplicity. The (logarithmic) Mahler measure of $f(t)$ was defined independently by Lehmer and Mahler in two different equivalent forms as

$$m(f(t)) = \log|s| + \sum_{|\lambda_i|>1} \log|\lambda_i|.$$

Example 6.3 (Algebraic Youzvinski formula). *An endomorphism φ of $\mathbb{Q}^n$ is uniquely determined by its $n \times n$ matrix over $\mathbb{Q}$ and its characteristic polynomial $f(t) = st^n + a_{n-1}t^{n-1} + \cdots + a_0$ considered as a primitive polynomial with integer coefficients. The algebraic entropy of φ con be computed by using the formula $h_{alg}(\varphi) = m(f)$, established in* [68] *under the name* Algebraic Youzvinski formula.

In particular, if φ is a non-zero endomorphism of $\mathbb{Q}$, then φ is the multiplication by some rational number a/b, with non-zero co-prime a, b, so the above formula gives $h_{alg}(\varphi) = \max\{\log|a|, \log|b|\}$. Therefore, for every non-zero endomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}$ one has $h_{alg}(\varphi) = \log|\varphi(1)|$.

The next theorem collects three fundamental properties of the algebraic entropy:

Theorem 6.4. [49] *Let φ be an endomorphism of an abelian group A.*

(1) [Invariance under conjugation] *If A_1 is a group and $\xi : A \to A_1$ an isomorphism, then for the endomorphism $\psi = \xi\varphi\xi^{-1}$ of A_1 one has $h_{alg}(\varphi) = h_{alg}(\psi)$.*

(2) [Continuity] *If A is direct limit of φ-invariant subgroups $\{A_i : i \in I\}$, then $h_{alg}(\varphi) = \sup_{i \in I} h_{alg}(\varphi \restriction_{A_i})$.*

(3) [Addition Theorem] *If N a φ-invariant subgroup of A and $\overline{\varphi} : A/N \to A/N$ the endomorphism induced by φ. Then*

$$h_{alg}(\varphi) = h_{alg}(\varphi \restriction_N) + h_{alg}(\overline{\varphi}).$$

6.1.2. Algebraic i-entropy for module endomorphisms. Let R be a ring and let Mod_R be the category of all left R-modules. An *invariant* of Mod_R is a function $i : \mathrm{Mod}_R \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ with $i(0) = 0$ and $i(M) = i(N)$ whenever $M \cong N$. The invariant is said to be *subadditive*, if

(a) $i(N_1 + N_2) \leq i(N_1) + i(N_2)$ for all submodules N_1, N_2 of M;

(b) $i(M/N) \leq i(M)$ for every submodule N of M;

Furthermore, the invariant i is called *discrete*, if the set I of all finite values $i(M)$ is a discrete subset of $\mathbb{R}$ (i.e., $I \cap [0, N]$ is finite for every $N > 0$), integer valued invariant are typical examples of discrete invariants. The invariant i is called *additive* if equality holds in (a) (this implies (b)). In case i is additive and

$$i(M) = \sup\{i(N) : N \text{ is a finitely generated submodule of } M\}$$

the invariant i is call a *length function*.

For a right R-module M, denote by $\mathcal{F}_i(M)$ the family of all submodules N of M with $i(N) < \infty$. The *algebraic i-entropy* ent_i was introduced by Salce and Zanardo [96] as follows.

Definition 6.5. [96] *Let i be a subadditive invariant, M be an R-module and $\varphi : M \to M$ an endomorphism. The* algebraic i-entropy of φ with respect to $N \in \mathcal{F}_i(M)$ *is*

$$\mathcal{H}_i(\varphi, N) := \lim_{n \to \infty} \frac{i(T_n(\varphi, N))}{n};$$

The algebraic i-entropy of φ is $\mathrm{ent}_i(\varphi) := \sup\{H_i(\varphi, N) : N \in \mathcal{F}_i(M)\}$.

Typical examples are:

   (a) $R = \mathbb{Z}$, so Mod_R is the category of abelian groups, now we have two examples:

   (a_1) $i(M) = \log |M|$, then ent_i coincides with ent;

   (a_2) $i(M) = r_0(M)$, this entropy, termed *rank-entropy*, was largely studied in [96, 73];

   (b) R is a field, so that Mod_R is simply the category of R-linear spaces, and $i(M) = \dim M$. This entropy $\mathrm{ent}_{\dim}$ was briefly mentioned in [96] and thoroughly studied in [15, 64].

Salce, Vámos and Virili [94] extensively studied the entropy function ent_i for a lenght function i.

6.2. **Intrinsic algebraic entropy.** Regardless of their nice properties, the entropies ent and h_{alg} have their disadvantages. The case of ent was discussed above, the case of h_{alg} has the obvious inconvenience to work with *subsets* rather than *subgroups*. An excellent remedy to both problems is provided by the *intrinsic algebraic entropy* defined below.

It was observed already in [53], that if H is a subgroup of an abelian group A, then for the increasing chain of subgroups $T_n := T_n(\varphi, H)$ of A each quotient T_{n+1}/T_n is an epic image of $H + \varphi(H)/H = T_2/T_1$ for every $n \in \mathbb{N}$. Therefore, when H is finite, the sequence $|T_{n+1}/T_n|$ stabilizes to a value that coincides with $\mathcal{H}_{alg}(\varphi, H)$. Clearly this occurs also when simply the quotient $H + \varphi(H)/H$ is finite, i.e., H is φ-inert. This simple but relevant observation allowed Dikranian, Giordano Bruno, Salce and Virili [52] to extend the definition of the entropy $\mathcal{H}_{alg}(\varphi, F)$ to φ-inert subgroups F of A. The *intrinsic algebraic entropy* is defined as

(2) $$\widetilde{\mathrm{ent}}\,(\varphi) := \sup\{\mathcal{H}_{alg}(\varphi, F) : F \text{ is a } \varphi\text{-inert subgroup of } A\}.$$

This dynamical invariant is better behaved since it allows to treat also endomorphisms of torsion-free abelian groups where the entropy ent vanishes completely for the lack of non-trivial finite subgroups. One can easily show that

$$\mathrm{ent}\,(\varphi) \leq \widetilde{\mathrm{ent}}\,(\varphi) \leq h_{alg}(\varphi),$$

where the first inequality becomes an equality when A is torsion ([52]). The function $\widetilde{\mathrm{ent}}$ retains many of the useful properties of ent: the Logarithmic Law (LL) for $n > 0$ [52, Lemma 3.12], all three properties (1)–(3) from Theorem 6.4 (these are Lemma 3.9, Lemma 3.14 and Corollary 5.10 respectively, from [52]), etc.

Here we recall the following counterpart of the Algebraic Youzvinski formula obtained in [52].

