TECHNOLOGICAL AND ECONOMIC DEVELOPMENT OF ECONOMY
Baltic Journal on Sustainability
2009
15(4): 646–669

FUZZY REAL OPTIONS VALUATION FOR OIL INVESTMENTS

İrem Uçal[1], Cengiz Kahraman

Istanbul Technical University, Department of Industrial Engineering,
Macka, Istanbul, 34367, Turkey

Received 26 March 2009; accepted 20 August 2009

Abstract. Traditional valuation methods are less viable under uncertainty. Hence, other methods such as real options valuation models, which can minimize uncertainty, have become more important. In this study, the hybrid approach suggested by Carlsson and Fuller is examined for the case of discrete compounding as this approach better models risky cash flows. A new real options valuation model that will evaluate the investment in a more realistic way is suggested by postponing the defuzzification of parameters in early stages. The suggested model has been applied to the data of an oil field investment and in conclusion the loss of information caused by early-defuzzification has been determined.

Keywords: Fuzzy Sets, Real Options Valuation, Fuzzy Real Options, Investment.

Reference to this paper should be made as follows: Uçal, İ.; Kahraman, C. 2009. Fuzzy real options valuation for oil investments, *Technological and Economic Development of Economy* 15(4): 646–669.

1. Introduction

The investment valuation methods that are currently in use try to provide decision makers with enough information for making investment decisions. In literature, there exist numerous methods for investment evaluation including traditional valuation methods and real options methods. Traditional valuation methodologies based on discounted cash flows (DCF) do not consider some of the intrinsic attributes of the asset or investment opportunity (Mun 2002). In the DCF methods, expected value and priced risk characteristics of cash flows summarize unknown future cash flows. The DCF model values the equity of an investment by discounting free cash flows from the operations of the investment to a present value.

[1] Corresponding author, e-mail: *ucal@itu.edu.tr* Phone: +902122931300

ISSN 1392-8619 print/ISSN 1822-3613 online
doi: 10.3846/1392-8619.2009.15.646-669
http://www.tede.vgtu.lt

The operating flexibility and strategic value aspects of various projects cannot be properly captured by traditional DCF techniques because of their discretionary asymmetric nature and their dependence on future events that are uncertain at the time of the initial decision (Trigeorgis 1996). Brach (2003) identified three fundamental differences between discounted cash flows methods and real options valuation methods. The first difference is that in the discounted cash flows method, decisions cannot be changed in the future, while in the real options valuation (ROV) method it is likely to perform directional changes propped up by obtaining new information. Secondly, in DCF method estimated cash flows group is considered as a base, whereas in ROV method, cash flows depend on the ambiguous conditions in the future. Thirdly, in the former method, sensitivity and scenario analyses are static; while in the latter there is a managerial flexibility in order to provide adaptability to changing conditions. As a result, it can be claimed that the differences arise in decision variation, dependencies, and dynamism.

Real options analysis has appeared as a tool to give more distinct results than traditional DCF analysis in investment projects by giving the option of postponing the project to a later time or abandoning the project whenever it is necessary. Mun (2002) explains real options as a systematic approach and integrated solution making use of financial theory, economic analysis, management science, decision sciences, statistics, and econometric modeling in applying options theory to valuing real physical assets, as opposed to financial assets defined as a dynamic and uncertain business environment where business decisions are flexible in the context of strategic capital investment decision making, valuing investment opportunities and project capital expenditures. The general analysis of real options as a strategic tool rather than a mere valuation tool, that is, proactive rather than just reactive flexibility represents an advance on current thinking in this area (Leslie and Michaels 1997). Real options theory provides a method to better valuate investment projects in the presence of managerial flexibilities as information option, waiting option and abandonment option (Rocha *et al.* 2007). ROV is practically the same as the valuation of financial options; yet, there exist a number of differences. A financial option gives the holder of the right to buy or sell a specified quantity of an underlying asset at a fixed price (it is also called a strike price or an exercise price) at or before the expiration date of the option. Since it is a right and not an obligation, the holder can prefer not to exercise the right and allow the option to expire. There are two types of options - call options and put options. In a call option, the buyer of the option has the right, but not the obligation, to buy an agreed quantity of a particular commodity or financial instrument from the seller of the option at a certain time for a certain price, and the seller is obligated to sell the commodity or financial instrument if the buyer exercises the option. The put option allows its buyer the right but not the obligation to sell a commodity or financial instrument to the seller of the option at a certain time for a certain price, and with respect to that the seller has the obligation to purchase the underlying asset at that strike price, if the buyer exercises the option. There are two main types of options: American options and European options. A primary distinction between American and European types of options is that American options can be exercised at any time prior to their expiry date, while European options can be exercised only at expiration. The possibility of early exercise makes American options more valuable than otherwise similar European options; it also makes them more difficult to value. There is one compensating factor that enables the former to be valued using models designed for the latter. In most cases, the time premium associated with the remaining life of an option and transactions costs makes early exercise sub-optimal. In other words, the holders of in the money options will generally get much more by selling the option to someone else than by exercising the options.

Mun (2002) emphasized nine distinctive characteristics of financial and real options. The first one is that the life span of financial options is shorter than real options. In financial options the main variable determining the value is the price of the financial asset, while in real options the main variables determining the value are the cash flows determined by the demand, management and competition. The values of the financial options are usually small, while the values of the real options are usually extremely large. The value of options cannot be controlled by changing the stock prices in financial options, while in real options the value of strategic options can be increased by managerial decisions and flexibility. In financial options market or competition effects are irrelevant in determining the value and pricing of options, whereas in real options, market or competition effects determine the strategic value of strategic options. Financial options have been used for the past 30 years, while real options are newly developed to be used in financial activities of the corporations for the last few years. Financial options are usually solved by applying closed formed partial differential equations and simulation, decreasing the variance methods, whereas real options are usually solved by applying closed formed equations and the simulation of the main variables with two binomial letters. Financial options are commercially secured and they can be marketed with price information and likewise, while real options are not commercially in use and they are naturally private and they do not have market resemblance. Lastly, management acceptance and movements are not influential on the valuation of financial options, while they determine the value of real options.

Option pricing theory has made vast strides since 1972, when Black and Scholes (1972) published their pioneering paper suggesting a model for valuing dividend-protected European options. Black and Scholes used a “replicating portfolio” to come up with their final formulation. Although their derivation is mathematically complicated, there is a simpler binomial model for valuing options that draws on the same logic.

In valuing real options, we often use the Black Scholes model which has fairly limited applicability. Most real options are not analogs of European options (Black and Scholes 1973). However, the Black Scholes partial differential equation itself has a wider applicability. Given the wide appropriate boundary conditions, this partial differential equation can be solved to evaluate many types of options, such as American and compound options.

The value of a real option is computed by (Leslie and Michaels 1997):

$$ROV = S_0 e^{-\delta T} N\left(d_1\right) - X e^{-rT} N\left(d_2\right), \quad (1)$$

where

$$d_1 = \frac{\ln(S_0 / X) + (r - \delta + \sigma^2 / 2)T}{\sigma\sqrt{T}}, \; d_2 = d_1 - \sigma\sqrt{T}, \quad (2)$$

where S: present value of expected cash flows, X: present value of fixed costs, δ: value lost over duration of the option, r: risk-free interest rate, σ: uncertainty of expected cash flows,

t: time to expiry, and $N(d)$: cumulative normal distribution function. Brach (2003) pointed out the equivalence of the real options parameters in financial options parameters as follows: the exercise price used in financial options indicates costs to acquire the asset, stock price accounts for the present value of the future cash flows from the asset, time to expiration stands for the length of time option viable, and the variance of stock returns is replaced by riskiness of the asset or, in other words, variance of the best and worst case scenario.

