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Quasi-Algebras versus Regular Algebras - Part I

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Abstract

Starting from quasi-Wajsberg algebras (which are generalizations of Wajsberg algebras), whose regular sets are Wajsberg algebras, we introduce a theory of quasi-algebras versus, in parallel, a theory of regular algebras. We introduce the quasi-RM, quasi-RML, quasi-BCI, (commutative, positive implicative, quasi-implicative, with product) quasi-BCK, quasi-Hilbert and quasi-Boolean algebras as generalizations of RM, RML, BCI, (commutative, positive implicative, implicative, with product) BCK, Hilbert and Boolean algebras respectively.

In Part I, the first part of the theory of quasi-algebras - versus the first part of a theory of regular algebras - is presented. We introduce the quasi-RM and the quasi-RML algebras and we present two equivalent definitions of quasi-BCI and of quasi-BCK algebras.

Keywords: quasi-MV algebra, quasi-Wajsberg algebra, MV algebra, Wajsberg algebra, BCI algebra, BCK algebra, RM algebra, RML algebra

1 Introduction

The quasi-MV algebras were introduced in 2006 [17], as generalizations of MV algebras introduced in 1958 [4], following an investigation into the foundations of quantum computing (see [7]). Since then, many papers investigated them [21], [1], [14], [13].

The quasi-Wajsberg algebras were introduced in 2010 [2], as generalizations of Wajsberg algebras introduced in 1984 [5]; they are term-equivalent

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to quasi-MV algebras, just as Wajsberg algebras are term equivalent to MV algebras. The regular set $R(A)$ of any quasi-Wajsberg algebra A is a Wajsberg algebra. Remark that any Wajsberg algebra A has in the signature an implication $\to$ and a constant 1 that verify the following two properties, among many others: for all $x \in A$,

(Re) $x \to x = 1$, (M) $1 \to x = x$,

while any quasi-Wajsberg algebra A has in the signature an implication $\to$ and a constant 1 that verify the following two properties, among many others: for all $x, y \in A$,

(Re) $x \to x = 1$, (qM) $1 \to (x \to y) = x \to y$.

Note that (M) implies (qM) and this is the most important reason why the quasi-Wajsberg algebras are generalizations of Wajsberg algebras.

We have introduced in 2013 [9] many new generalizations of BCI, of BCK and of Hilbert algebras, in a general investigation of algebras $(A, \to, 1)$ of type $(2, 0)$ that can verify properties in a given list of properties. Among the new generalizations, the most general one is the RM algebra, i.e. the algebra $(A, \to, 1)$ verifying the properties (Re), (M).

Based mainly on the results in [2] and in [9] and on the above remarks, we have developed a theory of quasi-algebras (including the lists qA, qB, qC of (basic, particular, quasi-negation, respectively) quasi-properties, with many connections) versus, in parallel, a theory of regular algebras (including the lists A, B, C of (basic, particular, negation, respectively) regular properties, with many connections). We have introduced new quasi-algebras: the quasi-RM, quasi-RML, quasi-BCI, (commutative, positive implicative, quasi-implicative, with product) quasi-BCK, quasi-Hilbert algebras and the quasi-Boolean algebras, as generalizations of the corresponding regular algebras: RM, RML, BCI, (commutative, positive implicative, implicative, with product) BCK, Hilbert and Boolean algebras. We have made the connection with the quasi-Wajsberg algebras.

In Part I, the first part of the theory of quasi-algebras is presented, including the *list qA of basic quasi-properties* and many connections - versus the first part of a theory of regular algebras, including the *list A of basic regular properties* and many connections. We introduce the quasi-order and the quasi-Hasse diagram - versus the regular order and the Hasse diagram - and we study the quasi-ordered algebras (structures). We introduce the quasi-RM algebras and the quasi-RML algebras and we present two equivalent definitions of quasi-BCI algebras and of quasi-BCK algebras.

This paper, Part I, is organized as follows: In Section 2, we present the definitions of quasi-algebras versus regular algebras. In Section 3, we present an introduction to the theory of regular algebras (part I), including the list A of basic regular properties and many connections. In Section 4, we present an introduction to the theory of quasi-algebras (part I), including the list qA of basic quasi-properties and many connections. In Section 5, we present the new quasi-algebras: the quasi-RM, quasi-RML, quasi-BCI and quasi-BCK algebras. In Section 6, we present some examples of finite quasi-algebras introduced in Section 5.

2 Quasi-Algebras (Quasi-Structures) vs Regular Algebras (Structures)

Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$. Let us introduce the following properties:
(M) $1 \to x = x$, for all $x \in A$,
(qM) $1 \to (x \to y) = x \to y$, for all $x, y \in A$,
(qM($1 \to x$)) $1 \to (1 \to x) = 1 \to x$, for all $x \in A$,
(11-1) $1 \to 1 = 1$,
and let us note that (M) $\Longrightarrow$ (11-1).
Then we have: $(M) \Longrightarrow (qM) \Longrightarrow (qM(1 \to x))$.
Indeed, for any $x, y \in A$, $1 \to (x \to y) \overset{(M)}{=} x \to y$, i.e. (qM) holds; then $1 \to (1 \to x) \overset{(qM)}{=} 1 \to x$, i.e. (qM($1 \to x$)) holds.

Besides the set A, let us define the following subsets of A:

$$U \overset{df.}{=} \{x \to y \mid x, y \in A\},\ V \overset{df.}{=} \{1 \to x \mid x \in A\},\ V_M \overset{df.}{=} \{x \in A \mid x \overset{(M)}{=} 1 \to x\}.$$

Then we have: $V_M \subseteq V \subseteq U \subseteq A$ (see Example 2.1).

• If property (M) holds, then $A \subseteq V_M$. Consequently, $V_M = V = U = A$. (see Example 2.2).

• If property (qM) holds, then $U \subseteq V_M$. Consequently, $V_M = V = U \subseteq A$ (see Example 2.3).

• If property (qM($1 \to x$)) holds, then $V \subseteq V_M$. Consequently, $V_M = V \subseteq U \subseteq A$ (see Example 2.4).

Example 2.1. Consider the set $A = \{a, b, c, d, e, f, 1\}$ and the algebra $\mathcal{A}_1 = (A, \to_1, 1)$ given by the following table:

$\mathcal{A}_1$

$\rightarrow_1$	**a**	**b**	**c**	d	e	f	**1**
a	**1**	**a**	**a**	a	1	1	**1**
b	**1**	**1**	**a**	1	1	1	**1**
c	**1**	**a**	**1**	a	1	1	**1**
d	1	1	a	1	1	1	1
e	1	a	a	a	1	1	1
f	a	b	c	b	**d**	1	1
1	**b**	**a**	**c**	b	a	1	**1**

Then $V_M = \{c, 1\} \subset V = \{a, b, c, 1\} \subset U = \{a, b, c, d, 1\} \subset A = \{a, b, c, d, e, f, 1\}$.

Example 2.2. Consider the same set $A = \{a, b, c, d, e, f, 1\}$ and the algebra $\mathcal{A}_2 = (A, \rightarrow_2, 1)$ given by the following table:

$\mathcal{A}_2$

$\rightarrow_2$	**a**	**b**	**c**	**d**	**e**	**f**	**1**
a	**1**	**1**	**1**	**1**	**1**	**1**	**1**
b	**a**	**1**	**c**	**1**	**1**	**1**	**1**
c	**a**	**b**	**1**	**1**	**1**	**1**	**1**
d	**a**	**b**	**c**	**1**	**e**	**1**	**1**
e	**a**	**b**	**c**	**d**	**1**	**1**	**1**
f	**a**	**b**	**c**	**d**	**e**	**1**	**1**
1	**a**	**b**	**c**	**d**	**e**	**f**	**1**

Then $V_M = V = U = A = \{a, b, c, d, e, f, 1\}$.

Example 2.3. Consider the same set $A = \{a, b, c, d, e, f, 1\}$ and the algebra $\mathcal{A}_3 = (A, \rightarrow_3, 1)$ given by the following table:

$\mathcal{A}_3$

$\rightarrow_3$	**a**	**b**	**c**	d	e	f	**1**
a	**1**	**a**	**a**	a	1	1	**1**
b	**1**	**1**	**a**	1	1	1	**1**
c	**1**	**a**	**1**	a	1	1	**1**
d	1	1	a	1	1	1	1
e	1	a	a	a	1	1	1
f	a	b	c	b	a	1	1
1	**a**	**b**	**c**	b	a	1	**1**

Then $V_M = V = U = \{a, b, c, 1\} \subset A = \{a, b, c, d, e, f, 1\}$.

Example 2.4. Consider the same set $A = \{a, b, c, d, e, f, 1\}$ and the algebra $\mathcal{A}_4 = (A, \rightarrow_4, 1)$ given by the following table:

$\mathcal{A}_4$

$\rightarrow_4$	**a**	**b**	**c**	d	e	f	**1**
a	**1**	**a**	**a**	a	1	1	**1**
b	**1**	**1**	**a**	1	1	1	**1**
c	**1**	**a**	**1**	a	1	1	**1**
d	1	1	a	1	1	1	1
e	1	a	a	a	1	1	1
f	a	b	c	b	**d**	1	1
1	**a**	**b**	**c**	b	a	1	**1**

Then $V_M = V = \{a, b, c, 1\} \subset U = \{a, b, c, d, 1\} \subset A = \{a, b, c, d, e, f, 1\}$.

Remarks 2.5.

(i) In all the above examples, the property (11-1) is fulfilled.

(ii) All the algebras of classical logics and of non-classical logics verify property (M). They will be called "regular algebras" in the sequel.

(iii) The quasi-Wajsberg algebras introduced in [2] verify properties (qM) and (11-1) and the quasi-MV algebras introduced in [17] are term equivalent to quasi-Wajsberg algebras. Therefore, we shall develop in the sequel a theory of quasi-algebras as those algebras having a subreduct $(A, \rightarrow, 1)$ verifying (qM) and (11-1) or term equivalent to such algebras. But note that someone can develop a theory of say "generalized-quasi-algebras", as those algebras having a subreduct $(A, \rightarrow, 1)$ verifying properties (qM$(1 \rightarrow x)$) and (11-1).

2.1 Regular Algebras (Structures) - Definition

Let $\mathcal{A} = (A, \rightarrow, 1)$ be an *algebra* of type $(2, 0)$ through this subsection, where a binary relation $\leq$ can be defined by: for all x, y,

$$x \leq y \overset{def.}{\Longleftrightarrow} x \rightarrow y = 1. \tag{1}$$

Equivalently,
let $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a *structure* where $\leq$ is a binary relation on A, $\rightarrow$ is a binary operation (an implication) on A and $1 \in A$, all connected by:

$$x \leq y \Longleftrightarrow x \rightarrow y = 1. \tag{2}$$

Let us consider the following properties:

(M) $\quad 1 \rightarrow x = x$, for all $x \in A$,

(11-1) $\quad 1 \rightarrow 1 = 1$,

where (M) $\Longrightarrow$ (11-1).

Definitions 2.6.

(1) The algebra $(A, \to, 1)$ (or, equivalently, the structure $(A, \leq, \to, 1)$) is called *regular*, if it satisfies the property (M).

(1') Any algebra (structure) $\mathcal{A}' = (A, \sigma)$ whose signature σ contains $\to$, 1 ($\leq$, $\to$, 1, respectively) is also called *regular*, if it satisfies the property (M).

(1") Any algebra (structure) $\mathcal{A}'' = (A, \tau)$ which is term equivalent to a regular algebra (structure) $\mathcal{A}' = (A, \sigma)$ is also called *regular*.

(2) The implication $\to$ from a regular algebra (structure) is called *regular implication*.

(3) The binary relation $\leq$ of a regular algebra (structure) is called *binary regular relation*.

Remark 2.7. By (M), we have that: $V_M = V = U = A$ and this is the basic, definable property of regular algebras (structures) - see Example 2.2.

Note that Boolean algebras, MV algebras, Wajsberg algebras, BL algebras, MTL algebras, residuated lattices, etc., Hilbert algebras, BCK algebras, BCI algebras, BCH, BCC, BZ, BE and pre-BCK algebras and all their generalizations introduced in [9], up to RM and RML algebras (the most general), are all regular algebras (because there exists a binary relation $\leq$ determined by an implication $\to$ and a constant 1 verifying (M)).

Note that, for example, lattices, in general, are not regular algebras (because the lattice order $\leq$, in general, is not determined by an implication and a constant 1); but there exist, in particular, the "regular lattices" (whose lattice order is determined by an implication and a constant 1), as we shall see in other paper.

2.2 Quasi-Algebras (Quasi-Structures) - Definition

Let $\mathcal{A} = (A, \rightsquigarrow, 1)$ be an *algebra* of type $(2, 0)$ through this subsection, where a binary relation $\preceq$ can be defined by: for all x, y,

$$x \preceq y \overset{def.}{\Longleftrightarrow} x \rightsquigarrow y = 1. \tag{3}$$

Equivalently,
let $\mathcal{A} = (A, \preceq, \rightsquigarrow, 1)$ be a *structure* where $\preceq$ is a binary relation on A, $\rightsquigarrow$ is a binary operation (an implication) on A and $1 \in A$, all connected by:

$$x \preceq y \Longleftrightarrow x \rightsquigarrow y = 1. \tag{4}$$

Let us consider the following properties:
(qM) $1 \rightsquigarrow (x \rightsquigarrow y) = x \rightsquigarrow y$, for all $x, y \in A$,
(11-1) $1 \rightsquigarrow 1 = 1$.

