A Precise Characterisation of Step Traces and Their Concurrent Histories

Ryszard Janicki[1], Jetty Kleijn[2] and Łukasz Mikulski[3]

Abstract

Step traces are an extension of Mazurkiewicz traces where each equivalence class (trace) consists of sequences of steps instead of sequences of atomic actions. Relations between the actions of the system are defined statically, as parameters of a concurrent step alphabet. By allowing only some of the possible relationships between actions, subclasses of step alphabets can be derived in a natural way. Properties of these classes can then be investigated in terms of invariant structures, i.e., the relational structures that represent the causal invariants that underlie the corresponding step traces.

In this paper, we refine an earlier classification of subclasses of step alphabets and add eight new subclasses to this hierarchy. We divide these eight classes into three families on basis of the absence of a specific behavioural relation and then characterise the corresponding invariant structures.

Keywords: step alphabet, trace of step sequences, simultaneity, serialisability, interleaving, classification, invariant structure

1 Introduction

Step traces [4] are an extension of the classical Mazurkiewicz traces, a basic and well-established model to represent concurrent behaviour [1, 12, 13, 14].

---

[1]Department of Computing and Software, McMaster University, Hamilton, ON, L8S 4K1, Canada, Email: `janicki@mcmaster.ca`

[2]LIACS, Leiden University, P.O.Box 9512, NL-2300 RA Leiden, The Netherlands, Email: `h.c.m.kleijn@liacs.leidenuniv.nl`

[3]Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Toruń, Chopina 12/18, Poland, Email: `lukasz.mikulski@mat.umk.pl`

Whereas a Mazurkiewicz trace is an equivalence class of sequences of action names that can be seen as representing all (sequential) observations of a run of a concurrent system, a step trace consists of step sequences. Such sequences are a concatenation not of single occurrences of actions, but rather of *steps*, i.e., sets of one or more actions that occur (or are observed as occurring) simultaneously. Some recent examples of the application of step traces can be found in computational biology [18], digital graphics [17], and model checking [10].

For Mazurkiewicz traces, the equivalence of sequences is based on a binary *independence* relation stating which pairs of actions are independent and thus can be observed in any order. Also the equivalence of step sequences forming a step trace is based on binary relations between a system's actions. First of all, there is *simultaneity* indicating that two actions may occur together in a step; secondly, *serialisability* specifies possible execution orders for potentially simultaneous actions; thirdly, *interleaving* declares for actions that cannot occur simultaneously that no specific ordering is required. The latter two relations can also be captured in terms of a single *sequentialisability* relation, see [4, 7]. This then leads to a notion of a *step alphabet* consisting of a finite set of symbols (action names) and two binary relations, simultaneity and sequentialisability.

The sequences forming a Mazurkiewicz trace, share an underlying acyclic structure based on the dependencies between their actions that are common to all elements of the trace. This dependence graph defines through its transitive closure a labelled partial order which can be seen as the (invariant) causality structure of the concurrent run captured by the trace [19]. Moreover, each partial order is the intersection of all its linearisations, i.e., saturations of the partial order that preserve the acyclicity. This latter property (Szpilrajn's property [20]) forms the link between the elements of a trace and its associated partial order: the linearisations of the partial order correspond exactly to the sequences forming the trace.

For step traces, obviously, more general dependence and causal structures are needed to describe the invariant relationships between action occurrences. The relational structures studied in [2, 3] and used in [4] to describe the causality in step traces, have — instead of a single strict partial order (causality) relation — two relations: a 'not later than' relation to represent *weak causality* (i.e., before or in the same step) and a 'mutual exclusion' *mutex* relation for pure interleaving (not allowed in the same step but not necessarily causally ordered). *Order structures* are (labelled)

relational structures satisfying certain additional properties, in particular a generalised acyclicity property. Moreover, the saturated versions of an order structure (i.e., the maximal extensions that respect this acyclicity property) correspond to step sequences. As demonstrated in [2, 3] there is a closure operator that, when applied to an order structure, yields an *invariant structure*. Invariant structures are order structures which can be considered as generalised partial orders in the sense that they satisfy a generalised Szpilrajn's property: each invariant structure is the intersection of all its saturations. In [4], it is moreover demonstrated how to capture the intrinsic dependencies in a step sequence (over a given step alphabet). Furthermore, equivalent step sequences generate the same dependence structure; every step sequence corresponds to a saturated version of its dependence structure; and, finally, all step sequences obtained by saturating a dependence structure are equivalent. Hence we may conclude that, in the context of step traces, dependence structures and invariant structures are the counterparts of the dependence graphs and partial orders of Mazurkiewicz traces.

As argued in [4], and following [8], invariant structures represent the most general concurrent histories satisfying Szpilrajn's property. So, step alphabets and step traces with their simultaneity and sequentialisability relations, are the most general in terms of their underlying order structures. On the other hand, the definition of step alphabets almost automatically leads to a hierarchy of step alphabets, depending on which combinations of simultaneity and sequentialisability are allowed. In [7], eight subclasses of step alphabets are distinguished. These include a class corresponding to Mazurkiewicz traces; step alphabets that combine the independence relation of Mazurkiewicz traces with step sequences (and simultaneity is the same as serialisability), and a class of step alphabets that leads to comtraces [9, 11]. For five of them, their corresponding invariant order structures have been characterised. In this paper, we apply a finer distinction and complete the picture by introducing eight more subclasses. We briefly discuss their semantical meaning and then single out three specific ones for further investigation.

The paper is organised as follows. After some preliminaries, we formally introduce step traces in Section 3. Next, in Section 4, we present the extended hierarchy of subclasses of step alphabets. In Section 5 we discuss general order structures and provide an axiomatic charaterisation of invariant structures and explain how they can also be obtained through closure of order structures. We demonstrate how to associate with each step trace a

unique order structure, the closure of which is the invariant structure of the trace. In Section 6 we divide the new subclasses of step alphabets into three families and characterise for each of these families the corresponding class of invariant structures. Finally in the concluding Section 7, we summarise what has been done and what is still to be done.

2 Preliminaries

In this section, we introduce basic terminology and notation that will be used later in this paper.

Given a binary relation $R \subseteq X \times X$, we denote the inverse of R by R^{-1}; its symmetric closure is $R^{sym} = R \cup R^{-1}$. Relation R is a strict partial order relation if it is irreflexive and transitive; R is a total order if it is a strict partial order relation such that $R^{sym} = (X \times X) \setminus id_X$. Here id_X denotes the identity relation over X. We define $R^0 = id_X$ and $R^n = R^{n-1} \circ R$, for all $n \geq 1$. Then: $R^+ = \bigcup_{i \geq 1} R^i$ and R is acyclic if R^+ is irreflexive; $R^* = \bigcup_{i \geq 0} R^i = R^+ \cup id_X$; $R^{\wedge} = R^+ \setminus id_X = R^* \setminus id_X$ is the irreflexive transitive closure of R; and $R^{\circledast} = R^* \cap (R^*)^{-1}$ is the largest equivalence relation contained in R^*.

A labelled *partial order* $\langle \Delta, R, \ell \rangle$ consists of a finite set Δ, a strict partial order relation R on Δ, and a labelling $\Delta \xrightarrow{\ell} \Sigma$ where Σ is a finite set of labels. We refer to Δ as the *domain* of the partial order. As we will deal only with labelled partial orders and strict partial order relations, we will mostly speak simply of partial orders and partial order relations.

Throughout the paper, $\Sigma \neq \varnothing$ is a finite *alphabet* of actions, $\mathbb{S} = 2^{\Sigma} \setminus \{\varnothing\}$ is the set of all *steps*, and $\mathbb{S}^*$ is the set of *step sequences* $u = A_1 \dots A_k$ with $A_j \in \mathbb{S}$ for $1 \leq j \leq k$. If $k = 0$, then u is the *empty step sequence*. A step containing a, b and c is denoted by (abc) rather than $\{a, b, c\}$.

Let $u = A_1 \dots A_k \in \mathbb{S}^*$ be a step sequence. For each action $a \in \Sigma$, let $\#(a, u)$ denote the number of occurrences of a in u. Then $occ(u) = \{\langle a, i \rangle \mid a \in \Sigma \wedge 1 \leq i \leq \#(a, u)\}$ is the set of *action occurrences* of u. We let $pos_u(\alpha)$ denote the *position* of action occurrence $\alpha = \langle a, i \rangle \in occ(u)$ in u, formally defined as the smallest index $j \leq k$ such that the number of occurrences of a within $A_1 \dots A_j$ is exactly i. Note that in case u is the empty step sequence, we have $\#(a, u) = 0$, for all $a \in \Sigma$, and $occ(u) = \varnothing$.

Let EQ be a finite set of equations on step sequences, with each equation being of the form $u = v$, where $u, v \in \mathbb{S}^*$ are both nonempty. Then EQ defines a relation $\approx$ on step sequences comprising all pairs $\langle tuw, tvw \rangle$ such

that either $t, w \in \mathbb{S}^*$, and $u = v$ or $v = u$ is an equation in EQ, or u and v are both the empty sequence. Then $\equiv\, =\, \approx^*$ is the equivalence relation on $\mathbb{S}^*$ induced by EQ.

3 Step Traces

Now we are ready to define step traces as equivalence classes of step sequences induced by equations based on two relations between actions: *simultaneity* (sim) and *sequentialisability* (seq). The first defines pairs of potentially simultaneous actions, i.e., pairs of actions that may occur together in a step. Note that sim does not enforce simultaneity; two actions that form a pair in sim may occur simultaneously, but they do not have to. The second relation, *sequentialisability* specifies pairs of actions whose simultaneous occurrence in a step means that they may also occur one after another in the order given and whose occurrences may be swapped in a step sequence (provided sequentialisability is symmetric for them).

A *step alphabet* is a triple $\theta = \langle \Sigma, \mathsf{sim}, \mathsf{seq} \rangle$, where $\mathsf{sim}, \mathsf{seq} \subseteq \Sigma \times \Sigma$ are irreflexive; moreover, sim and $\mathsf{seq} \setminus \mathsf{sim}$ are symmetric. The family of all step alphabets will be denoted by Θ. Simultaneity defines $\mathbb{S}_\theta = \{A \subseteq \Sigma \mid A \neq \varnothing \wedge (A \times A) \setminus id_\Sigma \subseteq \mathsf{sim}\}$, the set of (legal) *steps over* θ. The set $\mathsf{SSEQ}_\theta = \mathbb{S}_\theta^*$ consists of all *step sequences over* θ. Sequentialisability, on the other hand, identifies pairs of actions which can be interleaved and defines ways in which steps can be serialised. This leads to the following *equations over* θ, where $A, B \in \mathbb{S}_\theta$:

$$\begin{array}{llll} AB = BA & \text{if} & A \times B \subseteq \mathsf{seq} \cap \mathsf{seq}^{-1} & (interleaving), \\ AB = A \cup B & \text{if} & A \times B \subseteq \mathsf{sim} \cap \mathsf{seq} & (serialisability). \end{array}$$

Note that it follows from the irreflexivity of sim and seq, that the sets A and B in these equations are disjoint.