The aim of this study is to propose a new fuzzy real options valuation model which evaluates investments in a more realistic way. Since fuzzy logic fits best to the uncertain nature of the investment decisions, it has been utilized in the model. The proposed model postpones defuzzification to the very end stages of the process. This prevents the loss of information at the beginning of the process. Another superiority of our model is that the conversion of continuous compounding into discrete compounding provides more intelligibility, which increases its usability in practice.

The rest of the paper is organized as follows. Literature reviews on fuzzy real options valuation and real options valuation applications in oil investment valuations are given in Sections 2 and 3, respectively. In Section 4, we propose a model of fuzzy real options valuation. In Section 5, there is an application of this new model. In the last section we conclude the obtained results.

2. Fuzzy real options

In classical mathematics the binary valued logic and set theory are used. For an element that belongs to a set of all possible elements and to any given specific subset, it can be said exactly whether that element is or is not a member of it. For example, a person belongs to the set of all human beings and given a specific subset, such as all males, one can say whether or not each particular person belongs to this set. Unfortunately, not everything can be described using binary valued sets. The classifications of persons into males and females is easy, but it is problematic to classify them as being young or not. The set of young people is far more difficult to define as there is no distinct cut-off point at which age youngness begins or ends. This is not a measurement problem and measuring the age of all elements more precisely is not helpful. Such a problem is often distorted so that it can be described using the well-known existing methodology. Fuzzy logic gives a method to simulate the ability of human reasoning. With fuzzy logic an element could partially belong to a set and this is represented by the set membership. Unlike crisp theories, fuzzy logic enables vagueness and ambiguity as well as it avoids clear distinctions and limits.

Zadeh (1965) first founded the fuzzy set theory and he suggested that the notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which is parallel in many respects to the framework used in the case of ordinary sets; but it is more general than the latter and, potentially, may prove to have a much wider scope of applicability, particularly in the fields of pattern classification and information processing. Fuzzy set theory is a marvelous tool for modeling the kind of uncertainty associated with vagueness, with imprecision, and/or with a lack of information regarding a particular element of the problem at hand (Ross 1995). A fuzzy set is a set containing elements that have varying degrees of membership between 0 and 1 in the set, where 0 means the element is not a member of the set, 1 means the element is completely a member of the set which means full membership, and the values between these two values gives a partial membership of the element on the set. A fuzzy set is prescribed by vague or ambiguous properties hence its boundaries are ambiguously specified (Ross 1995).

A fuzzy number is a fuzzy set which is both normal and convex. Most common types of fuzzy numbers are triangular and trapezoidal. Other types of fuzzy numbers are possible, such as bell shaped or gaussian fuzzy numbers, yet these types of fuzzy numbers are rarely used in literature. Triangular fuzzy numbers are defined by three parameters, while trapezoidal require four parameters.

Fuzzy sets have been used for valuating real options in the literature. Carlsson *et al.* (2001) applied fuzzy real options on the project selection by identifying the optimal path of the dynamic decision trees with the biggest real option value in the end of the planning period. Carlsson and Fuller (2003) improved a fuzzy approach for real options valuation, which is one of the mostly used real options valuation approaches, by applying a heuristic real option rule in a fuzzy setting, where the present values of expected cash flows and expected costs are estimated by trapezoidal fuzzy numbers and they determined the optimal exercise time with the assistance of possibilistic mean value and variance of fuzzy numbers. $A = (a, b, \alpha, \beta)$ is a trapezoid fuzzy number which can be shown by

$$[A]^{\gamma} = [a-(1-\gamma)\alpha, b+(1-\gamma)\beta], \quad \forall\gamma \in [0,1] \tag{3}$$

which means the most possible values of expected A lie in the interval $[a, b]$ and $(a-\alpha)$ is the downward potential and $(b+\beta)$ is the upward potential for A. They suggested Eq. 4 for computing fuzzy real options values, where is the estimated present value of expected cash flows, $\tilde{X} = (x_1, x_2, \alpha', \beta')$ is estimated present value of expected costs, $E(\tilde{S}_0)$ denotes the possibilistic mean value of the present value of expected cash flows, $E(\tilde{X})$ denotes the possibilistic mean value of expected costs, and $\sigma = \sigma(\tilde{S}_0)$ is the possibilistic variance of the present value of expected cash flows.

$$FROV = \tilde{S}_0 e^{-\delta T} N(d_1) - \tilde{X} e^{-rT} N(d_2), \tag{4}$$

where

$$d_1 = \frac{\ln\left(\frac{E(\tilde{S}_0)}{E(\tilde{X})}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \tag{5}$$

$$d_2 = d_1 - \sigma\sqrt{T}. \tag{6}$$

Using Eq. 3 for arithmetic operations on trapezoidal fuzzy numbers they find the fuzzy real options formula as below:

$$\begin{aligned} FROV &= (s_1, s_2, \alpha, \beta) e^{-\delta T} N(d_1) - (x_1, x_2, \alpha', \beta') e^{-rT} N(d_2) = \\ &\left(s_1 e^{-\delta T} N(d_1) - x_2 e^{-rT} N(d_2), s_2 e^{-\delta T} N(d_1) - x_1 e^{-rT} N(d_2)\right. \\ &\left.\alpha e^{-\delta T} N(d_1) + \beta' e^{-rT} N(d_2), \beta e^{-\delta T} N(d_1) + \alpha' e^{-rT} N(d_2)\right). \end{aligned} \tag{7}$$

Garcia (2004) used the fuzzy real options valuation (FROV) model in a real investment project from the energy sector. In Garcia's paper, the model suggested by Carlsson and Fuller (2003) has been applied to timing decision and selection of power station for various investment alternatives. Han and Zheng (2005) applied fuzzy options to fail to perform risk analysis for municipal bonds in China. Zeng *et al.* (2007) compared the application of traditional net present value (NPV) method with real options in investment evaluations, analyzed the uncertainty of power grid investment project, and discussed how to make the investment decision when investment cost and cash flow are both fuzzy numbers. They also introduced fuzzy expectation and fuzzy variance concept to construct evaluation model and, finally, they performed a simulation model to show the validity. Tao *et al.* (2007) developed a comprehensive methodology based on fuzzy risk analysis and real options approach to evaluate information technologies investments in a nuclear power station. Allenotor and Thulasiram (2007) used a fuzzy trinomial real options model on pricing grid resources and proved the feasibility of the model through experiments. Kahraman and Ucal (2008) used the certainty equivalent approach for real options valuation in an oil investment with fuzzified data. Tolga and Kahraman (2008) considers the multidimensional and vague side of the R&D project selection process. The fuzzy analytic hierarchy process, which takes monetary (fuzzy real option value) and nonmonetary (capability, success probability, trends, etc.) criteria into account, was used to make this selection among alternative R&D projects. Based on fuzzy real option valuation, Majlender (2008) proposed the application of a possibilistic decision rule for optimal investment strategy. The proposed decision rule had to be reapplied every time when new information arrived during the deferral period to be able to analyze how the optimal investment strategy should be changed in the light of the new information.

3. Energy investments

Energy is one of the necessary elements that is required for human life and industrial production. Energy consumption increases sharply in parallel with the increase in population, standard of living, industrial and technologic developments. Planning of the exploition of energy resources has become a very crucial issue. Given insufficient existing resources, the importance of investment in new resources increases. The main energy resources are fossil fuels, coal, oil, natural gas, nuclear, and renewable energy resources: wind, solar, hydropower, biomass and geothermal.