Definitions 2.8.

(q1) The algebra $(A, \rightsquigarrow, 1)$ (or, equivalently, the structure $(A, \preceq, \rightsquigarrow, 1)$) is called *quasi-algebra* (*quasi-structure*, respectively) if it satisfies the properties (qM) and (11-1).

(q1') Any algebra (structure) $\mathcal{A}' = (A, \sigma)$ whose signature σ contains $\rightarrow$, 1 ($\leq$, $\rightarrow$, 1, respectively) is also called *quasi-algebra* (*quasi-structure*), if it satisfies the properties (qM) and (11-1).

(q1") Any algebra (structure) $\mathcal{A}'' = (A, \tau)$ which is term equivalent to a quasi-algebra (structure) $\mathcal{A}' = (A, \sigma)$, is also called *quasi-algebra* (*quasi-structure*).

(q2) The implication $\rightsquigarrow$ from a quasi-algebra (quasi-structure) is called *quasi-implication.*

(q3) The binary relation $\preceq$ of a quasi-algebra (quasi-structure) is called *binary quasi-relation.*

Note that the quasi-Wajsberg algebras [2] are quasi-algebras, by the above Definition 2.8 (q1'), while the quasi-MV algebras [17] are quasi-algebras, by the above Definition 2.8 (q1").

We shall introduce in this paper new quasi-algebras (quasi-structures).

Besides the set A, let us define now the following subsets of A:

$$U \overset{df.}{=} \{x \rightsquigarrow y \mid x, y \in A\},\ V \overset{df.}{=} \{1 \rightsquigarrow x \mid x \in A\},\ V_M \overset{df.}{=} \{x \in A \mid x \overset{(M)}{=} 1 \rightsquigarrow x\}.$$

Remarks 2.9.

(i) Note that:
- if (qM) holds, then $V_M = V = U \neq A$;
- if (M) holds, then $V_M = V = U = A$.
- (qM) implies (M) (i.e. (qM) coincides with (M)) if and only if $U = A$.
- (qM) is different of (M) if and only if $U \neq A$ (i.e. $\rightsquigarrow (A \times A) = U \neq A$), and this is the basic, definable property of quasi-algebras (quasi-structures) (see Example 2.3).

(ii) Since (M) implies (qM), it follows that any quasi-algebra (quasi-structure) will be a generalization of the corresponding regular algebra (structure). For example, the quasi-MV algebra introduced in [17] is a generalization of the MV algebra and the quasi-Wajsberg algebra introduced in [2] is a generalization of the Wajsberg algebra - the MV algebras and the Wajsberg algebras being called in this context *regular algebras*.

(iii) The above remarks show the central role played by the property (M): we obtain the general rule that, roughly speaking,

$$(M) + quasi - algebra\ (quasi - structure) = regular\ algebra\ (structure).$$

In the view of the above Remarks 2.9, we introduce the following definitions:

Definitions 2.10.

(1) For every quasi-algebra (quasi-structure) $\mathcal{A}$, the subset $V_M = V = U$ of A will be called the *regular* set of $\mathcal{A}$ and will be denoted by $R(A)$:

$$R(A) \overset{def.}{=} V_M = V = U \subseteq A.$$

The elements of $R(A)$ are called the *regular elements* of A.

(2) The quasi-algebra (quasi-structure) $\mathcal{A}$ is called *proper* if $R(A) \neq A$ (i.e. (M) $\not\Longleftarrow$ (qM)); otherwise, $\mathcal{A}$ is a regular algebra (structure).

Definition 2.11. For every proper quasi-algebra $\mathcal{A} = (A, \rightsquigarrow, 1)$ (or, equivalently, proper quasi-structure $\mathcal{A} = (A, \preceq, \rightsquigarrow, 1)$), any subset $S \subseteq A$ closed under $\rightsquigarrow$ and containing 1 is called a *quasi-subalgebra* (*quasi-substructure*) of $\mathcal{A}$.

We then have the following important result:

Theorem 2.12. *Let $\mathcal{A} = (A, \rightsquigarrow, 1)$ be a proper quasi-algebra (or, equivalently, let $\mathcal{A} = (A, \preceq, \rightsquigarrow, 1)$ be a proper quasi-structure). Then, $\mathcal{R}(\mathcal{A}) = (R(A), \rightarrow, 1)$ is a regular algebra (or, equivalently, $\mathcal{R}(\mathcal{A}) = (R(A), \leq, \rightarrow, 1)$ is a regular structure, respectively), where*

$$\rightarrow = \rightsquigarrow|_{R(A)}, \qquad \leq = \preceq|_{R(A)}\ .$$

Proof: First, we prove that the regular set $R(A)$ is closed under $\rightsquigarrow$ and that $1 \in R(A)$. Indeed, if $x, y \in R(A) \subset A$, then $x \rightsquigarrow y \overset{(qM)}{=} 1 \rightsquigarrow (x \rightsquigarrow y)$, hence $x \rightsquigarrow y \in R(A)$; since $1 \rightsquigarrow 1 = 1$ (by (11-1)), it follows that $1 \in R(A)$. Consequently, $R(A)$ is a quasi-subalgebra (quasi-substructure) of $\mathcal{A}$. Hence, $\mathcal{R}(\mathcal{A}) = (R(A), \rightsquigarrow, 1)$ is a quasi-algebra ($\mathcal{R}(\mathcal{A}) = (R(A), \preceq, \rightsquigarrow, 1)$ is a quasi-structure).

Moreover, (qM) coincides with (M) on $R(A)$, i.e. (qM)$|_{R(A)} \Longleftrightarrow$ (M), by the definition of $R(A)$. Consequently, $\mathcal{R}(\mathcal{A}) = (R(A), \rightarrow, 1)$ is a regular algebra (or, equivalently, $\mathcal{R}(\mathcal{A}) = (R(A), \leq, \rightarrow, 1)$ is a regular structure, respectively). $\Box$

Conventions: In order to simplify the writing,
- the *quasi-implication* $\rightsquigarrow$ of a quasi-algebra (quasi-structure) $\mathcal{A}$ and the corresponding (its restriction to $R(A)$) *regular implication* $\rightarrow$ of the regular algebra (structure) $\mathcal{R}(\mathcal{A})$ will be denoted the same in the sequel, namely by $\rightarrow$.
- the *binary quasi-relation* $\preceq$ of a quasi-algebra (quasi-structure) $\mathcal{A}$ and the corresponding (its restriction to $R(A)$) *binary regular relation* $\leq$ of $\mathcal{R}(\mathcal{A})$ will be also denoted the same in the sequel, namely by $\leq$.

3 Introduction to the Theory of Regular Algebras (Structures) - Part I

Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra (or, equivalently, let $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a structure) as before throught this section.

3.1 The List A of Basic Properties. Connections

Consider the following list A of basic properties (those from [9] plus two new properties: (#), (##)) that can be satisfied by $\mathcal{A}$ (in fact, the properties in the list A are the most important properties satisfied by a BCK algebra (see [9])), where each property is presented in two equivalent forms, determined by the corresponding two equivalent definitions of $\mathcal{A}$. We divide the list into two parts: the properties in Part 1 are those that will be generalized, when considering the quasi-algebras (quasi-structures).

List A, Part 1

(An) (Antisymmetry) $x \rightarrow y = 1 = y \rightarrow x \implies x = y$,
(An') (Antisymmetry) $x \leq y,\ y \leq x \implies x = y$;

(M) $1 \rightarrow x = x$;

(N) $1 \rightarrow x = 1 \implies x = 1$,
(N') $1 \leq x \implies x = 1$;

(Re) (Reflexivity) $x \to x = 1$ (we prefer here notation (Re) instead of notation (I) in the theory of BCI algebras),
(Re') (Reflexivity) $x \leq x$;

(L) (Last element) $x \to 1 = 1$,
(L') (Last element) $x \leq 1$.

List A, Part 2

(11-1) $1 \to 1 = 1$,
(11-1') $1 \leq 1$;

(B) $(y \to z) \to [(x \to y) \to (x \to z)] = 1$,
(B') $y \to z \leq (x \to y) \to (x \to z)$,

(BB) $(y \to z) \to [(z \to x) \to (y \to x)] = 1$,
(BB') $y \to z \leq (z \to x) \to (y \to x)$;

(*) $y \to z = 1 \implies (x \to y) \to (x \to z) = 1$,
(*') $y \leq z \implies x \to y \leq x \to z$;

(**) $y \to z = 1 \implies (z \to x) \to (y \to x) = 1$,
(**') $y \leq z \implies z \to x \leq y \to x$;

(C) $[x \to (y \to z)] \to [y \to (x \to z)] = 1$,
(C') $x \to (y \to z) \leq y \to (x \to z)$;

(D) $y \to [(y \to x) \to x] = 1$,
(D') $y \leq (y \to x) \to x$;

(Ex) (Exchange) $x \to (y \to z) = y \to (x \to z)$;

(K) $x \to (y \to x) = 1$,
(K') $x \leq y \to x$;

(S) $x = y \implies x \to y = 1$,
(S') $x = y \implies x \leq y$;

(Tr) (Transitivity) $x \to y = 1 = y \to z \implies x \to z = 1$,
(Tr') (Transitivity) $x \le y,\ y \le z \implies x \le z$;

(#) $x \to (y \to z) = 1 \Longrightarrow y \to (x \to z) = 1$,
(#') $x \le y \to z \Longrightarrow y \le x \to z$;

(##) $x \to (y \to z) = 1 \Longleftrightarrow y \to (x \to z) = 1$,
(##') $x \le y \to z \Longleftrightarrow y \le x \to z$.

Remarks 3.1.

(i) (M) $\Longrightarrow$ (11-1); (Re) $\Longrightarrow$ (11-1); (L) $\Longrightarrow$ (11-1).

(ii) The central role of property (M) in the study of regular algebras (structures) [9] is given by the fact that it determines that $V_M = V = U = A$, i.e. all the elements of A appear compulsory inside the table of $\to$.

(iii) Closely related (see the next section) with the property (M) are the properties (N), (An), (Re) and (L), therefore the all five form the Part 1 of the list A of properties.

(iv) Note that we have defined in [9] the most general algebra, called RM, as an algebra $(A, \to, 1)$ verifying (Re) and (M), hence as a generalization of BCI, BCK, BCC, BZ, BCH, BE and pre-BCK algebras. We have also defined in [9] the RML algebra, as a RM algebra verifying the property (L); thus, the RML algebra is a generalization of BCK, BCC, BE and pre-BCK algebras.

3.1.1 Connections between the Properties in the List A

We recall the connections found in [9] between the properties in list A.

Proposition 3.2. *[9] Let $(A, \to, 1)$ be an algebra of type $(2, 0)$. Then the following are true:*

(A0) (Re) $\Longrightarrow$ (S);
(A00) (M) $\Longrightarrow$ (N);
(A1) (L) + (An) $\Longrightarrow$ (N);
(A2) (K) + (An) $\Longrightarrow$ (N);
(A3) (C) + (An) $\Longrightarrow$ (Ex); *(A3') (Ex) + (Re) $\Longrightarrow$ (C);*
(A4) (Re) + (Ex) $\Longrightarrow$ (D); *(A4') (D) + (Re) + (An) $\Longrightarrow$ (N);*
(A5) (Re) + (Ex) + (An) $\Longrightarrow$ (M);
(A6) (Re) + (K) $\Longrightarrow$ (L);
(A7) (N) + (K) $\Longrightarrow$ (L); *(A7') (M) + (K) $\Longrightarrow$ (L);*

(A8) (Re) + (L) + (Ex) $\Longrightarrow$ (K);
(A9) (M) + (L) + (B) $\Longrightarrow$ (K); (A9') (M) + (L) + (**) $\Longrightarrow$ (K);
(A10) (Ex) $\Longrightarrow$ (B) $\Leftrightarrow$ (BB);
(A10') (Ex) + (B) $\Longrightarrow$ (BB); (A10") (Ex) + (BB) $\Longrightarrow$ (B);
(A11) (Re) + (Ex) + (*) $\Longrightarrow$ (BB);
(A12) (N) + (B) $\Longrightarrow$ (*); (A12') (M) + (B) $\Longrightarrow$ (*);
(A13) (N) + (*) $\Longrightarrow$ (Tr); (A13') (M) + (*) $\Longrightarrow$ (Tr);
(A14) (N) + (B) $\Longrightarrow$ (Tr); (A14') (M) + (B) $\Longrightarrow$ (Tr);
(A15) (N) + (BB) $\Longrightarrow$ (**); (A15') (M) + (BB) $\Longrightarrow$ (**);
(A16) (N) + (**) $\Longrightarrow$ (Tr); (A16') (M) + (**) $\Longrightarrow$ (Tr);
(A17) (N) + (BB) $\Longrightarrow$ (Tr); (A17') (M) + (BB) $\Longrightarrow$ (Tr);
(A18) (M) + (BB) $\Longrightarrow$ (Re); (A18') (M) + (BB) $\Longrightarrow$ (D);
(A19) (M) + (B) $\Longrightarrow$ (Re);
(A20) (BB) + (D) + (N) $\Longrightarrow$ (C); (A20') (M) + (BB) $\Longrightarrow$ (C);
(A21) (BB) + (D) + (N) + (An) $\Longrightarrow$ (Ex);
(A21') (BB) + (D) + (L) + (An) $\Longrightarrow$ (Ex);
(A21") (M) + (BB) + (An) $\Longrightarrow$ (Ex);
(A22) (K) + (Ex) + (M) $\Longrightarrow$ (Re);
(A23) (C) + (K) + (An) $\Longrightarrow$ (Re);
(A24) (Re) + (Ex) + (Tr) $\Longrightarrow$ (**).

