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## Completely simple endomorphism rings of modules

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### ABSTRACT

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*It is proved that if  $A_p$  is a countable elementary abelian  $p$ -group, then: (i) The ring  $\text{End}(A_p)$  does not admit a nondiscrete locally compact ring topology. (ii) Under (CH) the simple ring  $\text{End}(A_p)/I$ , where  $I$  is the ideal of  $\text{End}(A_p)$  consisting of all endomorphisms with finite images, does not admit a nondiscrete locally compact ring topology. (iii) The finite topology on  $\text{End}(A_p)$  is the only second metrizable ring topology on it. Moreover, a characterization of completely simple endomorphism rings of modules over commutative rings is obtained.*

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### 1. INTRODUCTION

The notion of associative simple ring can be extended for associative topological rings in several ways:

- (i) simple abstract ring endowed with a nondiscrete ring topology (for instance, the classification of nondiscrete locally compact division rings, see [25, Chapter IV] and [4, 15, 16]; we refer to some historical notes about locally compact division rings to [29]);

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- (ii) topological ring without nontrivial closed ideals (see [22, 31]).
- (iii) topological ring  $R$  with the property that if  $f : R \rightarrow S$  is a continuous homomorphism in a topological ring  $S$ , then either  $f = 0$  or  $f$  is a topological embedding of  $R$  into  $S$  (see [24]).

In all cases it is assumed that multiplication is not trivial.

I. Kaplansky has mentioned (see [20], p. 56) that the classification of locally compact simple rings in positive characteristic  $p$  is difficult. He proved that every simple nondiscrete locally compact simple torsion-free ring is a matrix ring over a locally compact division ring. However in [26] (see also [30]) has been constructed a nondiscrete locally compact simple ring of positive characteristic which is not a matrix ring over a division ring. Thereby the program of classification of nondiscrete locally compact simple rings was finished. Nevertheless it is interesting to look for new examples of locally compact simple rings.

If  $A_p$  is a countable elementary abelian  $p$ -group and  $I$  is the ideal of the ring  $\text{End}(A_p)$  consisting of endomorphisms with finite images, then the factor ring  $\text{End}(A_p)/I$  is a simple von Neumann regular ring. We prove that under (CH) this ring does not admit a nondiscrete locally compact ring topology.

S. Ulam (see [23, Problem 96, p. 181]) posed the following problem: "Can the group  $S_\infty$  of all permutations of integers so metrized that the group operation (composition of permutations) is a continuous function and the set  $S_\infty$  becomes, under this metric, a compact space? (locally compact?)". E.D. Gaughan (see [10]) has solved this problem in the negative.

We study in §3 an analogous problem for the endomorphism ring of a countable elementary abelian  $p$ -group, namely: "Does the endomorphism ring  $\text{End}(A_p)$  of a countable elementary  $p$ -group  $A_p$  admit a nondiscrete locally compact ring topology?". Similarly to the Ulam's problem we obtain a negative answer to this question. Moreover, we prove that  $\mathcal{T}_{fin}$  is the only ring topology  $\mathcal{T}$  on  $\text{End}(A_p)$  such that  $(\text{End}(A_p), \mathcal{T})$  is complete and second metrizable.

We classify in §4 the completely simple rings  $(\text{End}(M), \mathcal{T}_{fin})$  of vector spaces  $M$  over division rings. Corollary 4.4 gives a characterization of semisimple left linearly compact minimal rings. It should be mentioned that Corollary 4.4 is related to a result from [3] stating that any semisimple ring admits at most one linearly compact topology.

Furthermore, we obtain in §5 a description of completely simple rings of the form  $(\text{End}(M_R), \mathcal{T}_{fin})$  of modules  $M$  over a commutative ring  $R$ . We extend the result of [28] to topological rings  $(\text{End}(M_R), \mathcal{T}_{fin})$ .

## 2. NOTATION, CONVENTIONS AND PRELIMINARY RESULTS

Rings are assumed to be associative, not necessarily with identity. Topological spaces are assumed to be completely regular. The *weight* (see [8], p.12) of the space  $X$  is denoted by  $w(X)$ . The *pseudocharacter* of a point  $x \in X$  (see [8], p.135) is the smallest cardinal of the form  $|\mathcal{U}|$ , where  $\mathcal{U}$  is a family

of open subsets of  $X$  such that  $\cap \mathcal{U} = \{x\}$ . The closure of a subset  $A$  of the topological space  $X$  is denoted by  $\overline{A}$  and the interior by  $\text{Int}(A)$  (see [8], p.14). A topological space  $X$  is called a *Baire space* (see [8], p.198) if for each sequence  $\{X_1, X_2, \dots\}$  of open dense subsets of  $X$  the intersection  $\cap_{i=1}^{\infty} X_i$  is a dense set.

An abelian group  $A$  is called *elementary abelian  $p$ -group* ( $p$  prime) if  $pa = 0$  for all  $a \in A$ . Such group is a direct sum of copies of the cyclic group  $Z(p)$ . The subring of a ring  $R$  generated by a subset  $S$ , is denoted by  $\langle S \rangle$ . A ring  $R$  is called *locally finite* if every its finite subset is contained in a finite subring. A topological ring  $(R, \mathcal{T})$  is called *metrizable* if its underlying additive group satisfies the first axiom of countability. A ring  $R$  with 1 is called *Dedekind-finite* if each equality  $xy = 1$  implies  $yx = 1$ . It is well-known that every finite ring with identity is Dedekind-finite. Since every compact ring with identity is a subdirect product of finite rings, it follows that every compact ring with identity is Dedekind-finite. If  $A \subseteq R$ , then  $\text{Ann}_l(A) := \{x \in R \mid xA = 0\}$ . If  $X, Y$  are the subsets of  $R$ , then  $X \cdot Y := \{xy \mid x \in X, y \in Y\}$ . A topological ring  $R$  is called *compactly generated* (see [27, Chapter II]) if there exists a compact subset  $K$  such that  $R = \langle K \rangle$ . If  $(R, \mathcal{T})$  is a topological ring and  $I$  is an ideal of  $R$ , then the quotient topology of the factor ring  $R/I$  is denoted by  $\mathcal{T}/I$ . If  $K$  is a subset of an abelian group  $A$ , then set

$$T(K) = \{\alpha \in \text{End}(A) \mid \alpha(K) = 0\}.$$

When  $K$  runs over all finite subsets of  $A$ , the family  $\{T(K)\}$  defines a ring topology  $\mathcal{T}_{fin}$  on  $\text{End}(A)$ . This topology is called the *finite topology*.