The survey carried out by the International Energy Outlook 2008 (US.DoE 2008) shows that liquid fuels consumption increases at an average annual rate of 1.2 percent from 2005 to 2030. Renewable energy and coal are the fastest growing energy sources, with consumption increasing by 2.1 percent and 2.0 percent respectively, and it is estimated that the World energy consumption increments by 50% from 2005 to 2030.

According to the Worldwide Trends in Energy Use and Efficiency 2008 report (IEA 2008), oil products remained the most important final energy commodity with a global share of 37% in 2005, driven by their use in transport in which energy consumption has grown most quickly by rising passenger travel and freight transport. Due to a relative importance of the transport sector in the OECD countries, oil products accounted for 47% of total final energy use in 2005. Oil products also have the largest share of consumption in non-OECD countries, accounting on average for 28% of total final energy use in 2005. In many of these countries oil products are the most important natural resources. They are used as gasoline, jet fuel, and diesel fuel to run cars, trucks, aircraft, ships, and other vehicles. Home heat sources include oil, natural gas, and electricity, which in many areas are generated by burning natural gas. Petroleum and petroleum-based chemicals are important in manufacturing plastic, wax, fertilizers, lubricants, and many other goods. These data reveal how important and critical oil production is throughout the whole world.

A typical oil investment project has three main phases which are exploration, development, and extraction. The exploration phase comprises three main activities: scouting, concession and exploration. Development activity can be separated into two stages: appraisal and technical development activities and the two aspects of the extraction phase are particularly relevant to econometric modeling: operating expenditure and taxation (Favero, Pesaran 1991).

Investment in the power sector has three important characteristics; irreversibility, uncertainty, and flexibility. Traditional appraisal methodologies for project investment can hardly incorporate the above three characteristics. ROA is relatively new in assessing investments in climate change projects which handles the investment problem and uncertainty in a particular way. For example, it focuses on the timing of the investment decision whether to do the project or not (Yang and Blyth 2007).

As mentioned in the world energy outlook 2006 (Birol 2007), to increase geographic and fuel-supply diversity and to mitigate climate-destabilizing emissions, the need to curb the growth in fossil-energy demand is more urgent than ever. Global primary energy demand in the Reference Scenario is projected to increase by just over one-half between 2006 and 2030 – an average annual rate of 1.6%. Oil investment – three-quarters of which goes to the upstream – amounts to over $4 trillion in total over 2005–2030. With this huge demand oil investment decisions become more important in global economy.

According to world energy outlook 2007 (IEA 2007), the developing countries whose economies and populations are growing fastest contribute 74% of the increase in global primary energy use where China and India alone account for 45% of this increase.

In the literature there are lots of models which are used to valuate energy investments. Paddock *et al.* (1988) used option valuation theory to develop a new approach to value leases for offshore petroleum. Bergmann *et al.* (2006) estimated the magnitude of external costs and benefits such as landscape quality, wildlife and air quality for the case of renewable technologies in Scotland, a country which has set particularly ambitious targets for expanding renewable energy. C. Locatelli (2006) explored the investment strategies of oil companies of Russia. Chorn and Shokhor (2006) reported the union of a real options algorithm with the Bellman equation by allowing the analyst and management to avoid estimating outcome probabilities and computing expected values for investments with sequential stages using project experts' technical insight. Menegaki (2008) used environmental cost–benefit analysis for the evaluation of renewable energy projects. Dinca *et al.* (2007) aimed to select the optimal energetic scenario applied to a consumer with 100 000 inhabitants from the residential–tertiary sector, from the ecological, energetical and economic points of view. Kjærland (2007) presented a valuation study of hydropower investment opportunities in the Norwegian context. Mohn and

Misund (2009) presented a micro-econometric study of corporate investment and uncertainty in a period of market turbulence and restructuring in the international oil and gas industry. Rodríguez (2008) applied a real option model for the valuation of destination flexibility in long-term liquefied natural gas supplies. Abadie and Chamorro (2008) used Monte Carlo approach to evaluate natural gas resources investments. Lin *et al.* (2009) developed a hybrid interval-fuzzy two-stage stochastic energy systems planning model and applied the model to a hypothetical regional energy system.

There are some factors on the evaluation of energy investments: longer time of return of energy investments comparing to other investments, higher capital necessity and longevity of duration of management increase in the risk and ambiguity. The uncertainty and risks of the energy investments are also increasing due to the changes in the energy policies and escalating energy requirements of the world. Using fuzzy real options minimizes the effects of these uncertainties and risks on the investment.

4. Proposed model for real options valuation

In this study we suggest a new model based on Carlsson & Fuller's hybrid approach using discrete compounding instead of continuous compounding by defuzzifying the costs and revenues at a later stage than the based model and the based model has been improved by fuzzifying interest rates and probabilities.

4.1. Defuzzification methods used in the model

Defuzzification is the process of producing a quantifiable result in fuzzy logic. It is the conversion of a fuzzy quantity into a precise quantity, just as justification is the conversion of a precise quantity into a fuzzy quantity (Ross 1995). We use three defuzzification methods: *Total Integral Value Method*, *Centroid Method*, and *a defuzzification method for normal distribution*. They are summarized as follows:

4.1.1. Total Integral Value Method

Liou and Wang (1992) developed the total integral value method based on the mean of the integral value, and it ranks triangular fuzzy number $F = (a, b, c)$. The total integral value of F is defined as

$$I_T^{\alpha}(A) = 0.5\left[\alpha c + b + (1-\alpha)a\right], \qquad (8)$$

where α is an index of optimism that represents the degree of optimism of the decision-maker and has a value between [0,1]. The optimism coefficient also reflects the decision maker's risk taking trend. A lower optimism coeffiecient leads a risk-averse decision maker and likely a higher optimism coeffiecient leads a risk-seeking decision maker. This method has been used to defuzzify fuzzy triangular numbers.

4.1.2. Centroid Method

Centroid method is the most prevalent and physically appealing of all the defuzzification methods. This method determines the centre of the area of the combined membership functions and mimics the center of gravity approach in physics. Eq. 9 gives the algebric expression of this method (Ross 1995).

$$z^* = \frac{\int \mu_{\tilde{C}}(z) z dz}{\int \mu_{\tilde{C}}(z) dz}. \quad (9)$$

Although there are lots of other defuzzification methods available, the centroid method has been chosen to use in this paper for its simplicity and fast computation.

4.1.3. A defuzzification method for normal distribution

Figure 1 demonstrates the probability values that have a normal distribution. If the weights used to balance the probability values are taken respectively as t_1, t_2, t_3, and t_4 as determined by the experts, since a balancing must be applied towards the summit point of the bell curve that is observed in normal distribution. Then, for Fig. 1 these values have to provide the inequality of $t_1 > t_2 > t_3 > t_4$. Under these conditions, Eq. 10 is suggested in order to defuzzify the fuzzy numbers that have a normal distribution.

$$d = \frac{\tau_1 d_1 + \tau_2 d_2 + \tau_3 d_3 + \tau_4 d_4}{\tau_1 + \tau_2 + \tau_3 + \tau_4}. \quad (10)$$

4.2. Discrete compounding

Discrete compounding has more widespread usage in engineering economy community than continuous compounding. Carlsson & Fuller's (2003) real options formula is modified by applying discrete compounding. To do this, the equation $F = Pe^{rn} = P(1+i)^n$ has been considered. The discrete interest rate is obtained by the below operations.

$$e^{rn} = (1+i_r)^n \longrightarrow rn = \ln(1+i_r)^n \longrightarrow r = \ln(1+i_r) \longrightarrow i_r = e^r - 1, \quad (11)$$