Let us add the following new connections.

Proposition 3.3. *Let $(A, \to, 1)$ be an algebra of type $(2, 0)$. Then the following are true:*
(A9") (M) + (L) + (BB) $\Longrightarrow$ (K);
(A18") (M) + (D) $\Longrightarrow$ (Re);

(A25) (D) + (K) + (N) + (An) $\Longrightarrow$ (M);
(A26) (#) $\Longleftrightarrow$ (##) ;
(A27) (M) + (C) $\Longrightarrow$ (#);
(A28) (Ex) $\Longrightarrow$ (##);
(A29) (BB) + (#) $\Longrightarrow$ (B); (A29') (B) + (#) $\Longrightarrow$ (BB);
(A30) (Re) + (B) + (Tr) + (#) $\Longrightarrow$ (C);
(A31) (Re) + (#) $\Longrightarrow$ (D) (see (A4));
(A32) (Re) + (#) + (An) $\Longrightarrow$ (M) (see (A5)).

Proof:

(A9"): $x \to (y \to x) \overset{(M)}{=} (1 \to x) \to (y \to x) \overset{(M)}{=} 1 \to [(1 \to x) \to (y \to x)] \overset{(L)}{=} (y \to 1) \to [(1 \to x) \to (y \to x)] \overset{(BB)}{=} 1$, i.e. (K) holds.

(A18"): $1 \to ((1 \to x) \to x) \overset{(D)}{=} 1$, hence, by (M), $x \to x = 1$.

(A25): $1 \to [(1 \to x) \to x] \overset{(D)}{=} 1$, hence, by (N), $(1 \to x) \to x = 1$. On the other hand, $x \to (1 \to x) \overset{(K)}{=} 1$. Then, by (An), $1 \to x = x$.

(A26): Obviously.

(A27): Suppose that $x \to (y \to z) = 1$; then, (C) $([x \to (y \to z)] \to [y \to (x \to z)] = 1)$ gives $1 \to [y \to (x \to z)] = 1$; hence, by (M), $y \to (x \to z) = 1$; thus (#) holds.

(A28): $1 = x \to (y \to z) \overset{(Ex)}{=} y \to (x \to z) = 1$.

(A29): (see the proof of (qW32) from [2])
By (BB), $(x \to y) \to [(y \to z) \to (x \to z)] = 1$; then, by (#), $(y \to z) \to [(x \to y) \to (x \to z)] = 1$, i.e. (B) holds.

(A29'): Similarly, by (B), $(y \to z) \to [(x \to y) \to (x \to z)] = 1$; then, by (#), $(x \to y) \to [(y \to z) \to (x \to z)] = 1$, i.e. (BB) holds.

(A30): (see the proof of (qW33) from [2])
Since $(y \to z) \to (y \to z) \overset{(Re)}{=} 1$, it follows by (#) that $y \to [(y \to z) \to z] = 1$. On the other hand, $[(y \to z) \to z] \to [(x \to (y \to z)) \to (x \to z)] \overset{(B)}{=} 1$. Then, by (Tr), we obtain that $y \to [(x \to (y \to z)) \to (x \to z)] = 1$; hence, by (#), $[x \to (y \to z)] \to [y \to (x \to z)] = 1$, i.e. (C) holds.

(A31): $x \to y \overset{(Re)}{\leq} x \to y$ implies, by (#'), $x \leq (x \to y) \to y$.

(A32): First we prove: (a) $x \to (1 \to x) = 1$. Indeed, $1 \to (x \to x) \overset{(Re)}{=} 1 \to 1 \overset{(Re)}{=} 1$, hence by (#), we obtain (a).
Then, we prove: (b) $(1 \to x) \to x = 1$. Indeed, by (A31), (Re) + (#) $\Longrightarrow$ (D), and by (A4'), (D) + (Re) + (An) $\Longrightarrow$ (N); then $1 \to ((1 \to x) \to x) \overset{(D)}{=} 1$, hence by (N), we obtain (b).
Now, (a) + (b) + (An) imply (M); thus, (M) holds. □

Now recall the following four theorems from [9].

Theorem 3.4. *[9] (Generalization of ([3], Lemma 1.2 and Proposition 1.3))*
If properties (Re), (M), (Ex) hold, then: $(BB) \Leftrightarrow (B) \Leftrightarrow (*)$.

Theorem 3.5. *[9]*
If properties (Re), (M), (Ex) hold, then: $(**) \Leftrightarrow (Tr)$.

Theorem 3.6. *[9]*
If properties (M), (B), (An) hold, then: $(Ex) \Leftrightarrow (BB)$.

Theorem 3.7. *[9] (Michael Kinyon) In any algebra* $(A, \to, 1)$ *we have:*
(i) (M) + (BB) $\Longrightarrow$ *(B),*
(ii) (M) + (B) $\Longrightarrow$ *(**).*

By Kinyon's Theorem 3.7(i) and (A12'), we obtained immediately that:

Corollary 3.8. [9] (M) + (BB) $\Longrightarrow$ (*).

Concluding, by the above Kinyon's Theorem 3.7 and (A12'), (A13'), (A16'), we have obtained:

Corollary 3.9. [9] In any algebra $(A, \to, 1)$ verifying (M), we have:

$$(BB) \implies (B) \implies (*), (**) \implies (Tr).$$

3.2 Regular Order. Ordered Regular Algebras (Structures)

Let $\mathcal{A}$ be a regular algebra (structure) through this subsection, i.e. property (M) holds.

Definitions 3.10. Consider the following properties of $\to$ $(\leq)$: (Re), (An), (Tr) ((Re'), (An'), (Tr'), respectively). Then, we shall say that $\mathcal{A}$ is [9]:
- *reflexive*, if property (Re) (or (Re')) is satisfied;
- *antisymmetric*, if property (An) (or (An')) is satisfied;
- *transitive*, if property (Tr) (or (Tr')) is satisfied;
- *pre-ordered*, if it is reflexive and transitive;
- *ordered*, if it is reflexive, antisymmetric and transitive.

Among all the regular algebras (see[9]), for examples:
- the BE, RME, RM, RML algebras are only reflexive,
- the BCH, aBE, aRM, aRML algebras are reflexive and antisymmetric,
- the pre-BCK, pre-BCI, pre-BZ, pre-BCC algebras are pre-ordered,
- the BCI, BCK, BZ, BCC algebras - and also the Boolean algebras, MV algebras, Wajsberg algebras, BL algebras, MTL algebras, residuated lattices, Hilbert algebras - are ordered; some of them can be lattices, called *regular lattices* in a subsequent paper.

Note that the notion of partially ordered set (poset) $(A, \leq)$ is more general than the notion of (partially) ordered regular algebra (structure). Hence, the duality principle for posets is valid for regular ordered algebras (structures). Also the *Hasse diagram* for posets is valid for regular ordered algebras (structures), where an element of a regular ordered algebra (structure) will be represented by a bullet $\bullet$.

Remark 3.11. If the regular algebra (structure) is not ordered (i.e. it is only reflexive, or reflexive and transitive, or reflexive and antisymmetric), then a *Hasse-type diagram* is used, where an element is represented by a circ $\circ$, and if $a \leq b$ and $b \leq a$ and $a \neq b$ (i.e. a and b have the *same height*, or are *parallel*)), then a horizontal line will connect them. See for example the Hasse-type diagrams for some examples of regular RM and regular RML algebras from Figures 4 and 5, respectively.

The theory of regular algebras (structures) will be continued in the next papers.

3.3 Some Regular Algebras: the RM, RML, BCI and BCK Algebras

Recall now the following definitions:

Definition 3.12. An algebra $(A, \to, 1)$ is a:
- *RM algebra*, if it verifies the axioms (Re), (M) [9];
- *RML algebra*, if it verifies the axioms (Re), (M), (L) [9];
- *BCI algebra*, if it verifies the axioms (BB), (D), (Re), (An) [11], or, equivalently [9], (B), (C), (Re), (An);
- *BCK algebra*, if it verifies the axioms (BB), (D), (Re), (L), (An) [11], [8], [12], or, equivalently [9], (B), (C), (K), (An).

Note that, obviously, there are equivalent definitions as structures $(A, \leq, \to, 1)$.

Note also that these algebras are regular, since property (M) holds.

Recall that a BCI algebra $(A, \to, 1)$ is *p-semisimple* if for each $x \in A$, $x \leq 1$ implies $x = 1$ (see Remark 6.5). The p-semisimple BCI algebras are categorically equivalent with the commutative groups (see [19]).

Note that BCK algebras verify indeed all the properties in List A: (Ex), by (A3); (M), by (A5); (N), by (A00); (*), by (A12); (**), by (A15); (Tr) by (A14); (S), by (A0); (#), by (A27); (##), by (A28). See more about BCK algebras in the books [20], [10].

Remarks 3.13.

(i) Most of the algebras of logic [22], [18] (Boolean algebra, MV algebra, BL algebra, MTL algebra, divisible residuated lattice, residuated lattice, Hilbert algebra etc.) can be seen as particular cases of BCK algebras [10].

(ii) In [9], starting from BCI, BCK (and Hilbert) algebras and from some generalizations of BCI and of BCK algebras (BCC algebras, BZ algebras, BCH algebras, BE algebras, pre-BCK algebras), and based on the above connections between the properties in the list A, new generalizations of BCI, of BCK (and of Hilbert) algebras were introduced (the RM and RML algebras and many others) and the connections between all of them was shown.

Denote by **RM**, **RML**, **BCI**, **BCK** the classes of RM algebras, of RML algebras, of BCI algebras and of BCK algebras, respectively. We have then [9] the Hierarchy 1 from Figure 1.

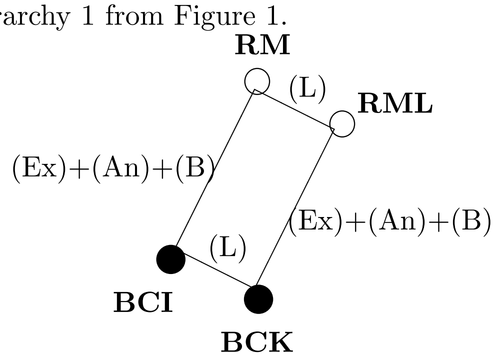


Figure 1: Hierarchy 1

4 Introduction to the Theory of Quasi-Algebras (Quasi-Structures) - Part I

Let $\mathcal{A} = (A, \to, 1)$ be an algebra (or, equivalently, let $\mathcal{A} = (A, \leq, \to, 1)$ be a structure) as before, throught this section too.

Definition 4.1. We call *proper quasi-properties* the following nine: (qAn), (qM), (qM($1 \to x$)), (qN), (qN($1 \to x$)), (qRe), (qRe($1 \to x$)), (qL), (qL($1 \to x$)) which form the Part 1 of List qA (corresponding to the five properties (An), (M), (N), (Re), (L) respectively, which form the Part 1 of List A).

4.1 The list qA of Basic Quasi-Properties. Connections

The list qA of “quasi-properties” that can be satisfied by $\mathcal{A}$ has also two parts, and follows closely the list A of properties. The proper quasi-properties in Part 1 of List qA are generalizations of the properties in Part 1 of List A, while the “quasi-properties” in Part 2 of List qA are both the properties in

Part 2 of List A and eight new specific properties ((qR) - (qI3)). We shall understand now which was the criterion by which a property was written in Part 1 or in Part 2 of the list.

List qA, Part 1

(qAn) (quasi-Antisymmetry) $x \to y = 1 = y \to x \implies 1 \to x = 1 \to y$,
(qAn') (quasi-Antisymmetry) $x \leq y,\ y \leq x \implies 1 \to x = 1 \to y$;

(qM) $1 \to (x \to y) = x \to y$;
(qM($1 \to x$)) $1 \to (1 \to x) = 1 \to x$;

(qN) $1 \to (x \to y) = 1 \implies x \to y = 1$,
(qN') $1 \leq x \to y \implies x \to y = 1$;
(qN($1 \to x$)) $1 \to (1 \to x) = 1 \implies 1 \to x = 1$,
(qN($1 \to x$)') $1 \leq 1 \to x \implies 1 \to x = 1$;

(qRe) (quasi-Reflexivity) $(x \to y) \to (x \to y) = 1$,
(qRe') (quasi-Reflexivity) $x \to y \leq x \to y$;
(qRe($1 \to x$)) $(1 \to x) \to (1 \to x) = 1$,
(qRe($1 \to x$)') $1 \to x \leq 1 \to x$;

(qL) $(x \to y) \to 1 = 1$,
(qL') $x \to y \leq 1$;
(qL($1 \to x$)) $(1 \to x) \to 1 = 1$,
(qL($1 \to x$)') $1 \to x \leq 1$.

List qA, Part 2

(11-1), (B), (BB), (*), (**), (C), (D), (Ex), (K), (S), (Tr); (#), (##);

(qR) $(x \to y) \to ((1 \to x) \to (1 \to y)) = 1$,
(qR1) $(1 \to x) \to x = 1$,
(qR2) $x \to (1 \to x) = 1$,
(qR3) $(x \to (1 \to y)) \to (1 \to (x \to y)) = 1$;

(qI) $x \to y = (1 \to x) \to (1 \to y)$,
(qI1) $x \to y = (1 \to x) \to y$,

(qI2) $x \to y = x \to (1 \to y)$,
(qI3) $(1 \to x) \to (1 \to y) = (1 \to x) \to y$;

(qrelI) $x \leq y \Leftrightarrow 1 \to x \leq 1 \to y$,
(qrelI1) $x \leq y \Leftrightarrow 1 \to x \leq y$,
(qrelI2) $x \leq y \Leftrightarrow x \leq 1 \to y$.

