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Partition Numbers of Finite Solvable Groups

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Abstract

A group partition is a group cover in which the elements have trivial pairwise intersection. Here we define the partition number of a group - the minimal number of subgroups necessary to form a partition - and examine some of its properties, including its relation to the covering number for solvable groups.

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1 Introduction

A group cover is a collection of proper subgroups whose union is the group. For brevity, we say that a group G is *coverable* if there exists some covering of G. Among the set of all coverable groups exists a certain subset, the *partitionable* groups, which admit a particular type of cover known as a *partition*. This is not a partition in the set-theoretic sense, but rather a collection of subgroups that cover the group and have trivial pairwise intersection:

Definition 1.1 *A collection of nontrivial proper subgroups*

$$\{H_1, H_2, \dots, H_n\}$$

is a partition of a group G *if* $G = \cup H_i$ *and* $H_i \cap H_j = \{e\}$ *whenever* $i \neq j$.

We will call a subgroup $H_i \in \{H_1, H_2, \ldots, H_n\}$ a *summand* of the partition.
The study of group partitions began with the work of Miller in 1906 (see [17]), and culminated in the complete classification of partitionable groups through the combined efforts of Baer, Kegel, and Suzuki in 1961 (see [3],[14],[19]):

Theorem 1.2 *A group* G *is partitionable if and only if* G *is isomorphic to one of:*

1. S_4,
2. *a noncyclic* p*-group with* $H_p(G) \neq G$*, where*
$$H_p(G) = \langle x \in G : x^p \neq 1 \rangle,$$
3. *a group of the Hughes-Thompson type,*
4. *a Frobenius group,*
5. $PSL(2, p^n)$*, with* $p^n \geqslant 4$*,*
6. $PGL(2, p^n)$*, with* $p^n \geqslant 5$ *and* p *odd,*
7. $Sz(2^{2n+1})$,

for some prime p *and some* $n \in \mathbb{N}$*.*

Definition 1.3 *The covering number of a group, denoted* $\sigma(G)$*, is the minimal number of subgroups necessary to form a cover of the group. If* G *is not coverable, then* $\sigma(G) = \infty$*.*

The covering number has received attention in the literature, and there are a modest number of known results. For example, no group has a covering number of 2 (discovered independently by Scorza, among others [13]); and a group has a covering number of 3 if and only if the group is homomorphic to the Klein four group (see [5],[13]). Other bounds and various minimality conditions have places upon $\sigma(G)$ for various classes of groups. See the work of Cohn [7], Maróti [16], Kappe [13], Garonzi [9],Tomkinson [20], Abdollahi [1],[2], among others.
We can extend the idea of the covering number and analogously define the partition number of a group.

Definition 1.4 *The partition number of a group, denoted* $\rho(G)$*, is the minimal number of subgroups necessary to form a partition of* G*. If* G *has no partition, then* $\rho(G) = \infty$*.*

Of course, making the definition presumes that it is an interesting enough number to warrant its own definition; in particular, that it is distinct from the covering number at least sometimes. Note that since all partitions are covers, then $\sigma(G) \leqslant \rho(G)$ for any partitionable group G. But for which partitionable groups is $\sigma(G) < \rho(G)$?, and for which partitionable groups $\sigma(G) = \rho(G)$? While we cannot completely answer these questions, it turns out that the dihedral groups, generalized dihedral groups, Frobenius groups and p-groups provide us with nice examples.

2 The dihedral groups and $\sigma(G) < \rho(G)$

In the discussion that follows, for the dihedral group D_n of order $2n$, we will use the presentation

$$D_n = \left\langle a, b : a^n = b^2 = e, ab = (ab)^{-1} \right\rangle.$$

We will use the following theorem of Tomkinson [20] to find the covering number of D_n and other solvable groups.

Theorem 2.1 (Tomkinson) *Let G be a finite solvable group and let p^n be the order of the smallest chief factor having more than one complement. Then $\sigma(G) = p^n + 1$.*

Thus by applying Tomkinson's Theorem to D_n we get the following Corollary.

Corollary 2.2 *If p is the smallest prime such that $p \mid n$, then*

$$\sigma(D_n) = \sigma(D_p) = p + 1.$$

It is known that the coverable groups are precisely the noncyclic groups [5], so groups of order 2, 3, 5, and 7 have no covers or partitions. Further, no group of order 4 or 6 can satisfy $\sigma(G) < \rho(G)$ due to the observation below:

Remark 2.3 *If $|G| = pq$ for some primes p and q, then $\sigma(G) = \rho(G)$, because two distinct subgroups of prime order have trivial intersection.*

Thus if a group G satisfies $\sigma(G) < \rho(G)$, then $|G| \geqslant 8$.