Theorem 6.6. [52, Theorem 4.2] *Let $m \in \mathbb{N}$ and let $\varphi : \mathbb{Q}^m \to \mathbb{Q}^m$ be an endomorphism. Then* $\widetilde{\mathrm{ent}}\,(\varphi) = \log s$, *where s is the positive leading coefficient of the characteristic primitive polynomial of φ over $\mathbb{Z}$.*

Further details and results can be found in [52] and in the survey paper [74].

The part *intrinsic* in the name of this new entropy is justified by the fact that in the defining formula (2) one makes recourse to a family of subgroups of A depending (intrinsically) on φ. Recently, Goldsmith and Salce [73] investigated the ground to consider this entropy really as an *intrinsic* one. To this end they introduced the so called *fully inert algebraic entropy*

$$\widetilde{\mathrm{ent}}\,(\varphi) = \sup\{\mathcal{H}(\varphi, F) : F \in \mathcal{I}(A)\},$$

where A is an abelian group and $\varphi \in \mathrm{End}(A)$. In the definition of this entropy the choice of the subgroups F does not depend on φ. Their aim was to compare this new entropy with the intrinsic one. They showed that $\widetilde{\mathrm{ent}}\,(\varphi)$ can be computed by using the fully inert algebraic entropy $\widetilde{\mathrm{ent}}\,(\bar{\varphi})$ of the induced endomorphism $\bar{\varphi}$ of φ-invariant sections of G [73, Theorem 5.5]. On the other hand, it is also shown that the fully inert algebraic entropy need not coincide with the intrinsic one in general and cases when they coincide (so "when the intrinsic entropy is not really intrisic") are examined ([73, Theorems 5.1, 5.3 and 6.4]).

6.2.1. **Intrinsic algebraic i-entropy.** Assume that i is a discrete invariant of R-modules. Then one can define intrinsic algebraic i-entropy for endomorphisms of R-modules following the pattern of the definition of intrinsic algebraic entropy for abelian groups. Namely, noticing that if F is a submodule of M, then each $T_n := T_n(\varphi, F)$ is a submodule of M and each quotient T_{n+1}/T_n is an epic image of T_n/T_{n-1}, so in particular of $F+\varphi(F)/F = T_2/T_1$. Therefore, when $i(F+\varphi(F)/F)$ is finite, the sequence $i(T_{n+1}/T_n)$ stabilizes to a value, say $\mathcal{H}_i(\varphi, F)$. This motivates the notion of an *i-φ-inert submodule* of M, namely a submodule F of M such that $i(F+\varphi(F)/F)$ is finite. Then the *intrinsic algebraic i-entropy* $\widetilde{\mathrm{ent}}\,_i$ is defined by

$$\widetilde{\mathrm{ent}}\,_i(\varphi) = \sup\{\mathcal{H}_i(\varphi, F) : F \text{ is a } i\text{-}\varphi\text{-inert submodule of } M\}.$$

In the sequel we shall use it for linear spaces and $i = \dim$. In this setting the intrinsic algebraic i-entropy was defined and studied in [15] (see also the recent paper [21] and the comment in §6.5 below). It was proved in [15], that for linear space endomorphisms the entropies $\mathrm{ent}\,_{\dim}$ and $\widetilde{\mathrm{ent}}\,_{\dim}$ coincide. The linear maps φ for which each subspace is dim-φ-inert have been treated in §4.3.

6.3. **Values and uniqueness of the algebraic entropy.** It follows from the definiton of ent and $\widetilde{\mathrm{ent}}$, that the finite values of these entropies belong to the (discrete) subset $\log \mathbb{N} = \{\log n : n \in \mathbb{N}, n > 0\}$ of the whole possible range $\mathbb{R}_{\geq 0}$.

As far as the entropy function h_{alg} is concerned, the question of understanding its precise range, i.e., set of values

$$\mathcal{E} = \{h_{alg}(\varphi) : \varphi \in End(A), A \text{ abelian group}\} \subseteq \mathbb{R}_{\geq 0} \cup \{\infty\}$$

is much more complicated and still open. In fact, it is not known whether $\mathcal{E} = \mathbb{R}_{\geq 0} \cup \{\infty\}$. More precisely, even the simpler question of whether $0 = \inf(\mathcal{E} \setminus \{0\})$ is open and equivalent to a long standing open problem of Lehmer (see Problem 6.7).

The Lehmer Problem concerns the range of the Mahler measure. Let us start by pointing out that the case of zero Mahler measure is completely determined by the a classical theorem due to Kronecker: $m(f(t)) = 0$ for a primitive $f(t) \in \mathbb{Z}[t] \setminus \{0\}$ if and only if $f(t)$ is cyclotomic (i.e., all the roots of $f(t)$ are roots of unity). Before going further let us recall, that Mahler measure is used also when referring to an algebraic integer α, taking into account the Mahler measure of the minimal polynomial of α. In these terms, the (still open) problem posed by Lehmer in 1933 aks whether the lower bound of all Mahler measures of algebraic integers that are not roots of unity is positive:

Problem 6.7 (Lehmer Problem). *Given any $\delta > 0$, is there any algebraic integer whose Mahler measure is strictly between 0 and δ?*

One can show that the positive solution of Lehmer Problem is equivalent to the fact that $\mathcal{E} = \mathbb{R}_{\geq 0} \cup \{\infty\}$ (as well as to the apparently much weaker property $0 \in \inf(\mathcal{E} \setminus \{0\})$, see [42] and [49, Theorem 1.5]).

The negative solution of Lehmer Problem has also a notable impact on the algebraic entropy (beyond the bold negation $\mathcal{E} \neq \mathbb{R}_{\geq 0} \cup \{\infty\}$)

Remark 6.8. *Let A be an abelian group and $\varphi \in \mathrm{End}(A)$ with $h_{alg}(\varphi) < \infty$. We shall briefly say that $h_{alg}(\varphi) < \infty$ realizes, if there exists a finite subset F of G such that $h_{alg}(\varphi) = h_{alg}(\varphi, F)$. It was shown in [49] that the negative solution of Lehmer Problem is equivalent to the assertion "the algebraic entropy of arbitrary endomorphisms of abelian groups realizes whenever it is finite".*

The importance of the Algebraic Youzvinski formula (see Example 6.3) and the Bernoulli shifts becomes clear from the Uniqueness Theorem for the algebraic entropy [49, Theorem 1.3], asserting that the algebraic entropy h_{alg} is the unique $\mathbb{R}_{\geq 0} \cup \{\infty\}$-valued function defined on endomorphisms of abelian groups having the three properties (1)–(3) from Theorem 6.4, satisfying the Algebraic Yuzvinski formula and the "normalization" condition, imposed by the values taken on the Bernoulli shifts β_F (see Example 6.2). Similar uniqueness theorem for $\widetilde{ent}$ is proved in [52, Theorem 6.1]. Both uniqueness theorems were inspired by the uniqueness theorem for the entropy function ent for endomorphisms of torsion abelian groups, previously established in [53].