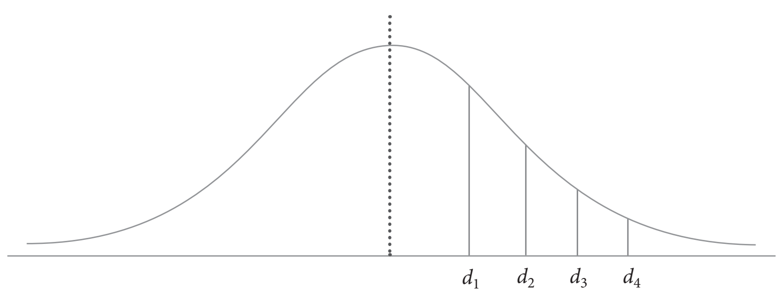

Fig. 1. Normal distribution

$$e^{\delta n} = (1+i_\delta)^n \longrightarrow \delta n = \ln(1+i_\delta)^n \longrightarrow \delta = \ln(1+i_\delta) \longrightarrow i_\delta = e^{\delta} - 1, \tag{12}$$

where i_r is risk-free interest rate in dicrete cases and i_δ is the percentage value lost over the duration of option in discrete cases. When the discrete interest rate formulas acquired in the fuzzy real options formula are used instead of continuous interest rate in the equations, below formulas are reached:

$$\tilde{d}_1 = \frac{\ln\left(\frac{E(\tilde{S})}{E(\tilde{X})}\right) + \left((\ln(1+i_r) - \ln(1+i_\delta) + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{E(\tilde{S})}{E(\tilde{X})}\right) + \ln\left(\frac{1+i_r}{1+i_\delta}\right)^T + \frac{T\sigma^2}{2}}{\sigma\sqrt{T}} \tag{13}$$

and

$$FROV = \tilde{S}_0 \left(1+i_\delta\right)^{-T} N\left(d_1\right) - \tilde{X}\left(1+i_r\right)^{-T} N\left(d_2\right), \tag{14}$$

where $\tilde{S}_0$ is the fuzzy present value of expected cash flows, $\tilde{X}$ is the fuzzy present value of fixed costs, and other symbols are the same as the ones in the crisp formulas.

For operational convenience, the variable w has been designated for Eq. 15 and after some simplifications Eq.s 17 and 16 have been reached.

$$w = \frac{1+i_r}{1+i_\delta}. \tag{15}$$

If we substitute Eq. 15 in Eq. 13, Eqs. 16 and 17 are obtained.

$$\tilde{d}_1 = \frac{\ln\left(\frac{E(\tilde{S})}{E(\tilde{X})}\right) + \ln\left(\frac{1+i_r}{1+i_\delta}\right)^T + \frac{T\sigma^2}{2}}{\sigma\sqrt{T}} = \frac{\ln\left(\left(\frac{E(\tilde{S})}{E(\tilde{X})}\right)w^T\right) + \ln e^{\frac{T\sigma^2}{2}}}{\sigma\sqrt{T}} = \tag{16}$$

$$\ln\left(\left(\left(\frac{E(\tilde{S})}{E(\tilde{X})}\right)w^T\right)^{\frac{1}{\sigma\sqrt{T}}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right).$$

$$\tilde{d}_1 = \ln\left(\left(\frac{E(\tilde{S})}{E(\tilde{X})}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right). \tag{17}$$

If we substitute Eq. 17 in Eq. 6, Eq. 18 is obtained.

$$\tilde{d}_2 = \ln\left(\left(\frac{E(\tilde{S})}{E(\tilde{X})}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right) - \sigma\sqrt{T}. \tag{18}$$

4.3. Fuzzifying the discrete interest rates

In this subsection i_r and i_δ are accepted as triangular fuzzy numbers. We prefer using Buckley's notation. Buckley's membership function for a future value $\tilde{F}$ is given by

$$\mu\left(x\middle|\tilde{M}\right)=\left(m_1,\frac{f_1\left(y\middle|\tilde{M}\right)}{m_2},\frac{m_2}{f_2\left(y\middle|\tilde{M}\right)},m_3\right), \tag{19}$$

which is determined by

$$f_{ni}\left(y\middle|\tilde{M}\right)=f_i\left(y\middle|\tilde{F}\right)\left(1+f_k\left(y\middle|\tilde{r}\right)\right)^{-n} \tag{20}$$

for i = 1,2 where $k = i$ for negative $\tilde{F}$, $k = 3 - i$ for positive $\tilde{F}$ (Buckley 1987).

The equations that demonstrate the right and left convergences towards the i_δ and i_r functions are given by Eqs. 21–24.

$$f_1\left(y\middle|\tilde{i}_r\right)=i_{r_L}+y\left(i_{r_M}-i_{r_L}\right), \tag{21}$$

$$f_2\left(y\middle|\tilde{i}_r\right)=i_{r_R}+y\left(i_{r_M}-i_{r_R}\right), \tag{22}$$

$$f_1\left(y\middle|\tilde{i}_\delta\right)=i_{\delta_L}+y\left(i_{\delta_M}-i_{\delta_L}\right), \tag{23}$$

$$f_2\left(y\middle|\tilde{i}_\delta\right)=i_{\delta_R}+y\left(i_{\delta_M}-i_{\delta_R}\right). \tag{24}$$

Fig. 2 demonstrates the membership functions of the interest rates i_δ and i_r.

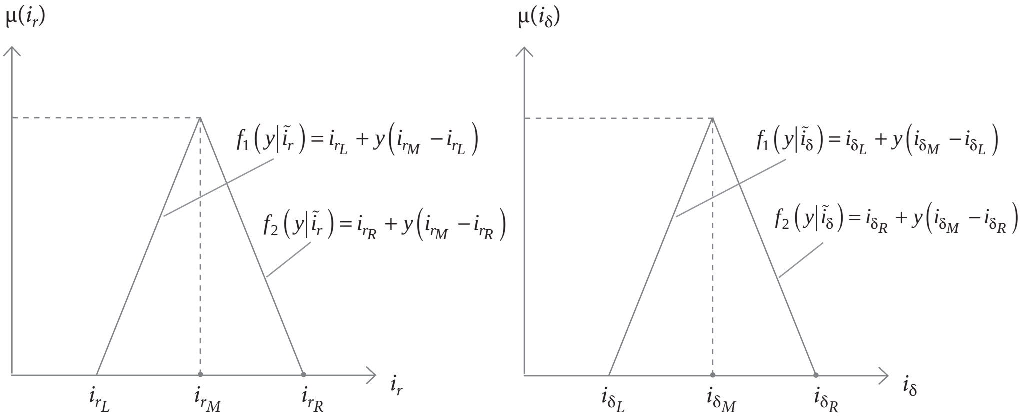

Fig. 2. Membership functions of i_δ and i_r

With substituting Eqs. 21–24 into Eq. 15, we obtain Eq. 25.

$$w=\frac{1+i_r}{1+i_\delta}=\frac{1+f_k\left(y\middle|i_r\right)}{1+f_i\left(y\middle|i_\delta\right)}, \tag{25}$$

where $k = 3 - i$ and i = 1,2.