**Lemma 2.1.** *For any abelian group  $A$  the ring  $(\text{End}(A), \mathcal{T}_{fin})$  is complete.*

*Proof.* See [27, Theorem 19.2]. □

**Lemma 2.2** (Cauchy's criterion). *In a Hausdorff complete commutative group  $G$ , in order that a family  $(x_\alpha)_{\alpha \in \Omega}$  should be summable it is necessary and sufficient that, for each neighborhood  $V$  of zero in  $G$ , there is a finite subset  $\Omega_0$  of  $\Omega$  such that  $\sum_{\alpha \in K} x_\alpha \in V$  for all finite subsets  $K$  of  $\Omega$  which do not meet  $\Omega_0$ .*

*Proof.* See [5], p.263. □

**Lemma 2.3.** *If  $(x_\alpha)_{\alpha \in \Omega}$  is a summable subset in  $(\text{End}(A), \mathcal{T}_{fin})$  then every subset  $\Delta$  of  $\Omega$  the family  $(x_\beta)_{\beta \in \Delta}$  is summable.*

*Proof.* Let  $V$  be a neighborhood of zero of  $(\text{End}(A), \mathcal{T}_{fin})$ . We can consider without loss of generality that  $V$  is a left ideal of  $\text{End}(A)$ . There exists a finite subset  $\Omega_0$  of  $\Omega$  such that  $\sum_{\alpha \in K} x_\alpha \in V$  for every finite subset  $K$  of  $\Omega$  for which  $K \cap \Omega_0 = \emptyset$ . Let  $F$  be a finite subset of  $\Delta$  such that  $F \cap (\Omega_0 \cap \Delta) = \emptyset$ . If  $\alpha \in F$ , then  $\alpha \notin \Omega_0$ , hence  $\sum_{\alpha \in F} x_\alpha \in V$ . By Cauchy's criterion the family  $(x_\beta)_{\beta \in \Delta}$  is summable. □

A topological ring  $(R, \mathcal{T})$  is called *minimal* (see, for instance, [7]) if there is no ring topology  $\mathcal{U}$  such that  $\mathcal{U} \leq \mathcal{T}$  and  $\mathcal{U} \neq \mathcal{T}$ . A topological ring  $(R, \mathcal{T})$  is called *simple* if  $R$  is simple as a ring without topology. A topological ring  $(R, \mathcal{T})$  is called *weakly simple* if  $R^2 \neq 0$  and every its closed ideal is either 0

or  $R$ . A topological ring  $(R, \mathcal{T})$  is called *completely simple* (see [24]) if  $R^2 \neq 0$  and for every continuous homomorphism  $f : (R, \mathcal{T}) \rightarrow (S, \mathcal{U})$  in a topological ring  $(S, \mathcal{U})$  either  $\ker(f) = R$  or  $f$  is a homeomorphism of  $(R, \mathcal{T})$  on  $\text{Im}(f)$ . Equivalently,  $R^2 \neq 0$  and  $(R, \mathcal{T})$  is weakly simple and minimal. Let  $M$  be a unitary right  $R$ -module over a commutative ring  $R$  with 1. The module  $M$  is called *divisible* if  $Mr = M$  for every  $0 \neq r \in R$ . A right  $R$ -module  $M$  is called *faithful* if  $Mr = 0$  implies  $r = 0$  ( $r \in R$ ). A right  $R$ -module  $M$  is called *torsion-free* if  $mr = 0$  implies that either  $m = 0$  or  $r = 0$ , where  $m \in M$  and  $r \in R$ . Recall that a submodule  $N$  of an  $R$ -module  $M$  is called *fully invariant*  $\alpha(N) \subseteq N$  for every endomorphism  $\alpha$  of  $M_R$ . We use in the sequel the notion and results from the books [8, 27].

*Remark 2.4.* If  $R$  is a von Neumann regular ring, then  $R^2 = R$ .

**Lemma 2.5.** *An ideal  $I$  of a von Neumann regular ring is von Neumann regular.*

*Proof.* Let  $i \in I$ . Thus there exists  $x \in R$  such that  $ixi = i$ . It follows that  $ixixi = i$  and  $xix \in I$ . □

**Corollary 2.6.** *If  $I$  an ideal of a von Neumann regular ring  $R$ , then any ideal  $H$  of  $I$  is an ideal of  $R$ , too.*

*Proof.*  $RH = RH^2 \subseteq IH \subseteq H$ . Similarly,  $HR \subseteq H$ . □

If  $A_p$  is a  $p$ -elementary countable group, then

$$I = \{\alpha \in \text{End}(A_p) \mid |\text{Im}(\alpha)| < \aleph_0\}.$$

Fix a linear basis  $\{v_i \mid i \in \mathbb{N}\}$  of  $A_p$  over the field  $\mathbb{F}_p$ . Using this fixed basis, we define the map  $e_i : A \rightarrow A$  such that

$$e_i(v_j) = \delta_{ij}v_j, \quad (i, j \in \mathbb{N})$$

where  $\delta_{ij}$  is the Kronecker delta.

**Lemma 2.7.** *We have for  $\text{End}(A_p)$ :*

- (i)  *$I$  is a von Neumann regular ring.*
- (ii)  *$I$  is a simple ring.*
- (iii) *The factor ring  $\text{End}(A_p)/I$  is simple von Neumann regular.*
- (iv)  *$I$  is a locally finite ring.*

*Proof.* (i): The ring  $\text{End}(A_p)$  is regular (see [21, Theorem 4.27, p. 63]), so  $I$  is von Neumann regular by Lemma 2.5.

(ii), (iii): The ideal  $I$  is the only nontrivial ideal of the ring  $\text{End}(A_p)$  (see [17, §17, Theorem 1, p. 93]). This means that  $\text{End}(A)/I$  is simple. It is regular by the part (i).

(iv) Since  $I$  is simple (see [17, §12, Proposition 1]), it suffices to show that  $I$  contains a nonzero locally finite right ideal.

Let us show that the left ideal  $Ie_1$  of  $I$  is locally finite as a ring (equivalently, as a  $\mathbb{F}_p$ -algebra). We have  $0 \neq e_1 \in Ie_1$ . If  $H$  is the left annihilator of  $Ie_1$ , then,

obviously,  $H$  is a locally finite ring, hence it is locally finite as a  $\mathbb{F}_p$ -algebra. We claim that  $Ie_1/H$  is finite. Define  $\beta_n \in H$  ( $n \geq 2$ ) in the following way

$$\beta_n(v_i) = \begin{cases} v_n, & \text{for } i = 1; \\ 0, & \text{for } i \neq 1. \end{cases}$$

Let us prove that  $Ie_1 = \mathbb{F}_p e_1 + \sum_{n=2}^{\infty} \mathbb{F}_p \beta_n$ .

If  $\alpha \in I$ , then  $\alpha(v_1) = r_1 v_1 + \dots + r_n v_n$ , where  $r_i \in \mathbb{F}_p$  and  $n \in \mathbb{N}$ , so

$$\begin{aligned} \alpha e_1(v_1) &= r_1 e_1(v_1) + r_2 \beta_2(v_1) + \dots + r_n \beta_n(v_1) \\ &= (r_1 e_1 + r_2 \beta_2 + \dots + r_n \beta_n)(v_1); \\ \alpha e_1(v_j) &= (r_1 e_1 + r_2 \beta_2 + \dots + r_n \beta_n)(v_j) \quad (j \neq 1). \end{aligned}$$

This yields

$$\alpha e_1 = r_1 e_1 + r_2 \beta_2 + \dots + r_n \beta_n$$

and so  $Ie_1 = \mathbb{F}_p e_1 + \sum_{n=2}^{\infty} \mathbb{F}_p \beta_n$ .