Remark 2.4 *By Corollary 2.2 we see that $\sigma(D_4) = 3$. Since subgroup $\langle a \rangle$ must be a summand in any partition of D_4, an inspection of the subgroup structure of D_4 reveals that $\rho(D_4) = 5$. Therefore, $\sigma(D_4) < \rho(D_4)$.*

It turns out that the partition number of D_n can be expressed as a closed formula that holds without restriction on n.

Lemma 2.5 *Let G be a group with a partition*

$$\{H_1, H_2, \ldots, H_n\}$$

such that $|H_1| = |G|/2$. If $x \in G - H_1$, then $|x| = 2$.

Proof — Since H_1 is a normal subgroup of index 2 in G, $x^2 \in H_1$ for all $x \in G$. If $x \in G - H_1$, then $x \in H_i$ for some $i \neq 1$, so $\langle x \rangle \cap H_1 = e$ which implies $|x| = 2$. □

Theorem 2.6 *For $n > 1$, $\rho(D_n) = n + 1$.*

Proof — Observe that any covering of D_n must include the subgroup $\langle a \rangle$, since this is the only proper subgroup that contains a. Thus by Lemma 2.5 since $\langle a \rangle$ has index 2 in D_n, the remaining subgroups in the partition will then have the form $\langle ba^i \rangle$, where $0 \leqslant i < n$. Each of these subgroups has order 2, and they intersect trivially. So the only partition we can form is

$$\Pi = \{\langle a \rangle\} \cup \left\{ \langle ba^i \rangle : 0 \leqslant i < n \right\},$$

and clearly, $|\Pi| = n + 1$. □

Remark 2.7 *Given $n \geqslant 3$ there is a partitionable group G with $\rho(G) = n$.*

Corollary 2.2 and Theorem 2.6 give us immediately a large class of groups for which the covering number is strictly less than the partition number.

Corollary 2.8 *If n is composite, then $\sigma(D_n) < \rho(D_n)$.*

Proof — If n is composite, then n has a prime divisor strictly less than n. Let p be the smallest such divisor. Then by Corollary 2.2 and Theorem 2.6, $\sigma(D_n) = p + 1 < n + 1 = \rho(D_n)$. □

Corollary 2.9 *$\sigma(D_n) = \rho(D_n)$ if and only if n is a prime.*

Proof — That $\rho(D_p) = p + 1$ for a prime p was established in Theorem 2.6. The other direction is immediate from Corollary 2.2 because, if p is prime, then p is the smallest prime divisor of p. □

3 The Frobenius groups and $\sigma(G) = \rho(G)$

In Remark 2.3 we noted that if $|G| = pq$ for some primes p and q, then $\sigma(G)$ is equal to $\rho(G)$. In this section we look at another family of groups with $\sigma(G) = \rho(G)$. In the discussion that follows, for a Frobenius group, we will denote the Frobenius complement by H and the Frobenius kernel by K. For more information about the structure of the Frobenius group see [18].

Theorem 3.1 *If G is a finite solvable Frobenius group with the Frobenius kernel K which is a minimal normal subgroup and an abelian Frobenius complement H, then $\sigma(G) = \rho(G)$.*

Proof — By 8.5.5 of [18], K has more than one complement, and since G is a finite solvable group, $|K| = p^n$ for p a prime. Since H is abelian, it is in fact cyclic, by Corollary 6.17 of [12]; thus K is the only chief factor with more than one complement. Thus, by Tomkinson's Theorem (Theorem 2.1),

$$\sigma(G) = p^n + 1.$$

On the other hand, by 8.5.5 of [18],

$$\{K, H^x \mid x \in K\}$$

is a partition of G with $p^n + 1$ summands. So $\sigma(G) = \rho(G)$. □

Example 3.2 below is also an immediate consequence of Remark 2.3.

Example 3.2 *If* $G = \langle a, b\rangle$ *is nonabelian of order* pq *for some primes* p *and* q *with* $\langle a\rangle \lhd G$*, then* G *is a finite solvable Frobenius group with the Frobenius kernel* $K = \langle a\rangle$ *a minimal normal subgroup, and an abelian Frobenius complement* $H = \langle b\rangle$*.*

Example 3.3 *The Frobenius group of order 20. In this case,* K *the Frobenius kernel is isomorphic to* $\mathbb{Z}_5$*, and the Frobenius complement* H *is isomorphic to* $\mathbb{Z}_4$*.*

Example 3.3 is a special case of Example 3.4.