6.4. **Growth function of an endomorphism.** Dikranjan and Giordano Bruno in [42, 47] have proposed an extension of the notions of entropy in the non-abelian context as follows.

If G is any group in multiplicative notation, F a finite subset of G and φ an endomorphism of G, one can consider the trajectory $T_n := T_n(\varphi, F) := F \cdot F^\varphi \cdots \cdots F^{\varphi^n}$ as the setwise product of the images F^{φ^i} for $i \leq n \in \mathbb{N}$. Clearly $|T_n| \leq |F|^n$. Furthermore, the limit $\mathcal{H}(\varphi, F) := \lim_n \frac{\log |T_n|}{n}$ exists by Remark 6.1 as the sequence $\log |T_n|$ is again subadditive and the entropy $h_{alg}(\varphi)$ is the sup of $\mathcal{H}(\varphi, F)$ when F ranges over all finite subsets of G. Now properties (1) and (2) from Theorem 6.4 remain true, but

(c) fails even for the identity automorphism, which may have infinite entropy in the non-abelian case. Thus, the Logarithmic Law (LL) fails for $n = 0$, but remains true for all $n \neq 0$ [42]. Yet, it implies $h_{alg}(id_G) \in \{0, \infty\}$.

The above described "anomalous" behavior of the identity automorphism in the non-abelian case can be understood under a more precise looking glass. Namely, considering the *growth function* $\gamma_{\varphi,F}(n) = |T_n(\varphi, F)|$ of the n-th trajectory rather than the limit $\mathcal{H}(\varphi, F) := \lim_n \frac{\log |T_n|}{n}$ which gives less information (see Lemma 6.9). Following [43, 42], we say that φ has polynomial (*respectively, exponential*) *growth* if $\gamma_{\varphi,F}(n)$ has such growth for all (respectively, for one) finite subsets F of G. Otherwise, we say that φ has intermediate growth. The next easy to check property (see [43, 42]) gives the basic relation between entropy and growth of an endomorphism.

Lemma 6.9. *Let φ be an endomorphism of a group G and let F be a finite subgroup of G. Then $\mathcal{H}(\varphi, F) > 0$ if and only if $\gamma_{\varphi,F}(n)$ has exponential growth. In particular, $h_{alg}(\varphi) > 0$ if and only if φ has exponential growth.*

The above terminology meets the existing one on growth of groups in the following sense. A group G has polynomial (respectively, exponential) growth (in the classical sense of geometric group theory [79]) precisely when the identity automorphism of G has polynomial (respectively, exponential) growth. Therefore, in a group G of exponential growth (e.g., a free non-abelian group) one has $h_{alg}(id_G) = \infty$.

This trend was triggered by the following result recalling relevant well known facts on the growth of finitely generated groups (according to the celebrated theorem of Milnor-Wolf every finitely generated soluble group has either polynomial or exponential growth [79]).

Theorem 6.10. [43] *All endomorphisms of an abelian group have either polynomial or exponential growth.*

One can see that *all inner automorphisms of a group have the same growth type* [66]. Moreover, these authors gave also a counterpart of Theorem 6.10 for locally finite groups.

Theorem 6.11. [66] *An endomorphism φ of a locally finite group has polynomial growth if and only if the algebraic entropy $h(\varphi)$ is zero.*

According to Lemma 6.9, this theorem implies that an endomorphism of a locally finite group has either polynomial or exponential growth. A simultaneous extension of this theorem and Theorem 6.10 to locally (soluble-by-finite) groups was obtained recently by Giordano Bruno and Spiga:

Theorem 6.12. [67] *An endomorphism of a locally (soluble-by-finite) group has either polynomial or exponential growth.*

One can define *intrinsic* entropy in the non-abelian case making use, again, of inert subgroups. The idea is that if H is a φ-inert subgroup of a group G, then the index $|H^\varphi : (H \cap H^\varphi)|$ is just the minimal size of a subset X of H such that $H^\varphi = (H \cap H^\varphi)X^\varphi$ (setwise product), hence $H^\varphi \subseteq HX^\varphi$. This will enable us to extend the definition of algebraic entropy (in the non-abelian setting) to an *intrinsic* one by

using the trajectories of finite transverals like X^φ above. We formulate this in the following immediate proposition:

Proposition 6.13. *Let φ be an endomorphism of a group G. If H is a φ-inert subgroup of G with $t := |H^\varphi : (H \cap H^\varphi)|$, then for every $n \in \mathbb{N}$, the setwise product $T_n := H \cdot H^\varphi \cdots H^{\varphi^n}$ is contained in a setwise product HY_n, where $Y_n \subseteq G$ is finite. If t_n is the smallest possible size of a finite $Y_n \subseteq G$ with $T_n \subseteq NY_n$, then $t_n \leq t^n$ and*

$$(3) \qquad \widetilde{\mathcal{H}}(\varphi, H) := \limsup \frac{\log t_n}{n}$$

exists, with $\widetilde{\mathcal{H}}(\varphi, H) \leq t$.

Proof. Fix a subset F with $|F| \leq t$ and $H^\varphi \subseteq HF^\varphi$. Then, by transforming this relation by φ^n, one has $H^{\varphi^{n+1}} \subseteq H^{\varphi^n} F^{\varphi^{n+1}}$. Let us show that the claim holds with $Y_n := F^\varphi \cdots F^{\varphi^n}$ arguing by induction. The case $n = 1$ is simply our hypothesis on F. Assume the statement holds for n. Now

$$T_{n+1} = T_n H^{\varphi^{n+1}} \subseteq T_n H^{\varphi^n} F^{\varphi^{n+1}} = T_n F^{\varphi^{n+1}} \subseteq HF^\varphi \cdots F^{\varphi^n} F^{\varphi^{n+1}}.$$

The inequality $|Y_n| \leq t^n$ is trivial. This proves $t_n \leq t^n$. The rest is obvious. $\qquad\square$

Call the limit in (3) *intrinsic entropy of φ with respect to N*. In case G is abelian, the $\limsup$ in (3) is a limit:

Problem 6.14. *Under which condition the $\limsup$ in (3) defining the intrinsic entropy $\widetilde{\mathcal{H}}(N, \varphi)$ is a limit ?*

6.4.1. **Limit-free computation of entropy.** The next rule for limit-free computation of entropy was proved in [45] in order to repair a wrong statement in [109]:

Theorem 6.15. [45, Algebraic Formula] *Let G be a locally finite group, $\varphi : G \to G$ an endomorphism and F a finite normal subgroup of G such that $\ker \varphi \cap T(\varphi, F)$ is finite. Then*

$$\mathcal{H}_{alg}(\varphi, F) = \log \left| \frac{T(\varphi, F)}{\varphi(T(\varphi, F))} \right| - \log |\ker \varphi \cap T(\varphi, F)|.$$

It was proved in [45, Lemma 4.1] that the restraint $|\ker \varphi \cap T(\varphi, F)| < \infty$ can be omitted in the abelian case; namely $\ker \varphi \cap T(\varphi, F)$ is finite whenever G is a torsion abelian group.