If it is substituted in Eq.s 17 and 18, Eqs. 26–27 are obtained.

$$\tilde{d}_1=\ln\left(\left(\frac{E(\tilde{S})}{E(\tilde{X})}\right)^{\frac{1}{\sigma\sqrt{T}}}\left(\frac{1+f_k\left(y\middle|i_r\right)}{1+f_i\left(y\middle|i_\delta\right)}\right)^{\frac{\sqrt{T}}{\sigma}}e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right)=$$

$$\ln\left(\left(\frac{E(\tilde{S})}{E(\tilde{X})}\right)^{\frac{1}{\sigma\sqrt{T}}}\left(\frac{1+i_{r_R}+y\left(i_{r_M}-i_{r_R}\right)}{1+i_{\delta_L}+y\left(i_{\delta_M}-i_{\delta_L}\right)}\right)^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right). \tag{26}$$

$$\tilde{d}_1=\left(\ln\left(\left(\frac{E(\tilde{S})}{E(\tilde{X})}\right)^{\frac{1}{\sigma\sqrt{T}}}\left(\frac{1+i_{r_R}}{1+i_{\delta_L}}\right)^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right);\ln\left(\left(\frac{E(\tilde{S})}{E(\tilde{X})}\right)^{\frac{1}{\sigma\sqrt{T}}}\left(\frac{1+i_{r_M}}{1+i_{\delta_M}}\right)^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right);\right.$$
$$\left.\ln\left(\left(\frac{E(\tilde{S})}{E(\tilde{X})}\right)^{\frac{1}{\sigma\sqrt{T}}}\left(\frac{1+i_{r_L}}{1+i_{\delta_R}}\right)^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right)\right). \tag{27}$$

$$\tilde{d}_2=\ln\left(\left(\frac{E(\tilde{S})}{E(\tilde{X})}\right)^{\frac{1}{\sigma\sqrt{T}}}\left(\frac{1+f_k\left(y|i_r\right)}{1+f_i\left(y|i_\delta\right)}\right)^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right)-\sigma\sqrt{T}=$$
$$\ln\left(\left(\frac{E(\tilde{S})}{E(\tilde{X})}\right)^{\frac{1}{\sigma\sqrt{T}}}\left(\frac{1+i_{r_R}+y\left(i_{r_M}-i_{r_R}\right)}{1+i_{\delta_L}+y\left(i_{\delta_M}-i_{\delta_L}\right)}\right)^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right)-\sigma\sqrt{T}. \tag{28}$$

$$\tilde{d}_2=\left(\ln\left(\left(\frac{E(\tilde{S})}{E(\tilde{X})}\right)^{\frac{1}{\sigma\sqrt{T}}}\left(\frac{1+i_{r_R}}{1+i_{\delta_L}}\right)^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right)-\sigma\sqrt{T}\right.$$
$$\ln\left(\left(\frac{E(\tilde{S})}{E(\tilde{X})}\right)^{\frac{1}{\sigma\sqrt{T}}}\left(\frac{1+i_{r_M}}{1+i_{\delta_M}}\right)^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right)-\sigma\sqrt{T}; \tag{29}$$
$$\left.\ln\left(\left(\frac{E(\tilde{S})}{E(\tilde{X})}\right)^{\frac{1}{\sigma\sqrt{T}}}\left(\frac{1+i_{r_L}}{1+i_{\delta_R}}\right)^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right)-\sigma\sqrt{T}\right).$$

FROV equation is adapted for fuzzy discrete interest rates as follows;

$$FROV=S_0\left(1+f_i\left(y|i_\delta\right)\right)^{-T}N\left(\tilde{d}_1\right)-\tilde{X}\left(1+f_i\left(y|i_r\right)\right)^{-T}N\left(\tilde{d}_2\right), \tag{30}$$

$$FROV = \left(S_0 \left(1+i_{\delta_L}\right)^{-T} N\left(\tilde{d}_1\right) - \tilde{X}\left(1+i_{r_R}\right)^{-T} N\left(\tilde{d}_2\right); S_0\left(1+i_{\delta_M}\right)^{-T}, \right.$$
$$\left. N\left(\tilde{d}_1\right) - \tilde{X}\left(1+i_{r_M}\right)^{-T} N\left(\tilde{d}_2\right);\ S_0\left(1+i_{\delta_R}\right)^{-T} N\left(\tilde{d}_1\right) - \tilde{X}\left(1+i_{r_L}\right)^{-T} N\left(\tilde{d}_2\right)\right). \quad (31)$$

Eq. 31 is developed for the situations that occur while dealing with fuzzy discrete parameters. It is helpful when there is incomplete information under fuzzy decision environment.

4.4. Postponing the defuzzification of costs and revenues

$\tilde{S}_0 = (s_1, s_2, \alpha, \beta)$ and $\tilde{X} = (x_1, x_2, \alpha', \beta')$ are trapezoidal fuzzy numbers. If $\ln\left(\frac{E\left[\tilde{S}_0\right]}{E\left[\tilde{X}\right]}\right)$ expression found in Eq. 5 is replaced by the fuzzy numbers $\tilde{S}_0$ and $\tilde{X}$, Eq. 32 is reached subsequently.

$$\ln\left(\frac{E\left[\tilde{S}_0\right]}{E\left[\tilde{X}\right]}\right) = \ln\left[\frac{s_1}{x_4}, \frac{s_2}{x_3}, \frac{s_3}{x_2}, \frac{s_4}{x_1}\right] = \left[\ln\frac{s_1}{x_4}, \ln\frac{s_2}{x_3}, \ln\frac{s_3}{x_2}, \ln\frac{s_4}{x_1}\right]. \quad (32)$$

If Eq.32 is placed in Eq. 6, Eq. 33 is reached.

$$d_1 = \left(\ln\left(\left(\frac{s_1}{x_4}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right), \ln\left(\left(\frac{s_2}{x_3}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right), \right.$$
$$\left. \ln\left(\left(\frac{s_3}{x_2}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right), \ln\left(\left(\frac{s_4}{x_1}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right)\right). \quad (33)$$

In $d_2 = d_1 - \sigma\sqrt{T}$ expression, $\widetilde{\sigma\sqrt{T}} = (\left(\sigma\sqrt{T}\right)_1, \left(\sigma\sqrt{T}\right)_2, \left(\sigma\sqrt{T}\right)_3, \left(\sigma\sqrt{T}\right)_4)$ can be stated and Eq. 34 is reached:

$$d_2 = \left(\ln\left(\left(\frac{s_1}{x_4}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right) - \sigma\sqrt{T},\ \ln\left(\left(\frac{s_2}{x_3}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right) - \sigma\sqrt{T}, \right.$$
$$\left. \ln\left(\left(\frac{s_3}{x_2}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right) - \sigma\sqrt{T}, \ln\left(\left(\frac{s_4}{x_1}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right) - \sigma\sqrt{T} \right). \quad (34)$$

Carlsson and Fuller (2003) made a defuzzification in the very early stages of their model. Eqs. 33 and 34 are developed for postponing the defuzzification to the very end stages of the model.