Example 3.4 *For every finite field* $GF(p^n)$ *with* $p^n (> 2)$ *elements, the group of invertible affine transformations acting naturally on* $GF(p^n)$ *is a Frobenius group with the Frobenius kernel* $K \simeq \mathbb{Z}_p^n$ *a minimal normal subgroup, and an abelian Frobenius complement* $H \simeq \mathbb{Z}_{p^n-1}$*.*

4 $\rho(G)$ for G an elementary abelian group

Similar to the dihedral groups, we can calculate $\rho(G)$ for G an elementary abelian group. We will denote the elementary abelian group of order p^n by E_{p^n}. The following result of Beutelspacher [4] gives us a lower bound for $\rho(E_{p^n})$.

Lemma 4.1 *If* p *is a prime, then* $1 + p^{\lceil \frac{n}{2} \rceil} \leqslant \rho(E_{p^n})$*.*

Remark 4.2 *Using the construction of Bu in* [6]*, given* $n > 1$ *and* p *an odd prime, one can construct a partition of* $G = E_{p^n}$ *with* $p^{n-1} + 1 < p^n + 1$ *summands, by viewing*

$$G = GF(p^{n-1}) \times GF(p)$$

as a vector space over $GF(p)$*. Also using Bu's construction, if* $d \mid n$ *one can construct a partition of* G *composed of* $(p^n - 1)/(p^d - 1)$ *subgroups isomorphic to* E_{p^d} *by viewing*

$$G = GF(p^d) \times \ldots \times GF(p^d)$$

as a vector space over $GF(p)$*.*

Remark 4.3 *Using the construction of Bu in [6] we see that*

$$1 + p^2 = \rho(E_{p^3}) = \rho(E_{p^4}).$$

These results are generalized in Theorem 4.5.

Lemma 4.4 *If* $K \leqslant G$ *are two partitionable groups such that* $\rho(K) = n$*, and* Π *is a partition of* G *with no* $H \in \Pi$ *containing* K*, then* $|\Pi| \geqslant n$*.*

Proof — Let $\Pi_K = \{H \cap K \mid H \in \Pi \text{ and } H \cap K \neq \{1\}\}$. Since there is no $H \in \Pi$ containing K, all subgroups in Π_K are proper. And since Π is a partition of G, Π_K is a partition of K. Thus $|\Pi| \geqslant |\Pi_K| \geqslant \rho(K) = n$. □

Theorem 4.5 *If* p *is a prime and* $n > 1$*, then* $1 + p^{\lceil \frac{n}{2} \rceil} = \rho(E_{p^n})$*.*

Proof — *Case 1.* If $n = 2d$ is even, then by [4]

$$1 + p^d \leqslant \rho(E_{p^n})$$

and by [6] there is a partition of E_{p^d} with

$$(p^n - 1)/(p^d - 1) = ((p^d)^2 - 1)/(p^d - 1) = p^d + 1$$

summands. So

$$1 + p^{\lceil \frac{n}{2} \rceil} = \rho(E_{p^n}).$$

Case 2. If $n = 2d - 1$ is odd, then by [4] $1 + p^d \leqslant \rho(E_{p^n})$ and by Case 1 above

$$1 + p^d = \rho(E_{p^{n+1}}).$$

Using Lemma 4.4, we see that

$$\rho(E_{p^n}) \leqslant \rho(E_{p^{n+1}}),$$

and so again that $1 + p^{\lceil \frac{n}{2} \rceil} = \rho(E_{p^n})$. □

Corollary 4.6 *If* $n > 2$*, then* $\sigma(E_{p^n}) < \rho(E_{p^n})$*.*

Proof — By Tomkinson's Theorem (Theorem 2.1) $\sigma(E_{p^n}) = 1 + p$ and by Theorem 4.5 $\rho(E_{p^n}) = 1 + p^{\lceil \frac{n}{2} \rceil}$. □

5 Generalized dihedral groups and nonabelian summands

Having seen some interesting results surrounding the dihedral groups, there are other interesting questions we might ask. One observation we may extract from our study of the dihedral groups is that the subgroups contained in the partitions we constructed were always abelian.