Example 1 *Consider $\theta_0 = \langle \{a, b, c, d, e, f\}, \mathsf{sim}, \mathsf{seq} \rangle$, a step alphabet with its simultaneity and sequentialisability relations as given in Figure 1 where each undirected edge stands for two arrows in opposite directions. The step alphabet θ_0 generates, e.g., the interleaving equations $ad = da$ and $d(ac) = (ac)d$; it also generates the serialisability equations $(ac) = ac$, $(ac) = ca$, and $(ab) = ba$. However, $(ab) = ab$ is not an equation over θ_0, since $(a, b) \notin \mathsf{seq}$.* ◇
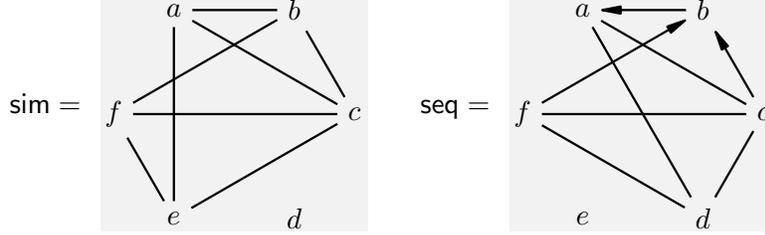
Figure 1:   The step alphabet θ_0.

These equations define a relation $\approx$ such that for two nonempty step sequences u, v we have $u \approx v$ if there exist $w, t \in \mathbb{S}^*$ and $A, B \in \mathbb{S}$ satisfying, respectively: (i) $u = wABt$ and $v = wBAt$ and $AB = BA$; or (ii) $u = wABt$ and $v = w(A \cup B)t$ or $v = wABt$ and $u = w(A \cup B)t$, and $AB = A \cup B$.

Let $\equiv \; = \approx^*$. The equivalence classes of $\equiv$ that contain a step sequence from SSEQ_θ are the *step traces over* θ. We denote by STR_θ the set of all step traces over θ. It is important to observe at this point that *all* step sequences forming a step trace are sequences of legal steps over θ, thus each such sequence is an element of $\mathbb{S}_\theta^*$ i.e., if $\tau \in \mathsf{STR}_\theta$ then $\tau \subseteq \mathsf{SSEQ}_\theta$. The trace containing $u \in \mathsf{SSEQ}_\theta$ will be denoted by $[\![u]\!]$. For a step trace $\tau = [\![u]\!] \in \mathsf{STR}_\theta$, where u is a step sequence over θ, we use $occ(\tau) = occ(u)$ to denote the set of action occurrences in τ (note that this is well-defined, as all step sequences in τ have the same set of action occurrences).

Example 2 *The step alphabet θ_0 is as before in Example 1. Thus we have that f and b can occur as a step (fb) and be sequentialised to the equivalent fb, but not to bf; and similarly for b and a. Since f and a are neither related by* sim *nor* seq *we thus obtain the first step trace $[\![(fb)a]\!]$ over θ_0 in the list below. The elements of the other step traces can be found in the same way, using* sim *and* seq.

$$
\begin{array}{llll}
[\![(fb)a]\!] & = & \{(fb)a, fba, f(ba)\} & [\![adf]\!] & = & \{adf, afd, daf\} \\
[\![acf]\!] & = & \{acf, caf, (ac)f, afc, a(cf)\} & [\![d(bc)]\!] & = & \{d(bc), dcb, cdb\} \\
[\![(abc)]\!] & = & \{(abc), c(ab), (bc)a, cba\} & [\![(ace)]\!] & = & \{(ace)\} \\
[\![acd]\!] & = & \{acd, adc, cad, cda, dac, dca, (ac)d, d(ac)\} \; .
\end{array}
$$

$\diamond$

4 Subclasses of Step Alphabets

Recall that step alphabets are triples $\theta = \langle \Sigma, \mathsf{sim}, \mathsf{seq} \rangle$ and that sim and $\mathsf{seq} \setminus \mathsf{sim}$ are symmetric. Consequently, $\mathsf{seq} \setminus \mathsf{sim} = \mathsf{seq}^{-1} \setminus \mathsf{sim}$ and $\mathsf{seq} \setminus (\mathsf{sim} \cup \mathsf{seq}^{-1}) = \mathsf{seq}^{-1} \setminus (\mathsf{sim} \cup \mathsf{seq}) = \varnothing$. In Figure 2 we sketch the partition of $\Sigma \times \Sigma$ in the form of a Venn diagram.
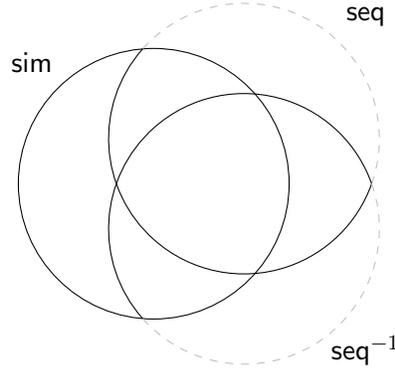

Figure 2: The relations defined by a step alphabet.

In [7], subfamilies of step traces have been investigated based on a classification of step alphabets, defined by assuming that one or more of the relations $\mathsf{sim} \setminus \mathsf{seq}$, $\mathsf{seq} \setminus \mathsf{sim}$, and $\mathsf{sim} \cap \mathsf{seq}$ are empty. This led to eight classes of step alphabets. In this paper, we extend this classification by refining the partition of $\Sigma \times \Sigma$. First, we distinguish six additional possible relationships between pairs of actions.

The six relations are the following:

- *strong simultaneity*: $\mathsf{ssi} = \mathsf{sim} \setminus (\mathsf{seq} \cup \mathsf{seq}^{-1})$,

- *semi-serialisability*: $\mathsf{sse} = \mathsf{seq} \setminus \mathsf{seq}^{-1}$,

- *weak dependence*: $\mathsf{wdp} = \mathsf{seq}^{-1} \setminus \mathsf{seq}$,

- *concurrency*: $\mathsf{con} = \mathsf{sim} \cap \mathsf{seq} \cap \mathsf{seq}^{-1}$,

- *interleaving*: $\mathsf{inl} = \mathsf{seq} \setminus \mathsf{sim}$,

- *rigid order*: $\mathsf{rig} = (\Sigma \times \Sigma) \setminus (\mathsf{sim} \cup \mathsf{seq})$.

As argued in [4], these relations have a clear semantical meaning. Strong simultaneity, ssi, allows a pair of actions to be executed simultaneously, but disallows to sequentialise them when they occur in a step. The relation sse, semi-serialisability, allows a pair of simultaneously executed actions to be executed sequentially in the order given, but not in the reverse order. Weak dependence, wdp, is the reverse of semi-serialisability, while the concurrency relation, con, allows actions to be executed simultaneously as well as in any order. The relation inl allows to swap occurrences of actions. Note that, as illustrated in Figure 2, $\mathsf{seq} \setminus \mathsf{sim} = (\mathsf{seq} \cap \mathsf{seq}^{-1}) \setminus \mathsf{sim}$ and 'interleaving' here refers to individual occurrences of actions, whereas the interleaving equations in Section 3 on the other hand, are based on $\mathsf{seq} \cap \mathsf{seq}^{-1}$ and relate to the swapping of complete steps. The last relation, rigid order rig, allows neither simultaneity nor changing the order of actions.

The technical usefulness of distinguishing such relations has been demonstrated e.g., in [15], where comtraces (i.e., step traces over step alphabets with an empty inl relation) are considered. The investigation of the other five relationships has led to an alternative compact version of the invariant structures representing the causal invariants that underlie step traces, based on projections of a step trace on binary subalphabets. In [6], this decomposition is used in an efficient procedure to check whether a labelled invariant order structure is the invariant structure of a step trace and to synthesize suitable step alphabets.

Example 3 *Consider once more the step alphabet θ_0 from Example 1. The corresponding partition of $\{a, b, c, d, e, f\} \times \{a, b, c, d, e, f\}$ is illustrated in Figure 3.*   ⋄

Again we build our classification by assuming for each subclass that one or more of these six possible relations are empty. Before discussing these classes, however, we simplify the picture a bit. We do not consider the case that $\mathsf{rig} = \varnothing$, as $id_\Sigma \subseteq \mathsf{rig}$ due to the irreflexivity of sim and seq. Furthermore $\mathsf{sse} = \mathsf{seq} \setminus \mathsf{seq}^{-1} = \varnothing$ if and only if $\mathsf{wdp} = \mathsf{seq}^{-1} \setminus \mathsf{seq} = \varnothing$. Hence the cases that only one of these relations is empty, do not occur. In what follows we will not refer to sse anymore and always use wdp.

All this leads to sixteen relevant subclasses of step alphabets as outlined next. The subscripts indicate which relations are empty. Thus, for example, $\Theta_{\mathsf{inl} \cup \mathsf{con}}$ comprises all step alphabets such that $\mathsf{inl} \cup \mathsf{con} = \varnothing$. The corresponding areas in the diagram are rendered in a lighter shade of grey.
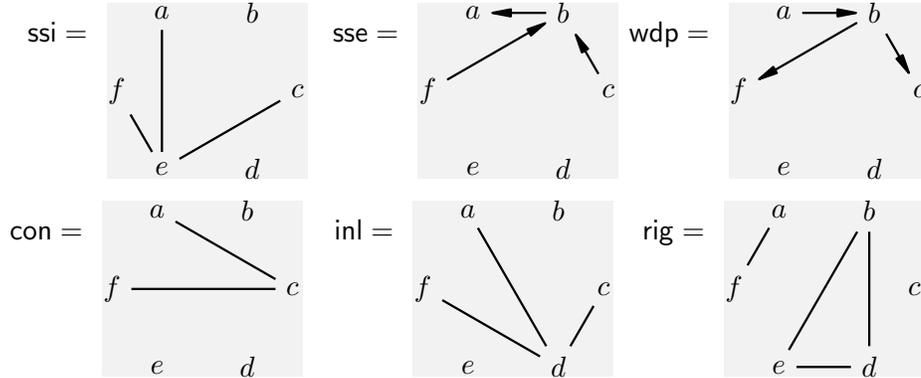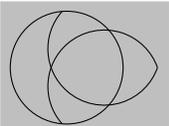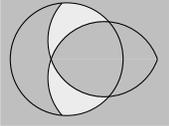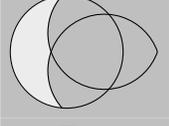
Figure 3:   The partition of step alphabet θ_0.