In particular,  $Ie_1 = \mathbb{F}_p e_1 + H$ , and so  $H$  has a finite index in  $Ie_1$ . Clearly,  $Ie_1$  is a locally finite  $\mathbb{F}_p$ -algebra (see [17, Proposition 1, p. 241]) and  $I$  is a locally finite  $\mathbb{F}_p$ -algebra (see [17, Proposition 2, p. 242]).  $\square$

The next result can be deduced from [27, Lemma 36.11].

**Lemma 2.8.** *Let  $A$  be a locally compact, compactly generated, and totally disconnected ring. If  $A$  contains a dense locally finite subring  $B$ , then  $A$  is compact.*

*Proof.* Let  $A = \langle V \rangle$ , where  $V$  is a compact symmetric neighborhood of zero. Since  $V$  is compact, the subset  $V + V + V \cdot V$  also is compact. Since  $B$  is dense,  $A = B + V$ . By compactness of  $V + V + V \cdot V$  there exists a finite subset  $H \subseteq B$  such that  $V + V + V \cdot V \subseteq H + V$ . Since  $B$  is a locally finite ring, we can assume without loss of generality that  $H$  is a subring. Let  $H \setminus \{0\} = \{h_1, \dots, h_k\}$ . The subset

$$H + h_1 V + \dots + h_k V + V$$

is an open subgroup of  $R(+)$ . Indeed, this subset is symmetric and

$$\begin{aligned} &(H + h_1 V + \dots + h_k V + V) + (H + h_1 V + \dots + h_k V + V) \\ &\subseteq H + h_1(V + V) + \dots + h_k(V + V) + V + V \\ &\subseteq H + h_1 V + \dots + h_k V + V. \end{aligned}$$

We prove by induction on  $m$  that

$$V^{[m]} \subseteq H + h_1 V + \dots + h_k V + V, \quad (m \in \mathbb{N})$$

where  $V^{[1]} = V$  and  $V^{[m]} = V^{[m-1]} \cdot V$  for all  $m$ .

The inclusion is obvious for  $m = 1$ .

Assume that the assertion has been proved for  $m \geq 1$ . Clearly,

$$\begin{aligned} V^{[m+1]} &= V^{[m]} \cdot V \subseteq H \cdot V + h_1(V \cdot V) + \dots + h_k(V \cdot V) + V \cdot V \subseteq \\ &h_1 V + \dots + h_k V + h_1(H + V) + \dots + h_k(H + V) + H + V \subseteq \\ &H + h_1 V + \dots + h_k V + V. \end{aligned}$$

Consequently,  $A = H + h_1V + \dots + h_kV + V$ , therefore  $A$  is compact.  $\square$

An element  $x$  of a topological ring is called *discrete* if there exists a neighborhood  $V$  of zero such that  $xV = 0$  (i.e., the right annihilator of  $x$  is open).

**Lemma 2.9.** *The set of all discrete elements of a topological ring is an ideal. A simple ring with identity does not contain nonzero discrete elements.*

3. LOCALLY COMPACT RING TOPOLOGIES ON  $\text{End}(A)$  OF A COUNTABLE ELEMENTARY ABELIAN  $p$ -GROUP  $A$

**Theorem 3.1.** *Let  $R$  be a simple, nondiscrete and locally compact ring of  $\text{char}(R) = p > 0$  and  $1 \in R$ . If  $V$  is a compact open subring of  $R$  and  $\{e_\alpha \mid \alpha \in \Omega\}$  is a set of orthogonal idempotents in  $R$ , then*

$$|\Omega| \leq w(V).$$

*Proof.* The ring  $R$  does not contain nonzero discrete elements by Lemma 2.9. Since  $R$  is locally compact and  $\text{char}(R) = p$ , it is totally disconnected. Additionally,  $R$  has a fundamental system of neighborhoods of zero consisting of compact open subrings by [19, Lemma 9].

If  $V$  is a compact open subring of  $R$ , then by continuity of the ring operations for each  $\alpha \in \Omega$  there exists an open ideal  $V_\alpha$  of  $V$  such that  $e_\alpha V_\alpha \subseteq V$ . Clearly, there exists  $y_\alpha \in V_\alpha$  for which  $e_\alpha y_\alpha \neq 0$  since  $R$  has no nonzero discrete elements.

We claim that hold the following two properties:

- (i)  $e_\alpha y_\alpha \notin \overline{\{e_\beta y_\beta \mid \beta \neq \alpha\}}$  for each  $\alpha \in \Omega$ ;
- (ii) the set  $X = \{e_\alpha y_\alpha \mid \alpha \in \Omega\}$  is a discrete subspace of  $V$ .

Indeed, if were  $e_\alpha y_\alpha \in \overline{\{e_\beta y_\beta \mid \beta \neq \alpha\}}$  for some  $\alpha \in \Omega$ , then

$$\begin{aligned} e_\alpha y_\alpha &= e_\alpha e_\alpha y_\alpha \in e_\alpha \overline{\{e_\beta y_\beta \mid \beta \neq \alpha\}} \\ &\subseteq \overline{\{e_\alpha e_\beta y_\beta \mid \beta \neq \alpha\}} \\ &= \{0\}, \end{aligned}$$

so  $e_\alpha y_\alpha = 0$ , a contradiction. The part (i) is proved.

(ii) Now, for each  $\alpha \in \Omega$  we have  $V \setminus \overline{\{e_\beta y_\beta \mid \beta \neq \alpha\}}$  is open and, consequently,

$$(V \setminus \overline{\{e_\beta y_\beta \mid \beta \neq \alpha\}}) \cap X = \{e_\alpha y_\alpha\},$$

by (i). Therefore the point  $e_\alpha y_\alpha (\alpha \in \Omega)$  of  $X$  is isolated. In other words, the subspace  $X$  of  $V$  is discrete.

Since  $X$  is discrete,  $|\Omega| = |X| = w(X) \leq w(V)$  (see [1, Exercises 98-99, p. 72]).  $\square$

**Theorem 3.2.** *Let  $A_p$  be a countable elementary abelian  $p$ -group. Then the ring*

$$I = \{\alpha \in \text{End}(A_p) \mid |\text{Im}(\alpha)| < \aleph_0\}$$

*does not admit a nondiscrete ring topology  $\mathcal{U}$  such that  $(I, \mathcal{U})$  is a Baire space.*

*Proof.* Put  $S_n = \{\alpha \in I \mid \alpha(A) \subseteq \mathbb{F}_p v_1 + \cdots + \mathbb{F}_p v_n\}$ , where  $n \in \mathbb{N}$ . Clearly,  $I = \cup_{n \in \mathbb{N}} S_n$  and

$$S_n = \{\alpha \in I \mid e_i \alpha = 0 \text{ for } i > n\} = \text{Ann}_r(\{e_k \mid k > n\}).$$

This yields that the subset  $S_n$  is closed due the continuity of the ring operations.

