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## A MATHEMATICAL MODEL OF TRAIN CONTINUOUS MOTION UPHILL

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**Abstract.** The determination problem of adhesion coefficient of locomotives' wheels with rails is described in this paper. The use of methods of mathematical statistics and theory of probability provides wider possibilities for determining the most acceptable (suitable) value of the adhesion coefficient. The random factors influencing on this value are analysed. The use of the adhesion coefficient values based on reliable research results could help maintain uninterrupted (continuous) traffic of heavy freight trains and increase not only the carrying capacity, but also the volume of the transported goods. The paper considers a mathematical model of steady traffic of heavy freight trains pulled by upgraded locomotives of the series 2M62M on railway line with a gradient. The research results show that the actual values of the adhesion coefficient and the total resistance are distributed according to the normal distribution law. A mode of locomotive motion largely depends on the relationship between the total train's running resistance and locomotive tractive force. Finally, basic conclusions are given.

**Keywords:** heavy freight trains, track line gradient, locomotive tractive force, train continuous motion, coefficient of adhesion, running resistance.

### 1. Introduction

Adhesion coefficient of the locomotive wheels with the rails  $\psi_s$  should be determined based on the empirical results of the locomotive performance, while its value taken for traction calculation should ensure the most effective use of the locomotive power and steady traffic of freight trains. This problem has considerably grown in importance, since the number of heavy (4 000–5 000 tons) freight trains has largely increased on Lithuanian railways in recent years (2007–2008 m. prekinį traukinių... 2007).

The use of methods of mathematical statistics and theory of probability provides wider possibilities for determining the most acceptable (suitable) value of the adhesion coefficient  $\psi_{adh}$ . Their application allows us to determine not only the average values of the coefficient  $\psi_s$ , but to estimate the whole range of experimental results. However, usually, only the values of the adhesion coefficient exceeding its average values were determined by Bureika and Mikaliūnas (2002). Under the conditions of the intensive use of locomotives, it is of paramount importance to analyze and precisely evaluate factors influencing the reduction of the coefficient  $\psi_{adh}$ . When, in calculating the locomotive speed, the adhesion coefficient is represented by a single curve  $\psi_{adh}(v)$  and a single value  $\psi_{adh}$ , though the external factors of its maintenance are changing in a

wide range, this can hardly ensure the most rational choice of the tractive force of a locomotive depending on the adhesion value  $F_{tr}^{calc}$  and the locomotive mass for a particular route. The use of the adhesion coefficient values based on reliable research results could help maintain uninterrupted (continuous) traffic of heavy freight trains and increase not only the carrying capacity, but also the volume of the transported goods (Handbook... 2006).

To determine the most suitable adhesion coefficient  $\psi_{adh}$  value for particular operating conditions, the factors influencing this value should be analysed. On Lithuanian railways, the main problem for heavy freight trains is running uphill incline of 8.1 % and 3 km-long track gradient on the railway line between the stations of Telšiai and Lieplaukė in Lithuania. The paper considers a mathematical model of regular traffic of heavy freight trains pulled by upgraded freight locomotives of the series 2M62M on this line.

### 2. Random factors influencing steady train motion uphill

The locomotive, rolling stock and the road make a complicated mechanical system influenced by a great number of random external and internal factors, e.g. unevenness of the rails or non-uniformly worn-out wheel-set rolling surfaces as well as different rolling radii of one of

the wheel-set tyres, variation of the road curve radius, dirty or wet rails, interaction of wagons, etc. Two main concepts concerning train motion up the vertical grade may be defined as continuous and non-continuous (unsteady) motion. When the motion is continuous, the speed of the train  $v_{calc}$  can be calculated irrespective of the influence of random factors. When the motion of the train is non-continuous (unsteady), its speed cannot be maintained as specified and decreases, though the locomotive operator takes some actions to control it.

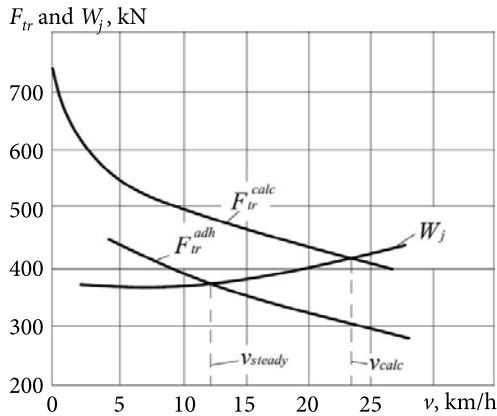
The condition of steady train motion uphill is expressed as follows:

$$F_{tr}^{adh} \geq W_j \quad \text{and} \quad v_{steady} \geq v_{calc}, \quad (1)$$

where:  $F_{tr}^{adh}$  is tractive force according to adhesion, kN;  $W_j$  is running resistance, kN;  $v_{steady}$  is train speed in steady motion, km/h;  $v_{calc}$  is the calculated train velocity, km/h.