- Θ is the family of all step alphabets.

- Θ_{wdp} comprises alphabets where serialisability is symmetric.

- Θ_{ssi} comprises alphabets where every step with two elements may be split.
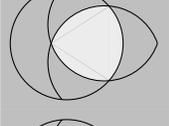
- Θ_{con} comprises alphabets where steps commute only through interleaving.
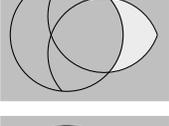
- Θ_{inl} comprises alphabets without true interleaving. Alphabets in Θ_{inl} (after dropping the empty relation inl) are also known as *comtrace alphabets* [8].
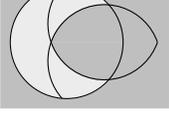
- $\Theta_{wdp\cup ssi}$ comprises alphabets with serialisability rich enough to split every step in each possible way.

- $\Theta_{\mathsf{wdp}\cup\mathsf{con}}$
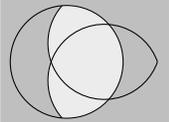
  comprises alphabets with only inter-leaving.

- $\Theta_{\mathsf{ssi}\cup\mathsf{con}}$
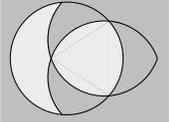
  comprises alphabets where every step with two elements may be split only in a unique way and commuting is only through interleaving.

- $\Theta_{\mathsf{wdp}\cup\mathsf{inl}}$
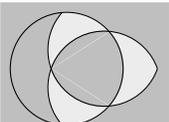
  comprises alphabets with symmetric serialisability and without true inter-leaving.

- $\Theta_{\mathsf{ssi}\cup\mathsf{inl}}$
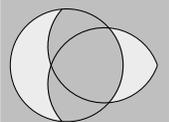
  comprises alphabets without true in-terleaving, where every step with two elements may be split.

- $\Theta_{\mathsf{con}\cup\mathsf{inl}}$
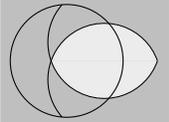
  comprises alphabets without commu-tativity.

- $\Theta_{\mathsf{wdp}\cup\mathsf{ssi}\cup\mathsf{con}}$
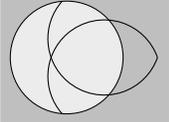
  comprises alphabets which do not al-low steps with two or more elements and know no serialisablity either. Al-phabets in $\Theta_{\mathsf{ssi}\cup\mathsf{wdp}\cup\mathsf{con}}$ after dropping the empty sim relation, correspond to *Mazurkiewicz concurrency alphabets* with inl as their independence rela-tion.

- $\Theta_{\mathsf{wdp}\cup\mathsf{ssi}\cup\mathsf{inl}}$
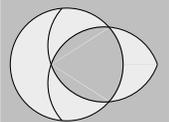
  comprises alphabets without inter-leaving equations, but serialisability is rich enough to split and reorder steps in every possible way. Al-phabets in $\Theta_{\mathsf{ssi}\cup\mathsf{wdp}\cup\mathsf{inl}}$ correspond to *Mazurkiewicz concurrency alphabets for step sequences*.

- $\Theta_{\mathsf{wdp}\cup\mathsf{con}\cup\mathsf{inl}}$
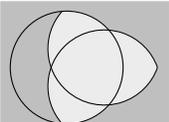
  comprises alphabets which generate step traces consisting of a single step sequence.

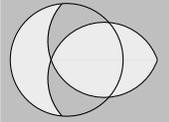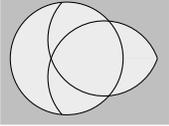- $\Theta_{\mathsf{ssi}\cup\mathsf{con}\cup\mathsf{inl}}$ comprises alphabets without commutation where every step with two elements may be split.

- $\Theta_{\mathsf{wdp}\cup\mathsf{ssi}\cup\mathsf{con}\cup\mathsf{inl}}$ comprises alphabets defining step traces consisting of a single sequence.

      Figure 4 shows all subclasses. The alphabets in $\Theta_{\mathsf{wdp}\cup\mathsf{ssi}\cup\mathsf{con}\cup\mathsf{inl}}$ and $\Theta_{\mathsf{wdp}\cup\mathsf{con}\cup\mathsf{inl}}$ are of little interest. Those in Θ, Θ_{inl}, $\Theta_{\mathsf{wdp}\cup\mathsf{ssi}}$, $\Theta_{\mathsf{wdp}\cup\mathsf{con}}$, $\Theta_{\mathsf{wdp}\cup\mathsf{ssi}\cup\mathsf{con}}$ and $\Theta_{\mathsf{wdp}\cup\mathsf{ssi}\cup\mathsf{inl}}$ are the ones that were studied in [7]. The eight others are as new subclasses, our subject of investigation in the next section.

5   Invariant structures

The relational structures $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ underlying the order theoretic treatment of step traces are determined by a finite set Δ, two binary relations $\rightleftharpoons$ and $\sqsubset$ on Δ, and a labelling $\Delta \xrightarrow{\ell} \Sigma$. The elements of Δ, called the *domain* of or, represent events (occurrences of actions) which are labelled by the name of their corresponding action. The relation $\rightleftharpoons$ is called the *mutex* relation and, intuitively, $x \rightleftharpoons y$ indicates that x and y did not occur simultaneously. The second relation $\sqsubset$ is *weak causality*: if $x \sqsubset y$, then x did not occur later than y (in other words x was before or simultaneous with y). All this implies that if both $x \rightleftharpoons y$ and $x \sqsubset y$, then x occurred before y and we will denote this also as $x \prec y$. We write $\prec$ for $\rightleftharpoons \cap \sqsubset$ and refer to this relation as *(strong) causality*.

      A relational structure $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ is an *order (relational) structure* if it is (*i*) *separable*, meaning that $\rightleftharpoons$ is symmetric, $\sqsubset$ is irreflexive, and $\rightleftharpoons \cap \sqsubset^{\circledast} = \varnothing$ (which implies that $\rightleftharpoons$ is also irreflexive); and (*ii*) *label-ordered*, meaning that any two distinct events x and y such that $\ell(x) = \ell(y)$ are related by both $\rightleftharpoons$ and $\sqsubset^{sym}$. Note that in this way we obtain a nice graphical representation of an order structure (with Δ as nodes, $\rightleftharpoons/\sqsubset$ as two types of edges/arcs and ℓ as a function which assigns labels to nodes). Intuitively, separability guarantees that any two elements that are in a cycle of weak causalities ($\sqsubset$) cannot be mutually exclusive ($\rightleftharpoons$).

      The *class of all order structures* will be denoted by OR.
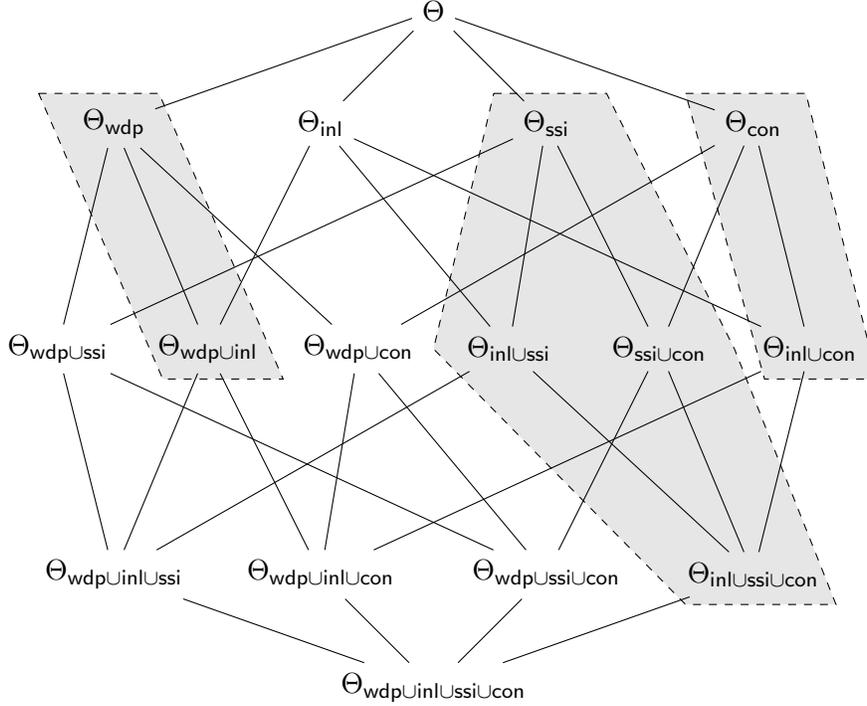
Figure 4: Hierarchy of the sixteen subclasses of step alphabets. When two classes are linked, the one higher up encompasses the lower one. The grey polygons show the three families of subclasses studied in Section 6.

There is a natural way (based on set theoretical inclusions) to order and intersect relational structures over the same Δ and with the same ℓ. We write $or_1 \leq or_2$ whenever $or_1 = \langle \Delta, \rightleftharpoons_1, \sqsubset_1, \ell \rangle$, $or_2 = \langle \Delta, \rightleftharpoons_2, \sqsubset_2, \ell \rangle$, and $\rightleftharpoons_1 \subseteq \rightleftharpoons_2$ and $\sqsubset_1 \subseteq \sqsubset_2$ (in other words or_2 is an extension of or_1). Moreover, $or_1 \cap or_2 = \langle \Delta, \rightleftharpoons_1 \cap \rightleftharpoons_2, \sqsubset_1 \cap \sqsubset_2, \ell \rangle$. We can now identify maximal elements w.r.t. $\leq$. These maximal order structures are referred to as *saturated (relational) structures*. The intuition behind this name is that adding any additional edge/arc to the graph representation of a saturated structure would destroy its separability. It is important to notice at this point that saturated structures correspond to step sequences (see e.g., [4]).

A relational structure $ir = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ is an *invariant (relational)*

structure if it satisfies, for all $x, y, z, z' \in \Delta$, each of the following axioms:

$$
\begin{array}{rclcll}
 & & & & x \not\sqsubset x & \text{(I1)} \\
x \neq y & \wedge & x \sqsubset z \sqsubset y & \implies & x \sqsubset y & \text{(I2)} \\
 & & x \rightleftharpoons y & \implies & y \rightleftharpoons x \neq y & \text{(I3)} \\
x \prec z \sqsubset y & \vee & x \sqsubset z \prec y & \implies & x \rightleftharpoons y & \text{(I4)} \\
z \rightleftharpoons y & \wedge & z \sqsubset x \sqsubset z & \implies & x \rightleftharpoons y & \text{(I5)} \\
z \rightleftharpoons z' \wedge x \sqsubset z \sqsubset y & \wedge & x \sqsubset z' \sqsubset y & \implies & x \rightleftharpoons y & \text{(I6)} \\
x \neq y & \wedge & \ell(x) = \ell(y) & \implies & x \prec^{sym} y & \text{(I7)}
\end{array}
$$

The *class of all invariant structures* will be denoted by IR.