In the discussion that follows, we will denote by $E_{p^n} \rtimes \mathbb{Z}_2$ for p an odd prime the generalized dihedral group, formed from the semidirect product of the elementary abelian group of order p^n with a cyclic group of order two acting via the inverse map.

Theorem 5.1 *$G = E_{p^n} \rtimes \mathbb{Z}_2$ for p an odd prime and $n \geqslant 2$ does contain a nonabelian summand in a partition.*

Proof — Let H be a proper nonabelian subgroup in G , since every non-trivial element in G has order 2 or p,

$$\Pi = \{H\} \bigcup \{\langle x \rangle \,|x \in G - H\}$$

is a partition of G. □

Corollary 5.2 *$G = E_9 \rtimes \mathbb{Z}_2$ does contain a nonabelian summand in a partition.*

Remark 5.3 *Using GAP [8] one can show that $G = E_9 \rtimes \mathbb{Z}_2$ is the smallest example of a group that contains a nonabelian summand in a partition.*

Theorem 5.4 *For p an odd prime,*

$$\sigma(E_{p^2} \rtimes \mathbb{Z}_2) = p + 1 < \rho(E_{p^2} \rtimes \mathbb{Z}_2) = p^2 + 1.$$

Proof — By Tomkinson's Theorem (Theorem 2.1), $\sigma(E_{p^2} \rtimes \mathbb{Z}_2) = p + 1$.

In any partition $\Pi = \{H_1, H_2, \dots, H_n\}$ of $E_{p^2} \rtimes \mathbb{Z}_2$,

$$|\,H_i\,| \in \{2, 2p, p, p^2\}.$$

Note that G has

$$(p^2 - 1)/(p - 1) = p + 1$$

subgroups of order p and and p^2 subgroups of order 2. So if all subgroups in Π have order 2 or p, then

$$|\Pi| = p^2 + p + 1.$$

Case 1. There is a subgroup in Π isomorphic to E_{p^2}, then by Lemma 2.5

$$|\Pi| = p^2 + 1.$$

Case 2. There is a subgroup in Π isomorphic to D_p, then without loss of generality, we may assume that $H_1 = \langle x, y_1 \rangle$. Note that the order of any other summand must be $< 2p$. Thus,

$$\Pi = \left\{ \langle x, y_1 \rangle, \{C_p(G) - \langle y_1 \rangle\}, \left\{ C_2(G) - \left\{ \langle x \rangle, \langle xy_1 \rangle, \left\langle xy_1^2 \right\rangle, \dots, \left\langle xy_1^{p-1} \right\rangle \right\}\right\}\right\},$$

where as before,

$$C_p(G) = \{\text{subgroups of } G \text{ with order } p\}$$

and

$$C_2(G) = \{\text{subgroups of } G \text{ with order } 2\}.$$

Also in this case, $|\Pi| = 1 + p + (p^2 - p) = p^2 + 1$. □

Lemma 5.5 *If* Π *is a partition of* $G = E_{p^n} \rtimes \mathbb{Z}_2$ *containing a subgroup isomorphic to* E_{p^n} *or* $E_{p^{n-1}} \rtimes \mathbb{Z}_2$*, then* $|\Pi| = p^n + 1$.

Proof — *Case 1.* Π contains a subgroup isomorphic to E_{p^n}. We may assume that $H_1 \simeq E_{p^n}$; then the other subgroups in Π are $C_2(G)$, so

$$|\Pi| = p^n + 1.$$

Case 2. Π contains a subgroup isomorphic to $E_{p^{n-1}} \rtimes \mathbb{Z}_2$. Here, we may assume that

$$H_1 \simeq E_{p^{n-1}} \rtimes \mathbb{Z}_2$$

and

$$H_1 = \langle x, y_1 \dots y_{n-1} \rangle .$$

Note that any other summand must have order 2 or p. Now

$$|C_2(G) - C_2(H_1)| = p^n - p^{n-1}$$

and

$$|C_p(G) - C_p(H_1)| = \sum_{k=0}^{n-1} p^k - \sum_{k=0}^{n-2} p^k = p^{n-1},$$

so $|\Pi| = p^n + 1$. □

Lemma 5.6 *For* p *an odd prime and* $n \geqslant 2$*,* $\rho(E_{p^{n-1}} \rtimes \mathbb{Z}_2) \leqslant \rho(E_{p^n} \rtimes \mathbb{Z}_2)$.