Since  $I$  is a Baire space, there exists  $n \in \mathbb{N}$  such that  $\text{Int}(S_n) \neq \emptyset$ , hence  $S_n$  is an open subgroup.

Set  $\beta \in I$  such that

$$\beta(v_i) = \begin{cases} v_{n+i}, & \text{for } i = 1, \dots, n; \\ 0, & \text{for } i > n. \end{cases}$$

Let  $W \subseteq S_n$  be a neighborhood of zero of  $(I, \mathcal{U})$  such that  $\beta W \subseteq S_n$ . If  $w \in W \setminus \{0\}$ , then there exist  $a \in A$  and  $r_1, \dots, r_n \in \mathbb{F}_p$  such that

$$0 \neq w(a) = \sum_{i=1}^n r_i v_i \quad \text{and} \quad \beta(w(a)) = \sum_{i=1}^n r_i v_{n+i}.$$

There exists  $j \in 1, \dots, n$  such that  $r_j \neq 0$ . Then

$$e_{n+j} \beta w(a) = r_j v_{n+j} \neq 0,$$

hence  $e_{n+j} \beta w \neq 0$  and so  $\beta w \notin S_n$ , a contradiction.  $\square$

**Corollary 3.3.** *Under the notation of Theorem 3.2 the ring  $I$  does not admit a nondiscrete locally compact ring topology.*

*Proof.* This follows from the fact that each locally compact space is a Baire space (see [6, Theorem 1, p. 117]).  $\square$

Our main result is the following.

**Theorem 3.4.** *The endomorphism ring  $\text{End}(A_p)$  of a countable elementary abelian  $p$ -group  $A_p$  does not admit a nondiscrete locally compact ring topology.*

*Proof.* We use the notation and results from section 2. Denote  $R = \text{End}(A_p)$ . Assume on the contrary that there exists on  $R$  a nondiscrete locally compact ring topology  $\mathcal{T}$ .

**Fact 1.** The ring  $(R, \mathcal{T})$  has a fundamental system of neighborhoods of zero consisting of compact open subrings.

Since the additive group of the ring  $R$  has exponent  $p$ , it is totally disconnected (this follows from [12, Theorem 9.14, p. 95]). By I. Kaplansky's result (see [19, Lemma 9]), the ring  $(R, \mathcal{T})$  has a fundamental system of neighborhoods of zero consisting of compact open subrings.

**Fact 2.** The group  $Re_i$  is countable for each  $i \in \mathbb{N}$ .

We claim that  $Re_i$  is infinite. Indeed, for each  $j \in \mathbb{N}$  put  $\beta_j \in R$  such that

$$\beta_j(v_k) = \begin{cases} v_j, & \text{for } k = i; \\ 0, & \text{for } k \neq i. \end{cases}$$

If  $j \neq s$ , then  $\beta_j e_i(v_i) = \beta_j(v_i) = v_j$  and  $\beta_s e_i(v_i) = \beta_s(v_i) = v_s$ , hence  $\beta_j e_i \neq \beta_s e_i$ , so  $Re_i$  is infinite.

The ring  $Re_i$  is countable. Indeed, consider the mapping  $\psi : Re_i \rightarrow A_p^{\mathbb{F}_p v_i}$ , where

$$\psi(\alpha e_i)(rv_i) = \alpha(rv_i) \quad \text{for all } r \in \mathbb{F}_p.$$

If  $\alpha e_i \neq \beta e_i$  ( $\alpha, \beta \in R$ ), then there exists an element  $x = \sum_j r_j v_j \in A_p$  such that  $\alpha e_i(x) \neq \beta e_i(x)$ , hence,  $\alpha(rv_i) \neq \beta(rv_i)$ . Thus

$$\psi(\alpha e_i)(rv_i) = \alpha(rv_i) \neq \beta(rv_i) = \psi(\beta e_i)(rv_i).$$

The latter means that  $\psi$  is an injective mapping of  $Re_i$  into  $A_p^{\mathbb{F}_p v_i}$ . Since  $A_p^{\mathbb{F}_p v_i}$  is countable,  $Re_i$  is countable, too.

**Fact 3.**  $I$  is a closed ideal of  $R$ . We claim that  $I$  is not dense in the topological ring  $(R, \mathcal{T})$ . Assume the contrary. Since  $I$  is locally finite and is a maximal ideal,  $(R, \mathcal{T})$  is topologically locally finite by Lemma 2.8. The ring  $\overline{R}$  contains two elements  $x, y$  such that  $xy = 1$  and  $yx \neq 1$ . The subring  $\langle x, y \rangle$  is compact, hence Dedekind-finite, a contradiction. We obtained that  $(R/I, \mathcal{T}/I)$  is a nondiscrete metrizable locally compact ring.

**Fact 4.**  $I$  is a discrete ideal of  $R$ .

This follows from Theorem 3.2.

**Fact 5.**  $Re_i$  is a discrete left ideal of  $R$  for every  $i \in \mathbb{N}$ .

Indeed,  $Re_i \subseteq I$  and  $I$  is discrete by Fact 4 for every  $i \in \mathbb{N}$ .

**Fact 6.**  $\text{Ann}_l(e_i)$  is open in  $R$  for every  $i \in \mathbb{N}$ .

Indeed, the group homomorphism  $q : R \rightarrow Re_i, r \mapsto re_i$ , is continuous. Since  $Re_i$  is discrete  $q^{-1}(0) = \text{Ann}_l(e_i)$  is open.

**Fact 7.**  $\cap_i \text{Ann}_l(e_i) = 0$ .

Obvious.

**Fact 8.**  $\mathcal{T} \geq \mathcal{T}_{fin}$ .

We notice that  $\text{Ann}_l(e_i) = T(\{v_i\})$  for every  $i \in \mathbb{N}$ . For, if  $\alpha e_i = 0$ , then  $\alpha(v_i) = \alpha e_i(v_i) = 0$ , i.e.,  $\alpha \in T(\{v_i\})$ . Conversely, if  $\alpha \in T(\{v_i\})$ , then  $\alpha e_i(v_i) = \alpha(v_i) = 0$ . If  $j \neq i$  then  $\alpha e_i(v_j) = 0$ . Therefore  $\alpha e_i = 0$ . Moreover

$$T(\{v_1, \dots, v_n\}) = \cap_{i=1}^n T(\{v_i\}) = \cap_{i=1}^n \text{Ann}_l(e_i) \in \mathcal{T} \quad (\forall n \in \mathbb{N}).$$

Since the family  $\{T(\{v_1, \dots, v_n\})\}$  forms a fundamental system of neighborhoods of zero of  $(R, \mathcal{T}_{fin})$ , we get that  $\mathcal{T}_{fin} \leq \mathcal{T}$ .