The axiomatic characterisation of invariant order structures (for the unlabelled case) was introduced originally in [2, 3]. Like in [4], here we add axiom *(I7)* to capture label-orderedness. Note that by axiom *(I2)*, if $x \neq y$, $x \sqsubset^* y$, and $x \rightleftharpoons y$, then $x \prec y$. Hence instead of explicitly requiring label-orderedness, we use the more compact formulation of axiom *(I7)*.

Invariant structures are order structures (see [4, 7]). Actually, every order structure can be 'closed' by adding pairs to their mutex and weak causality relations to obtain the unique invariant structure that has the same set of saturated extensions. Intuitively, to obtain an invariant structure from an order structure, one should apply the implications in the axioms *(I2)* and *(I4)-(I6)* until these axioms are satisfied. Note that, because of the label-orderedness of order structures, axiom *(I7)* is satisfied from the start. Also the axioms *(I1)* and *(I3)* are satisfied initially and adding relations according to *(I2)* and *(I4)-(I6)* will preserve *(I1)* and *(I3)* thanks to the irreflexivity of $\sqsubset$, the symmetry of $\rightleftharpoons$, and separability.

However, the closure of an order structure can also be described by defining a new mutex and a new weak causality relation directly in terms of the original relations of the order structure. Axiom *(I2)* is then satisfied by the standard irreflexive and transitive closure of the weak causality relation $\sqsubset$ of the order structure. Simultaneously guaranteeing the three axioms*(I4)-(I6)* for the mutex relation is however more involved. In the closure mapping defined below, the first component of the new mutex relation contains the original mutex relation and guarantees axiom *(I5)*. Combining axiom *(I6)* with the transitive closure of weak causality gives rise to the new *cross* relation. It is not difficult to see that *cross* also guarantees that axiom *(I4)* will hold.

The *order structure closure* $\mathsf{OR} \xrightarrow{\text{or2ir}} \mathsf{IR}$ is a mapping, for every structure

$or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in$ OR, defined by:

$$\mathsf{or2ir}(or) = \langle \Delta, \sqsubset^{\circledast} \circ \rightleftharpoons \circ \sqsubset^{\circledast} \cup cross^{sym}, \sqsubset^{\wedge}, \ell \rangle$$

where $cross = \{\langle x, y \rangle \mid \exists z, z' : z \rightleftharpoons z' \wedge x \sqsubset^* z \sqsubset^* y \wedge x \sqsubset^* z' \sqsubset^* y\}$.

5.1   Step Traces and Invariant Structures

Given a step alphabet $\theta = \langle \Sigma, \mathsf{sim}, \mathsf{seq} \rangle$, the dependencies between the events underlying a step sequence in SSEQ_θ are given by the mapping $\mathsf{SSEQ}_\theta \xrightarrow{\mathsf{sseq2or}_\theta} \mathsf{OR}$ which is defined, for all $u \in \mathsf{SSEQ}_\theta$, by $\mathsf{sseq2or}_\theta(u) = \langle occ(u), \rightleftharpoons, \sqsubset, \ell \rangle$ where for all $\alpha, \beta \in occ(u)$ with $pos_u(\alpha) = k$, $pos_u(\beta) = m$:

$$
\begin{aligned}
\alpha \rightleftharpoons \beta \quad & if \quad \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{sim} \cap \mathsf{seq} \quad && \wedge \quad k < m \\
& or \quad \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{sim} \cap \mathsf{seq}^{-1} \quad && \wedge \quad k > m \\
\alpha \sqsubset \beta \quad & if \quad \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{seq} \cap \mathsf{seq}^{-1} \quad && \wedge \quad k < m \\
& or \quad \langle \ell(\alpha), \ell(\beta) \rangle \in \mathsf{sim} \setminus \mathsf{seq}^{-1} \quad && \wedge \quad k = m \, .
\end{aligned}
\tag{1}
$$

The relational structure $\mathsf{sseq2or}_\theta(u)$ is the *dependence structure of u (over θ)*. Later in the paper we want to be able to discuss relational structures that are relational structures associated with step sequences over a specific subclass X of step alphabets. Hence we define

$$\mathsf{OR}_X = \mathsf{sseq2or}_X(\mathsf{SSEQ}) = \bigcup_{\theta \in X} \mathsf{sseq2or}_\theta(\mathsf{SSEQ}_\theta).$$

and $\mathsf{IR}_X = \mathsf{or2ir}(\mathsf{OR}_X)$.

Finally, for a given step alphabet θ and a step sequence u over θ, order structure closure (the mapping $\mathsf{or2ir}$) can be used to obtain an invariant structure for u. As was proven in [3], $\mathsf{or2ir}(\mathsf{sseq2or}_\theta(u))$, the *invariant structure of u (over θ),* is not only the same for all step sequences forming the step trace $[\![u]\!]$, but also determines $[\![u]\!]$ as every saturated extension of $\mathsf{or2ir}(\mathsf{sseq2or}_\theta(u))$ corresponds to a step sequence equivalent with u and every step sequence equivalent with u corresponds to a saturated extension of $\mathsf{or2ir}(\mathsf{sseq2or}_\theta(u))$. Moreover, every invariant structure satisfies a generalised version of Szpilrajn theorem [20]: it is the intersection of its saturated extensions.

Example 4 *Consider the step alphabet θ_0 from Example 1 and Figure 1 and the step sequence $u = (ef)dcba$ over this alphabet. Then $occ(u) =$*
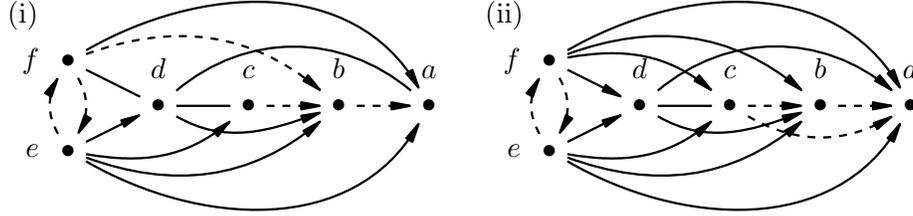
Figure 5: Dependence structure (i) $\mathsf{sseq2or}_{\theta_0}(u)$ and invariant structure (ii) $\mathsf{or2ir}(\mathsf{sseq2or}_{\theta_0}(u))$ of the step sequence $u = (ef)dcba$ for the step alphabet θ_0 from Figure 1. The relation $\rightleftharpoons$ is depicted using solid (undirected) edges, $\sqsubset$ by dashed arcs, and $\prec\, =\, \rightleftharpoons \cap \sqsubset$ by solid arcs.

$\{\langle a, 1\rangle, \langle b, 1\rangle, \langle c, 1\rangle, \langle d, 1\rangle, \langle e, 1\rangle, \langle f, 1\rangle\}$ *with* $pos_u(\langle a, 1\rangle) = 5$, $pos_u(\langle b, 1\rangle) = 4$, $pos_u(\langle c, 1\rangle) = 3$, $pos_u(\langle d, 1\rangle) = 2$ *and* $pos_u(\langle e, 1\rangle) = pos_u(\langle f, 1\rangle) = 1$.

The dependence structure of u is depicted in Figure 5(i). In this structure we have for instance $\langle f, 1\rangle \rightleftharpoons \langle d, 1\rangle$ *because* $\langle f, 1\rangle$ *is before* $\langle d, 1\rangle$ *in u, in other words* $pos_u(\langle f, 1\rangle) < pos_u(\langle d, 1\rangle)$, *while* $\langle f, d\rangle \notin \mathsf{sim} \cap \mathsf{seq}$ *and hence this ordering of the two occurrences cannot correspond to a simultaneous occurrence of f and d that was sequentialised.*

Similarly we have $\langle f, 1\rangle \rightleftharpoons \langle a, 1\rangle$*; moreover,* $\langle f, 1\rangle \sqsubset \langle a, 1\rangle$ *because* $\langle f, 1\rangle$ *is before* $\langle a, 1\rangle$ *and* $\langle f, a\rangle \notin \mathsf{seq} \cap \mathsf{seq}^{-1}$ *implying that these occurrences of f and a cannot be swapped. So we obtain* $\langle f, 1\rangle \prec \langle a, 1\rangle$.

Note that $\langle f, 1\rangle \sqsubset \langle e, 1\rangle \sqsubset \langle f, 1\rangle$ *because* $\langle f, 1\rangle$ *and* $\langle e, 1\rangle$ *are in the same step while* (e, f) *and* (f, e) *are in* $\mathsf{sim} \setminus \mathsf{seq}^{-1}$.

Next we consider the invariant structure of u obtained from its dependence structure by order structure closure. It is depicted in Figure 5(ii).

We now have $\langle f, 1\rangle \prec \langle d, 1\rangle$ *instead of only* $\langle f, 1\rangle \rightleftharpoons \langle d, 1\rangle$, *because the closure operator introduces* $\langle f, 1\rangle \prec \langle d, 1\rangle$ *from* $\langle f, 1\rangle \sqsubset \langle e, 1\rangle \sqsubset \langle d, 1\rangle$ *in the dependence structure.*

Another new relation is $\langle f, 1\rangle \rightleftharpoons \langle b, 1\rangle$. *This is added because we have in the dependence structure* $\langle f, 1\rangle \sqsubset \langle e, 1\rangle \prec \langle b, 1\rangle$ *which implies that* $\langle f, 1\rangle$ *cannot be later than* $\langle e, 1\rangle$ *which in its turn is earlier than* $\langle b, 1\rangle$*; thus having these occurrences of f and b in the same step is impossible.*

Finally, note that because $(f, c) \in \mathsf{sim} \cap \mathsf{seq}$, *we have no* $\sqsubset$ *relationship nor a* $\rightleftharpoons$ *relationship between the occurrences of f and c in the dependence structure of u. Order structure closure however adds* $\langle f, 1\rangle \prec \langle c, 1\rangle$, *because*

$\langle f, 1 \rangle \sqsubset \langle c, 1 \rangle$ *since* $\langle f, 1 \rangle \sqsubset \langle e, 1 \rangle \sqsubset \langle c, 1 \rangle$ *in the dependence structure; and* $\langle f, 1 \rangle \rightleftharpoons \langle c, 1 \rangle$ *since* $\langle f, 1 \rangle \sqsubset^{\circledast} \langle e, 1 \rangle \rightleftharpoons \langle c, 1 \rangle$ *in the dependence structure.* ⋄

6   Subclasses of Step Alphabets and Their Invariant Structures

In this section, we investigate what invariant structures underlie the step traces defined by the eight new subclasses of step alphabets identified in Section 4. We divide them into three families as indicated by the grey polygons in Figure 4.