Proof — Note that by Lemma 5.5

$$\rho(E_{p^{n-1}} \rtimes \mathbb{Z}_2) \leqslant p^{n-1} + 1.$$

Let Π be a partition of $E_{p^n} \rtimes \mathbb{Z}_2$ with $|\Pi| = \rho(E_{p^n} \rtimes \mathbb{Z}_2)$.
Case 1. Π contains a subgroup isomorphic to $E_{p^{n-1}} \rtimes \mathbb{Z}_2$ so

$$\rho(E_{p^n} \rtimes \mathbb{Z}_2) = p^n + 1 > \rho(E_{p^{n-1}} \rtimes \mathbb{Z}_2)$$

by Lemma 5.5.
Case 2. Π does not contains a subgroup isomorphic to $E_{p^{n-1}} \rtimes \mathbb{Z}_2$, so by Lemma 4.4 $\rho(E_{p^{n-1}} \rtimes \mathbb{Z}_2) \leqslant \rho(E_{p^n} \rtimes \mathbb{Z}_2)$. □

Remark 5.7 *Let* $G = E_{p^n} \rtimes \mathbb{Z}_2$ *and denote by* M_t *any subgroup of* G *isomorphic to* $E_{p^t} \rtimes \mathbb{Z}_2$. *Note that if* Π *is a partition of* G, *then*

$$\Pi = \{E_{p^{i_1}}, \ldots, E_{p^{i_k}}, M_{t_1}, \ldots, M_{t_m}, \langle x\rangle^{g_1}, \ldots, \langle x\rangle^{g_l}\}$$

and

$$|\Pi| = k + m + [p^n - (p^{t_1} + \ldots + p^{t_m})].$$

Also note that:

$$\{E_{p^{i_1}}, \ldots, E_{p^{i_k}}, Syl_p(M_{t_1}), \ldots, Syl_p(M_{t_m})\}$$

is a partition of E_{p^n}.

Lemma 5.8 *For* p *an odd prime and* Π *a partition of* $E_{p^n} \rtimes \mathbb{Z}_2$, *then* $|\Pi| \geqslant \rho(E_{p^n})$.

Proof — If Π is a partition of $G = E_{p^n} \rtimes \mathbb{Z}_2$ then

$$|\Pi| = k + m + [p^n - (p^{t_1} + \ldots + p^{t_m})] \geqslant k + m \geqslant \rho(E_{p^n}).$$

The statement is proved. □

Corollary 5.9 *For* p *an odd prime*

$$\rho(E_{p^n}) \leqslant \rho(E_{p^n} \rtimes \mathbb{Z}_2) \leqslant p^n + 1.$$

Theorem 5.10 *For* p *an odd prime and* $n \geqslant 3$, *then*

$$1 + p^2 \leqslant 1 + p^{\lceil \frac{n}{2} \rceil} = \rho(E_{p^n}) \leqslant \rho(E_{p^n} \rtimes \mathbb{Z}_2) \leqslant p^n + 1$$

and

$$\rho(E_{p^{n-1}} \rtimes \mathbb{Z}_2) \leqslant \rho(E_{p^n} \rtimes \mathbb{Z}_2).$$

Proof — Theorem 4.5, Lemmas 5.6 and Corollary 5.9. □

Remark 5.11 *For* p *an odd prime one can find examples where*

$$1 + p^{\lceil \frac{n}{2} \rceil} < \rho(E_{p^n} \rtimes \mathbb{Z}_2) < p^n + 1.$$

Let $p = n = 3$ *using* GAP *[8] we see that*

$$10 = 1 + 3^{\lceil \frac{3}{2} \rceil} = 1 + 3^2 < \rho(E_{3^3} \rtimes \mathbb{Z}_2) = \frac{3^3 - 1}{3 - 1} = 13 < 3^3 + 1$$

this partition consists of nine subgroups of order 6 *isomorphic to* D_3 *and four of order* 3.

Lemma 5.12 *For* p *a prime, then* $\rho(E_{p^n} \rtimes \mathbb{Z}_2) \equiv 1 \mod p$.