The next corollary was formulated by Yuzvinski's [109] who omitted the condition for injectivity of the endomorphism:

Corollary 6.16. [45, Corollary 1.1] *Let G be a locally finite group, $\varphi : G \to G$ an injective endomorphism and F a finite normal subgroup of G. Then*

$$\mathcal{H}_{alg}(\varphi, F) = \log \left| \frac{T(\varphi, F)}{\varphi(T(\varphi, F))} \right|.$$

Corollary 6.17. [45, Corollary 2.3] *Let G be a locally finite group, $\varphi : G \to G$ an endomorphism and F a finite normal subgroup of G such that $G = T(\varphi, F)$. If $\ker \varphi$ is finite, then*

$$\mathcal{H}_{alg}(\varphi, F) = \log |\operatorname{coker} \varphi| - \log |\ker \varphi|.$$

In particular, $|\operatorname{coker} \varphi| \geq |\ker \varphi|$.

For limit-free computation of intrinsic entropy see [69].

6.5. **Inert subgroups of topological groups.** There are some circumstances when inert subgroups naturally appear in the framework of topological groups and their continuous endomorphisms. In the sequel we denote by $C\mathrm{End}(G)$ the family of all continuous endomorphisms of a topological group G.

First we recall the definition of algebraic entropy of a continuous endomorphism φ of a locally compact group G. Let μ be a Haar measure of G and let $\mathcal{U}_G$ be the local base at 1_G formed by all compact neighborhoods of 1_G. Then $T_n(\varphi, U)$ is a compact set of G, so $\mu(T_n(\varphi, U)) < \infty$. Call

$$\mathcal{H}_{alg}(\varphi, U) := \limsup_n \frac{\log \mu(T_n(\varphi, U))}{n} < \infty$$

algebraic entropy of φ with respect to U and let $h_{alg}(\varphi) = \sup\{H_{alg}(\varphi, U) : U \in \mathcal{U}_G\}$, *algebraic entropy* of φ. For a discrete G this coincides with the definition given in §6.3, so using the same notation does not lead to confusion.

One can show that $h_{alg}(\varphi)$ does not depend on the choice of the Haar measure μ. In the case of abelian locally compact groups, this definition is due to Virili [102], the above definition is from [42].

Let G be a locally compact totally disconnected group. It is a well known fact, due to van Dantzig [30], that the filter base $\mathcal{B}_G$ consisting of all open compact subgroups of G forms a base of the neighborhood filter $\mathcal{U}_G$ at the identity of G. Here comes a remarkable connection to inertiality: the ring $C\mathrm{End}(G)$ is a subring of the ring $\mathcal{I}_{\mathcal{B}_G}\mathrm{End}(G)$ of all $\mathcal{B}_G$-inertial endomorphisms of G, in other words:

Lemma 6.18. *Every $U \in \mathcal{B}_G$ is φ-inert for every $\varphi \in C\mathrm{End}(G)$.*

Indeed, $\varphi(U)$ is compact, as a continuous image of the compact subgroup U, whereas $U \cap \varphi(U)$ is an open subgroup of the compact group $\varphi(U)$. This entails finiteness of the index $|\varphi(U) : (U \cap \varphi(U)|$, so U is is φ-inert.

We do not know if one can invert Lemma 6.18, i.e., does the equality $\mathcal{I}_{\mathcal{B}_G}\mathrm{End}(G) = C\mathrm{End}(G)$ hold true?

Question 6.19. *If G be a locally compact totally disconnected group, are then all $\mathcal{B}_G$-inertial endomorphisms of G necessarily continuous?*

One can apply Proposition 6.13 and Lemma 6.18 to get a a measure-free formula for the algebraic entropy in totally disconnected locally compact groups.

Proposition 6.20. *Let G be a totally disconnected locally compact group, $\varphi : G \to G$ a continuous endomorphism and $U \in \mathcal{B}(G)$. Then $\mathcal{H}_{alg}(\varphi, U) = \widetilde{\mathcal{H}}(\varphi, U)$.*

Proof. By Lemma 6.18, U is φ-inert so by Proposition 6.13 one can define $\widetilde{\mathcal{H}}(\varphi, U)$. Then one can argue as in the proof of [42, Proposition 4.5.3] (see also [42, Proposition 5.6.4]). □

6.5.1. **Entropy in locally linearly compact vector spaces.** In the sequel we consider topological vector spaces over a discrete field K, endowed with linear topologies, i.e., having a local base of neighborhoods of 0 formed by open subspaces. Such a space V is said to be *linearly compact*, if every filter base in V formed by cosets of closed subspaces of V has a non-empty intersection. A topological vector space space V is said to be *locally linearly compact*, if V has an open linearly compact subspace. These notions were introduced by Lefschetz in 1942.

In analogy to the situation observed in locally compact groups, when V is a locally linearly compact vector space and φ is a continuous vector space endomorphism of V, every open linearly compact subspace U of V is dim-φ-inert in the sense of §6.2.1 (i.e., $\dim(\varphi(U)/(U \cap \varphi(U))) < \infty$). The argument to see this is similar to that of Lemma 6.18: the subspace $\varphi(U)$ is linearly compact, as a continuous image of the linearly compact subspace U, whereas $U \cap \varphi(U)$ is an open subspace of the the linearly compact subspace $\varphi(U)$. This entails finiteness of the dimension of the (discrete) quotient $\varphi(U)/(U \cap \varphi(U))$. Therefore, U is dim-φ-inert, so one can define the intrinsic dim-entropy of φ. This entropy was deeply investigated in [21] (see also [20, 22] for its counterpart – the topological entropy).

6.6. **Epilogue.** Here we only mention very briefly three more relevant topics that were not touched so far. Before doing that let us mention that the algebraic entropy, so far defined for endomorphisms of abelian groups, was extended recently also to amenable actions of semigroups in [41] (a semigroup S is *right amenable* if for every finite subset $K \subseteq S$ and every $\varepsilon > 0$ there exists finite non-empty subset $F \subseteq S$, such that $|Fx \setminus F| \leq \varepsilon |F|$ for every $x \in K$). More precisely, if S is a right amenable cancellative semigroup with a left action $S \overset{\alpha}{\curvearrowright} A$ of S on an abelian group A, then the algebraic entropy $h_{alg}(S \overset{\alpha}{\curvearrowright} A)$ is defined in such a way that if $f \in \text{End}(A)$ and the action $\mathbb{N} \overset{\alpha}{\curvearrowright} A$ is defined by $\alpha(n)(x) = f^n(x)$ for $n \in \mathbb{N}$ and $x \in A$, then $h_{alg}(\mathbb{N} \overset{\alpha}{\curvearrowright} A)$ coincides with the algebraic entropy $h_{alg}(f)$ already defined. (The definition uses a generalization of Fekete Lemma from [23].) This entropy shares many of the properties of the algebriac entropy defined for single endomorphisms, for more detail see [41] (see also [103] for algebraic entropy of actions of amenable groups on locally compact abelian groups).