The step alphabets that belong to the family consisting of Θ_{wdp} and $\Theta_{\mathsf{wdp} \cup \mathsf{inl}}$ know no weak dependence (nor semi-serialisability sse). It appears that only those two behave as regular as the classes investigated in [7]. It turns out that we can characterise the invariant structures associated with the step traces defined by these step alphabets (actually simplifying the axiomatisation of general invariant structures).

The second family of step alphabets we discuss, excludes strong simultaneity (but allows weak dependence). Exactly half of our eight new subclasses of alphabets belong to this family. For this family we identify two properties crucial in the definition of the dependence structures and invariant structures associated with their step traces.

Thirdly we consider step alphabets which may have non-empty weak dependence and strong simultaneity relations, but allow no true concurrent events, i.e., $\mathsf{con} = \varnothing$. The two subclasses of step alphabets considered are Θ_{con} and $\Theta_{\mathsf{con} \cup \mathsf{inl}}$. In these cases we establish an additional property for the dependence structures and invariant structures of their step traces.

6.1   Alphabets Without Weak Dependence

We present the main properties of alphabet classes from this family on the example of Θ_{wdp}.

A step alphabet $\theta \in \Theta_{\mathsf{wdp}}$ has $\mathsf{wdp} = \varnothing$, hence also $\mathsf{sse} = \varnothing$. As a result, we get the case where all the remaining relations between actions are symmetric.
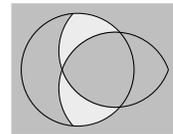
Example 5 *Consider $\theta_1 = \langle \{a, c, d, e, f\}, \mathsf{sim}, \mathsf{seq} \rangle$, a step alphabet with its simultaneity and sequentialisability relations given in Figure 6 where each undirected edge stands for two arrows in opposite*

directions. Some step traces over θ_1 are:

$$
\begin{aligned}
[\![acf]\!] &= \{acf, caf, (ac)f, afc, a(cf)\} & [\![adf]\!] &= \{adf, afd, daf\} \\
[\![cde]\!] &= \{cde, dce\} & [\![(ace)]\!] &= \{(ace)\} \\
[\![acd]\!] &= \{acd, adc, cad, cda, dac, dca, (ac)d, d(ac)\} \; .
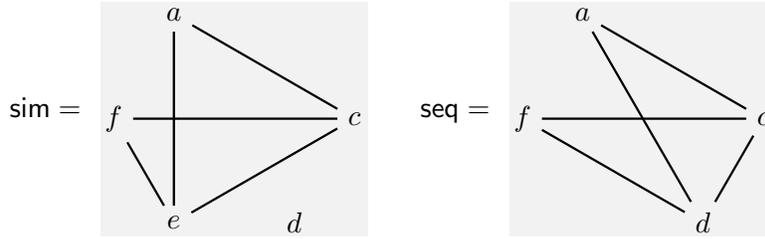\end{aligned}
$$

◇

Figure 6: The step alphabet θ_1.

On the level of dependence structures and events, this symmetry implies that we cannot observe one-direction weak causality by itself. Formally, we have $\mathsf{seq} = \mathsf{seq}^{-1}$ and so $\mathsf{sim} \cap \mathsf{seq} = \mathsf{sim} \cap \mathsf{seq}^{-1} = \mathsf{sim} \cap \mathsf{seq} \cap \mathsf{seq}^{-1} = \mathsf{con}$. Moreover, $\mathsf{inl} = (\Sigma \times \Sigma) \setminus (\mathsf{sim} \setminus \mathsf{seq}) = \mathsf{inl}$. This leads to the following version of (1):

$$
\begin{aligned}
\alpha \rightleftharpoons \beta \quad & \text{if} \quad \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{con} & \wedge \quad k \neq m \; , \\
\alpha \sqsubset \beta \quad & \text{if} \quad \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{con} \cup \mathsf{inl} \; \wedge \; k \leq m \; .
\end{aligned}
\tag{2}
$$

In other words, two occurrences that are not in the same step are in the mutex relation if they are not concurrent; and two occurrences are ordered by $\sqsubset$ whenever they are not concurrent nor can be interleaved. Hence, the resulting dependence order structures have the property $x \sqsubset y \implies y \sqsubset x \vee x \rightleftharpoons y$. Let us consider $\mathsf{OR_{wdp}}$ consisting of all order structures $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ that satisfy this additional property.

We propose the following axiomatisation for the corresponding invariant structures. A relational structure $\langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ belongs to $\mathsf{IR_{wdp}}$ if, for all

$x, y, z \in \Delta$:

$$
\begin{aligned}
& & & x \not\sqsubset x & \text{(A1)} \\
x \neq y &\;\wedge\; & x \sqsubset z \sqsubset y &\implies& x \sqsubset y & \text{(A2)} \\
& & x \rightleftharpoons y &\implies& y \rightleftharpoons x \neq y & \text{(A3)} \\
& & x \prec z \prec y &\implies& x \rightleftharpoons y & \text{(A4)} \\
z \rightleftharpoons y &\;\wedge\; & z \sqsubset x \sqsubset z &\implies& x \rightleftharpoons y & \text{(A5)} \\
& & x \sqsubset y &\implies& x \rightleftharpoons y \vee y \sqsubset x & \text{(A6)} \\
x \neq y &\;\wedge\; & \ell(x) = \ell(y) &\implies& x \prec^{sym} y & \text{(A7)}
\end{aligned}
$$

Note that only the axioms *(A4)* and *(A6) are new. The other five are equal to (I1), (I2), (I3),(I5), and (I7), respectively.*

Moreover, since axioms *(I4)* and *(I6)* are replaced by the more restrictive *(A4)* and *(A6)*, we can simplify the procedure of computing the closure by changing the involved *cross* relation into mixed transitivity (based on $\prec$ and weaker $\sqsubset$). In this way we have $\mathsf{or2ir_{wdp}}$ such that for any $or \in \mathsf{OR_{wdp}}$

$$\mathsf{or2ir_{wdp}}(or) = \langle \Delta, \sqsubset^{\circledast} \circ \rightleftharpoons \circ \sqsubset^{\circledast} \; \cup \; (\sqsubset^{*} \circ \prec \circ \sqsubset^{*})^{sym}, \sqsubset^{\wedge}, \ell \rangle.$$

Lemma 1 $\mathsf{IR_{wdp}} \subseteq \mathsf{IR}$.

Proof: First of all, recall that *(I1)=(A1)*, *(I2)=(A2)*, *(I3)=(A3)*, *(I5)=(A5)*, and *(I7)=(A7)*. Hence, we need to prove that *(I4)* and *(I6)* are satisfied for every invariant structure $ir \in \mathsf{IR_{wdp}}$.

Let $x \prec z \sqsubset y$. Then, by *(A6)* $z \rightleftharpoons y$ or $y \sqsubset z$. In the first case we have $x \prec z \prec y$, and so, by *(A4)*, $x \rightleftharpoons y$. In the second case we have $z \sqsubset y \sqsubset z$ and $x \rightleftharpoons z$, hence, by *(A3)* (used twice) and *(A5)*, $z \rightleftharpoons x$, $y \rightleftharpoons x$ and finally $x \rightleftharpoons y$. If $x \sqsubset z \prec y$ we proceed similarly. As a result, every $ir \in \mathsf{IR_{wdp}}$ satisfy *(I4)*.

Let $z \rightleftharpoons z'$ and $x \sqsubset z \sqsubset y$ and $x \sqsubset z' \sqsubset y$. Then, by *(A6)*, $z \rightleftharpoons x$ or $z \sqsubset x$. In the first case we have $x \prec z \sqsubset y$, and so, by *(I4)*, $x \rightleftharpoons y$. In the second case we have $z \sqsubset x \sqsubset z$ and so, by *(A6)* we get $x \rightleftharpoons z'$. Note that it is impossible to get $z' \sqsubset x$ as in this case we would have $z' \sqsubset x \sqsubset z \sqsubset x \sqsubset z'$ which together with $z \rightleftharpoons z'$ denies separability. Then we have, $x \prec z' \sqsubset y$ and, by *(I4)*, $x \rightleftharpoons y$. As a result, every $ir \in \mathsf{IR_{wdp}}$ satisfies *(I6)*, which ends the proof. $\qquad\qquad\square$

Lemma 2 $\mathsf{IR_{wdp}} \subseteq \mathsf{OR_{wdp}}$.

Proof: Let $ir \in \mathsf{IR_{wdp}}$. By Lemma 1 we have that $ir \in \mathsf{IR}$, hence $ir \in \mathsf{OR}$. Moreover, by *(A6)* ir satisfies the condition $x \sqsubset y \implies y \sqsubset x \vee x \rightleftharpoons y$. Hence $ir \in \mathsf{OR_{wdp}}$. $\qquad\qquad\square$

Lemma 3 $\mathsf{or2ir_{wdp}}(\mathsf{OR_{wdp}}) \subseteq \mathsf{IR_{wdp}}$.

Proof: Let $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ and $ir = \mathsf{or2ir_{wdp}}(or) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell \rangle$.

To show *(A1)*, suppose that $x \mathrel{\widehat{\sqsubset}} x$. But this means $x \sqsubset^{\curlywedge} x$, which is impossible by irreflexivity of $\sqsubset^{\curlywedge}$.

To show *(A2)*, assume that $x \neq y$ and $x \mathrel{\widehat{\sqsubset}} z \mathrel{\widehat{\sqsubset}} y$. Then $x \sqsubset^{\curlywedge} y$, since $x \sqsubset^* y$ and $x \neq y$, and so $x \mathrel{\widehat{\sqsubset}} y$.

To show *(A3)*, assume that $x \mathrel{\widehat{\rightleftharpoons}} y$. Then $x \sqsubset^{\circledast} \circ \rightleftharpoons \circ \sqsubset^{\circledast} y$ or $x(\sqsubset^* \circ \prec \circ \sqsubset^*)^{sym} y$. In the first case, by symmetry of $\sqsubset^{\circledast}$ and $\rightleftharpoons$, $y \sqsubset^{\circledast} \circ \rightleftharpoons \circ \sqsubset^{\circledast} x$ hence $y \mathrel{\widehat{\rightleftharpoons}} x$. In the second case, $(\sqsubset^* \circ \prec \circ \sqsubset^*)^{sym}$ is symmetric as it is a symmetric closure, hence also $y \mathrel{\widehat{\rightleftharpoons}} x$. Suppose that $x = y$, which means that $x \sqsubset^{\circledast} \circ \rightleftharpoons \circ \sqsubset^{\circledast} x$ or $x(\sqsubset^* \circ \prec \circ \sqsubset^*)^{sym} x$. In both cases we have $x \sqsubset^* y \rightleftharpoons z \sqsubset^* x$, hence $y \sqsubset^* x \sqsubset^* z$ and $y \rightleftharpoons z$, which means that or is not separable, giving an obvious contradiction, and so $x \neq y$.