4.1.1 Connections between the Properties in List A, Part 1 and the Proper Quasi-Properties in List qA, Part 1

Theorem 4.2. *Let $(A, \to, 1)$ be an algebra of type $(2, 0)$. Then the following are true:*

(i) (An) $\Longrightarrow$ (qAn);
(ii) (M) $\Longrightarrow$ (qM) $\Longrightarrow$ (qM$(1 \to x)$);
(iii) (N) $\Longrightarrow$ (qN) $\Longrightarrow$ (qN$(1 \to x)$);
(iv) (Re) $\Longrightarrow$ (qRe) $\Longrightarrow$ (qRe$(1 \to x)$);
(v) (L) $\Longrightarrow$ (qL) $\Longrightarrow$ (qL$(1 \to x)$);
(vi) (M) + (qAn) $\Longrightarrow$ (An);
(vii) (M) + (qRe$(1 \to x)$) $\Longrightarrow$ (Re);
(viii) (M) + (qL$(1 \to x)$) $\Longrightarrow$ (L);
(ix) (qM) + (qRe$(1 \to x)$) $\Longrightarrow$ (qRe);
(x) (qM) + (qL$(1 \to x)$) $\Longrightarrow$ (qL).

Proof:

(i): Suppose that for any $x, y \in A$, $x \to y = 1 = y \to x$; then, by (An), $x = y$, hence $1 \to x = 1 \to y$; thus, (qAn) holds.

(ii): Obviously.

(iii): For any $x, y \in A$, $1 \to (x \to y) = 1$ implies, by (N), that $x \to y = 1$, i.e. (qN) holds; then $1 \to (1 \to x) = 1$ implies, by (qN), that $1 \to x = 1$.

(iv): For any $x, y \in A$, $(x \to y) \to (x \to y) \overset{(Re)}{=} 1$, i.e. (qRe) holds; then $(1 \to x) \to (1 \to x) \overset{(qRe)}{=} 1$, i.e. (qRe$(1 \to x)$) holds.

(v): For any $x, y \in A$, $(x \to y) \to 1 \overset{(L)}{=} 1$, i.e. (qL) holds; then $(1 \to x) \to 1 \overset{(qL)}{=} 1$, i.e. (qL$(1 \to x)$) holds.

(vi): For any $x, y \in A$, suppose that $x \to y = 1 = y \to x$; then, by (qAn), $1 \to x = 1 \to y$; finally, by (M), we obtain $x = y$. Thus, (An) holds.

(vii): For any $y \in A$, suppose that $(1 \to x) \to (1 \to x) = 1$, i.e. (qRe$(1 \to x)$) holds; then, $x \to x \overset{(M)}{=} (1 \to x) \to (1 \to x) = 1$.

(viii): For any $y \in A$, suppose that $(1 \to x) \to 1 = 1$, i.e. (qL$(1 \to x)$) holds; then, $y \to 1 \overset{(M)}{=} (1 \to y) \to 1 = 1$, i.e. (L) holds.

(ix): (See the proof of (qW11) from [2])

For any $x, y \in A$, $(x \to y) \to (x \to y) \overset{(qM)}{=} [1 \to (x \to y)] \to [1 \to (x \to y)] \overset{(qRe(1\to z))}{=} 1$, i.e. (qRe) holds.

(x): (See the proof of (qW14) from [2])

For any $x, y \in A$, $(x \to y) \to 1 \overset{(qM)}{=} [1 \to (x \to y)] \to 1 \overset{(qL(1\to z))}{=} 1$. □

Remarks 4.3.

(i) This theorem, (i) - (v), show that the proper quasi-properties: (qAn), (qM), (qM$(1 \to x)$), (qN), (qN$(1 \to x)$), (qRe), (qRe$(1 \to x)$), (qL), (qL$(1 \to x)$) are generalizations of the corresponding - call them *regular properties* - (An), (M), (N), (Re), (L) respectively.

(ii) This theorem (vi) - (viii) show the importance of property (M).

Remark 4.4. Note that if (M) holds, then:
- (qR1) $\equiv$ (qR2) $\equiv$ (Re), while (qR)$\equiv$(qRe) and (qR3)$\equiv$(qRe), and this motivates their names;
- (qI), (qI1), (qI2), (qI3) are identically satisfied, and this motivates their names.

4.1.2 Connections between the Quasi-Properties in List qA

Looking at the connections between the properties in List A, we generalize easily and obtain the following corresponding connections between the quasi-properties in List qA:

Proposition 4.5. *(See Proposition 3.2)*

Let $(A, \to, 1)$ be an algebra of type $(2, 0)$. Then the following are true (following the numbering from Proposition 3.2):

(qA00) (qM) $\Longrightarrow$ (qN); (qA00') (qM$(1 \to x)$) $\Longrightarrow$ (qN $(1 \to x)$);
(qA3) (C) + (qM) + (qAn) $\Longrightarrow$ (Ex);
(qA4) (Ex) + (qRe) $\Longrightarrow$ (D);
(qA7) (qN) + (K) $\Longrightarrow$ (L); (qA7') (qM) + (K) $\Longrightarrow$ (L);
(qA12) (qN) + (B) $\Longrightarrow$ (*); (qA12') (qM) + (B) $\Longrightarrow$ (*);
(qA13) (qN) + (*) $\Longrightarrow$ (Tr); (qA13') (qM) + (*) $\Longrightarrow$ (Tr);
(qA14) (qN) + (B) $\Longrightarrow$ (Tr); (qA14') (qM) + (B) $\Longrightarrow$ (Tr);
(qA15) (qN) + (BB) $\Longrightarrow$ (**); (qA15') (qM) + (BB) $\Longrightarrow$ (**);
(qA16) (qN) + (**) $\Longrightarrow$ (Tr); (qA16') (qM) + (**) $\Longrightarrow$ (Tr);

(qA17) (qN) + (BB) $\Longrightarrow$ (Tr); (qA17') (qM) + (BB) $\Longrightarrow$ (Tr);
(qA18) (qM) + (BB) $\Longrightarrow$ (qRe($1 \to x$));
(qA19) (qM) + (B) $\Longrightarrow$ (qRe($1 \to x$));
(qA20) (BB) + (D) + (qN) $\Longrightarrow$ (C);
(qA20') (BB) + (D) + (qM) $\Longrightarrow$ (C);
(qA21) (BB) + (D) + (qM) + (qAn) $\Longrightarrow$ (Ex);
(qA22) (K) + (Ex) + (qM) $\Longrightarrow$ (Re);
(qA23) (C) + (K) + (qM) + (qAn) $\Longrightarrow$ (Re).

Proof:

(qA00): Suppose $1 \to (x \to y) = 1$. Then, by (qM), we get $x \to y = 1$.

(qA00'): Suppose $1 \to (1 \to x) = 1$. Then, by (qM($1 \to x$)), we get $1 \to x = 1$, i.e. (qN($1 \to x$)) holds.

(qA3): By (C), we have: $[x \to (y \to z)] \to [y \to (x \to z)] = 1$ and also $[y \to (x \to z)] \to [x \to (y \to z)] = 1$; hence, by (qAn), we obtain that: $1 \to [x \to (y \to z)] = 1 \to [y \to (x \to z)]$; then, by (qM), we get that $x \to (y \to z)] = y \to (x \to z)$, i.e. (Ex) holds.

(qA4): $y \to [(y \to x) \to x] \overset{(Ex)}{=} (y \to x) \to (y \to x) \overset{(qRe)}{=} 1$.

(qA7): By (K), we have: $1 \to (x \to 1) = 1$; hence, by (qN), we obtain that $x \to 1 = 1$, i.e. (L) holds.

(qA7'): $x \to 1 \overset{(qM)}{=} 1 \to (x \to 1) \overset{(K)}{=} 1$; hence, $x \to 1 = 1$, i.e. (L) holds.

(qA12): Suppose $y \to z = 1$; then, by (B) ($(y \to z) \to [(x \to y) \to (x \to z)] = 1$), it follows that $1 \to [(x \to y) \to (x \to z)] = 1$; hence, by (qN), we obtain that $(x \to y) \to (x \to z) = 1$, i.e. (*) holds.

(qA12'): By (q00), (qM) implies (qN); then apply above (qA12).

(qA13): Suppose $x \leq y$ and $y \leq z$; then, by (*'), we obtain: $x \to y \leq x \to z$ i.e $(x \to y) \to (x \to z) = 1 = 1 \to (x \to z)$; hence by (qN), we obtain: $x \to z = 1$, i.e. (Tr) holds.

(qA13'): By (q00), (qM) implies (qN); then apply (qA13).

(qA14): Suppose $x \to y = 1$ and $y \to z = 1$; then, by (B) ($(y \to z) \to [(x \to y) \to (x \to z)] = 1$), we obtain: $1 \to [1 \to (x \to z)] = 1$; then, by (qN), we obtain: $1 \to (x \to z) = 1$; by (qN) again, we obtain: $x \to z = 1$.

(qA14'): By (q00), (qM) implies (qN); then, apply (qA14).

(qA15): Suppose $y \to z = 1$; then, by (BB) ($(y \to z) \to [(z \to x) \to (y \to x)] = 1$), it follows that $1 \to [(z \to x) \to (y \to x)] = 1$; hence, by (qN), we obtain $(z \to x) \to (y \to x) = 1$, i.e. (**) holds.

(qA15'): By (q00), (qM) implies (qN); then apply (qA15).

(qA16): Suppose $y \leq z$ and $z \leq x$; then, by (**'), we obtain: $z \to x \leq y \to x$, i.e. $(z \to x) \to (y \to x) = 1 = 1 \to (y \to x)$; hence, by (qN), we obtain: $y \to x = 1$, i.e. (Tr) holds.

(qA16'): By (q00), (qM) implies (qN); then apply (qA16).

(qA17): Suppose $y \to z = 1$ and $z \to x = 1$; then, by (BB) ($(y \to z) \to [(z \to x) \to (y \to x)] = 1$), we obtain: $1 \to [1 \to (y \to x)] = 1$; then, applying (qN) twice, we obtain $y \to x = 1$. Thus, (Tr) holds.

(qA17'): By (q00), (qM) implies (qN); then apply (qA17).

(qA18): In (BB) ($(y \to z) \to [(z \to x) \to (y \to x)] = 1$), take $y = z = 1$; we obtain: $(1 \to 1) \to [(1 \to x) \to (1 \to x)] = 1$, hence, by the subsequent (qAA1), $1 \to [(1 \to x) \to (1 \to x)] = 1$; then, by (qM), $(1 \to x) \to (1 \to x) = 1$, i.e. (qRe$(1 \to x)$) holds.

(qA19): In (B) ($(y \to z) \to [(x \to y) \to (x \to z)] = 1$), take $x = y = 1$; we obtain: $(1 \to z) \to [(1 \to 1) \to (1 \to z)] = 1$, hence, by the subsequent (qAA1'), $(1 \to z) \to [1 \to (1 \to z)] = 1$; now apply (qM) to obtain, $(1 \to z) \to (1 \to z) = 1$; thus, (qRe$(1 \to x)$) holds.

(qA20): First, by (qA15), (BB) + (qN) imply (**).
Then, by (BB') ($Y \to Z \leq (Z \to X) \to (Y \to X)$), for $X = u \to x$, $Y = y$, $Z = z \to x$, we obtain:
$y \to (z \to x) \leq ((z \to x) \to (u \to x)) \to (y \to (u \to x))$.
Then, by (**'), we obtain: $V \overset{notation}{=}$
$[((z \to x) \to (u \to x)) \to (y \to (u \to x))] \to [(u \to z) \to (y \to (u \to x))] \leq$
$(y \to (z \to x)) \to [(u \to z) \to (y \to (u \to x))] \overset{notation}{=} W$.
But, the left side $V = 1$; indeed, by (BB'), we have:
$u \to z \leq (z \to x) \to (u \to x)$; then, by (**'), we obtain:
$((z \to x) \to (u \to x)) \to (y \to (u \to x)) \leq (u \to z) \to (y \to (u \to x))$,
i.e. $V = 1$. Then, by (qN'), $W = 1$, i.e.
$y \to (z \to x) \leq (u \to z) \to (y \to (u \to x))$,
which for $z = y \to x$ and $u = z$ gives:
$y \to ((y \to x) \to x) \leq (z \to (y \to x)) \to (y \to (z \to x))$;
but, by (D), the left side $y \to ((y \to x) \to x) = 1$; hence, by (qN'),
$(z \to (y \to x)) \to (y \to (z \to x)) = 1$, i.e (C) holds.

(qA20'): By the above (q00), (qM) implies (qN); then apply (qA20).

(qA21): By (qA20'), (BB) + (D) + (qM) imply (C); and by above (qA3), (C) + (qM) + (qAn) imply (Ex).

(qA22): $x \to x \overset{(qM)}{=} 1 \to (x \to x) \overset{(Ex)}{=} x \to (1 \to x) \overset{(K)}{=} 1$, hence $x \to x = 1$, i.e. (Re) holds.