**Fact 9.** The ring  $(R, \mathcal{T})$  is metrizable.

Since  $\cap_{i \in \mathbb{N}} \text{Ann}_l(e_i) = 0$ , the pseudocharacter of  $(R, \mathcal{T})$  is  $\aleph_0$ . If  $V$  is a compact open subring of  $(R, \mathcal{T})$  (see Fact 1), then the pseudocharacter of  $V$  also is  $\aleph_0$ . However in every compact space the pseudocharacter of a point coincides with its character. Therefore  $(R, \mathcal{T})$  is metrizable.

**Fact 10.**  $(R/I, \mathcal{T}/I)$  has an open compact subring.

Indeed, it is well-known (see [19]) that every totally disconnected ring has a fundamental system of neighborhood of zero consisting of compact open subrings. Henceforth  $V$  is a fixed open compact subring of  $(R/I, \mathcal{T}/I)$ .

**Fact 11.**  $R/I$  contains a family of orthogonal idempotents of cardinality  $2^{\aleph_0}$ .

Indeed, the family  $\{e_i\}_{i \in \mathbb{N}}$  of idempotents of the ring  $(R, \mathcal{T}_{fin})$  is summable and  $1_A = \sum_{n \in \mathbb{N}} e_n$ , where  $1_A$  is the identity of  $R$ .



The first ordinal number of cardinality  $\mathfrak{c}$  of continuum is denoted by  $\omega(\mathfrak{c})$ . Let  $\{\mathbb{N}(\alpha) \mid \alpha < \omega(\mathfrak{c})\}$  be a family of infinite almost disjoint subsets of  $\mathbb{N}$  (see [8, Example 3.6.18, p. 175–176]). Put  $f_{\mathbb{N}(\alpha)} = \sum_{i \in \mathbb{N}(\alpha)} e_i$  for each  $\alpha < \omega(\mathfrak{c})$ . The element  $f_{\mathbb{N}(\alpha)}$  exists by Lemma 2.3. Then:

- (i)  $f_{\mathbb{N}(\alpha)} \notin I$  for every  $\alpha < \omega(\mathfrak{c})$ ;
- (ii)  $f_{\mathbb{N}(\alpha)} f_{\mathbb{N}(\beta)} \in I$  for each  $\alpha, \beta < \omega(\mathfrak{c})$  and  $\alpha \neq \beta$ .

If  $g_\alpha = f_{\mathbb{N}(\alpha)} + I$  for each  $\alpha < \omega(\mathfrak{c})$ , then  $\{g_\alpha \mid \alpha < \omega(\mathfrak{c})\}$  is the required system of orthogonal idempotents.

The subring  $V$  is metrizable (by Fact 9). Since  $V$  is compact and  $R/I$  is a simple von Neumann regular ring by Lemma 2.7 and  $w(V) \leq \aleph_0$ , we obtain a contradiction to Theorem 3.1.  $\square$

**Theorem 3.5.** (CH) *Under the notation of Theorem 3.4, the ring  $R/I$  does not admit a nondiscrete locally compact ring topology.*

*Proof.* Assume on the contrary that the factor ring  $R/I$  admits a nondiscrete locally compact ring topology  $\mathcal{T}$ , so  $(R/I, \mathcal{T})$  contains an open compact subring  $V$ . Since the cardinality of  $R/I$  is continuum and  $V$  is infinite, the power of  $V$  is continuum. Since we have assumed (CH), the subring  $V$  is metrizable, hence second metrizable (see [14, 18]). However we have proved in Theorem 3.4 that the ring  $R/I$  contains a family of orthogonal idempotents of cardinality  $\mathfrak{c}$ , a contradiction with Theorem 3.1.  $\square$

**Theorem 3.6.** *The finite topology  $\mathcal{T}_{fin}$  is the only second metrizable ring topology  $\mathcal{T}$  on  $R$  for which  $(R, \mathcal{T}_{fin})$  is complete.*

*Proof.* Let  $K = \langle F \rangle$ , where  $F$  is a finite subset of  $A$ . Clearly, there exists a subgroup  $A'$  of  $A$  such that  $A = K \oplus A'$ . Choose  $e_F \in R$  such that  $e_F \upharpoonright_K = \text{id}_K$  and  $e_F(A') = 0$ . Clearly,

$$T(K) = R(1 - e_F)$$

and  $\alpha K = 0$  if and only if  $\alpha \in R(1 - e_F)$ , so the family  $\{R(1 - e_F)\}$ , where  $F$  runs over all finite subset of  $A$ , forms a fundamental system of neighborhoods of zero for  $(R, \mathcal{T}_{fin})$ .

There exists an injective map of  $Re_F$  to  $\text{Hom}(K, A)$ , so the left ideal  $Re_F$  is countable, due to countability  $\text{Hom}(K, A)$ . Since  $e_F^2 = e_F$ , the Peirce decomposition

$$R = Re_F \oplus R(1 - e_F)$$

of  $R$  with respect to the idempotent  $e_F$  is a decomposition of the topological group  $(R, +, \mathcal{T})$ . It follows that  $Re_F$  is discrete, hence  $R(1 - e_F)$  is open (in the topology  $\mathcal{T}$ ). Hence  $\mathcal{T} \geq \mathcal{T}_{fin}$ , so  $\mathcal{T} = \mathcal{T}_{fin}$  (see [9, Theorem 30] or [11]).  $\square$

#### 4. COMPLETELY SIMPLE TOPOLOGICAL ENDOMORPHISM RINGS OF VECTOR SPACES

**Theorem 4.1.** *Let  $A_F$  be a right vector space over a division ring  $F$  and  $S = \text{End}(A_F)$ . The following conditions are equivalent:*

- (i)  $(S, \mathcal{T}_{fin})$  is a completely simple topological ring.

(ii)  $\dim(A_F) = \infty$  or  $\dim(A_F) < \infty$  and  $F$  does not admit a nondiscrete ring topology.

*Proof.* (i) $\Rightarrow$ (ii): If  $A_F$  is finite-dimensional, then  $S$  is discrete and isomorphic to the matrix ring  $M(n, F)$ , where  $n$  is the dimension of  $A_F$ . Then, obviously,  $F$  does not admit a nondiscrete ring topology.

(ii) $\Rightarrow$ (i): If  $\dim(A_F) = n < \infty$ , then  $S \cong M(n, F)$ . Since  $F$  does not admit nondiscrete ring topologies, the same holds for  $M(n, F)$ .