To show *(A4)*, assume that $x \mathrel{\widehat{\prec}} z \mathrel{\widehat{\prec}} y$. Since $x \mathrel{\widehat{\prec}} z$, we have $x \sqsubset^{\curlywedge} z \sqsubset^{\curlywedge} y$ and $x \mathrel{\widehat{\rightleftharpoons}} z$. Hence $x \sqsubset^* z$ and $x \sqsubset^* y$. According to the definition of $\mathsf{or2ir_{wdp}}$, there are two cases/reasons for $x \mathrel{\widehat{\rightleftharpoons}} z$. In the first case we directly have $x \sqsubset^* \circ \prec \circ \sqsubset^* z$. For the second case assume that $x \sqsubset^{\circledast} t' \rightleftharpoons t \sqsubset^{\circledast} z$ and suppose $z \sqsubset^* x$. Then $t' \sqsubset^* x \sqsubset^* z \sqsubset^* t \sqsubset^* z \sqsubset^* x \sqsubset^* t'$ and $t' \rightleftharpoons t$, which means that or is not separable, giving an obvious contradiction. Hence $z \not\sqsubset^* x$ and $x \sqsubset^* z$. Since $or \in \mathsf{OR_{wdp}}$ ($x \sqsubset z \implies z \sqsubset x \vee x \rightleftharpoons z$), there exist p, q such that $x \sqsubset^* p \prec q \sqsubset^* z$. As a result, in both cases, we get $x \sqsubset^* p \prec q \sqsubset^* z \sqsubset^* y$ and so $x \mathrel{\widehat{\rightleftharpoons}} y$.

To show *(A5)*, assume that $z \mathrel{\widehat{\rightleftharpoons}} y$ and $z \mathrel{\widehat{\sqsubset}} x \mathrel{\widehat{\sqsubset}} z$. First note that $x \sqsubset^{\circledast} z$. Moreover, $z \sqsubset^{\circledast} \circ \rightleftharpoons \circ \sqsubset^{\circledast} y$ or $z(\sqsubset^* \circ \prec \circ \sqsubset^*)^{sym} y$. In the first case, $x \sqsubset^{\circledast} z \sqsubset^{\circledast} \circ \rightleftharpoons \circ \sqsubset^{\circledast} y$ and so $x \mathrel{\widehat{\rightleftharpoons}} y$. In the second case, as both $x \sqsubset^* z$ and $z \sqsubset^* x$, we have $x(\sqsubset^* \circ \prec \circ \sqsubset^*)^{sym} y$ hence $x \mathrel{\widehat{\rightleftharpoons}} y$.

To show *(A6)*, assume that $x \mathrel{\widehat{\sqsubset}} y$, but $y \mathrel{\widehat{\not\sqsubset}} x$. Then $x \neq y$ and $x \sqsubset^* y$ but $y \sqsubset^* x$ does not hold. This means, that there exist z, t such that $x \sqsubset^* z \sqsubset t \sqsubset^* y$ and $t \not\sqsubset z$. But $or \in \mathsf{OR_{wdp}}$, hence $z \rightleftharpoons t$ and we have $x \sqsubset^* z \prec t \sqsubset^* y$. Therefore $x \mathrel{\widehat{\rightleftharpoons}} y$.

Finally, *(A7)* follows directly from the label-orderedness of or. $\square$

Theorem 1

$$
\begin{array}{ccccc}
\mathsf{OR_{\Theta_{wdp}}} & \subset & \mathsf{OR_{wdp}} & \subset & \mathsf{OR} \\
\cup & & \cup & & \cup \\
\mathsf{IR_{\Theta_{wdp}}} & \subset & \mathsf{IR_{wdp}} & \subset & \mathsf{IR}
\end{array}
$$

Proof: Let us consider one by one all the inclusions:

- $\mathsf{IR} \subset \mathsf{OR}$ follows from the general results proven in [4] and

$$or = \left\langle \begin{array}{c} \{x,y,z\}, \{\langle x,y\rangle, \langle y,x\rangle, \langle y,z\rangle, \langle z,y\rangle, \langle x,z\rangle, \langle z,x\rangle\}, \\ \{\langle x,y\rangle, \langle y,z\rangle\}, \{x \mapsto a, y \mapsto b, z \mapsto c\} \end{array} \right\rangle \in \mathsf{OR} \backslash \mathsf{IR} \ .$$

- $\mathsf{IR}_{\mathsf{wdp}} \subset \mathsf{OR}_{\mathsf{wdp}}$ follows from $or \in \mathsf{OR}_{\mathsf{wdp}} \setminus \mathsf{IR}_{\mathsf{wdp}}$ and Lemma 2.

- $\mathsf{IR}_{\Theta_{\mathsf{wdp}}} \subset \mathsf{OR}_{\Theta_{\mathsf{wdp}}}$ follows from $or \in \mathsf{OR}_{\Theta_{\mathsf{wdp}}} \setminus \mathsf{IR}_{\Theta_{\mathsf{wdp}}}$ and the general results proven in [4].

- $\mathsf{OR}_{\mathsf{wdp}} \subset \mathsf{OR}$ follows from the definition of $\mathsf{OR}_{\mathsf{wdp}}$ and

$$or' = \langle \{x,y\}, \varnothing, \{\langle x,y\rangle\}, \{x \mapsto a, y \mapsto b\}\rangle \in \mathsf{OR} \setminus \mathsf{OR}_{\mathsf{wdp}} \ .$$

- $\mathsf{IR}_{\mathsf{wdp}} \subset \mathsf{IR}$ follows from $or' \in \mathsf{IR} \setminus \mathsf{IR}_{\mathsf{wdp}}$ and Lemma 1.

- $\mathsf{OR}_{\Theta_{\mathsf{wdp}}} \subset \mathsf{OR}_{\mathsf{wdp}}$ can be proven by taking $\theta \in \Theta_{\mathsf{wdp}}$, $u \in \mathsf{SSEQ}_\theta$, and $or = \mathsf{sseq2or}_\theta(u)$. We know that $or \in \mathsf{OR}$. Suppose that $\alpha, \beta \in occ(u)$ and $\alpha \sqsubset \beta$. We consider two cases: $pos_u(\alpha) < pos_u(\beta)$ and $pos_u(\alpha) = pos_u(\beta)$. In the first case, since $\mathsf{wdp} = \mathsf{sse} = \varnothing$, by (1), $\langle \ell(\alpha), \ell(\beta)\rangle \in \mathsf{rig} \cup \mathsf{sse}$ and so $\langle \alpha, \beta\rangle \in \rightleftharpoons$. In the second case, once more by $\mathsf{wdp} = \mathsf{sse} = \varnothing$ and (1), $\langle \ell(\alpha), \ell(\beta)\rangle \in \mathsf{sse}$, hence $\beta \sqsubset \alpha$. Therefore $or \in \mathsf{OR}_{\mathsf{wdp}}$. Moreover, we note that

$$or'' = \left\langle \begin{array}{c} \{x,y,z\}, \{\langle x,y\rangle, \langle y,x\rangle, \langle x,z\rangle, \langle z,x\rangle, \langle y,z\rangle, \langle z,y\rangle\}, \\ \{\langle x,y\rangle, \langle x,z\rangle\}, \{x \mapsto a, y \mapsto a, z \mapsto b\} \end{array} \right\rangle$$
$$\in \mathsf{OR}_{\mathsf{wdp}} \setminus \mathsf{OR}_{\Theta_{\mathsf{wdp}}}.$$

- $\mathsf{IR}_{\Theta_{\mathsf{wdp}}} \subset \mathsf{IR}_{\mathsf{wdp}}$ follows from $or'' \in \mathsf{IR}_{\mathsf{wdp}} \setminus \mathsf{IR}_{\Theta_{\mathsf{wdp}}}$, $\mathsf{OR}_{\Theta_{\mathsf{wdp}}} \subseteq \mathsf{OR}_{\mathsf{wdp}}$ and Lemma 3.

Moreover, note that $or \in \mathsf{OR}_{\mathsf{wdp}} \setminus \mathsf{IR}$ and $or' \in \mathsf{IR} \setminus \mathsf{OR}_{\mathsf{wdp}}$ which justifies that IR and $\mathsf{OR}_{\mathsf{wdp}}$ are not related. Similarly, there is no inclusion between $\mathsf{IR}_{\mathsf{wdp}}$ and $\mathsf{OR}_{\Theta_{\mathsf{wdp}}}$ since $or \in \mathsf{OR}_{\Theta_{\mathsf{wdp}}} \setminus \mathsf{IR}_{\mathsf{wdp}}$ and $or'' \in \mathsf{IR}_{\mathsf{wdp}} \setminus \mathsf{OR}_{\Theta_{\mathsf{wdp}}}$. $\square$

In addition, we give three properties that fully justify the proposed characterisation for the invariant structures associated with Θ_{wdp}. The first one shows that invariant structures satisfying the set of proposed axioms are precisely those invariant structures that satisfy the defining property of $\mathsf{OR}_{\mathsf{wdp}}$: $x \sqsubset y \implies y \sqsubset x \vee x \rightleftharpoons y$. The second shows that all dependence structures of step sequences compatible with any step alphabet from Θ_{wdp}

are precisely those satisfying the defining property of $\mathsf{OR_{wdp}}$. And, most importantly, there are no fake invariant structures in $\mathsf{IR_{wdp}}$, meaning that the invariant structures over any alphabet from Θ_{wdp} are precisely those that can be obtained as invariant structures for any step alphabet and simultaneously satisfy the defining property of $\mathsf{OR_{wdp}}$.

Proposition 1 *For every relational structure $ir = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$,*

$$ir \in \mathsf{IR_{wdp}} \quad \Longleftrightarrow \quad (ir \in \mathsf{IR} \;\wedge\; \forall x, y \in \Delta : x \sqsubset y \Longrightarrow x \rightleftharpoons y \vee y \sqsubset x)\,.$$

Proof: ($\Longrightarrow$) Follows from Theorem 1 and *(A6)*.

($\Longleftarrow$) Note that *(A6)* is the additional property; *(I1)* and *(A1)*, *(I2)* and *(A2)*, *(I3)* and *(A3)*, *(I5)* and *(A5)*, and *(I7)* and *(A7)* are pairs of the same axioms; while *(A4)* follows directly from *(I4)*.   □

Proposition 2 *For every relational structure $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$,*

$$or \in \mathsf{sseq2or}_{\Theta_{\mathsf{wdp}}}(\mathsf{SSEQ}) \quad \Longleftrightarrow$$
$$(or \in \mathsf{sseq2or}_{\Theta}(\mathsf{SSEQ}) \;\wedge\; \forall x, y \in \Delta : x \sqsubset y \Longrightarrow x \rightleftharpoons y \vee y \sqsubset x)\,.$$

Proof: ($\Longrightarrow$) Follows from (2). Note that if $\alpha \sqsubset \beta$ and $k < m$ then $\alpha \rightleftharpoons \beta$. Moreover, if $\alpha \sqsubset \beta$ and $k = m$ then $\beta \sqsubset \alpha$.