(qA23): By the above (qA3), (C) + (qM) + (qAn) imply (Ex). Then, by the above (qA22), (K) + (Ex) + (qM) imply (Re); thus, (Re) holds. □

Proposition 4.6. *(See Proposition 3.3)*
Let $(A, \rightarrow, 1)$ be an algebra of type $(2, 0)$. We have the additional quasi-properties (following the numbering from Proposition 3.3):
(qA18") (qM) + (D) $\Longrightarrow$ (qR1);
(qA25) (D) + (K) + (qN) + (qAn) $\Longrightarrow$ (qM($1 \rightarrow x$));
(qA27) (qM) + (C) $\Longrightarrow$ (#).

Proof:
(qA18"): $1 \rightarrow ((1 \rightarrow x) \rightarrow x) \overset{(D)}{=} 1$; then, by (qM), $(1 \rightarrow x) \rightarrow x = 1$.
(qA25): $1 \rightarrow [(1 \rightarrow x) \rightarrow x] \overset{(D)}{=} 1$; now, by (qN), we obtain: $(1 \rightarrow x) \rightarrow x = 1$. On the other hand, $x \rightarrow (1 \rightarrow x) \overset{(K)}{=} 1$. Consequently, by (qAn), we obtain: $1 \rightarrow (1 \rightarrow x) = 1 \rightarrow x$, i.e. (qM($1 \rightarrow x$)) holds.
(qA27): By Theorem 4.2 (ii), (M) $\Longrightarrow$ (qM); then use (A27). □

Now we prove the corresponding theorems of (above recalled from [9]) Theorems 3.4, 3.5, 3.6, 3.7.

Theorem 4.7. *(See Theorem 3.4)*
If properties (Re), (qM), (Ex) hold, then: $(BB) \Leftrightarrow (B) \Leftrightarrow (*)$.

Proof: By (A11), (Re) + (Ex) + (*) $\Longrightarrow$ (BB). By (A10), (Ex) implies that (B) $\Leftrightarrow$ (BB). By (qA12'), (qM) + (B) $\Longrightarrow$ (*). Hence, we have: $(*) \Longrightarrow (BB) \Leftrightarrow (B) \Longrightarrow (*)$, thus $(BB) \Leftrightarrow (B) \Leftrightarrow (*)$. □

Theorem 4.8. *(See Theorem 3.5)*
If properties (Re), (qM), (Ex) hold, then: $(**) \Leftrightarrow (Tr)$.

Proof: By (qA16'), (qM) + (**) imply (Tr). By (A24), (Re) + (Ex) + (Tr) imply (**). □

Theorem 4.9. *(See Theorem 3.6)*
If properties (B), (D), (qM), (qAn) hold, then: $(Ex) \Leftrightarrow (BB)$.

Proof: By (A10'), (B) + (Ex) imply (BB). By (qA21), (qM) + (qAn) + (BB) + (D) imply (Ex). □

Theorem 4.10. *(See Theorem 3.7) In any algebra $(A, \rightarrow, 1)$ we have:*
(i) (qM) + (BB) + (D) imply (B),
(ii) (qM) + (B) imply (**).

Proof: (i): In (BB) $((x \to y) \to [(y \to z) \to (x \to z)] = 1)$, set $x = u$ and $y = (u \to v) \to v$, to get:
$(u \to [(u \to v) \to v]) \to [(((u \to v) \to v) \to z) \to (u \to z)] \overset{(D)}{=}$
$1 \to [(((u \to v) \to v) \to z) \to (u \to z)] \overset{(qM)}{=}$
$(((u \to v) \to v) \to z) \to (u \to z) = 1$.
After renaming variables, we get:
(a) $\quad (((x \to y) \to y) \to z) \to (x \to z) = 1$.
Next, in (BB) set $x = u \to v$ and $y = (v \to w) \to (u \to w)$, to get:
$((u \to v) \to [(v \to w) \to (u \to w)]) \to [(((v \to w) \to (u \to w)) \to z) \to ((u \to v) \to z)] \overset{(BB)}{=} 1 \to [(((v \to w) \to (u \to w)) \to z) \to ((u \to v) \to z)] \overset{(qM)}{=} (((v \to w) \to (u \to w)) \to z) \to ((u \to v) \to z) = 1$.
After renaming variables, we get:
(b) $\quad (((x \to y) \to (u \to y)) \to z) \to ((u \to x) \to z) = 1$.
Taking $z = u \to y$ in (b), we get:
(c) $\quad (((x \to y) \to (u \to y)) \to (u \to y)) \to ((u \to x) \to (u \to y)) = 1$.
Now, in (a) set $x = v \to w$, $y = t \to w$, $z = (t \to v) \to (t \to w)$ to get:
$[(((v \to w) \to (t \to w)) \to (t \to w)) \to ((t \to v) \to (t \to w))] \to ((v \to w) \to ((t \to v) \to (t \to w))) \overset{(c)}{=}$
$1 \to ((v \to w) \to ((t \to v) \to (t \to w))) \overset{(qM)}{=}$
$(v \to w) \to ((t \to v) \to (t \to w)) = 1$, i.e. (B) holds.

(ii): Suppose (B) is $(y \to z) \to [(x \to y) \to (x \to z)] = 1$. If $x \to y = 1$ in (B), then we get: $(y \to z) \to [1 \to (x \to z)] \overset{(qM)}{=} (y \to z) \to (x \to z) = 1$, i.e. (**) holds. □

Concluding, by the above Theorem 4.10 and (qA12'), (qA13'), (qA16'), we immediately obtain:

Corollary 4.11. (See Corollary 3.9)
In any algebra $(A, \to, 1)$ verifying (qM), we have:

$$(BB)\ +\ (D) \implies (B) \implies (*), (**) \implies (Tr).$$

Proposition 4.12. *Let $(A, \to, 1)$ be an algebra of type $(2, 0)$. Then we have the following additional quasi-properties (with an independent numbering):*
(qAA1) (qM) + (BB) $\implies$ (11-1); (qAA1') (qM) + (B) $\implies$ (11-1);
(qAA1") (qM) + (K) $\implies$ (11-1);
(qAA2) (Ex) + (qRe($1 \to x$)) + (qM) $\implies$ (qR1);
(qAA3) (B) $\implies$ (qR);

(qAA4) $(K) \Longrightarrow (qR2)$;
(qAA5) $(qR1) + (BB) \Longrightarrow (qR3)$;
(qAA6) $(qR1) + (K) + (**) + (qM) + (qAn) \Longrightarrow (qI1)$;
(qAA7) $(qRe(1 \to x)) + (Ex) + (K) + (**) + (qM) + (qAn) \Longrightarrow (qI1)$;
(qAA7') $(D) + (K) + (**) + (qM) + (qAn) \Longrightarrow (qI1)$;
(qAA8) $(Ex) + (qM) \Longrightarrow (qI2) + (qI3)$;
(qAA9) $(qI1) + (qI3) \Longrightarrow (qI)$;
(qAA10) $(qR1) + (qR2) + (BB) + (qM) + (qAn) \Longrightarrow (qI1) + (qI2)$;
(qAA11) $(qR3) + (K) + (*) + (qM) + (qAn) \Longrightarrow (qI2)$;
(qAA12) $(qI1) + (qI2) \Longrightarrow (qI)$;
(qAA13) $(qI) + (BB) + (qM) \Longrightarrow (Re)$; *(see (A18), (qA18))*
(qAA14) $(qI1) + (BB) + (L) + (qM) \Longrightarrow (K)$; *(see (A9"))*
(qAA15) $(B) + (Ex) + (K) + (**) + (qM) + (qAn) \Longrightarrow (qI)$;
(qAA15') $(qRe(1 \to x)) + (Ex) + (K) + (**) + (qM) + (qAn) \Longrightarrow (qI)$;
(qAA15") $(Re) + (Tr) + (Ex) + (L) + (qM) + (qAn) \Leftrightarrow (qRe(1 \to x)) + (Ex) + (K) + (**) + (qM) + (qAn)$;
(qAA15"') $(Re) + (Tr) + (Ex) + (L) + (qM) + (qAn) \Longrightarrow (qI)$;
(qAA16) $(qI) + (qRe(1 \to x)) \Longrightarrow (Re)$;
(qAA17) $(qI) \Longrightarrow ((qRe(1 \to x)) \Leftrightarrow (Re))$;
(qAA18) $(\#) + (qM) + (qR1) \Longrightarrow (qRe(1 \to x))$;
(qAA18') $(\#) + (qM) + (qRe(1 \to x)) \Longrightarrow (qR1)$;
(qAA18") $(\#) + (qM) \Longrightarrow ((qR1) \Leftrightarrow (qRe(1 \to x)))$;
(qAA19) $(qI) \Longrightarrow (qrelI)$;
(qAA19') $(qI1) \Longrightarrow (qrelI1)$;
(qAA19") $(qI2) \Longrightarrow (qrelI2)$.

Proof:

(qAA1): (See the proof of (qW9) from [2])
$1 \to 1 \overset{(BB)}{=} 1 \to [(x \to y) \to [(y \to z) \to (x \to z)]] \overset{(qM)}{=}$
$(x \to y) \to [(y \to z) \to (x \to z)] \overset{(BB)}{=} 1$, i.e. (11-1) holds.

(qAA1'): $1 \to 1 \overset{(B)}{=} 1 \to [(y \to z) \to [(x \to y) \to (x \to z)]] \overset{(qM)}{=}$
$(y \to z) \to [(x \to y) \to (x \to z)] \overset{(B)}{=} 1$, i.e. (11-1) holds.

(qAA1"): $1 \to 1 \overset{(qM)}{=} 1 \to (1 \to 1) \overset{(K)}{=} 1$, i.e. (11-1) holds.

(qAA2): $(1 \to x) \to x \overset{(qM)}{=} 1 \to [(1 \to x) \to x] \overset{(Ex)}{=}$
$(1 \to x) \to (1 \to x) \overset{(qRe(1\to x))}{=} 1$. Thus, (qR1) holds.

(qAA3): Obviously.

(qAA4): Obviously.

(qAA5): (See the proof of (qW16) from [2])
$[x \to (1 \to y)] \to [1 \to (x \to y)] \overset{(qR1)}{=}$
$[x \to (1 \to y)] \to [((1 \to y) \to y) \to (x \to y)] \overset{(BB)}{=} 1$, i.e. (qR3) holds.

(qAA6): $x \overset{(K')}{\leq} 1 \to x$, hence $(1 \to x) \to y \overset{(**')}{\leq} x \to y$. On the other hand, $1 \to x \overset{(qR1)}{\leq} x$, hence $x \to y \overset{(**')}{\leq} (1 \to x) \to y$. Consequently, $x \to y \overset{(qM)}{=} 1 \to (x \to y) \overset{(qAn)}{=} 1 \to [(1 \to x) \to y] \overset{(qM)}{=} (1 \to x) \to y$.

(qAA7): By (qAA2), (Ex) + (qRe($1 \to x$)) + (qM) imply (qR1); then apply (qAA6).

(qAA7'): By (qA18"), (D) + (qM) imply (qR1); then apply (qAA6).

(qAA8): $x \to y \overset{(qM)}{=} 1 \to (x \to y) \overset{(Ex)}{=} x \to (1 \to y)$; thus, (qI2) holds; $(1 \to x) \to (1 \to y) \overset{(Ex)}{=} 1 \to [(1 \to x) \to y] \overset{(qM)}{=} (1 \to x) \to y$.

(qAA9): Obviously.

(qAA10): We prove that (qI1) holds (see the proof of (qW26) from [2]):
On the one hand, $((1 \to x) \to y) \to (x \to y) \overset{(qM)}{=} 1 \to [((1 \to x) \to y) \to (x \to y)] \overset{(qR2)}{=} [x \to (1 \to x)] \to [((1 \to x) \to y) \to (x \to y)] \overset{(BB)}{=} 1$.
On the other hand, $(x \to y) \to ((1 \to x) \to y) \overset{(qM)}{=} 1 \to [(x \to y) \to ((1 \to x) \to y)] \overset{(qR1)}{=} [(1 \to x) \to x] \to [(x \to y) \to ((1 \to x) \to y)] \overset{(BB)}{=} 1$.
Now, by (qAn), we obtain $x \to y \overset{(qM)}{=} 1 \to (x \to y) = 1 \to ((1 \to x) \to y) \overset{(qM)}{=} (1 \to x) \to y$, i.e. (qI1) holds.

We prove that (qI2) holds (see the proof of (qW27) from [2]):
On the one hand, $(x \to (1 \to y)) \to (x \to y) \overset{(qM)}{=} (x \to (1 \to y)) \to (1 \to (x \to y)) \overset{(qR1)}{=} (x \to (1 \to y)) \to (((1 \to y) \to y) \to (x \to y)) \overset{(BB)}{=} 1$.
On the other hand, $(x \to y) \to (x \to (1 \to y)) \overset{(qM)}{=} (x \to y) \to [1 \to (x \to (1 \to y))] \overset{(qR2)}{=} (x \to y) \to [(y \to (1 \to y)) \to (x \to (1 \to y))] \overset{(BB)}{=} 1$.
Now, by (qAn) and (qM), we obtain that $x \to (1 \to y) = x \to y$.

(qAA11): On the one hand, (qR3) + (qM) implies that $(x \to (1 \to y)) \to (x \to y) = 1$. On the other hand, $y \to (1 \to y) \overset{(K)}{=} 1$ implies by (*) that $(x \to y) \to (x \to (1 \to y)) = 1$. Now, by (qAn), (qM) we obtain that $x \to y = x \to (1 \to y)$, i.e. (qI2) holds.