Let  $A_F$  be infinite dimensional. Fix a basis  $\{x_\alpha\}_{\alpha < \tau}$  over  $F$ , where  $\tau$  is an infinite ordinal number. It is well-known that the topological ring  $(S, \mathcal{T}_{fin})$  is weakly simple (see [22, Satz 12, p. 258]) and the family  $\{T(x_\alpha)\}_{\alpha < \tau}$  is a prebase at zero for the finite topology  $\mathcal{T}_{fin}$  of  $S$ .

Assume on the contrary that there exists a Hausdorff ring topology  $\mathcal{T}$ , coarser than  $\mathcal{T}_{fin}$  and different from it. Let  $e_\alpha \in S$  such that  $e_\alpha^2 = e_\alpha$  and  $e_\alpha(x_\beta) = \delta_{\alpha\beta}x_\alpha$  for each  $\alpha < \tau$ , where  $\delta_{\alpha\beta}$  is the Kronecker delta.

**Fact 1.**  $T(x_\alpha) = \text{Ann}_l(e_\alpha)$  for each  $\alpha < \tau$ .

Indeed, if  $p \in T(x_\alpha)$ , then  $pe_\alpha(x_\alpha) = p(x_\alpha) = 0$ . If  $\beta \neq \alpha$ , then  $e_\alpha(x_\beta) = 0$ , hence  $pe_\alpha = 0$ , i.e.  $p \in \text{Ann}_l(e_\alpha)$ . Conversely, if  $pe_\alpha = 0$ , then we have  $p(x_\alpha) = pe_\alpha(x_\alpha) = 0$ , i.e.  $p \in T(x_\alpha)$ .

**Fact 2.** There exists  $\alpha_0 < \tau$  for which  $Se_{\alpha_0}$  is nondiscrete in  $(S, \mathcal{T})$ .

Assume on the contrary that for every  $\alpha < \tau$  there exists a neighborhood  $V_\alpha$  of zero of  $(S, \mathcal{T})$  such that  $Se_\alpha \cap V_\alpha = 0$ . If  $U_\alpha$  is a neighborhood of zero of  $(S, \mathcal{T})$  such that  $U_\alpha e_\alpha \subseteq V_\alpha$ , then  $U_\alpha e_\alpha = 0$ , hence  $\text{Ann}_l(e_\alpha) = T(x_\alpha)$  is open in  $(S, \mathcal{T})$ . Hence  $\mathcal{T}_{fin} \leq \mathcal{T}$  and  $\mathcal{T} = \mathcal{T}_{fin}$ , a contradiction.

**Fact 3.**  $(Se_{\alpha_0} \cap V)x_{\alpha_0} \not\subseteq \bigoplus_{\beta \in K} x_\beta F$  for any neighborhood  $V$  of zero of  $(S, \mathcal{T})$  and any finite subset  $K$  of the set  $[0, \tau)$  of all ordinal numbers less than  $\tau$ .

Assume on the contrary that there exists a finite subset  $K$  of  $[0, \tau)$  and a neighborhood  $V$  of zero of  $(S, \mathcal{T})$  such that

$$(4.1) \quad (Se_{\alpha_0} \cap V)x_{\alpha_0} \subseteq \bigoplus_{\beta \in K} x_\beta F.$$

Fix  $\gamma \in [0, \tau) \setminus K$ . For each  $\beta \in K$  define  $q_\beta \in S$  such that  $q_\beta(x_\beta) = x_\gamma$  and  $q(x_\delta) = 0$  for  $\delta \neq \beta$ .

Let  $V_0$  be a neighborhood of zero of  $(S, \mathcal{T})$  such that  $V_0 \subseteq V$  and  $q_\beta V_0 \subseteq V$  for all  $\beta \in K$ . There exists  $0 \neq h \in Se_{\alpha_0} \cap V_0$  by Fact 2 and  $hx_{\alpha_0} \neq 0$  by Fact 1. Since  $Se_{\alpha_0} \cap V_0 \subseteq Se_{\alpha_0} \cap V$ , we obtain that  $hx_{\alpha_0} = \sum_{\beta \in K} x_\beta f_\beta$ , ( $f_\beta \in F$ ) by (4.1). There exists  $\beta_0 \in K$  such that  $f_{\beta_0} \neq 0$  (because  $hx_{\alpha_0} \neq 0$ ), so

$$q_{\beta_0} h = q_{\beta_0}(\sum_{\beta \in K} x_\beta f_\beta) = r_{\beta_0} x_\gamma \notin \bigoplus_{\beta \in K} x_\beta F,$$

a contradiction. Therefore Fact 3 is proved.

Now let  $V$  be a neighborhood of zero of  $(S, \mathcal{T})$ . Pick up a neighborhood  $V_0$  of zero of  $(S, \mathcal{T})$  such that  $V_0 \cdot V_0 \subseteq V$ . Since  $\mathcal{T} \leq \mathcal{T}_{fin}$ , there exists a finite subset  $K$  of  $[0, \tau)$  such that

$$T(\{x_\beta \mid \beta \in K\}) \subseteq V_0.$$

We have  $(Se_{\alpha_0} \cap V_0)x_{\alpha_0} \notin \oplus_{\beta \in K} x_{\beta}F$  by Fact 3. It follows that there exists  $q \in Se_{\alpha_0} \cap V_0$  such that

$$q(x_{\alpha_0}) \notin \oplus_{\beta \in K} x_{\beta}F.$$

Clearly,  $q(x_{\alpha_0}) \in A_F$ , so it can be written as  $q(x_{\alpha_0}) = \sum_{\alpha < \tau} x_{\alpha}f_{\alpha}$ , where  $f_{\alpha} \in F$  and there exists  $\beta_0 \notin K$  such that  $f_{\beta_0} \neq 0$ .

Consider the element  $s \in S$  such that  $s(x_{\beta_0}) = x_{\alpha_0}f_{\beta_0}^{-1}$  and  $s(x_{\lambda}) = 0$  for  $\lambda \neq \beta_0$ . Evidently,  $s \in T(K)$ , hence

$$sq \in T(K) \cdot V_0 \subseteq V_0 \cdot V_0 \subseteq V.$$

Moreover,  $sq(x_{\alpha_0}) = s(x_{\beta_0}f_{\beta_0} + \dots) = x_{\alpha_0}$ . Since  $q \in Se_{\alpha_0}$ , we obtain that  $sq(x_{\beta}) = 0$  for  $\beta \neq \alpha_0$ . Consequently,  $e_{\alpha_0} = sq \in V$  for every neighborhood  $V$  of zero of  $(S, \mathcal{T})$ , a contradiction.  $\square$

*Remark 4.2.* The question of existence of an uncountable division ring which does not admit a nondiscrete Hausdorff ring topology is open. Several results on this topic can be found in Chapter 5 of [2].