($\Longleftarrow$) Suppose that there exists $\theta \in \Theta$ with $\langle a, b \rangle \in \mathsf{wdp}$ (and so $\langle b, a \rangle \in \mathsf{sse}$). Let us consider $seq \in \mathsf{SSEQ}$ consisting both a and b. Let $\alpha, \beta \in occ_u$ such that $\ell(\alpha) = a$ and $\ell(\beta) = b$, $pos(\alpha) = k$ and $pos(\beta) = m$. Then, if $k < m$ then $\alpha \prec \beta$; if $k \geq m$ then $\beta \sqsubset \alpha$, but $\alpha \not\sqsubset \beta$ and $\beta \neq \alpha$ which gives contradiction with the assumption that $\forall x, y \in \Delta : x \sqsubset y \Longrightarrow x \rightleftharpoons y \vee y \sqsubset x)$. As a result, in every $or \in \Theta_\theta$ all occurrences labelled by a have to causally preceded all occurrences labelled by b and one can consider θ' which is θ with $\langle a, b \rangle \in \mathsf{rig}$ instead of $\langle a, b \rangle \in \mathsf{wdp}$. Note that $u \in \mathsf{SSEQ}_{\theta'}$, and $\mathsf{sseq2or}_{\theta}(u) = \mathsf{sseq2or}_{\theta'}(u)$. Hence $or \in \mathsf{sseq2or}_{\Theta_{\mathsf{wdp}}}(\mathsf{SSEQ})$.
□

Proposition 3 *For every relational structure $ir = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$,*

$$ir \in \mathsf{IR}_{\Theta_{\mathsf{wdp}}} \quad \Longleftrightarrow \quad (ir \in \mathsf{IR}_{\Theta} \;\wedge\; \forall x, y \in \Delta : x \sqsubset y \Longrightarrow x \rightleftharpoons y \vee y \sqsubset x)\,.$$

Proof: ($\Longrightarrow$) Follows directly from Lemma 3.
($\Longleftarrow$) Let $\theta \in \Theta$ be a step alphabet with $\langle a, b \rangle \in \mathsf{wdp}$ (and so also $\langle b, a \rangle \in \mathsf{sse}$). Let us consider $u \in \mathsf{SSEQ}$ containing both a and b, and $ir = \mathsf{or2ir}(or)$, where

$or = \mathsf{sseq2or}(u)$. Let $\alpha, \beta \in occ(u)$ such that $\ell(\alpha) = a$ and $\ell(\beta) = b$, $pos(\alpha) = k$ and $pos(\beta) = m$. We consider three possible orders of k and m. In the second and third cases we proceed by contradiction excluding one of two possible conclusions given by the property that defines $\mathsf{OR_{wdp}}$ (namely $\forall x, y \in \Delta : x \sqsubset y \Longrightarrow x \rightleftharpoons y \vee y \sqsubset x$).

If $k < m$ then, by (1), $\alpha \prec_{or} \beta$ hence also $\alpha \prec \beta$.

If $k > m$ then $\beta \sqsubset_{or} \alpha$ and hence also $\beta \sqsubset \alpha$. Note that, by the assumption, $\alpha \sqsubset \beta$ or $\beta \rightleftharpoons \alpha$. Suppose that $\alpha \sqsubset \beta$. Then $\alpha \sqsubset_{or}^* \beta$ and there exist δ, γ such that $\alpha \sqsubset_{or}^* \gamma \sqsubset_{or} \delta \sqsubset_{or}^* \beta$ with $pos(\delta) < pos(\gamma)$ which is in contradiction with (1). Hence if $k > m$ then $\beta \prec \alpha$.

If $k = m$ then $\beta \sqsubset_{or} \alpha$ and naturally $\beta \sqsubset \alpha$. Note that, by the assumption, $\alpha \sqsubset \beta$ or $\beta \rightleftharpoons \alpha$. Suppose that $\beta \rightleftharpoons \alpha$, then, by the definition of $\mathsf{or2ir}$, $\beta \sqsubset_{or}^{\circledast} \gamma \rightleftharpoons_{or} \delta \sqsubset_{or}^{\circledast} \alpha$ or $\beta \, cross^{sym} \, \alpha$, where $cross = \{\langle x, y \rangle \mid \exists z, w : z \rightleftharpoons_{or} w \wedge x \sqsubset_{or}^* z \sqsubset_{or}^* y \wedge x \sqsubset_{or}^* w \sqsubset_{or}^* y\}$. In the first case we get $pos(\gamma) = pos(\beta) = pos(\alpha) = pos(\delta)$, hence $u \notin \mathsf{SSEQ}_\theta$ (we cannot observe mutex between events in the same step). In the second case we also get $pos(x) \leq pos(z) \leq pos(y) = pos(x)$ and $pos(x) \leq pos(w) \leq pos(y) = pos(x)$ and so $pos(z) = pos(w)$ and $u \notin \mathsf{SSEQ}_\theta$. Therefore, if $k = m$ then $\beta \sqsubset \alpha \sqsubset \beta$.

Finally, we get $\alpha \prec \beta$ if $k < m$, $\beta \prec \alpha$ if $m < k$ and $\beta \sqsubset \alpha \sqsubset \beta$ if $m = k$, hence referring to (1) one can consider θ' which is θ with $\langle a, b \rangle \in \mathsf{ssi}$ instead of $\langle a, b \rangle \in \mathsf{wdp}$. Note that $u \in \mathsf{SSEQ}_{\theta'}$, and $\mathsf{sseq2or}_\theta(u) = \mathsf{sseq2or}_{\theta'}(u)$. Hence $ir = \mathsf{or2ir_{wdp}}(\mathsf{sseq2or}_{\theta'}(u))$, which completes the proof. $\qquad\square$

6.2 Alphabets Without Strong Simultaneity

In this subsection we consider step alphabets from Θ_{ssi}, $\Theta_{\mathsf{ssi\cup con}}$, $\Theta_{\mathsf{ssi\cup inl}}$, and $\Theta_{\mathsf{ssi\cup con\cup inl}}$.

These alphabets know no strong simultaneity, but may nevertheless still have indivisible steps – as a result of cycles comprising weakly dependent and simultaneous events. However, there are no indivisible steps of size two. We briefly present the main properties of alphabet classes from this family for the example of Θ_{ssi}.
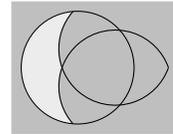
Example 6 *Consider* $\theta_2 = \langle \{a, b, c, d, f\}, \mathsf{sim}, \mathsf{seq} \rangle$, *a step alphabet with simultaneity and sequentialisation relations given in Figure 7. Some step*

traces over θ_1 are:

$$
\begin{aligned}
[\![fba]\!] &= \{fba, (fb)a, f(ab)\} & [\![adf]\!] &= \{adf, afd, daf\} \\
[\![acf]\!] &= \{acf, afc, caf, (ac)f, a(cf)\} & [\![d(bc)]\!] &= \{d(bc), dcb, cdb\} \\
[\![acd]\!] &= \{acd, adc, cad, cda, dac, dca, (ac)d, d(ac)\} \ .
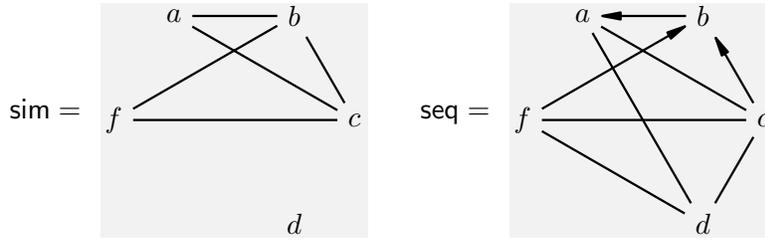\end{aligned}
$$

$\diamond$

Figure 7:   The step alphabet θ_2.

The assumption of an empty ssi relation gives an almost unnoticable simplification of (1):

$$
\begin{aligned}
\alpha \rightleftharpoons \beta \quad &\text{if} \quad \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{con} \cup \mathsf{sse} \quad \wedge \quad k < m \ , \\
&\text{or} \quad \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{con} \cup \mathsf{wdp} \quad \wedge \quad k > m \ , \\
\alpha \sqsubset \beta \quad &\text{if} \quad \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{con} \cup \mathsf{inl} \quad \wedge \quad k < m \ , \\
&\text{or} \quad \langle \ell(\alpha), \ell(\beta) \rangle \in \mathsf{sse} \qquad \quad \wedge \quad k = m \ .
\end{aligned}
\tag{3}
$$

Since $\sqsubset$ observed in a single step is implied by sse on the level of actions, the current dependence structures satisfy the property $x \sqsubset y \Longrightarrow y \not\sqsubset x$. Let us consider $\mathsf{OR_{ssi}}$ consisting of all order structures $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ that satisfy this additional property.

Since by *(I2)* we would add backward $\sqsubset$-links in the case of events that form a $\sqsubset$-cycle, this newly proposed property cannot be added to the axiomatisation of invariant structures. We can try and replace it by $x \sqsubset y \sqsubset x \Longrightarrow \exists_t \ x \sqsubset t \sqsubset y$. This, however, while reducing the set of considered invariant structures, cannot be used to simplify the remaining set of axioms. Conversely, we need to keep all seven axioms *(I1)-(I7)* calling them *(B1)-(B7)* and add another one of a different nature (with an existential quantifier). A relational structure $\langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ belongs to $\mathsf{IR_{ssi}}$ if, for all

$x, y, z, z' \in \Delta$:

$$
\begin{array}{rcll}
 & & x \not\sqsubset x & \text{(B1)} \\
x \neq y \ \wedge \ x \sqsubset z \sqsubset y & \Longrightarrow & x \sqsubset y & \text{(B2)} \\
x \rightleftharpoons y & \Longrightarrow & y \rightleftharpoons x \neq y & \text{(B3)} \\
x \prec z \sqsubset y \ \vee \ x \sqsubset z \prec y & \Longrightarrow & x \rightleftharpoons y & \text{(B4)} \\
z \rightleftharpoons y \ \wedge \ z \sqsubset x \sqsubset z & \Longrightarrow & x \rightleftharpoons y & \text{(B5)} \\
z \rightleftharpoons z' \ \wedge \ x \sqsubset z \sqsubset y \ \wedge \ x \sqsubset z' \sqsubset y & \Longrightarrow & x \rightleftharpoons y & \text{(B6)} \\
x \neq y \ \wedge \ \ell(x) = \ell(y) & \Longrightarrow & x \prec^{sym} y & \text{(B7)} \\
x \sqsubset y \sqsubset x & \Longrightarrow & \exists_t \ x \sqsubset t \sqsubset y & \text{(B8)}
\end{array}
$$

The closure operation cannot be simplified and we leave $\mathsf{or2ir_{ssi}} = \mathsf{or2ir}$. Moreover, similarly to the case of Θ_{wdp}, $\mathsf{IR_{ssi}} \subseteq \mathsf{IR}$ and $\mathsf{or2ir_{ssi}}(\mathsf{OR_{ssi}}) \subseteq \mathsf{IR_{ssi}}$ (see Lemma 1 and Lemma 3).