4.5. Fuzzy probabilities

After d_1 and d_2 values obtained in Section 4.4 were defuzzified by Eq. 10, probability values could be obtained from the normal distribution table. By postponing the defuzzification, operations could be carried out with fuzzy probabilities. Below, formulas have been developed to be used in the case of carrying on with fuzzy probabilities.

$$
\begin{aligned}
FROV = {} & \tilde{S}_0\left(1+i_\delta\right)^{-T} N\left(\ln\left(\left(\frac{s_1}{x_4}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right), \ln\left(\left(\frac{s_2}{x_3}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right),\right. \\
& \left.\ln\left(\left(\frac{s_3}{x_2}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right), \ln\left(\left(\frac{s_4}{x_1}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right)\right) - \\
& \tilde{X}\left(1+i_r\right)^{-T} N\left(\ln\left(\left(\frac{s_1}{x_4}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right) - \sigma\sqrt{T}, \ln\left(\left(\frac{s_2}{x_3}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right) - \sigma\sqrt{T},\right. \\
& \left.\ln\left(\left(\frac{s_3}{x_2}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right) - \sigma\sqrt{T}, \ln\left(\left(\frac{s_4}{x_1}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right) - \sigma\sqrt{T}\right).
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
FROV = {} & \left(1+i_\delta\right)^{-T}\left[s_1 N\left(\ln\left(\left(\frac{s_1}{x_4}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right)\right), s_2 N\left(\ln\left(\left(\frac{s_2}{x_3}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right)\right),\right. \\
& \left. s_3 N\left(\ln\left(\left(\frac{s_3}{x_2}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right)\right), s_4 N\left(\ln\left(\left(\frac{s_4}{x_1}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right)\right)\right] - \\
& \left(1+i_r\right)^{-T}\left[\left(x_1 N \ln\left(\left(\frac{s_1}{x_4}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right) - \sigma\sqrt{T}\right), x_2 N\left(\ln\left(\left(\frac{s_2}{x_3}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right) - \sigma\sqrt{T}\right),\right. \\
& \left. x_3 N\left(\ln\left(\left(\frac{s_3}{x_2}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right) - \sigma\sqrt{T}\right), x_4 N\left(\ln\left(\left(\frac{s_4}{x_1}\right)^{\frac{1}{\sigma\sqrt{T}}} w^{\frac{\sqrt{T}}{\sigma}} e^{\left(\frac{\sigma\sqrt{T}}{2}\right)}\right) - \sigma\sqrt{T}\right)\right].
\end{aligned} \tag{36}
$$

5. Application

Grafström and Lundquist (2002) examined whether the value of an undeveloped oilfield differs depending on whether real options valuation or discounted cash flows analysis is used. In that application the present value of the costs is found as $X = 188777796\$$ and the present value of the revenues is found as $S_0 = 241188460\$$. Table 1 reports financial parameters used in *ROV* where δ is convenience yield, and r is instantaneous risk-free interest rate.

Table 1. Parameters used in Real Options Valuation

δ	*R*	σ_δ	σ_S
5.20%	5.00%	67.18%	33.66%

The current values of the revenues and costs are fuzzified in order to obtain the standart deviation (33.66%) and the fuzzy real options value is calculated through their application to Eq.s 5–7.

$$S_0 = 241188460\$, \quad \tilde{S}_0 = (229129037; 253247883; 176197913; 176197913),$$

$$X = 188777796\$, \quad \tilde{X} = (179338906; 198216686; 100000000; 100000000),$$

$$E\left(\tilde{S}_0\right) = \frac{229129037 + 253247883}{2} + \frac{176197913 - 176197913}{6} = 241188460\$,$$

$$s_2 - s_1 = 253247883 - 229129037 = 24118846,$$

$$\alpha + \beta = 176197913 + 176197913 = 352395826,$$

$$\sigma\left(\tilde{S}_0\right) = \frac{\sqrt{\frac{(s_2 - s_1)}{4} + \frac{(s_2 - s_1)(\alpha + \beta)}{6} + \frac{(\alpha + \beta)^2}{24}}}{E\left(\tilde{S}_0\right)} = \frac{81184035}{241188460} = 33.66\%,$$

$$E\left(\tilde{X}\right) = \frac{(179338906 + 198216686)}{2} + \frac{100000000 + 100000000}{6} = 188777796\$,$$

$$d_1 = \frac{\ln\left(\frac{241188460}{188777796}\right) + \left(0.05 - 0.052 + \frac{0.3366^2}{2}\right)12}{0.3366\sqrt{12}} = 0.7725,$$

$$N\left(d_1\right) = 0.7801,$$

$$d_2 = 0.7725 - 0.3366\sqrt{12} = -0.3935,$$

$$N\left(d_2\right) = 0.3445,$$

$$FROV = (229129037; 253247883; 176197913; 176197193) \cdot e^{-0.052 \cdot 12} \cdot 0.7801 -$$
$$(179338906; 198216686; 100000000; 100000000) \cdot e^{-0.05 \cdot 12} \cdot 0.3445 =$$
$$(58294298; 71944517; 92552920; 92552920)\$.$$

In the defuzzification process, the centroid method based on the centers of gravity of the possibility distribution is used.

$$z^* = \frac{\int_{-34258622}^{58294298} z^2 dz + \int_{58294298}^{71944517} zdz + \int_{71944517}^{164497437} \left(1 - \frac{z - 71944517}{92552920}\right) zdz}{\int_{-34258622}^{58294298} zdz + \int_{58294298}^{71944517} dz + \int_{71944517}^{164497437} \left(1 - \frac{z - 71944517}{92552920}\right) dz} = 71415953.66\$.$$

The fuzzy real options value of the investment calculated through Carlsson and Fuller approach is found as 71415953.66$ after being defuzzified through the centroid method.

Below, the interest rates for the case of discrete compounding are obtained by applying the data of oil field investment example to Eq.s 11 and 12.

$$i_r = e^r - 1 = e^{0.05} - 1 = 0.0513,$$

$$i_\delta = e^\delta - 1 = e^{0.052} - 1 = 0.0534,$$

$$w = \frac{1 + i_r}{1 + i_\delta} = 0.998.$$

Fuzzy revenues and costs are:

$$\tilde{S}_0 = (59931124; 229129037; 253247883; 429445796),$$

$$\tilde{X} = (79338906; 179338906; 198216686; 298216686).$$

The expected values of the revenues and costs are given below.

$$E\left(\tilde{S}_0\right) = \frac{229129037 + 253247883}{2} + \frac{(229129037 - 52931124) - (429445796 - 253247883)}{6},$$

$$E\left(\tilde{S}_0\right) = 241188460\$,$$

$$E\left(\tilde{X}\right) = \frac{179338906 + 198216686}{2} + \frac{(179338906 - 79338906) - (298216686 - 198216686)}{6},$$

$$E\left(\tilde{X}\right) = 188777796\$.$$

When the values are placed in Eqs. 17 and 18, the fuzzy real options value is calculated as below:

$$d_1 = \ln\left(\left(\frac{241188460}{188777796} \right)^{\frac{1}{0.3366\sqrt{12}}} 0.998^{\frac{\sqrt{12}}{0.3366}} e^{\frac{0.3366\sqrt{12}}{2}} \right) = 0.7726,$$

$$d_2 = 0.7726 - 0.3366\sqrt{12} = -0.3934,$$

$$N(d_1) = 0.78012,$$

$$N(d_2) = 0.34701,$$

$$FROV = (52931124; 229129037; 253247883; 429445796) \cdot 1.0534^{-12} \cdot 0.78012 - (79338906; 179338906; 198216686; 298216686) \cdot 1.0513^{-12} \cdot 0.34701 = (-34660540; 58012568; 71686162; 164359270).$$

$$FROV = (-34660540; 58012568; 71686162; 164359270).$$

The fuzzy real options value of the investment is defuzzified through the centroid method and found to be 72971941\$.

i_δ and i_r are fuzzified at the rate of ±%5 to calculate the real options value of the investment in the condition of fuzzifying discrete interest rates.

$$i_r = (0.0487; 0.0513; 0.0539),$$

$$i_\delta = (0.0507; 0.0534; 0.0561).$$

When the data are applied to Eq.s 21 and 22, the below membership functions for i_r are obtained.

$$f_1(y|\tilde{i}_r) = i_{r_L} + y(i_{r_M} - i_{r_L}) = 0.0487 + y(0.0513 - 0.0487) = 0.0487 + 0.0026y,$$

$$f_2(y|\tilde{i}_r) = i_{r_R} + y(i_{r_M} - i_{r_R}) = 0.0539 + y(0.0513 - 0.0539) = 0.0539 - 0.0026y.$$

When approached from the left: for $y = 0$ $f_1(y|\tilde{i}_r) = 0.0487$, for $y = 1$ $f_1(y|\tilde{i}_r) = 0.0513$, and when approached from the right: for $y = 0$ $f_2(y|\tilde{i}_r) = 0.0539$ and for $y = 1$ $f_2(y|\tilde{i}_r) = 0.0513$ values are obtained.