6.6.1. **Adjoint entropy and intrinsic adjoint entropy.** In analogy to the algebraic entropy ent, in [50] the adjoint algebraic entropy of endomorphisms of abelian groups A was introduced "replacing" the family of all finite subgroups of G by the family of all finite-index subgroups of A as follows. For a finite-index subgroup H of A, an endomorphism $\varphi : A \to A$ and $n \in \mathbb{N}_+$ the subgroup $C_n(\varphi, H) = H \cap \varphi^{-1}(H) \cap \cdots \cap \varphi^{-n+1}(H)$ still has a finite index. We call it *n-th φ-cotrajectory of H* and we call *φ-cotrajectory of H* the subgroup $C(\varphi, H) = \bigcap_{n \in \mathbb{N}} \varphi^{-n}(H)$. Clearly, it is the maximum φ-invariant subgroup of H.

(a) The *adjoint algebraic entropy of φ with respect to H* is

$$(4) \qquad \mathcal{H}^{\star}(\varphi, H) = \lim_{n \to \infty} \frac{\log |A : C_n(\varphi, H)|}{n}.$$

(b) The *adjoint algebraic entropy of* φ is

$$\mathrm{ent}^{\star}(\varphi) = \sup\{\mathcal{H}^{\star}(\varphi, H) : H \text{ is finite-index subgroup of } A)\}.$$

The adjoint algebraic entropy, although a very natural counterpart of the algebraic one, does not have equally nice properties for the simple reason that many groups fail to have sufficiently many finite-index subgroups. Consequently, this entropy is not monotone with respect to taking restrictions to invariant subgroups.

The remedy is provided again by the inert subgroups. A careful analysis of the limit in (a) shows that the sequence of the sizes of the finite quotient groups $\frac{C_n(\varphi, H)}{C_{n+1}(\varphi, H)}$ is stationary [50]. So, in order to define the limit one only needs the finiteness of the quotients. So similarly to the case of the intrinsic entropy, it is enough to have a φ-inert subgroup H in order to be able to define *intrinsic adjoint entropy* $\widetilde{\mathcal{H}}^{\star}(\varphi, H)$ of φ with respect to H by $\widetilde{\mathcal{H}}^{\star}(\varphi, H) = \lim_{n\to\infty} \frac{\log[H:C_n(\varphi, H)]}{n}$. The *intrinsic adjoint entropy* of φ is

$$\widetilde{\mathrm{ent}}^{\star}(\varphi) = \sup\{\widetilde{\mathcal{H}}^{\star}(\varphi, H) : H \in \mathcal{I}_{\varphi}(A)\}.$$

This entropy has better properties than the adjoint entropy [65]. More details on the adjoint entropy and the intrinsic adjoint entropy can be found in [50, 65].

6.6.2. **Entropy in normed semigroups and a unified approach to all entropies.** In the expository paper [47] a unifying approach for many known entropies in many branches in mathematics, algebra, topology, measure theory, set theory, etc. is described.

The basis step of this approach is a notion of semigroup entropy $h_{\mathfrak{S}}$ in the category $\mathfrak{S}$ of normed semigroups and contractive homomorphisms. It has a number of natural properties (as the Logarithmic Law, Invariance under Conjugation, etc.). The second step is to define, for a specific category $\mathfrak{X}$, a functor $F : \mathfrak{X} \to \mathfrak{S}$ and then defined the entropy h_F as the composition $h_F = h_{\mathfrak{S}} \cdot F$. More precisely, for every $\mathfrak{X}$ morphism φ one takes $h_F := h_{\mathfrak{S}}(F\varphi)$. This provides, under certain natural restrains on F, the same (or the counterparts of the) properties proved for $h_{\mathfrak{S}}$. This general scheme permits to obtain many of the known entropies as h_F, for appropriately chosen categories $\mathfrak{X}$ and functors $F : \mathfrak{X} \to \mathfrak{S}$. The fundamental properties of the algebraic entropy for group endomorphisms easily follow using this setting.

6.6.3. **Topological entropy and its relation to algebraic entropy.** In §6.5 we defined the algebraic entropy h_{alg} for continuous endomorphisms of a locally compact group G, endowed with a right Haar measure μ. In what follows we discuss the *topological entropy* h_{top} for such endomorphisms.

For $n \in \mathbb{N}_+$ and $U \in \mathcal{U}_G$, the *n-th ψ-cotrajectory* $C_n(\psi, U) = U \cap \psi^{-1}(U) \cap \cdots \cap \psi^{-n+1}(U)$ is compact, so measurable. Let

$$(5) \qquad \mathcal{H}_{top}(\psi, U) = \limsup_{n\to\infty} -\frac{\log \mu(C_n(\psi, U))}{n}.$$

The topological entropy is defined by

$$(6) \qquad h_{top}(\psi) = \sup\{\mathcal{H}_{top}(\psi, U) : U \in \mathcal{U}_G\}$$

(see [42, 55]). Moreover, if G totally disconnected locally compact group and $U \in \mathcal{B}(G)$, then $C_n(\psi, U)$ is an open subgroup of the compact group U, hence $|G : C_n(\psi, U)| < \infty$ and one can obtain the following measure-free version of the formula (5)

$$\mathcal{H}_{top}(\psi, U) = \lim_{n \to \infty} \frac{\log |U : C_n(\psi, U)|}{n}.$$

Finally, if G is also compact (i.e., profinite), then one can take $U = G$ obtaining directly this convenient counterpart of (5)

$$H_{top}(\psi, U) = \lim_{n \to \infty} \frac{\log |G : C_n(\psi, U)|}{n}.$$

The utmost interest in topological entropy stems from topological dynamics, this is why the relevance of the algebraic entropy is motivated by its close connection to the topological one which we discuss now. Following [44], we call *Bridge Theorem* statements of the form $h_{top}(\varphi) = h_{alg}(\widehat{\varphi})$ for a continuous endomorphism $\varphi : G \to G$ of a locally compact abelian group and its Pontryagin dual $\widehat{\varphi} : \widehat{G} \to \widehat{G}$. The Bridge Theorem was proved in [44] for G compact, and previously by Weiss [106] when G is profinite and by Peters [89], when G is compact metrizable and φ is a topological automorphism. In [48] the Bridge Theorem is proved for totally disconnected locally compact abelian groups. A proof of the Bridge theorem for topological automorphisms of locally compact abelian groups can be found in [103].

6.6.4. **Entropy and scale function.** The scale function was introduced by Willis [107, 108] as follows.