(qAA12): (See the proof of (qW28) from [2])
$(1 \to x) \to (1 \to y) \overset{(qI1)}{=} x \to (1 \to y) \overset{(qI2)}{=} x \to y$, i.e. (qI) holds.

(qAA13): (See the proof of (qW29) from [2])
By (qAA1), (qM) + (BB) $\Longrightarrow$ (11-1). Then, $x \to x \overset{(qI)}{=}$ $(1 \to x) \to (1 \to x) \overset{(qM)}{=} 1 \to [(1 \to x) \to (1 \to x)] \overset{(11-1)}{=}$ $(1 \to 1) \to [(1 \to x) \to (1 \to x)] \overset{(BB)}{=} 1$, i.e. (Re) holds.

(qAA14): (See the proof of (qW30) from [2])
$x \to (y \to x) \overset{(qI1)}{=} (1 \to x) \to (y \to x) \overset{(qM)}{=} 1 \to [(1 \to x) \to (y \to x)] \overset{(L)}{=}$ $(y \to 1) \to [(1 \to x) \to (y \to x)] \overset{(BB)}{=} 1$, i.e. (K) holds.

(qAA15): $x \to y \overset{(B')}{\leq} (1 \to x) \to (1 \to y)$. On the other hand, $x \overset{(K)}{\leq} 1 \to x$ implies $(1 \to x) \to y \overset{(**')}{\leq} x \to y$, hence $(1 \to x) \to (1 \to y) \overset{(Ex)}{=}$ $1 \to [(1 \to x) \to y] \overset{(qM)}{=} (1 \to x) \to y \leq x \to y$. Consequently, $x \to y \overset{(qM)}{=}$ $1 \to (x \to y) \overset{(qAn)}{=} 1 \to [(1 \to x) \to (1 \to y)] \overset{(qM)}{=} (1 \to x) \to (1 \to y)$.

(qAA15'): By (qAA7), (qRe($1 \to x$)) + (Ex) + (K) + (**) + (qM) + (qAn) $\Longrightarrow$ (qI1); by (qAA8), (Ex) + (qM) $\Longrightarrow$ (qI3); by (qAA9), (qI1) + (qI3) $\Longrightarrow$ (qI); thus, (qI) holds.

(qAA15"): $\Longrightarrow$: by (A8), (Re) + (Ex) + (L) imply (K); by Theorem 4.8, (Re) + (Ex) + (qM) imply (Tr) $\Leftrightarrow$ (**), hence (**) holds; by Theorem 4.2 (iv), (Re) implies (qRe($1 \to x$)).
$\Longleftarrow$: By (qA7')), (K) + (qM) imply (L), by (qA22), (K) + (Ex) + (qM) imply (Re) and by Theorem 4.8, (Re) + (Ex) + (qM) imply (Tr) $\Leftrightarrow$ (**), hence (Tr) holds too.

(qAA15"'): By (qAA15') and (qAA15").

(qAA16): $x \to x \overset{(qI)}{=} (1 \to x) \to (1 \to x) \overset{(qRe(1 \to x))}{=} 1$, i.e. (Re) holds.

(qAA17): By Theorem 4.2 (iv), (Re) $\Longrightarrow$ (qRe($1 \to x$)); by the above (qAA16), (qI) + (qRe($1 \to x$)) $\Longrightarrow$ (Re).

(qAA18): $1 \overset{(qR1)}{=} (1 \to x) \to x \overset{(qM)}{=} 1 \to [(1 \to x) \to x]$; then, by (#), we obtain $(1 \to x) \to (1 \to x) = 1$, i.e. (qRe($1 \to x$)) holds.

(qAA18'): $1 \overset{(qRe(1 \to x))}{=} (1 \to x) \to (1 \to x)$ implies, by (#), that $1 \to [(1 \to x) \to x] = 1$; hence, by (qM), we obtain $(1 \to x) \to x = 1$.

(qAA18"): By (qAA18) and (qAA18').

(qAA19): $x \leq y \Leftrightarrow x \to y = 1 \overset{(qI)}{\Leftrightarrow} (1 \to x) \to (1 \to y) = 1 \Leftrightarrow 1 \to x \leq$ $1 \to y$, i.e. (qrelI) holds.

(qAA19'): similarly, by (qI1). (qAA19"): similarly, by (qI2). $\square$

Remark 4.13. Since
(M) $\overset{(A00)}{\Longrightarrow}$ (N) $\overset{(iii)}{\Longrightarrow}$(qN) $\overset{(iii)}{\Longrightarrow}$ (qN($1 \to x$)) and
(M) $\overset{(ii)}{\Longrightarrow}$ (qM) $\overset{(qA00)}{\Longrightarrow}$ (qN) and
(qM) $\overset{(ii)}{\Longrightarrow}$ (qM($1 \to x$)) $\overset{(qA00')}{\Longrightarrow}$ (qN($1 \to x$)),
then we can picture the situation as follows:

$$
\begin{array}{ccccccc}
 & & & (qN(1 \to x)) & & & \\
 & & \nearrow & & \nwarrow & & \\
 & (qN) & & & & (qM(1 \to x)) & \\
 \nearrow & & \nwarrow & & \nearrow & & \\
(N) & & & (\mathbf{qM}) & & & \\
 \nwarrow & & \nearrow & & & & \\
 & (\mathbf{M}) & & & & &
\end{array}
$$

4.2 Quasi-Order. Quasi-Ordered Algebras (Structures)

Let $\mathcal{A}$ be a proper quasi-algebra (quasi-structure) (i.e. (qM), which differs from (M), and (11-1) hold) through this subsection; then its subalgebra $\mathcal{R}(\mathcal{A})$ is a regular algebra (structure) (i.e. (M) holds), by Theorem 2.12.

Definitions 4.14. Consider the following properties of $\to$ ($\leq$): (Re), (qAn), (Tr) ((Re'), (qAn'), (Tr'), respectively). Then, we shall say that $\mathcal{A}$ is:
- *reflexive*, if property (Re) (or (Re')) is satisfied,
- *quasi-antisymmetric*, if property (qAn) (or (qAn')) is satisfied,
- *transitive*, if property (Tr) (or (Tr')) is satisfied;
- *quasi-pre-ordered*, and $\leq$ is a *quasi-pre-order*, if it is reflexive and transitive;
- *quasi-ordered*, and $\leq$ is a *quasi-order* (or a *q-order* for short), if it is reflexive, quasi-antisymmetric and transitive.

A quasi-ordered quasi-algebra (quasi-structure) will be simply called "a quasi-ordered algebra (structure)".

Remarks 4.15.

(i) Recall that by Theorem 4.2 (vi), (M) + (qAn) $\Longrightarrow$ (An). Consequently, in the presence of (M), any q-order becomes a regular order.

(ii) The above remark (i) shows again the central role played by the property (M): we obtain the general rule that, roughly speaking,

$$(M){+}quasi{-}ordered\ algebra\ (struct.) = ordered\ regular\ algebra\ (struct.).$$

Note that both quasi-MV algebras and quasi-Wajsberg algebras are in fact *quasi-ordered algebras* and that both MV algebras and Wajsberg algebras are in fact *ordered regular algebras*.

We now present an important remark:

Remark 4.16. Let $\mathcal{A} = (A, \rightarrow, 1)$ be a quasi-ordered algebra (or $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a quasi-ordered structure). Since (qM) coincides with (M) on $R(A)$, by Theorem 2.12, it follows that (qAn) coincides with (An) on $R(A)$. Consequently,
- the q-order relation $\leq$ on A becomes an order relation on $R(A)$;
- $\mathcal{R}(\mathcal{A}) = (R(A), \rightarrow, 1)$ is an ordered regular algebra (or, equivalently, $\mathcal{R}(\mathcal{A}) = (R(A), \leq, \rightarrow, 1)$ is an ordered regular structure, respectively).

4.2.1 The Duality Principle. Elements of the Same Height. The Quasi-Hasse Diagram

Let $\mathcal{A} = (A, \rightarrow, 1)$ be a quasi-ordered algebra (or, equivalently, let $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a quasi-ordered structure) and $\leq$ be the quasi-order of $\mathcal{A}$ (i.e. (qM), (Re) (hence (11-1)), (qAn), (Tr) hold) through this subsubsection.

The relation $\geq$, defined on A as follows: for every $x, y \in A$,

$$x \geq y \overset{def.}{\Longleftrightarrow} y \leq x,$$

is also a q-order, called the *inverse q-order relation* or the *dual q-order relation* of the q-order relation $\leq$.

The duality principle for quasi-ordered algebras (structures) is the following:
"every statement (definition of a notion, proposition, theorem, etc.) concerning the quasi-ordered algebra $\mathcal{A} = (A, \rightarrow, 1)$ (quasi-ordered structure $\mathcal{A} = (A, \leq, \rightarrow, 1)$) remains valid if we replace everywhere inside it the q-order relation $\leq$ with the inverse q-order relation, $\geq$".

The algebra $(A, \rightarrow, 1)$ (the structure $(A, \geq, \rightarrow, 1)$) obtained in this way is also a quasi-ordered algebra (quasi-ordered structure), called the *dual* of $\mathcal{A}$. The statement obtained in this way (definition of a notion, proposition, theorem, etc.) is the *dual* statement of the first statement (definition of a notion, proposition, theorem, etc.). We say also that the two quasi-ordered algebras (structures) or statements are *dual to each other* or simple *dual*.

Definition 4.17. We say that $a, b \in A$ have *the same height*, or are *parallel*, and we denote this by $a \parallel b$, if $a \to b = 1$ and $b \to a = 1$ (or, equivalently, $a \leq b$ and $b \leq a$).

Note that if (M) holds, then $a \parallel b \Leftrightarrow a = b$, by (An).
Note also that if $a \parallel b$, then $1 \to a = 1 \to b$, by (qAn).

Corollary 4.18. If (Ex), (L) hold, then

$$a \parallel b \quad \Longleftrightarrow \; 1 \to a = 1 \to b.$$

Proof: By (qAA15"'), (qM) + (Re) + (qAn) + (Tr) + (Ex) + (L) imply (qI), hence

$$a \to b \overset{(qI)}{=} (1 \to a) \to (1 \to b), \quad b \to a \overset{(qI)}{=} (1 \to b) \to (1 \to a).$$

If $1 \to a = 1 \to b$, then $(1 \to a) \to (1 \to b) \overset{(Re)}{=} 1$ and
$(1 \to b) \to (1 \to a) \overset{(Re)}{=} 1$, hence $a \to b = 1$ and $b \to a = 1$, i.e. $a \parallel b$.
The other implication follows by (qAn). □

Proposition 4.19. *The relation $\parallel$ is an equivalence relation of $\mathcal{A}$.*

Proof: For all $a \in A$, $a \parallel a$ means $a \to a = 1$, which is true by (Re); thus $\parallel$ is reflexive.
For all $a, b \in A$, if $a \parallel b$, i.e. $a \to b = 1$ and $b \to a = 1$, then obviously $b \to a = 1$ and $a \to b = 1$, i.e. $b \parallel a$; thus $\parallel$ is symmetric.
For all $a, b, c \in A$, if $a \parallel b$ and $b \parallel c$, i.e. $a \to b = 1$, $b \to a = 1$ and $b \to c = 1$, $c \to b = 1$, then $a \to b = 1$ and $b \to c = 1$ imply $a \to c = 1$, by (Tr), and also $c \to b = 1$ and $b \to a = 1$ imply $c \to a$=1, by (Tr); so $a \parallel c$ and thus $\parallel$ is transitive. □

Proposition 4.20. *If properties (*), (**) also hold, then $\parallel$ is a congruence relation of $\mathcal{A}$.*

Proof: We must prove that $\parallel$ is compatible with $\to$.
Indeed, if $a \parallel b$ and $x \parallel y$, i.e. $a \leq b$, $b \leq a$ and $x \leq y$, $y \leq x$, then
$a \to x \overset{(*')}{\leq} a \to y \overset{(**')}{\leq} b \to y$ and $b \to y \overset{(*')}{\leq} b \to x \overset{(**')}{\leq} a \to x$.
Then, by (Tr'), we obtain that $a \to x \leq b \to y$ and $b \to y \leq a \to x$, i.e. $a \to x \parallel b \to y$. □

Proposition 4.21. *If (*), (**) also hold and if* $b \parallel a$*, then, in the table of* $\to$*, we have:*
(i) the row of b *coincides with the row of* a*;*
(ii) the column of b *coincides with the column of* a*.*

Proof: Suppose $b \parallel a$, i.e. $b \leq a$ and $a \leq b$. Then,

(i): By (**'), we obtain: for each $z \in A$, $a \to z \leq b \to z$ and $b \to z \leq a \to z$. Then, by (qAn), $1 \to (b \to z) = 1 \to (a \to z)$, hence, by (qM), $b \to z = a \to z$.

(ii): By (*'), we obtain: for each $z \in A$, $z \to b \leq z \to a$ and $z \to a \leq z \to b$. Then, by (qAn), $1 \to (z \to b) = 1 \to (z \to a)$, hence, by (qM), $z \to b = z \to a$. □

For each $x \in A$, we denote its equivalence class by

$$\mid x \mid \overset{notation}{=} \{y \in A \mid y \parallel x\}$$

and we denote by $A/\parallel$ the quotient set (i.e. the set of all equivalence classes):

$$A/\parallel \overset{notation}{=} \{\mid x \mid \mid x \in A\}.$$

Note that in the particular case of quasi-MV algebras [17], the equivalence relation $\parallel$ is denoted by χ and the equivalence classes determined by χ are called *clouds*; it is proved there (Lemma 19) that each cloud contains exactly one regular element. We believe that our notation $\parallel$ is more appropriate, but we shall also use the name "cloud" for an equivalence class; we have, for each $x \in A$:

$$C(x) = \mid x \mid .$$

We have, more generally:

Lemma 4.22. *If properties (Ex), (L) also hold, then every cloud in a quasi-ordered algebra (structure)* $\mathcal{A}$ *contains exactly one regular element.*

Proof: First, by (qAA15"'), (qM) + (Re) + (qAn) + (Tr) + (Ex) + (L) imply (qI).