Proof — Let Π be a partition of $E_{p^n} \rtimes \mathbb{Z}_2$ then

$$\Pi = \{E_{p^{i_1}}, \ldots, E_{p^{i_k}}, M_{t_1}, \ldots, M_{t_m}, \langle x\rangle^{g_1}, \ldots, \langle x\rangle^{g_l}\}$$

so

$$|\Pi| = k + m + [p^n - (p^{t_1} + \ldots + p^{t_m})].$$

By the *first packing condition* in the Introduction of [10] we see that $k + m \equiv 1 \mod p$, so $|\Pi| \equiv 1 \mod p$. □

6 Partitionable p-groups with $\sigma(G) < \rho(G)$

Remark 6.1 *Note that a partitionable nilpotent group is a* p*-group.*

Hughes conjectured in [11] that if $G > |H_p(G)| > 1$, then

$$H_p(G) = \langle x \in G : x^p \neq 1\rangle$$

has index p in G. While the conjecture is false, it is true for large families of groups (see [15]).

Definition 6.2 *We will call a partitionable* p*-group of* Hughes type *If* $H_p(G)$ *has index* p *in* G.

Lemma 6.3 *If* G *is a* p*-group with partition* Π *and* $|H_p(G)| > 1$*, then* $H_p(G)$ *lies in one summand.*

Proof — Let $x \in Z(G)$ with order p. Let y be an element of order greater than p (note that $y \in H_p(G)$). Assume $x \in H_1$ and that $y \in H_i$. Note that $y^p \in H_i$ and $y^p = (xy)^p$; thus $xy \in H_i$ since Π is a partition. Now since H_i is a group, $x \in H_i$, which implies that $i = 1$, and thus $H_p(G) \subseteq H_1$. □

Lemma 6.4 *If* G *is a group of exponent* p *and order* p^3*, then*

$$1 + p = \sigma(G) < 1 + p^2 = \rho(G).$$

Proof — Note that if Π is a partition, then either there is one summand of order p^2 or all summands have order p. If all summands have order p, then

$$|\Pi| = 1 + p + p^2.$$

If there is a summand of order p^2, then $|\Pi| = 1 + p^2$. □

Lemma 6.5 *If* G *is a* p*-group with partition* Π *which has a summand of order*

$$\frac{|G|}{p} = p^{n-1},$$

then $|\Pi| = 1 + p^{n-1}$.

Proof — Note that all but one summand have order p, so $|\Pi| = 1 + p^{n-1}$. □

Lemma 6.6 *If* G *is a finite* p*-group and*

$$1 < |H_p(G)| < \frac{|G|}{p},$$

then for every partition Π *of* G *there exists a subgroup* K_Π *such that* $H_p(G)$ *has index* p *in* K_Π *and for all* $H \in \Pi$, $K_\Pi \nsubseteq H$.

Proof — By Lemma 6.3 $H_p(G)$ lies in one summand (say H_1), and every other summand must have exponent p. Let x be in $H_2 - \{1\}$ and let $K_\Pi = H_p(G)\langle x\rangle$. □

Theorem 6.7 *If* G *is a finite* p*-group with partition and* $|G| \geqslant p^3$, *then* $\sigma(G) < \rho(G)$.

Proof — By Tomkinson's Theorem (Theorem 2.1), $\sigma(G) = p + 1$.

Note that by Lemmas 6.4 and 6.5 if G is p-group of order p^3 with a partition, then $\rho(G) = 1 + p^2$.

Case 1. For G an exponent p group we will use induction. From the remarks above if $|G| = p^3$, then $\rho(G) = 1 + p^2 \geqslant 1 + p^2$.

Assume that

$$\rho(G) \geqslant 1 + p^2$$

for any group G of order p^k and exponent p, $k \geqslant 3$. Consider the case $k + 1$. Let Π be a partition of G with $|\Pi| = \rho(G)$, since $|G| = p^{k+1} > p^3$ there is a maximal subgroup M which is not contained in any summand (G has more than one maximal subgroup and any two of them intersect nontrivially). Thus by Lemma 4.4 we have $\rho(G) \geqslant \rho(M) \geqslant 1 + p^2 \geqslant \sigma(G) = p + 1$.

Case 2. If G is a partitionable p-group of Hughes type, then $H_p(G)$ is a maximal subgroup of G, and by Lemma 6.3 it is a summand, thus by Lemma 6.5, $\rho(G) = 1 + p^{n-1} > \sigma(G)$.

Case 3. If

$$1 < |H_p(G)| < \frac{|G|}{p}$$

and Π be a partition of G with $|\Pi| = \rho(G)$, then by Lemma 6.6 there exists $K_\Pi \leqslant G$ a partitionable p-group of Hughes type which is not contained in any summand of Π. From case above $1 + p^2 \leqslant \rho(K_\Pi)$, so by Lemma 4.4, $1 + p^2 \leqslant \rho(G)$. □

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