Let G be a locally compact totally disconnected group. According to Lemma 6.18, every $U \in \mathcal{B}_G$ is φ-inert for every $\varphi \in C\mathrm{End}(G)$. This means that $s_G(\varphi, U) := |U^\varphi : (U^\varphi \cap U)|$ is a positive natural number. The *scale* of ϕ is

$$(7) \qquad\qquad s_G(\phi) = \min\{s_G(\phi, U) : U \in \mathcal{B}_G\}.$$

We use the notations $s(\phi, U)$ and $s(\phi)$ when the group G is clear from the context.

Since the scale function is defined as a minimum, there exists $U \in \mathcal{B}_G$ for which this minimum realizes, that is $s(\phi) = s(\phi, U)$, and such U is called *minimizing* for ϕ in [108]. Let $\mathcal{M}(G, \phi)$ be the subfamily of $\mathcal{B}_G$ consisting of all minimizing subgroups for ϕ. In [107] the scale function was defined only for internal automorphisms of G and its definition was motivated by the fact that in the non-commutative case, a locally compact totally disconnected group G may fail to have sufficiently many compact open normal subgroups, in particular, $\mathcal{B}_G$ may contain no normal subgroups at all, when G is neither compact nor discrete.

It is easy to see that $\mathcal{M}(G, \phi)$ contains all ϕ-invariant; in particular, $s(\phi) = 1$ precisely when there exists a ϕ-invariant $U \in \mathcal{B}_G$. In particular, $s(\phi) = 1$ for all φ when G is compact (as $s(\phi, G) = 1$).

For a topological automorphism $\phi : G \to G$ of a totally disconnected locally compact group G, the subgroup

$$(8) \qquad\qquad \mathrm{nub}(\phi) = \bigcap\{U \in \mathcal{B}_G : U \in \mathcal{M}(G, \phi)\}.$$

is ϕ-stable and compact.

The following theorem, proved in [10], answered a question posed by T. Weigel in 2011, about the relation between the scale function and the entropy.

Theorem 6.21. *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism. Then $\log s(\phi) \leq h_{top}(\phi)$, the equality $\log s(\phi) = h_{top}(\phi)$ holds if and only if $\mathrm{nub}(\phi) = \{e_G\}$.*

For the relation of the topological entropy with the scale see also the recent paper [70].

Acknowledgments

References

[1] R. L. Adler, A. G. Konheim and M. H. McAndrew, Topological entropy, *Trans. Amer. Math. Soc.*, **114** (1965) 309–319.

[2] R. Baer, Automorphismengruppen von Gruppen mit endlichen Bahnen gleichmssig beschrnkter Mchtigkeit, *J. Reine. Angew. Math.*, **262-263** (1973) 93–119.

[3] V. V. Belyaev, Inert subgroups in infinite simple groups, *Sibirsk. Mat. Zh.* **34** (1993) 17–23; English translation in: *Siberian Math. J.*, **34** (1993) 606–611.

[4] V. V. Belyaev, Locally finite groups containing a finite inseparable subgroup, *Sib. Mat. Zh. 34*, (1993) 23-41; English translation in: *Siberian Math. J.*, **34** (1993) 218–232.

[5] V. V. Belyaev, Inert subgroups in simple locally finite groups, *Finite and locally finite groups*, Istanbul, 1994 213-218, *NATO Adv. Sci.Inst. Ser. C Math. Phys. Sci.*, **471**, Kluwer Acad. Publ., Dordrecht, 1995.

[6] V. V. Belyaev, M. Kuzucuoglu and E. Seckin, Totally inert groups, *Rend. Sem. Mat. Univ. Padova*, **102** (1999) 151–156.

[7] V. V. Belyaev and D. A. Shved, Finitary automorphisms of groups, *Proc. Steklov Inst. Math.*, **267** (2009) S49–S56.

[8] V. V. Belyaev and D. A. Shved, Groups of outer finitary automorphisms, *Mat. Zametki*, **89** (2011) 635–636; translation in: *Math. Notes*, **89** (2011) 596–597.

[9] G. M. Bergman, H. W. Lenstra Jr., Subgroups close to normal subgroups, *J. Algebra*, **127** (1989) 80–97.

[10] F. Berlai, D. Dikranjan and A. Giordano Bruno, Scale function vs Topological entropy, *Topology Appl.*, **160** (2013) 2314–2334.

[11] J. C. Beidleman, H. Heineken, A Note on I-Automorphisms, *J. Algebra*, **234** (2000) 694–706.

[12] S. Breaz and G. Calugareanu, Strongly inert subgroups of Abelian groups, to appear on *Rend. Sem. Mat. Univ. Padova*, **1xx** (201x).

[13] J. T. Buckley, J. C. Lennox, B. H. Neumann, H. Smith and J. Wiegold, Groups with all subgroups normal-by-finite, *J. Austral. Math. Soc. Ser. A*, **59** (1995) 384–398.

[14] G. Calugareanu, Strongly invariant subgroups, *Glasgow Math. J.*, **57** (2015) 431–443.

[15] G. Carraro, *Invarianti nella categoria dei flussi per spazi vettoriali*, Master Thesis, Padua University, 2014.

[16] C. Casolo, Groups with finite conjugacy classes of subnormal subgroups, *Rend. Sem. Mat. Univ. Padova*, **81** (1989) 107–149.

[17] C. Casolo, Groups in which all subgroups are subnormal-by-finite, *Adv. Group Theory Appl.*, **1** (2016) 33–45.

[18] C. Casolo, U. Dardano and S. Rinauro, *Groups in which each subgroup is commensurable with a normal subgroup*, submitted, arXiv:1705.02360.

[19] C. Casolo and O. Puglisi, Hirsch-Plotkin radical of stability groups, *J. Algebra*, **370** (2012) 133–151.

[20] I. Castellano, *Topological entropy for linearly compact vector spaces*, corank and Bernoulli shifts, work in progress.

[21] I. Castellano and A. Giordano Bruno, *Algebraic entropy in locally linearly compact vector spaces*, M. Fontana et al. (eds.), Rings, Polynomials, and Modules, Springer International Publishing AG 2017,

[22] I. Castellano and A. Giordano Bruno, Topological entropy on locally linearly compact vector spaces, *Topology Appl.*, to appear.

[23] T. Ceccherini-Silberstein, M. Coornaert and F. Krieger, An analogue of Fekete's lemma for subadditive functions on cancellative amenable semigroups, *J. Anal. Math.*, **124** (2014) 59–81.

[24] A. R. Chekhlov, Fully inert subgroups of completely decomposable finite rank groups and their commensurability, (in Russian) *Vestn. Tomsk. Gos. Univ. Mat. Mekh.*, (2016) **41** 42–50.

[25] A. R. Chekhlov, On Fully Inert Subgroups of Completely Decomposable Groups, *Mathematical Notes*, **101** (2017) 365–373.

[26] C. D. Cooper, Power automorphisms of a group, *Math.Z.*, **107** (1968) 335–356.