Let C be a cloud, i.e. there exists $c \in A$ such that $C = \mid c \mid$.

• If $c \in R(A)$, then $c = 1 \to c$. Let $a \in C$ (i.e. $a \parallel c$) such that $a \neq c$; we prove that $a \notin R(A)$. Indeed, if, by absurdum hypothesis, $a \in R(A)$, then $a = 1 \to a$; then $a \to c = 1$ and $c \to a = 1$ imply $1 \to a = 1 \to c$, by (qAn), hence $a = c$: contradiction.

• If $c \notin R(A)$, we put $b = 1 \to c$. Then $b = 1 \to c \overset{(qM)}{=} 1 \to (1 \to c) = 1 \to b$, hence $b \in R(A)$.

We prove now that $b \in C$. Indeed, $b \to c \overset{(qI)}{=} (1 \to b) \to (1 \to c) = b \to b \overset{(Re)}{=} 1$ and $c \to b \overset{(qI)}{=} (1 \to c) \to (1 \to b) = b \to b \overset{(Re)}{=} 1$, hence $b \parallel c$, i.e. $b \in \mid c \mid = C$.
We prove now that any other element a of C, such that $a \neq b$, is not a regular element. Indeed, if, by absurdum hypothesis, $a \in R(A)$, i.e. $a = 1 \to a$, then, since $a \parallel c$, it follows that:
$1 = a \to c \overset{(qI)}{=} (1 \to a) \to (1 \to c) = a \to b$ and
$1 = c \to a \overset{(qI)}{=} (1 \to c) \to (1 \to a) = b \to a$;
hence, by (qAn), $1 \to a = 1 \to b$, i.e. $a = b$: contradiction. □

We define an implication of clouds as follows: for all $x, y \in A$,

$$\mid x \mid \to \mid y \mid \overset{def.}{=} \mid x \to y \mid .$$

Then, we have the following expected result:

Proposition 4.23. *If (Ex), (L) also hold, then the quotient algebra* $(A/\parallel, \to, \mid 1 \mid)$ *is a regular ordered algebra, isomorphic to* $(R(A), \to, 1)$.

Proof: Routine, by Lemma 4.22. □

Remarks 4.24.

(i) Given a finite regular ordered algebra $(X, \to, 1)$, we can obtain, **in general**, an infinity of finite quasi-ordered algebras: $(A_1, \to, 1)$, $(A_2, \to, 1)$, ... such that $R(A_1) = R(A_2) = ... = X$, by adding one or more elements parallel with some (all) elements of X (see the examples in Section 6).
We can quickly draw the table of $\to$ for such a finite quasi-ordered algebra $(A, \to, 1)$ with $R(A) = X$ by using either:
- property (qI), if (Ex), (L) hold or
- Proposition 4.21, if (*), (**) hold.

(ii) **In particular**, given a finite p-semisimple BCI algebra $(G, \to, 1)$, with $n \geq 2$ regular elements, then we cannot obtain any quasi-BCI algebra $(A, \to, 1)$, such that $R(A) = G$, by adding elements parallel to m elements of G, if $1 \leq m < n$ (see Remark 6.5).

A quasi-order relation $\leq$ on A will be represented graphically by a *quasi-Hasse diagram*, i.e.:

- a regular element is represented by a bullet $\bullet$,
- an element parallel with a regular element is represented by a big circ $\bigcirc$,
- the fact that $x < y$ (i.e. $x \leq y$ and $x \neq y$) and there is no z with $x < z < y$ is represented by:
$\cdot$ a line connecting the two points, y being higher than x, if the elements x, y are regular,
$\cdot$ a horizontal line connecting the two points, if the elements x, y have the same height (are parallel).

Consequently, the regular order relation $\leq$ on $R(A)$ will be represented graphically by a Hasse diagram.

Examples. Let us consider the following two quasi-algebras $\mathcal{A}_1 = (A_1 = \{0, x, 1\}, \rightarrow, 1)$ and $\mathcal{A}_2 = (A_2 = \{0, x, y, 1\}, \rightarrow, 1)$ given by the following tables of $\rightarrow$:

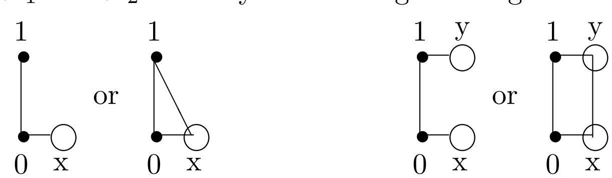
$\mathcal{A}_1$

$\rightarrow$	**0**	x	**1**
0	**1**	1	**1**
x	1	1	1
1	**0**	0	**1**

$\mathcal{A}_2$

$\rightarrow$	**0**	x	y	**1**
0	**1**	1	1	**1**
x	1	1	1	1
y	0	0	1	1
1	**0**	0	1	**1**

Then the quasi-Hasse diagrams from Figure 2 present for each of the quasi-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ two ways of drawing the diagrams.



$\mathcal{A}_1$ $\mathcal{A}_2$

Figure 2: The quasi-Hasse diagrams of quasi-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$

The quasi-Hasse diagram is useful for recognizing the properties of a quasi-order relation - just as the Hasse diagram is useful for recognizing the properties of an (regular) order relation.

Remark 4.25. If the quasi-algebra (quasi-structure) $\mathcal{A}$ is not ordered (i.e. it is only reflexive, or reflexive and transitive, or reflexive and quasi-anti-symmetric), then we shall use a *quasi-Hasse-type diagram*, with regular elements (parallel or not parallel) represented by a circ $\circ$ and with the parallel quasi-elements represented by a big circ $\bigcirc$. See for example the quasi-Hasse-type diagrams for the examples of quasi-RM algebra and quasi-RML algebra from Figures 4 and 5, respectively.

The theory of quasi-algebras (quasi-structures) will be continued in the next papers.

4.3 Quasi-Subtractive Varieties and Quasi-Subtractive Quasi-Algebras

The quasi-subtractive varieties were introduced in [16] and studied more deeply in [15].

Definition 4.26. [16] A variety $\mathcal{V}$ whose signature σ includes a nullary term 1 and a unary term $\Box$ is called *quasi-subtractive* with respect to 1 and $\Box$ if there is a binary term $\to$ of signature σ such that $\mathcal{V}$ satisfies the equations:
(Q1) $\Box(x) \to x = 1$,
(Q2) $1 \to x = \Box(x)$,
(Q3) $\Box(x \to y) = x \to y$,
(Q4) $\Box(x \to y) \to (\Box(x) \to \Box(y)) = 1$.

According to [16], the variety of quasi-MV algebras is quasi-subtractive, witness the terms $x' \oplus y$, $x \oplus 0$ and 1. According to F. Paoli (in a personal e-mail discussion), the variety of quasi-Wajsberg algebras is quasi-subtractive for $\Box(x) \overset{def.}{=} 1 \to x$.

Remark indeed, that taking $\Box(x) \overset{def.}{=} 1 \to x$, then the above equations (Q1) - (Q4) become:
(Q1') $(1 \to x) \to x = 1$,
(Q2') $1 \to x = 1 \to x$,
(Q3') $1 \to (x \to y) = x \to y$,
(Q4') $(1 \to (x \to y)) \to ((1 \to x) \to (1 \to y)) = 1$,
and note that (Q1') is (qR1), (Q2') is always true, (Q3') is (qM) and that (Q4') $\overset{(qM)}{=}$ (qR) (see (qA18"), (qAA2), (qAA3)). Hence, we have the following result.

Theorem 4.27. *Any variety of quasi-algebras satisfying (qR1) and (qR) (besides (qM) and (11-1)) is a quasi-subtractive variety with respect to* 1 *and* $\Box(x) = 1 \to x$.

We shall generalize now Definition 4.26 for any quasi-algebra (i.e. verifying (qM) and (11-1)).

Definition 4.28. A quasi-algebra is called *quasi-subtractive*, if it satisfies the properties (qR1) and (qR).

It is an open problem to find which are the results from the theory of quasi-subtractive varieties [16], [15] that remain true for the quasi-subtractive quasi-algebras.

5 New Quasi-Algebras

Starting from the study of quasi-MV algebras [17] and of their term equivalent quasi-Wajsberg algebras [2] (which verify (qM), (11-1), (Re), (L) - among others), we shall begin here the study of quasi-RM algebras and of quasi-RML algebras, of quasi-BCI algebras and of quasi-BCK algebras.

Definition 5.1. Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$ (or, equivalently, let $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a structure) as before. Then,
(1) $\mathcal{A}$ is a *quasi-RM algebra*, if quasi-properties (qM) and (Re) hold.
(2) $\mathcal{A}$ is a *quasi-RML algebra*, if quasi-properties (qM), (Re) and (L) hold.

Note that:
- the quasi-RM algebras and the quasi-RML algebras are quasi-algebras, since properties (qM) and (11-1) hold (by Remarks 3.1 (i));
- for any proper quasi-RM algebra (quasi-RML algebra) $\mathcal{A}$ (i.e. (qM) is not (M)), the regular algebra $\mathcal{R}(\mathcal{A})$ is a regular RM algebra (RML algebra, respectively), by Theorem 2.12.

Remarks 5.2.
(i) We shall call the quasi-algebras verifying (Re) and (L) as being of *order 1*.
(ii) Note that we can define more general algebras if property (qRe) or even more, (qRe$(1 \rightarrow x)$), is satisfied, instead of property (Re), and if property (qL) or even more, (qL$(1 \rightarrow x)$), is satisfied, instead of property (L).

We shall present now two equivalent definitions of the quasi-algebras: quasi-BCI algebras, quasi-BCK algebras.

5.1 Quasi-BCI Algebras

Theorem 5.3. *Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$ (or, equivalently, let $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a structure) as before. Then, the following*

two groups of quasi-properties are equivalent:
(qBCI-1) *(BB), (D), (Re), (qM), (qAn) and*
(qBCI-2) *(B), (C), (Re), (qM), (qAn).*

Proof:
(qBCI-1) $\Longrightarrow$ (qBCI-2): It is sufficient to prove that (B), (C) hold. Indeed,
- by (qA00), (qM) $\Longrightarrow$ (qN); by (qA20), (BB) + (D) + (qN) $\Longrightarrow$ (C);
- by (qA3), (C) + (qM) + (qAn) $\Longrightarrow$ (Ex);
- by (A10"), (Ex) + (BB) $\Longrightarrow$ (B).

(qBCI-2) $\Longrightarrow$ (qBCI-1): It is sufficient to prove that (BB), (D) hold. Indeed,
- by (qA3), (C) + (qM) + (qAn) $\Longrightarrow$ (Ex); by (A10'), (Ex) + (B) $\Longrightarrow$ (BB);
- by (A4), (Re) + (Ex) $\Longrightarrow$ (D). $\square$

Hence, we have the following definition:

Definition 5.4. Let $\mathcal{A} = (A, \rightarrow, 1)$ be an algebra of type $(2, 0)$ (or, equivalently, let $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a structure). $\mathcal{A}$ is called a *quasi-BCI algebra* (or a *qBCI algebra*, for short) if one of the two above equivalent groups of properties is satisfied: (qBCI-1) or (qBCI-2).

Definition 5.5. A quasi-BCI algebra $(A, \rightarrow, 1)$ is *p-semisimple* if for each $x \in A$, $x \leq 1$ implies $x \parallel 1$.

Note that:
- a quasi-BCI algebra is a quasi-algebra, since (qM) and (11-1) hold;
- a quasi-BCI algebra is quasi-subtractive, by (qA18"), (qAA3);
- if condition (M) holds, then a (p-semisimple) quasi-BCI algebra is a (p-semisimple) BCI algebra, by Theorem 4.2 (vi);
- for any proper (p-semisimple) quasi-BCI algebra $\mathcal{A}$ (i.e. (qM) is not (M)), the regular algebra $\mathcal{R}(\mathcal{A})$ is a regular (p-semisimple) BCI algebra, by Theorem 2.12.

It remains an open problem to check if the p-semisimple quasi-BCI algebras are categorically equivalent with the commutative quasi-groups (introduced in [6]), as happens in the regular case.

We do not develop further this subsection, we let this for future research.

5.2 Quasi-BCK Algebras

Theorem 5.6. *Let* $\mathcal{A} = (A, \rightarrow, 1)$ *be an algebra of type* $(2, 0)$ *(or, equivalently, let* $\mathcal{A} = (A, \leq, \rightarrow, 1)$ *be a structure), as before. Then, the following*

two groups of quasi-properties are equivalent:
(qBCK-1) (BB), (D), (Re), (L), (qM), (qAn) and
(qBCK-2) (B), (C), (K), (qM), (qAn).

Proof:

(qBCK-1) $\Longrightarrow$ (qBCK-2): It is sufficient to prove that (B), (C), (K) hold. Indeed,
- by (qA20'), (BB) + (D) + (qM) $\Longrightarrow$ (C);
- by (qA3), (C) + (qM) + (qAn) $\Longrightarrow$ (Ex);
- by (A8), (Re) + (L) + (Ex) $\Longrightarrow$ (K);
- by (A10"), (BB) + (Ex) $\Longrightarrow$(B).