When the data are applied to Eq.s 23 and 24, the below membership functions for i_δ are obtained.

$$f_1(y|\tilde{i}_\delta) = i_{\delta_L} + y(i_{\delta_M} - i_{\delta_L}) = 0.0507 + y(0.0534 - 0.0507) = 0.0507 + 0.0027y,$$

$$f_2(y|\tilde{i}_\delta) = i_{\delta_R} + y(i_{\delta_M} - i_{\delta_R}) = 0.0561 + y(0.0534 - 0.0561) = 0.0561 - 0.0027y.$$

When approached from the left: for $y = 0$ $f_1(y|\tilde{i}_\delta) = 0.0507$, for $y = 1$ $f_1(y|\tilde{i}_\delta) = 0.0534$, and when approached from right: for $y = 0$ $f_2(y|\tilde{i}_\delta) = 0.0561$ and for $y = 1$ $f_2(y|\tilde{i}_\delta) = 0.0534$ values are obtained,

The fuzzy number $\tilde{d}_1$ is calculated below by the help of Eq. 27.

$$d_1 = \left(\ln\left(\left(\frac{241188460}{188777796}\right)^{\frac{1}{0.3366\sqrt{12}}} \left(\frac{1+0.0487}{1+0.0561}\right)^{\frac{\sqrt{12}}{0.3366}} e^{\left(\frac{0.3366\sqrt{12}}{2}\right)} \right); \right.$$
$$\ln\left(\left(\frac{241188460}{188777796}\right)^{\frac{1}{0.3366\sqrt{12}}} \left(\frac{1+0.0513}{1+0.0534}\right)^{\frac{\sqrt{12}}{0.3366}} e^{\left(\frac{0.3366\sqrt{12}}{2}\right)} \right);$$
$$\left. \ln\left(\left(\frac{241188460}{188777796}\right)^{\frac{1}{0.3366\sqrt{12}}} \left(\frac{1+0.0539}{1+0.0507}\right)^{\frac{\sqrt{12}}{0.3366}} e^{\left(\frac{0.3366\sqrt{12}}{2}\right)} \right) \right),$$

$$\tilde{d}_1 = (0.720767; 0.772595; 0.824428).$$

Using the defuzzification method suggested for the fuzzy numbers which have normal distribution, $\tilde{d}_1 = (0.720767; 0.772595; 0.824428)$ is defuzzified and d_2 is calculated. If $\tau_{11} = 0.50; \tau_{12} = 0.30; \tau_{13} = 0.20$ is accepted;

$$d_1 = \frac{0.50 \cdot 0.720767 + 0.30 \cdot 0.772595 + 0.20 \cdot 0.824428}{0.50 + 0.30 + 0.20} = 0.757048,$$

$$d_2 = 0.757048 - 0.3366\sqrt{12} = -0.408968.$$

The areas under the standard normal distribution curve are given below for d_1 and d_2:

$$N(d_1) = 0.775489,$$

$$N(d_2) = 0.341281.$$

The fuzzy real options value is calculated with the help of Eq. 31.

$$FROV = (FROV_1; FROV_2; FROV_3) = \left(S_0 \left(1 + i_{\delta_L}\right)^{-T} N\left(\tilde{d}_1\right) - \tilde{X}\left(1 + i_{r_R}\right)^{-T} N\left(\tilde{d}_2\right); \right.$$
$$\left. S_0 \left(1 + i_{\delta_M}\right)^{-T} N\left(\tilde{d}_1\right) - \tilde{X}\left(1 + i_{r_M}\right)^{-T} N\left(\tilde{d}_2\right); S_0 \left(1 + i_{\delta_R}\right)^{-T} N\left(\tilde{d}_1\right) - \tilde{X}\left(1 + i_{r_L}\right)^{-T} N\left(\tilde{d}_2\right) \right);$$

$$FROV_1 = (52931124; 229129037; 253247883; 429445796)(1 + 0.0507)^{-12} \cdot 0.775489 -$$
$$(79338906; 179338906; 198216686; 298216686)(1 + 0.0539)^{-12} \cdot 0.341281 =$$
$$(-31531751; 62125072; 75888533; 169545362).$$

$FROV_1$ is found by defuzzying $(-31531751; 62125072; 75888533; 169545362)$ through the centroid method and found to be \$ 63082694.

$$FROV_2 = (52931124; 229129037; 253247883; 429445796)(1+0.0534)^{-12} \cdot 0.775489 - (79338906; 179338906; 198216686; 298216686)(1+0.0513)^{-12} \cdot 0.341281 = (-33850206; 58064338; 71617684; 163532227).$$

$FROV_2$ is found by defuzzying $(-33850206; 58064338; 71617684; 163532227)$ through the centroid method and found to be \$ 70256880.

$$FROV_3 = (52931124; 229129037; 253247883; 429445796)(1+0.0561)^{-12} \cdot 0.775489 - (79338906; 179338906; 198216686; 298216686)(1+0.0487)^{-12} \cdot 0.341281 = (-36199402; 54065743; 67422616; 157687761).$$

$FROV_3$ is found by defuzzying $(-36199402; 54065743; 67422616; 157687761)$ through the centroid method and found to be \$ 84940120.

$$FROV = (63082694; 70256880; 84940120)\$.$$

$FROV$ is found by defuzzying $(63082694; 70256880; 84940120)\$$ through the centroid method and found to be \$ 66734120.

The data in the example of oil field investment are applied to Eq.s 33 and 34, and d_1 and d_2 values are found by postponing the defuzzification.

$$d_1 = \left(\ln\left(\left(\frac{52931124}{298216686}\right)^{\frac{1}{0.3366\sqrt{12}}} \left(\frac{1.0513}{1.0534}\right)^{\frac{\sqrt{12}}{0.3366}} e^{\left(\frac{0.3366\sqrt{12}}{2}\right)} \right), \ln\left(\left(\frac{229129037}{198216686}\right)^{\frac{1}{\sigma\sqrt{T}}} \left(\frac{1.0513}{1.0534}\right)^{\frac{\sqrt{12}}{0.3366}} e^{\left(\frac{0.3366\sqrt{12}}{2}\right)} \right), \ln\left(\left(\frac{253247883}{179338906}\right)^{\frac{1}{\sigma\sqrt{T}}} \left(\frac{1.0513}{1.0534}\right)^{\frac{\sqrt{12}}{0.3366}} e^{\left(\frac{0.3366\sqrt{12}}{2}\right)} \right), \ln\left(\left(\frac{429445796}{79338906}\right)^{\frac{1}{\sigma\sqrt{T}}} \left(\frac{1.0513}{1.0534}\right)^{\frac{\sqrt{12}}{0.3366}} e^{\left(\frac{0.3366\sqrt{12}}{2}\right)} \right) \right);$$

$$d_1 = (-1.041530; 0.565438; 0.737106; 1.889470);$$

$$d_2 = (-1.041530 - 0.3366\sqrt{12}; 0.565438 - 0.3366\sqrt{12}; 0.737106 - 0.3366\sqrt{12}; 1.889470 - 0.3366\sqrt{12});$$

$$d_2 = (-2.207547; -0.598500; -0.428911; 0.723453).$$

In the model suggested for the defuzzification of the membership functions showing normal distribution: using the weights $\tau_{11} = 0.11;\ \tau_{12} = 0.41;\ \tau_{13} = 0.39;\ \tau_{14} = 0.09$; the fuzzy number $d_1 = (-1.041530; 0.565438; 0.737106; 1.889470)$ is defuzzyfied, and using the weights $\tau_{21} = 0.08;\ \tau_{22} = 0.35;\ \tau_{23} = 0.40;\ \tau_{24} = 0{,}17$; the fuzzy number $d_2 = (-2.207547; -0.598500; -0.428911; 0.723453)$ is defuzzified.