[27] M. Curzio, S. Franciosi and F. de Giovanni, On automorphisms fixing infinite subgroups of groups, *Arch. Math. (Basel)*, **54** (1990) 4–13.

[28] G. Cutolo, Quasi-power automorphisms of infinite groups, *Comm. Algebra*, **21** (1993) 1009–1022.

[29] G. Cutolo, E. I. Khukhro, J. C. Lennox, S. Rinauro, H. Smith and J. Wiegold, Locally finite groups all of whose subgroups are boundedly finite over their cores, *Bull. London Math. Soc.*, **29** (1997) 563–570.

[30] D. van Dantzig, *Studien over topologische Algebra*, Dissertation, Amsterdam, 1931.

[31] U. Dardano, On groups with many maximal subgroups, *Ricerche Mat.*, **38** (1989) 261–271.

[32] U. Dardano and C. Franchi, On group automorphisms fixing subnormal subgroups setwise, *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)*, **3** (2000) 811–820.

[33] U. Dardano and C. Franchi, A note on groups paralyzing a subgroup series, *Rend. Circ. Mat. Palermo (2)*, **50** (2001) 165–170.

[34] U. Dardano and S. Rinauro, Inertial automorphisms of an abelian group, *Rend. Sem. Mat. Univ. Padova*, **127** (2012) 213–233.

[35] U. Dardano, S. Rinauro, On the ring of inertial endomorphisms of an abelian group, *Ricerche Mat.*, **63** (2014) S103-S115.

[36] U. Dardano and S. Rinauro, On groups whose subnormal subgroups are inert, *Int. J. Group Theory*, **4** no. 2 (2015) 17–24.

[37] U. Dardano and S. Rinauro, Inertial endomorphisms of an abelian group, *Ann. Mat. Pura Appl. (4)*, **195** (2016) 219–234.

[38] U. Dardano and S. Rinauro, A group of generalized finitary automorphisms of an abelian group, *J. Group Theory*, **20** (2017) 347–369.

[39] M. De Falco, F. de Giovanni, C. Musella and N. Trabelsi, Strongly inertial groups, *Comm. Algebra*, **41** (2013) 2213–2227.

[40] M. De Falco, F. de Giovanni, C. Musella and Y. P. Sysak, *Weakly power automorphisms of groups*, Communications in Algebra, (2017).

[41] D. Dikranjan, A. Fornasiero and A. Giordano Bruno, *Algebraic entropy for amenable semigroup actions*, work in progress.

[42] D. Dikranjan and A. Giordano Bruno, *Topological entropy and algebraic entropy for group endomorphisms*, in: Proceedings of the International Conference on Topology and Its Applications, (ICTA 2011), Cambridge Scientific Publishers, Cambridge, 2012 133–214.

[43] D. Dikranjan and A. Giordano Bruno, The Pinsker subgroup of an algebraic flow, *Jour. Pure Appl. Algebra*, **216** (2012) 364–376.

[44] D. Dikranjan and A. Giordano Bruno, The connection between topological and algebraic entropy, *Topology Appl.*, **159** (2012) 2980–2989.

[45] D. Dikranjan and A. Giordano Bruno, Limit free computation of entropy, *Rend. Istit. Mat. Univ. Trieste*, **44** (2012) 297–312.

[46] D. Dikranjan and A. Giordano Bruno, Entropy in a category, *Appl. Categ. Structures*, **21** (2013) 67–101

[47] D. Dikranjan and A. Giordano Bruno, Discrete dynamical systems in group theory, *Note Mat.*, **33** (2013) 1–48.

[48] D. Dikranjan and A. Giordano Bruno, The Bridge Theorem for totally disconnected locally compact groups, *Topology Appl.*, **169** (2014) 21–32.

[49] D. Dikranjan and A. Giordano Bruno, Entropy on abelian groups, *Advances in Mathematics*, **298** (2016) 612–653.

[50] D. Dikranjan, A. Giordano Bruno and L. Salce, Adjoint algebraic entropy, *J. Algebra*, **324** (2010) 442–463.

[51] D. Dikranjan, A. Giordano Bruno, L. Salce and S. Virili, Fully inert subgroups of divisible Abelian groups, *J. Group Theory*, **16** (2013) 915–939.

[52] D. Dikranjan, A. Giordano Bruno, L. Salce and S. Virili, Intrinsic algebraic entropy, *J. Pure Appl. Algebra*, **219** (2015) 2933–2961.

[53] D. Dikranjan, B. Goldsmith, L. Salce and P. Zanardo, Algebraic entropy of endomorphisms of abelian groups, *Trans. Amer. Math. Soc.*, **361** (2009) 3401–3434.

[54] D. Dikranjan, L. Salce and P. Zanardo, Fully inert subgroups of free Abelian groups, *Period. Math. Hungar.*, **69** (2014) 69–78.

[55] D. Dikranjan, M. Sanchis and S. Virili, New and old facts about entropy in uniform spaces and topological groups, *Topology Appl.*, **159** (2012) 1916–1942.

[56] M. R. Dixon, M. J. Evans and H. Smith, Groups with all proper subgroups soluble-by-finite rank, *J. Algebra* **298** (2005) 135–147.

[57] M. Dixon and H. Smith, Embedding groups in locally (soluble-by-finite) simple groups, *J. Group Theory*, **9** (2006) 383–395.

[58] M. R. Dixon, M. Evans and A. Tortora, On totally inert simple groups, *Cent. Eur. J. Math*, **8** (2010) 22–25.

[59] S. Franciosi and F. de Giovanni, Groups in which every infinite subnormal subgroup is normal, *J. Algebra*, **96** (1985) 566–580.

[60] S. Franciosi and F. de Giovanni, Groups whose subnormal nonnormal subgroups have finite index, *Rend. Accad. Naz. Sci. XL Mem. Mat. (5)*, **17** (1993) 241–251.

[61] S. Franciosi, F. de Giovanni and M. L. Newell, Groups whose subnormal subgroups are normal-by-finite, *Comm. Alg.*, **23** (1995) 5483–5497.

[62] L. Fuchs, *Infinite Abelian Groups*, Academic Press, New York - London, 1970–1973.

[63] W. Gaschütz, Gruppen, in denen das Normalteilersein transitiv ist, *J. Reine Angew. Math.*, **198** (1957) 87–92.

[64] A. Giordano Bruno and L. Salce, A soft introduction to algebraic entropy, *Arabian J. of Math.*, **1** (2012) 69–87.

[65] A. Giordano Bruno and L. Salce, Adjoint intrinsic algebraic entropy, work in progress.

[66] A. Giordano Bruno and P. Spiga, Some properties of the growth and of the algebraic entropy of group endomorphisms, *J. Group Theory*, **20** (2017) 763–774.

[67] A. Giordano Bruno and P. Spiga, Milnor-Wolf theorem for the growth of endomorphisms of locally virtually soluble groups, submitted.