**Theorem 4.3.** *Let  $\prod_{\alpha \in \Omega} R_{\alpha}$  be a family of compact rings with identity. Then the product  $(\prod_{\alpha \in \Omega} R_{\alpha}, \prod_{\alpha \in \Omega} \mathcal{T}_{\alpha})$  is a minimal ring if and only if every  $(R_{\alpha}, \mathcal{T}_{\alpha})$  is a minimal topological ring. (Here  $\prod_{\alpha \in \Omega} \mathcal{T}_{\alpha}$  is the product topology on the ring  $\prod_{\alpha \in \Omega} R_{\alpha}$ .)*

*Proof.*  $\Rightarrow$ : Assume on the contrary that there exists  $\beta \in \Omega$  and a ring topology  $\mathcal{T}'$  on  $R_{\beta}$  such that  $\mathcal{T}' \leq \mathcal{T}_{\beta}$  and  $\mathcal{T}' \neq \mathcal{T}_{\beta}$ . Consider the product topology  $\mathcal{U}$  on  $\prod_{\alpha \in \Omega} R_{\alpha}$ , where  $R_{\alpha}$  is endowed with  $\mathcal{T}_{\alpha}$  when  $\alpha \neq \beta$  and  $R_{\beta}$  is endowed with  $\mathcal{T}'$ . Obviously,  $\mathcal{U} \leq \prod_{\alpha \in \Omega} \mathcal{T}_{\alpha}$  and  $\mathcal{U} \neq \prod_{\alpha \in \Omega} \mathcal{T}_{\alpha}$ , a contradiction.

$\Leftarrow$ : Denote by  $\pi_{\alpha} (\alpha \in \Omega)$  the projection of  $\prod_{\alpha \in \Omega} R_{\alpha}$  on  $R_{\alpha}$ . By definition of the product topology,  $\prod_{\alpha \in \Omega} \mathcal{T}_{\alpha}$  is the coarsest topology on  $\prod_{\alpha \in \Omega} R_{\alpha}$  for which the projections  $\pi_{\alpha} (\alpha \in \Omega)$  are continuous.

Let  $\mathcal{U}$  be a ring topology on  $\prod_{\alpha \in \Omega} R_{\alpha}$ ,  $\mathcal{U} \leq \prod_{\alpha \in \Omega} \mathcal{T}_{\alpha}$  and  $\beta \in \Omega$ . Since

$$\mathcal{U} \upharpoonright_{R_{\beta} \times \prod_{\gamma \neq \beta} \{0_{\gamma}\}} \leq \left( \prod_{\alpha \in \Omega} \mathcal{T}_{\alpha} \right) \upharpoonright_{R_{\beta} \times \prod_{\gamma \neq \beta} \{0_{\gamma}\}},$$

it follows that  $\mathcal{U} \upharpoonright_{R_{\beta} \times \prod_{\gamma \neq \beta} \{0_{\gamma}\}} = \left( \prod_{\alpha \in \Omega} \mathcal{T}_{\alpha} \right) \upharpoonright_{R_{\beta} \times \prod_{\gamma \neq \beta} \{0_{\gamma}\}}$  by minimality of  $(R_{\beta}, \mathcal{T}_{\beta})$ .

Then the family  $\{V \times \prod_{\gamma \neq \beta} \{0_{\gamma}\}\}$  when  $V$  runs all neighborhoods of zero of  $(R_{\beta}, \mathcal{T}_{\beta})$  is a fundamental system of neighborhoods of zero of

$$\left( R_{\beta} \times \prod_{\gamma \neq \beta} \{0_{\gamma}\}, \mathcal{U} \upharpoonright_{R_{\beta} \times \prod_{\gamma \neq \beta} \{0_{\gamma}\}} \right).$$

Since  $R_{\beta} \times \prod_{\gamma \neq \beta} \{0_{\gamma}\}$  is an ideal with identity of  $\prod_{\alpha \in \Omega} R_{\alpha}$ , the topological ring  $(\prod_{\alpha \in \Omega} R_{\alpha}, \mathcal{U})$  is a direct sum of ideals  $R_{\beta} \times \prod_{\gamma \neq \beta} \{0_{\gamma}\}$  and  $\{0_{\beta}\} \times \prod_{\gamma \neq \beta} R_{\gamma}$ . Let  $V$  be a neighborhood of zero of  $(R_{\beta}, \mathcal{T}_{\beta})$ . Then  $V \times \prod_{\gamma \neq \beta} R_{\gamma}$  be a neighborhood of zero of  $(\prod_{\alpha \in \Omega} R_{\alpha}, \mathcal{U})$  and  $\pi_{\beta}(V \times \prod_{\gamma \neq \beta} R_{\gamma}) = V$ .

We have proved that  $\pi_{\beta}$  is a continuous function from  $(\prod_{\alpha \in \Omega} R_{\alpha}, \mathcal{U})$  to  $(R_{\beta}, \mathcal{T}_{\beta})$ . It follows that  $\prod_{\alpha \in \Omega} \mathcal{T}_{\alpha} \leq \mathcal{U}$  and so  $\mathcal{U} = \prod_{\alpha \in \Omega} \mathcal{T}_{\alpha}$ .  $\square$

**Corollary 4.4.** *A left linearly compact semisimple ring is minimal if and only if has no direct summands of the form  $M(n, \Delta)$ , where  $\Delta$  is a division ring which does not admit a nondiscrete Hausdorff ring topology.*

*Proof.* This follows from Theorems 4.1, 4.3 and the Theorem of Leptin (see [22, Theorem 13, p. 258]) about the structure of left linearly compact semisimple rings.  $\square$

**Corollary 4.5.** *A semisimple linearly compact ring  $(R, \mathcal{T})$  having no ideals isomorphic to matrix rings over infinite division rings is minimal.*

### 5. COMPLETELY SIMPLE ENDOMORPHISM RINGS OF MODULES

The endomorphism ring of a right  $R$ -module  $M$  is denoted by  $\text{End}(M_R)$ .

**Lemma 5.1.** *Let  $M$  be a divisible, torsion-free module over a commutative domain  $R$  and  $K$  the field of fractions of  $R$ . The additive group of  $M$  has a structure of a vector  $K$ -space such that  $R$ -endomorphisms of  $M$  are exactly the  $K$ -linear transformations.*

*Proof.* We define a structure of a right vector  $K$ -space as follows: if  $\frac{a}{b} \in K$  and  $m \in M$ , then there exists a unique  $x \in M$  such that  $ma = xb$ ; set  $m \circ \frac{a}{b} = x$ . Moreover, if  $\frac{a}{b} = \frac{c}{d}$  and  $0 \neq m \in M$ , then  $m \circ \frac{a}{b} = m \circ \frac{c}{d}$ . Indeed, if  $m \circ \frac{a}{b} = x$  and  $m \circ \frac{c}{d} = y$ , then  $mad = xbd$  and  $mbc = ybd$  which means that  $xbd = ybd$ , hence  $x = y$ .

Let  $\alpha \in \text{End}(M_R)$ ,  $\frac{a}{b} \in K$ ,  $m \in M$ . By definition,  $am = b(\frac{a}{b} \circ m)$ , hence,  $a\alpha(m) = b\alpha(\frac{a}{b} \circ m)$ , which means that  $\alpha(\frac{a}{b} \circ m) = \frac{a}{b} \circ \alpha(m)$ , so  $\alpha$  is a  $K$ -linear transformation. Note that, if  $a \in R$  and  $m \in M$ , then  $m \circ \frac{a}{1} = ma$ .