If the conditions of wheel adhesion with the rails ensure the application of the tractive force equal to or exceeding the total resistance, train motion is continuous or steady.

The curves showing the dependence of the specified calculated tractive force  $F_{tr}^{calc}$ , actual tractive force based on the adhesion  $F_{tr}^{adh}$  and resistance to motion  $W_j$  on the train speed  $v$  are presented in Fig. 1.



**Fig. 1.** The dependence of calculated tractive force  $F_{tr}^{calc}$ , actual tractive force based on adhesion  $F_{tr}^{adh}$  and running resistance  $W_j$  on the locomotive velocity  $v$ :  $v_{steady}$  is locomotive travelling steady velocity;  $v_{calc}$  is calculated velocity

The intersection point A of the curves of the tractive force  $F_{tr}^{calc}$  and resistance  $W_j$  (Fig. 1) indicates the calculated velocity  $v_{calc}$  of the train. If the actually applied tractive force based on the adhesion  $F_{tr}^{adh}$  is lower than the calculated tractive force  $F_{tr}^{calc}$ , then, the values of running resistance  $W_j$  are decreasing according to the respective law, when the train speed is decreasing and the adhesion force  $F_{adh}$  is increasing. When a certain value of the decreasing speed  $v_{steady}$  is reached, the equilibrium between the locomotive tractive force and running resistance is established, and the motion becomes steady.

The transient process is over, and the train goes further uphill at a lower speed.

Thus, given the train mass  $m_{train}$ , and the defined dependence  $W_j(v)$ , in each case, the actual adhesion force  $F_{tr}^{adh}$  corresponds to the altered steady speed value. In some critical situations, the unsteady mode of locomotive motion is characterized by speed reduction to 0, i. e. the train stops “stretched”, implying that there are no gaps between the automatic clutches of the train wagons. These situations are possible but rarely found in practice.

### 3. Determining the conditions of steady locomotive motion

A mode of locomotive motion largely depends on the relationship between the total train running resistance  $W_j$  and tractive force  $F_{tr}^{adh}$ . Therefore, the problem of determining steady motion conditions is the analysis of the system of two random quantities –  $F_{tr}^{adh}$  and  $W_j$ . The distribution of their values is determined by a large number of factors which are not closely interrelated, each of them making a small contribution to the joint effect. This allows us to assume that the quantities  $F_{tr}^{adh}$  and  $W_j$  are distributed according to the normal law. Similar assumptions are made by Grebenyuk (Гребенюк 2003) when investigating locomotive traction.

The quantities  $F_{tr}^{adh}$  and  $W_j$  are the speed functions:

$$F_{tr}^{adh}(v) = 1000 \cdot P \cdot g \cdot \Psi_{adh}(v), \quad (2)$$

$$W_j(v) = Q \cdot g \cdot (i + w_0''(v)) + P \cdot g \cdot (i + w_0'(v)), \quad (3)$$

where:  $P$  is total mass of locomotive, t;  $Q$  is total mass of wagons, t;  $i$  is uphill gradient, %;  $g$  is gravitational acceleration,  $m/s^2$ ;  $w_0'$  is the locomotive's running main relative resistance, N/t;  $w_0''$  is the wagons' running main relative resistance, N/t.

Adhesion coefficients of locomotives and the main relative running resistance are found by the empirical formulas, see researches by Wickens (2003) and Grebenyuk (Гребенюк 2003):

$$\Psi_{adh} = a + \frac{b}{c + d \cdot v} - e \cdot v; \quad (4)$$

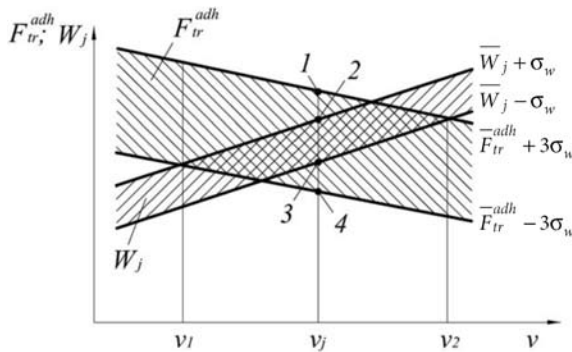
$$w_0' = l + p \cdot v + f \cdot v^2; \quad (5)$$

$$w_0'' = h + \frac{r + m \cdot v + n \cdot v^2}{q_a}, \quad (6)$$

where:  $a, b, c, d, e, f, h, l, m, n, p, r$  are constant empirical coefficients of particular trains;  $q_a$  is wagon axle load, tf.