$$d_1 = \frac{0.11 \cdot (-1.041530) + 0.41 \cdot 0.565438 + 0.39 \cdot 0.737106 + 0.09 \cdot 1.889470}{0.11 + 0.41 + 0.39 + 0.09} = 0.5748,$$

$$d_2 = \frac{0.08 \cdot (-2.207547) + 0.35 \cdot (-0.598500) + 0.40 \cdot (-0.428911) + 0.17 \cdot 0.723453}{0.08 + 0.17 + 0.35 + 0.40} = -0.4346,$$

$$N(d_1) = 0.7173,$$

$$N(d_2) = 0.3319.$$

When the values are placed in Eq. 14, the fuzzy real options value in the case of postponing the defuzzification of costs and revenues with defuzzifying the probabilities is calculated as below:

$$FROV = \tilde{S}_0 (1.0534)^{-12}\, 0.7173 - \tilde{X} (1.0513)^{-12}\, 0.3319,$$

$$\begin{aligned} FROV = {} & (52931124; 229129037; 253247883; 429445796) \cdot 0.3842 - \\ & (79338906; 179338906; 198216686; 298216686) \cdot 0.1821 = \\ & (-33965177; 51942806; 64647237; 150555219)\$. \end{aligned}$$

The real options value of the investment is found by defuzzifying the calculated value through the centroid method and found to be \$ 77408863.

Using Eq. 36, the fuzzy real options value is calculated without defuzzifying the possibilities.

$$\begin{aligned} FROV = {} & (1.0534)^{-12} \cdot (52931124 \cdot N(-1.04153); 229129037 \cdot N(0.5654); \\ & 253247883 \cdot N(0.7371); 429445796 \cdot N(1.8895) - \\ & (1.0513)^{-12} (79338906 \cdot N(-2.207547); 179338906 \cdot \leq N(-0.598500); \\ & 198216686 \cdot N(-0.428911); 298216686 \cdot N(0.723453)); \end{aligned}$$

$$FROV = (-120991904; 51323816; 77347074; 222672378).$$

FROV is found by defuzzfying $(-120991904; 51323816; 77347074; 222672378)$ through the centroid method and found to be \$ 105867783.

Table 2. The results of the application with suggested model

	Fuzzy Result	Defuzzified Result
Discrete Compounding	(–34660540; 58012568; 71686162; 164359270)$	72971941$
Fuzzifying the discrete interest rates	(63082694; 70256880; 84940120)$	66834120$
Postponing the defuzzification of costs and revenues with defuzzifying the probabilities	(–33965177; 51942806; 64647237; 150555219)$	77408863$
Postponing the defuzzification of costs and revenues without defuzzifying the probabilities	(–120991904; 51323816; 77347074; 222672278)$	105867783$

Table 2 represents the application results of the suggested model together. The last achieved value is the most sensitive value which means that the case of postponing the defuzzification of costs and revenues without defuzzifying the probabilites shows more information about the investment. Consequently, the result of the case of postponing the defuzzification of costs and revenues without defuzzifing the probabilites is the fuzzy number which has the widest range.

6. Conclusion

Real options valuation method, which is vastly different from the traditional investment valuation methods, makes more exact assessments since it considers future uncertainties as well as dependencies and dynamism. By using the real options valuation method particularly to analyse the risky investments, wrong decisions could be easily avoided. When examined by real options, an investment, that is rejected because its current value is found to be negative by using the discounted cash flow analysis, may deliver positive values in case of postponing the investment and, therefore, a method that does not consider the postponing option could miss the investment opportunity.

Due to the fact that estimations made by people have multilateral structures that do not have sharp certainties, fuzzy logic is more expressive than classical mathematics. Therefore, on consideration of a fuzzy manner, the parameters used in the real options valuation method will lead to more realistic results which concern human reasoning. Thus, the results obtained can be more trusted.

Using real options valuation methods to analyse an investment decision reduces the uncertainty to minimum and it ensures that the investment assessment is made in as the most realistic way as possible. The model suggested by Carlsson and Fuller (2003) has been found to be the most frequently used method amongst the fuzzy real options assessment methods during literature researches. In this model, however, it is observed that the fuzzy revenue and expenditure values were defuzzified at a relatively early stage. Early defuzzification of the fuzzy parameters causes information loss. In this study, a new model has been suggested; it postpones the defuzzification of fuzzy parameters in the real options valuation in order to avoid this information loss. Carlsson and Fuller's model has been re-arranged for discrete compounding and then the defuzzification of revenue and expenditures has been postponed and relavant equations have been formed for the cases of early defuzzifying probabilities and defuzzifying them at the last stage. On the other hand, the difference between the applications of this new model and of the early defuzzifying model has been found to be the information loss due to the early defuzzification.

For further research, it is suggested that the information loss caused by uncertainty should be measured by fuzzifying the other real options valuation methods that have a broad area of usage such as game theory, binomial, and trinomial valuation methods.

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REALIŲ ĮVERČIŲ METODO TAIKYMAS INVESTICIJOMS Į NAFTOS VERSLĄ VERTINTI

İ. Uçal, C. Kahraman

Santrauka

Tradiciniai vertinimo metodai yra mažiau patikimi esant neapibrėžtumams. Vadinasi, kiti metodai, tokie kaip realių pasirinkčių vertinimo modeliai, kurie gali minimizuoti neapibrėžtumus, tampa svarbesni. Šiame straipsnyje nagrinėjamas hibridinis Carlsson ir Fuller metodas, kuris buvo panaudotas diskrečiajam rizikingų pinigų srautų modeliavimui. Pasiūlytas naujas realių pasirinkčių vertinimo modelis, kuris realistiškiau įvertins investicijas, rodiklius apibūdinančią neapibrėžtą informaciją apdorojant ankstyvojoje stadijoje. Pasiūlytas modelis buvo pritaikytas investicijoms į naftos verslą modeliuoti, nustatytas informacijos nuostolis, kuris atsiranda dėl ankstyvo neapibrėžtų duomenų apdorojimo.

Reikšminiai žodžiai: neapibrėžtos aibės, realių pasirinkčių vertinimas, neapibrėžtos pasirinktys, investavimas.

İrem UÇAL. MSc, Research Assistant. Department of Industrial Engineering. Istanbul Technical University. First degree in environmental engineering, Istanbul Technical University (2004). Master of Science in engineering management, Istanbul Technical University (2008). PhD in industrial engineering (continued), Istanbul Technical University. Research interests: engineering economics, multiple criteria decision making and energy resources.

Cengiz KAHRAMAN. Prof. Department of Industrial Engineering. Istanbul Technical University. First degree in Industrial Engineering Istanbul Technical University (1988). Master of Science (1990). Doctor (1996). Assoc. Prof. (1997). Professor (2003). Author of about 70 scientific articles. Research interests: engineering economics, multiple criteria decision making, quality management and control, statistical decision making, fuzzy sets applications to decision making.