(qBCK-2) $\Longrightarrow$ (qBCK-1): It is sufficient to prove that (BB), (D), (Re), (L) hold. Indeed,
- by (qA7'), (K) + (qM) $\Longrightarrow$ (L);
- by (qA3), (C) + (qM) + (qAn) $\Longrightarrow$ (Ex); by (A10'), (B) + (Ex) $\Longrightarrow$ (BB);
- by (qA22), (K) + (Ex) + (qM) $\Longrightarrow$ (Re);
- by (A4), (Re) + (Ex) $\Longrightarrow$ (D). $\square$

Hence, we have the following definition:

Definition 5.7. Let $\mathcal{A} = (A, \to, 1)$ be an algebra of type $(2, 0)$ (or, equivalently, let $\mathcal{A} = (A, \leq, \to, 1)$ be a structure). $\mathcal{A}$ is called a *quasi-BCK algebra* (or a *qBCK algebra*, for short) if one of the two above equivalent groups of properties is satisfied: (qBCK-1) or (qBCK-2).

Note that quasi-BCK algebras verify all the quasi-properties in List qA; indeed, we have (qM($1 \to x$)), by Theorem 4.2 (ii); (qN), by (qA00); (qN($1 \to x$)), by Theorem 4.2 (iii); (qRe), (qRe($1 \to x$)), by Theorem 4.2 (iv); (qL), (qL($1 \to x$)), by Theorem 4.2 (v);

(11-1), by (qAA1); (*), by (qA12): (**), by (qA15); (Ex), by (qA3); (S), by (A0); (Tr), by (qA14); (#), by (qA27); (##), by (A28); (qR), by (qAA3); (qR1), by (qA18"); (qR2), by (qAA4); (qR3), by (qAA5); (qI1), by (qAA7'); (qI2), by (qAA8); (qI3), by (qAA8); (qI), by (qAA9).

Note that:
- a quasi-BCK algebra is a quasi-algebra, since (qM) and (11-1) hold;
- a quasi-BCK algebra is quasi-subtractive, by (qA18"), (qAA3);
- if condition (M) holds, then a quasi-BCK algebra is a BCK algebra, by Theorem 4.2 (vi);
- for any proper quasi-BCK algebra $\mathcal{A}$ (i.e. (qM) is not (M)), the regular algebra $\mathcal{R}(\mathcal{A})$ is a regular BCK algebra, by Theorem 2.12.

Denote by **qRM**, **qRML**, **qBCI**, **qBCK** the classes of quasi-RM algebras, of quasi-RML algebras, of quasi-BCI algebras and of quasi-BCK algebras, respectively. We have then the expected Hierarchy 2 from Figure 3.

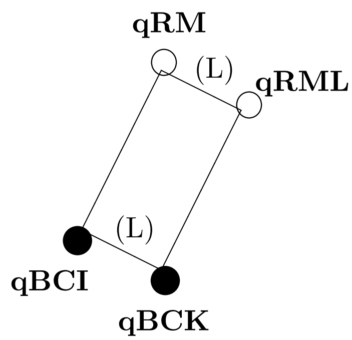


Figure 3: Hierarchy 2

6 Examples of Finite Quasi-Algebras

6.1 Examples of Quasi-RM and Quasi-RML Algebras

Example 6.1. Starting from the regular RM algebra $\mathcal{A} = (A = \{a, b, 1\}, \rightarrow, 1)$ from [9], represented by the Hasse-type diagram from Figure 4 and with the table of $\rightarrow$ given below, we obtain the proper quasi-RM algebra $\mathcal{A}' = (A' = \{a, b, x, 1\}, \rightarrow, 1)$, represented by the quasi-Hasse-type diagram given also in Figure 4 and with the table of $\rightarrow$ given below. Note that only properties (qM) and (Re) hold; (L) does not hold for a; (Ex) does not hold for a, b, a; (qAn) does not hold for a, b; (BB), (**), (B), (*), (Tr) do not hold for $a, b, 1$; (D) does not hold for $1, a$. Note that $R(A') = A$.

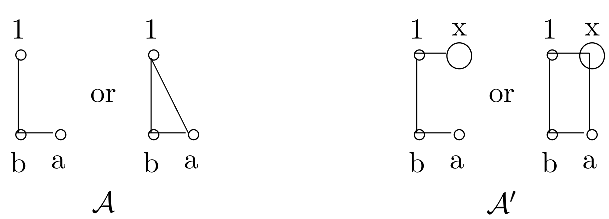


Figure 4: The Hasse-type diagram of the regular RM algebra $\mathcal{A}$ and the quasi-Hasse-type diagram of the quasi-RM algebra $\mathcal{A}'$

$\mathcal{A}$

$\to$	a	b	1
a	1	1	b
b	1	1	1
1	a	b	1

$\mathcal{A}'$

$\to$	a	b	x	1
a	1	1	b	b
b	1	1	1	1
x	a	b	1	1
1	a	b	1	1

Example 6.2. Starting from the regular RML algebra $\mathcal{A} = (A = \{a, b, c, 1\}, \to, 1)$ from [9], represented by the Hasse-type diagram from Figure 5 and with the table of $\to$ given below, we obtain the proper quasi-RML algebra $\mathcal{A}" = (A" = \{a, b, c, x, 1\}, \to, 1)$, represented by the quasi-Hasse-type diagram given also in Figure 5 and with the table of $\to$ given below. Note that properties (qM), (Re), (L), (D) hold; (Ex) does not hold for a, b, b; (qAn) does not hold for b, c; (BB), (**) do not hold for a, b, c; (B), (*), (Tr) do not hold for b, c, a. Note that $R(A") = A$.

Figure 5: The Hasse-type diagram of the regular RML algebra $\mathcal{A}$ and the quasi-Hasse-type diagram of the quasi-RML algebra $\mathcal{A}"$

$\mathcal{A}$

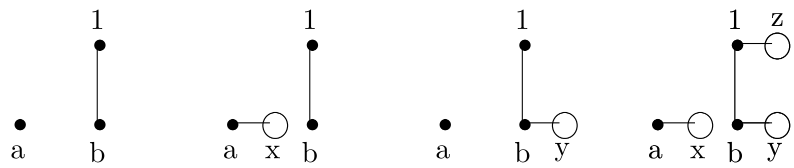
$\to$	a	b	c	1
a	1	a	a	1
b	a	1	1	1
c	1	1	1	1
1	a	b	c	1

$\mathcal{A}"$

$\to$	a	b	c	x	1
a	1	a	a	1	1
b	a	1	1	1	1
c	1	1	1	1	1
x	a	b	c	1	1
1	a	b	c	1	1

6.2 Examples of Quasi-BCI Algebras

Example 6.3. Starting from the regular BCI algebra $\mathcal{A} = (A = \{a, b, 1\}, \to, 1)$, represented by the Hasse diagram given in Figure 6 and with the table of $\to$ given below, we obtain the quasi-BCI algebras $\mathcal{A}^1 = (A^1 = \{a, b, x, 1\}, \to, 1)$, $\mathcal{A}^2 = (A^2 = \{a, b, y, 1\}, \to, 1)$, $\mathcal{A}^3 = (A^3 = \{a, b, x, y, z, 1\}, \to, 1)$, represented by the quasi-Hasse diagrams given also in Figure 6 and with the corresponding tables of $\to$ given below. Note that $R(A^1) = R(A^2) = R(A^3) = A$.



$\mathcal{A}$ BCI algebra $\mathcal{A}^1$ qBCI algebra $\mathcal{A}^2$ qBCI algebra $\mathcal{A}^3$ qBCI algebra

Figure 6: The BCI algebra $\mathcal{A}$ and the quasi-BCI algebras $\mathcal{A}^1$, $\mathcal{A}^2$, $\mathcal{A}^3$

$\mathcal{A}$

$\rightarrow$	**a**	**b**	**1**
a	**1**	**a**	**a**
b	**a**	**1**	**1**
1	**a**	**b**	**1**

$\rightarrow$	**a**	**b**	x	**1**
a	**1**	**a**	1	**a**
b	**a**	**1**	a	**1**
x	1	a	1	a
1	**a**	**b**	a	**1**

$\mathcal{A}^1$

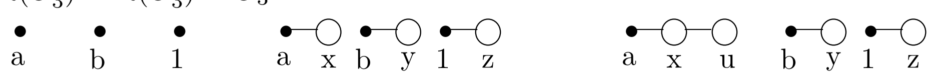
$\rightarrow$	**a**	**b**	y	**1**
a	**1**	**a**	a	**a**
b	**a**	**1**	1	**1**
y	a	1	1	1
1	**a**	**b**	b	**1**

$\mathcal{A}^2$

$\rightarrow$	**a**	**b**	x	y	z	**1**
a	**1**	**a**	1	a	a	**a**
b	**a**	**1**	a	1	1	**1**
x	1	a	1	a	a	a
y	a	1	a	1	1	1
z	a	b	a	b	1	1
1	**a**	**b**	a	b	1	**1**

$\mathcal{A}^3$

Example 6.4. Starting from the regular p-semisimple BCI algebra $\mathcal{G}_3 = (G_3 = \{a, b, 1\}, \rightarrow, 1)$, represented by the Hasse diagram given in Figure 7 and with the table of $\rightarrow$ given below, we obtain the p-semisimple quasi-BCI algebras $\mathcal{G}_3^1 = (G_3^1 = \{a, b, x, y, z, 1\}, \rightarrow, 1)$ and $\mathcal{G}_3^2 = (G_3^2 = \{a, b, x, u, y, z, 1\}, \rightarrow, 1)$, represented by the quasi-Hasse diagrams given also in Figure 7 and with the corresponding tables of $\rightarrow$ given below. Note that $R(G_3^1) = R(G_3^2) = G_3$.



$\mathcal{G}_3$ BCI algebra $\mathcal{G}_3^1$ qBCI algebra $\mathcal{G}_3^2$ qBCI algebra

Figure 7: The p-semisimple BCI algebra $\mathcal{G}_3$ and the associated p-semisimple qBCI algebras $\mathcal{G}_3^1$, $\mathcal{G}_3^2$

→	**a**	**b**	**1**
a	**1**	**a**	**b**
b	**b**	**1**	**a**
1	**a**	**b**	**1**

$\mathcal{G}_3$

→	**a**	**b**	x	y	z	**1**
a	**1**	**a**	1	a	b	**b**
b	**b**	**1**	b	1	a	**a**
x	1	a	1	a	b	b
y	b	1	b	1	a	a
z	a	b	a	b	1	1
1	**a**	**b**	a	b	1	**1**

$\mathcal{G}_3^1$

→	**a**	**b**	x	u	y	z	**1**
a	**1**	**a**	1	1	a	b	**b**
b	**b**	**1**	b	b	1	a	**a**
x	1	a	1	1	a	b	b
u	1	a	1	1	a	b	b
y	b	1	b	b	1	a	a
z	a	b	a	a	b	1	1
1	**a**	**b**	a	a	b	1	**1**

$\mathcal{G}_3^2$

Remark 6.5. If we start from the above p-semisimple regular BCI algebra $\mathcal{G}_3$, having three regular elements, then note that **we cannot obtain any quasi-BCI algebra by adding elements parallel only to one or two of the elements of** G_3, because the property (BB), for example, is no more verified.

6.3 Example of Quasi-BCK Algebra

Example 6.6. Starting from the regular BCK algebra $\mathcal{A} = (A = \{a, b, c, d, 1\}, \to, 1)$, represented by the Hasse diagram given in Figure 8 and with the table of $\to$ given below, we obtain the associated quasi-BCK algebra $\mathcal{A}^1 = (A^1 = \{a, b, c, d, m, 1\}, \to, 1)$, represented by the quasi-Hasse diagram given also in Figure 8 and with the table of $\to$ given below. Note that $R(A^1) = A$.

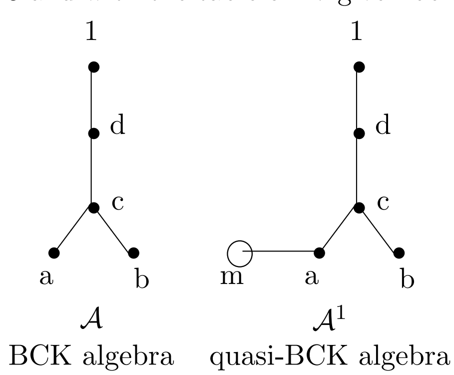


Figure 8: The regular BCK algebra $\mathcal{A}$ and an associated quasi-BCK algebra, $\mathcal{A}^1$

$\mathcal{A}$

$\rightarrow$	**a**	**b**	**c**	**d**	**1**
a	**1**	**b**	**1**	**1**	**1**
b	**a**	**1**	**1**	**1**	**1**
c	**a**	**b**	**1**	**1**	**1**
d	**a**	**b**	**d**	**1**	**1**
1	**a**	**b**	**c**	**d**	**1**

$\mathcal{A}^1$

$\rightarrow$	**a**	**b**	**c**	**d**	m	**1**
a	**1**	**b**	**1**	**1**	1	**1**
b	**a**	**1**	**1**	**1**	a	**1**
c	**a**	**b**	**1**	**1**	a	**1**
d	**a**	**b**	**d**	**1**	a	**1**
m	1	b	1	1	1	1
1	**a**	**b**	**c**	**d**	a	**1**

Other examples of quasi-BCK algebras will be presented in the next papers.

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