Conversely, if  $\alpha$  is a  $K$ -linear transformation,  $a \in R$ ,  $m \in M$ , then

$$\alpha(\frac{a}{1} \circ m) = \frac{a}{1} \circ \alpha m,$$

i.e.  $\alpha(am) = a\alpha(m)$ . We have proved that every  $K$ -linear transformation is an right  $R$ -module homomorphism.  $\square$

*Remark 5.2.* The center  $Z(R)$  of a weakly simple ring  $R$  is a domain.

*Remark 5.3.* For every right  $R$ -module  $M$  the underlying group  $M(+)$  is a discrete left topological  $(\text{End}(M_R), \mathcal{T}_{fin})$ -module.

Indeed,  $T(m)(m) = 0$  for every  $m \in M$ . Moreover,  $\text{End}(M_R)\{0\} = \{0\}$ , so  $M$  is a discrete left topological  $(\text{End}(M_R), \mathcal{T}_{fin})$ -module.

**Theorem 5.4.** *Let  $M_R$  be a module over a commutative ring  $R$ .*

*If the topological ring  $(\text{End}(M_R), \mathcal{T}_{fin})$  is weakly simple, then:*

- (i)  $P = \{r \in R \mid Mr = 0\}$  is a prime ideal of  $R$ .
- (ii)  $M$  is a vector space over the field  $K$  of fractions of  $R/P$  and the  $R$ -endomorphisms of  $M$  are exactly the  $K$ -linear transformations.

*Conversely, if  $M_R$  is an  $R$ -module and are satisfied (i) and (ii), then the ring  $(\text{End}(M_R), \mathcal{T}_{fin})$  is a weakly simple topological ring.*

*Proof.*  $\Rightarrow$ : If  $(\text{End}(M_R), \mathcal{T}_{fin})$  is weakly simple, then the mapping:

$$(5.1) \quad \alpha_r : M \rightarrow M, \quad m \mapsto mr \quad (r \in R)$$

is an  $R$ -module homomorphism and  $\alpha_r \in Z (= \text{the center of } \text{End}(M_R))$ .

First we show that the part (i) holds. Indeed, if  $a, b \in R$  and  $ab = 0$ , then  $\alpha_a \alpha_b = 0$  (see (5.1)). Thus  $(\text{End}(M_R)\alpha_a) \cdot (\text{End}(M_R)\alpha_b) = 0$ , so

$$\overline{(\text{End}(M_R)\alpha_a)} \cdot \overline{(\text{End}(M_R)\alpha_b)} = 0.$$

Since  $\text{End}(M_R)$  is weakly simple, one of them, say  $\overline{\text{End}(M_R)\alpha_a}$ , is zero. This implies that  $\alpha_a = 0$ , hence  $a \in P$ .

(ii) The structure of  $R/P$ -module on  $M$  is defined as follows: if  $r \in R$  and  $m \in M$ , then put  $M(r + P) = mr$ .

Note that  $M$  is a torsion-free right  $R/P$ -module. Assume that  $m(r + P) = 0$ , where  $0 \neq r + P \in R/P$  and  $0 \neq m \in M$ . Then  $mr = 0 = \alpha_r(m)$  (see (5.1)). Thus  $\text{End}(M_R)\alpha_r(m) = 0$ . It follows that  $\overline{(\text{End}(M_R)\alpha_r)}(m) = 0$  by Remark 5.3. Since  $\text{End}(M_R)$  is weakly simple

$$\overline{\text{End}(M_R)\alpha_r} = \text{End}(M_R).$$

We obtained that  $\text{End}(M_R)(m) = 0$ , so  $m = 0$ , a contradiction.

Under this convention  $R$ -submodules are exactly  $R/P$ -submodules and  $R$ -endomorphisms are exactly  $R/P$ -endomorphisms.

The module  $M$  is a divisible  $R/P$ -module. Indeed, if  $0 \neq r + P \in R/P$ , then  $0 \neq M(r + P) = Mr$ . Suppose that  $Mr \neq M$ . Consider

$$I = \{\alpha \in \text{End}(M_R) \mid \alpha(M) \subseteq Mr\}.$$

Since  $Mr$  is a fully invariant submodule,  $I$  is a two-sided ideal of the ring  $(\text{End}(M_R), \mathcal{T}_{fin})$ .

The ideal  $I$  is closed. Indeed, let  $\alpha \in \bar{I}$ . If  $m \in M$ , then there exists  $\beta \in I$  such that  $\alpha - \beta \in T(m)$ . Clearly,  $\alpha(m) = \beta(m) \in Mr$  and so  $\alpha \in I$ . We have proved that  $I$  is closed.

Since  $1_M \notin I$ ,  $I = 0$ . It follows that  $\alpha_r = 0$  (see (5.1)), a contradiction.

The module  $M$  has a structure of a right  $K$ -vector space and  $\text{End}(M_R)$  is exactly the ring of endomorphisms of  $M$  by Lemma 5.1.

The converse follows from Theorem 4.1. □

A characterization of completely simple topological ring  $\text{End}(M_R)$  is given by the following.

**Theorem 5.5.** *Let  $M_R$  be a module over a commutative ring  $R$ . The topological ring  $(\text{End}(M_R), \mathcal{T}_{fin})$  is completely simple if and only are satisfied the conditions (i) and (ii) of Theorem 5.4 and either*

- (i)  $M$  is finite or
- (ii)  $M$  is infinite and the dimension of  $M$  over the field  $K$  is infinite.

*Proof.*  $\Rightarrow$ : According to Theorem 5.4, the ideal  $P$  is prime and the topology of  $\text{End}(M_R)$  coincide with the finite topology of  $\text{End}(M_K)$ , where  $K$  is the field of fractions of  $R/P$ . If  $M$  is finite, we have the part (i). Assume that

$M$  is infinite. If  $R/P$  is finite, then the dimension of  $M$  over  $K$  is infinite. Suppose that  $R/P$  is infinite and  $\dim_K(M) = n < \aleph_0$ . Then  $M$  is isomorphic to  $M(n, K)$ . Since  $K$  is an infinite field, it admits a nondiscrete ring topology (see [13]) and we obtain a contradiction because  $\text{End}(M_R)$  is a discrete ring. Consequently  $\dim_K(M)$  is infinite.

⇐ This follows from Theorems 4.1 and 5.4. □

**Corollary 5.6.** *The topological ring  $(\text{End}(A), \mathcal{T}_{fin})$  of an abelian group  $A$  is completely simple if and only one of the following conditions holds:*

- (i)  $A$  is a elementary abelian  $p$ -group.
- (ii)  $A$  is a divisible torsion-free group of infinite rank.

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