Based on the actual values of the adhesion coefficient  $\Psi_{adh}$  and the specified calculated speed  $v_{calc}$ , the mass of the train moving at steady speed along the uphill's determinant gradient  $i_{det}$  is calculated for a locomotive of the considered series according to the condition  $F_{tr}^{adh} = W_j$ . In this case, average values of the main relative resistance  $w_0''$  and  $w_0'$  are used:

$$(P + Q) = \frac{1000 \cdot (P \cdot g) \cdot \Psi_{adh} - (P \cdot g) \cdot (w_0' + i_{det})}{w_0'' + i_{det}}. \quad (7)$$



**Fig. 2.** Tractive forces based on adhesion  $F_{tr}^{adh}$  and the variation range of actual total resistance  $W_j$  values depending on locomotive motion speed  $v$ : in the range between point 1 and point 2 – steady (continuous) motion; between point 3 and point 4 – unsteady (non-continuous) motion; between point 2 and point 3 – indeterminate train’s motion

Here, the specified (calculated) values of  $F_{tr}^{adh}$  and  $W_j$  are taken. In fact, these are random quantities, therefore, two ranges of values, rather than two speed dependence values, are obtained (see Fig. 2).

The limits of the variation range of force values are defined, taking into account mathematical expectation dependence on speed and standard deviations  $\sigma_F$ ,  $\sigma_W$ . According to the predetermined probability value, the expected values of  $F_{tr}^{adh}$  and  $W_j$  are within these ranges.

In each case, the locomotive adhesion tractive force  $F_{tr}^{adh}$  in the range of values from  $F_{tr}^{adh} - \sigma_F$  to  $F_{tr}^{adh} + \sigma_F$  can be applied. At the same time, the actual tractive force  $F_{tr}^{act}$ , can be equal to the total resistance  $W_j$  or be smaller or larger than its value. When the minimum tractive force (the lower variation range limit of  $F_{tr}^{adh}$ ) is used at the maximum total resistance (the upper limit of the values’ variation range), the steady speed will be  $v_2$ . Thus, for any ratio of tractive force to total resistance, the travelling speed of a locomotive will be between  $v_1$  and  $v_2$ . Therefore, steady train motion can be only achieved in the above-mentioned speed range between  $v_1$  and  $v_2$ .

#### 4. Developing a mathematical model of steady running uphill of the train

The calculated speed  $v_{calc}$  provided will be used, when tractive force based on the value of adhesion  $F_{tr}^{adh}$  will be in the range between points 1 and 2 (Fig. 2). If the value  $F_{tr}^{adh}$  is found between the points 3 and 4, the running of the train will be unsteady. In the range from point 2 to point 3, running stability depends on the ratio of force  $F_{tr}^{adh}$  to resistance  $W_j$ . A criterion describing the stability of running may be the difference between the above forces  $\Delta_{calc}$ , computed for the calculated velocity:

$$\Delta_{calc} = F_{tr}^{adh}(v_{calc}) - W_j(v_{calc}). \quad (8)$$

If  $\Delta_{calc} \geq 0$ , the running of the train is steady. Otherwise, the steady travelling speed of the train will be lower than the calculated speed  $v_{calc}$ .

In general, smoothly adjusting the tractive force and speed of a locomotive, we can achieve a tractive force

corresponding to actual dependence of  $F_{tr}^{adh}(v)$ . Then, the steady speed  $v_{steady}$ , can be expressed as:

$$v_{steady} = f(F_{tr}^{adh}(v) - W_j(v)) = f(v),$$

$$\Delta = F_{tr}^{adh} - W_j(v). \quad (9)$$

The above steady (uniform) speed  $v_{steady}$  is the function of the difference  $\Delta$ . By making some substitutions, we will obtain a 3rd order equation  $a_0 \cdot v_{steady}^3 + a_1 \cdot v_{steady}^2 + a_2 \cdot v_{steady} = 0$ , which should be solved.

When adhesion coefficient and random deviations from the total running resistance are not taken into consideration, the calculated speed and weight of the train as well as time intervals between the running trains are not optimal. As a result, the train schedule can hardly be reliable in this case.

#### 5. Conclusions

1. Solving the problem of steady train running uphill, its tractive force based on adhesion and the total resistance should be analysed as random quantities, using the methods of mathematical statistics and theory of probability.
2. The research results show that the actual values of the adhesion coefficient  $\Psi_{adh}$  and the total motion resistance  $W_j$  are distributed according to the normal distribution law, while the actual values differ from the specified calculated values by about 3–4 %.
3. When specifying the actual calculated velocity  $v_{calc}^{act}$  for Lithuanian freight trains running Lieplaukė uphill gradient of 8.1 %, it should be decreased by about 8–10 %. This could ensure steady running of freight trains uphill as well as steady traffic between these stations.

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