[68] A. Giordano Bruno and S. Virili, The Algebraic Yuzvinski Formula, *J. Algebra*, **423** (2015) 114–147.

[69] A. Giordano Bruno and S. Virili, On the Algebraic Yuzvinski Formula, *Topol. Algebra and its Appl.*, **3** (2015) 86–103.

[70] A. Giordano Bruno and S. Virili, Topological entropy in totally disconnected locally compact groups, *Ergodic Theory Dynam. Systems*, **37** (2017) 2163–2186.

[71] F. de Giovanni, Some trends in the theory of groups with restricted conjugacy classes, *Note Mat.*, **33** (2013) 71–87.

[72] R. Göbel and L. Salce, Endomorphism rings with different rank-entropy supports, *Q. J. Math.*, **63** (2012) 381–397.

[73] B. Goldsmith and L. Salce, When the intrinsic algebraic entropy is not really intrinsic, *Topol. Algebra Appl.*, **3** (2015) 45–56.

[74] B. Goldsmith and L. Salce, *Algebraic entropies for Abelian groups with applications to the structure of their endomorphism rings: a survey*, in Groups, Modules, and Model Theory - Surveys and Recent Developments, Springer 2017 135–175.

[75] B. Goldsmith and L. Salce, 's realization theorems from the viewpoint of algebraic entropy, Proceedings 2016 Brixen/Graz Conferences, Springer, 2017.

[76] B. Goldsmith, L. Salce and P. Zanardo, Fully inert subgroups of Abelian *p*-groups, *J. Algebra*, **419** (2014) 332–349.

[77] B. Goldsmith, L. Salce and P. Zanardo, Fully inert submodules of torsion-free modules over the ring of p-adic integers, *Colloq. Math.*, **136** (2014) 169–178.

[78] J. I. Hall, Finitary linear transformation groups and elements of finite local degree, *Arch. Math. (Basel)*, **50** (1988) 315–318.

[79] P. de la Harpe, *Topics in geometric group theory*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2000,

[80] H. Heineken, Groups with neighbourhood conditions for certain lattices, *Note Mat.*, **16** (1996) 131–143.

[81] H. Heineken and L. A. Kurdachenko, Groups with finitely many classes of almost equal normal subgroups, *Algebra Colloq.*, **4** (1997) 329–344.

[82] O. Kegel and D. Schmidt, *Existentially closed finitary linear groups*, Groups-St. Andrews, 1989, **2** 353–362, London Math. Soc. Lecture Note Ser., **160**, Cambridge Univ. Press, Cambridge, 1991.

[83] J. C. Lennox and D. J. S. Robinson, *The theory of infinite soluble groups*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2004.

[84] U. Meierfrankenfeld, R. E. Phillips and O. Puglisi, Locally solvable finitary linear groups, *J. London Math. Soc.*, **47** (1993) 31–40.

[85] F. Menegazzo and D. J. S. Robinson, A finiteness condition on automorphism groups, *Rend. Sem. Mat. Univ. Padova*, **78** (1987) 267–277.

[86] W. Möhres, Torsionsgruppen, deren Untergruppen alle subnormal sind, *Geom. Dedicata*, **31** (1989) 237–244.

[87] B. H. Neumann, Groups with finite classes of conjugate subgroups, *Math. Z.*, **63** (1955) 76–96.

[88] D. Palacin and F. O. Wagner, A Fitting theorem for simple theories, *Bull. Lond. Math. Soc.*, **48** (2016) 472–482.

[89] J. Peters, Entropy on discrete Abelian groups, *Adv. Math.*, **33** (1979) 1–13.

[90] R. E. Phillips, The structure of groups of finitary transformations, *J. Algebra*, **119** (1988) 400–448.

[91] D. J. S. Robinson, Groups in which normality is a transitive relation, *Proc. Cambridge Philos. Soc.*, **60** (1964) 21–38.

[92] D. J. S. Robinson, *A Course in the Theory of Groups*, Graduate Texts in Mathematics, **80** Springer-Verlag, New York, 1996.

[93] D. J. S. Robinson, On inert subgroups of a group, *Rend. Sem. Mat. Univ. Padova*, **115** (2006) 137–159.

[94] L. Salce, P. Vámos and S. Virili, Length functions, multiplicities and algebraic entropy, *Forum Math.*, **25** (2013) 255–282.

[95] L. Salce and S. Virili, Two new proofs concerning the intrinsic algebraic entropy, *Comm. Algebra*, to appear.

[96] L. Salce and P. Zanardo, A general notion of algebraic entropy and the rank entropy, *Forum Math.*, **21** (2009) 579–599.

[97] G. Schlichting, Operationen mit periodischen Stabilisatoren. *Arch. Math. (Basel)*, **34** (1980) 97–99.

[98] D. Shved, On the structure of groups of virtually trivial automorphisms, *Comm. Algebra*, **45** (2017) 184–1852.

[99] H. Smith and J. Wiegold, Locally graded groups with all subgroups normal-by- finite, *J. Austral. Math. Soc. Ser. A* , **60** (1996) 222–227.

[100] W. Specht and H. Heineken, Gruppen mit endlicher Komponentenzahl fastgleicher Untergruppen, *Math. Nachr.*, **134** 73–82.

[101] I. Ya. Subbotin, On the ZD-coradical of a KI-group, (Russian) *Vychisl. Prikl. Mat. (Kiev)*, **75** (1991) 120–124; translation in *J. Math. Sci.*, **72** (1994) 3149–3151.

[102] S. Virili, Entropy for endomorphisms of LCA groups, *Topology Appl.*, **159** (2012) 2546–2556.

[103] S. Virili, *Algebraic and topological entropy of group actions*, preprint.

[104] B. A. F. Wehrfritz, Finite-finitary groups of automorphisms, *J. Algebra Appl.*, **1** (2002) 375–389.

[105] B. A. F. Wehrfritz, On generalized finitary groups, *J. Algebra*, **247** (2002) 707–727.

[106] M. D. Weiss, Algebraic and other entropies of group endomorphisms, *Math. Systems Theory*, **8** (1974/75) 243–248.

[107] G. A. Willis, The structure of totally disconnected locally compact groups, *Math. Ann.*, **300** (1994) 341–363.

[108] G. A. Willis, Further properties of the scale function on a totally disconnected group, *J. Algebra*, **237** (2001) 142–164.

[109] S. Yuzvinski, Metric properties of endomorphisms of compact groups, English Translation: *Amer. Math. Soc. Transl. (2)*, **66** (1968) 63–98.

[110] G. Zacher, Una caratterizzazione reticolare della finitezza dell'indice di un sottogruppo in un gruppo, *Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)*, **69** (1980) 317–323.

[111] A. E. Zalesskii, Groups of bounded automorphisms of groups (in Russian), *Dokl. Akad. Nauk BSSR* **19** (1975) 681–684.

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