However, $\mathsf{IR_{ssi}} \not\subseteq \mathsf{OR_{ssi}}$. To justify this statement, let us consider $\Sigma = \{a, b, c\}$, step alphabet $\theta = \langle \Sigma, (\Sigma \times \Sigma) \setminus id_\Sigma, \{(a, b), (b, c), (c, a)\} \rangle$ from Θ_{ssi}, and a step sequence (abc) over this alphabet. By (3), the dependence structure of (abc) is an order structure $or \in \mathsf{OR_{ssi}}$ with empty $\rightleftharpoons$ and $x \sqsubset y \sqsubset z \sqsubset x$, where $\ell(x) = a$, $\ell(y) = b$ and $\ell(z) = c$. However, $ir = \mathsf{or2ir_{ssi}}(or)$ gives us also $x \sqsubset z \sqsubset y \sqsubset x$. Hence we have $x \sqsubset y \sqsubset x$ and so $ir \notin \mathsf{OR_{ssi}}$. One can verify that $ir \in \mathsf{IR_{ssi}}$, which ends the reasoning giving a non-trivial counterexample.

As a result, we obtain a diagram like the one in the statement of Theorem 1:

$$
\begin{array}{ccccc}
\mathsf{OR}_{\Theta_{\mathsf{ssi}}} & \subset & \mathsf{OR_{ssi}} & \subset & \mathsf{OR} \\
\cup & & & & \cup \\
\mathsf{IR}_{\Theta_{\mathsf{ssi}}} & \subset & \mathsf{IR_{ssi}} & \subset & \mathsf{IR}
\end{array}
$$

Note that the separation of the additional properties defining dependence structures and invariant structures are the counterparts of the three propositions stated for Θ_{wdp}. To sum up, for every relational structure $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$,

$$
or \in \mathsf{IR_{ssi}} \quad \Longleftrightarrow \quad (or \in \mathsf{IR} \ \wedge \ \forall x, y \in \Delta : x \sqsubset y \sqsubset x \Longrightarrow \exists_t x \sqsubset t \sqsubset y)
$$

and

$$
or \in \mathsf{sseq2or}_{\Theta_{\mathsf{ssi}}}(\mathsf{SSEQ}) \quad \Longleftrightarrow
$$
$$
(or \in \mathsf{sseq2or}_{\Theta}(\mathsf{SSEQ}) \ \wedge \ \forall x, y \in \Delta : x \sqsubset y \Longleftrightarrow y \not\sqsubset x) ,
$$

but

$$or \in \mathsf{IR}_{\Theta_{ssi}} \quad \Longrightarrow \quad (or \in \mathsf{IR}_{\Theta} \ \wedge \ \forall x, y \in \Delta : x \sqsubset y \Longleftrightarrow \exists_t x \sqsubset t \sqsubset y) \,.$$

The proof of the first equivalence stated above for $or \in \mathsf{IR}_{ssi}$ is similar to the proof of Proposition 1. Also the proof of the second equivalence given for $or \in \mathsf{IR}_{ssi}$ is very close to the proof of Proposition 2. The proof of the third statement, an implication when $or \in \mathsf{IR}_{\Theta_{ssi}}$ is similar to the proof of the left-to-right implication in Proposition 3. The reverse implication however, does not always hold, as we demonstrate next.

Let us consider the invariant order structure $ir \in \mathsf{IR}_{\Theta}$ of the step sequence $(abc)cabc$ over the same step alphabet from Θ_{ssi} as before: $\theta = \langle \Sigma, (\Sigma \times \Sigma) \setminus id_{\Sigma}, \{(a,b), (b,c), (c,a)\} \rangle$ where $\Sigma = \{a, b, c\}$. Then ir has domain $\Delta = \{a_1, a_2, b_1, b_2, c_1, c_2, c_3\}$ with $\ell(x_i) = x$ (see Figure 8).
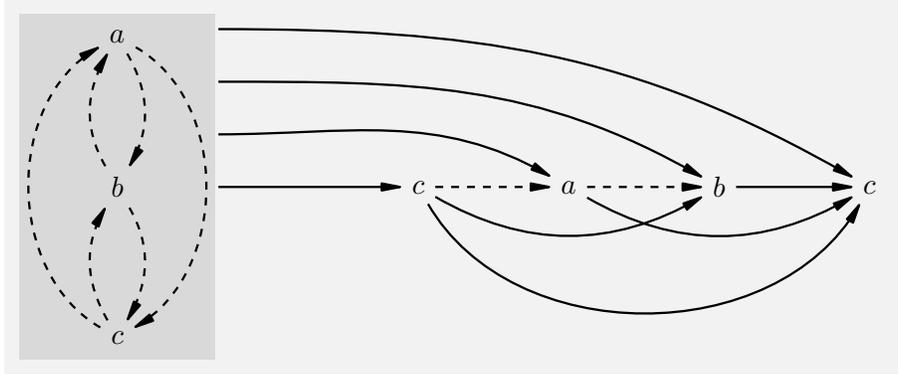

Figure 8:   The invariant order structure ir of the step sequence $(abc)cabc$ with the underlying occurrences ordered from left to right.

Suppose that $\theta' \in \Theta$ is such that ir is an invariant structure over θ'. Since b_1 and c_1 are in a weak causality cycle, it must be the case that (b, c) is in the simultaneity relation of θ'. Moreover, $b_2 \prec c_3$ and this relationship is not introduced by closure. Hence $(b, c) \notin \mathsf{sse}$ and $(b, c) \notin \mathsf{con}$. Since $c_2 \sqsubset a_2$ and $a_2 \sqsubset b_2$ (and those relationships cannot be introduced by closure), we know that $(c, a) \in \mathsf{sse}$ and $(a, b) \in \mathsf{sse}$. As a result in any dependence structure we have $c_1 \sqsubset a_1$ and $a_1 \sqsubset b_1$ but $a_1 \not\sqsubset c_1$ and $b_1 \not\sqsubset a_1$. Therefore $c_1 \sqsubset b_1$ cannot be introduced by closure and needs to be present in the dependence structure. Finally, $(b, c) \notin \mathsf{wdp}$ and the only remaining

possibility is $(b, c) \in \mathsf{ssi}$ which means that $\theta' \notin \Theta_{\mathsf{ssi}}$. This shows that the last implication indeed cannot be reversed.

6.3   Alphabets Without True Concurrency

Next we turn to the subclasses of Θ_{con} and $\Theta_{\mathsf{con} \cup \mathsf{inl}}$. As we show on the example of Θ_{con}, in these cases we are able to provide a nice additional property for both dependence structures and invariant structures. However, this property does not describe the extraction of invariant structures. The reason is similar to the situation described in the previous subsection. Information collected from more than one pair of events labelled in a specific manner, might individually mimic appropriate behaviour, but be inconsistent when considered together.

A step alphabet $\theta \in \Theta_{\mathsf{con}}$ has $\mathsf{con} = \varnothing$. As a result, every pair of events in a run is related (either causally or by mutex). From the point of view of all related order structures, there exists an arc or edge between any pair of nodes.
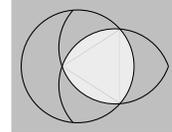
Example 7 *Consider $\theta_3 = \langle \{a, b, d, e, f\}, \mathsf{sim}, \mathsf{seq} \rangle$, a step alphabet with its simultaneity and sequentialisation relations given in Figure 9 where each undirected edge stands for two arrows in opposite directions. Some step traces over θ_3 are:*

$$
\begin{array}{rclrcl}
\llbracket fba \rrbracket & = & \{fba, (fb)a, f(ab)\} & \llbracket adf \rrbracket & = & \{adf, afd, daf\} \\
\llbracket a(ef) \rrbracket & = & \{a(ef)\} & \llbracket bde \rrbracket & = & \{bde\} \, .
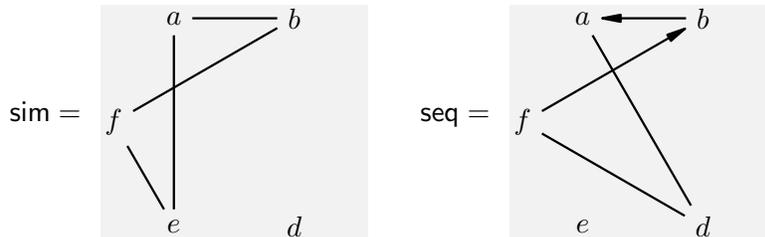\end{array}
$$

$\diamond$

Figure 9:   The step alphabet θ_3.

In the case of an empty con relation we obtain the following simplification of (1):

$$
\begin{aligned}
\alpha \rightleftharpoons \beta \quad &if \quad \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{sse} \quad &&\wedge \;\; k < m \,, \\
&or \quad \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{wdp} \quad &&\wedge \;\; k > m \,, \\
\alpha \sqsubset \beta \quad &if \quad \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{inl} \quad &&\wedge \;\; k < m \,, \\
&or \quad \langle \ell(\alpha), \ell(\beta) \rangle \in \mathsf{ssi} \cup \mathsf{sse} \quad &&\wedge \;\; k = m \,.
\end{aligned}
\tag{4}
$$

The property that every pair of events is related can be formally expressed as $x \neq y \Longrightarrow x \sqsubset y \vee y \sqsubset x \vee x \rightleftharpoons y$. We investigate $\mathsf{OR}_{\mathsf{con}}$ consisting of all order structures that satisfy this property.

We can use this property to reduce the number of invariant structures. However, it cannot be used to simplify the remaining axioms. A relational structure $\langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ belongs to $\mathsf{IR}_{\mathsf{con}}$ if, for all $x, y, z, z' \in \Delta$:

$$
\begin{aligned}
&&&&&& x \not\sqsubset x && (C1) \\
& x \neq y & \wedge && x \sqsubset z \sqsubset y & \Longrightarrow & x \sqsubset y && (C2) \\
&&&& x \rightleftharpoons y & \Longrightarrow & y \rightleftharpoons x \neq y && (C3) \\
& x \prec z \sqsubset y & \vee && x \sqsubset z \prec y & \Longrightarrow & x \rightleftharpoons y && (C4) \\
& z \rightleftharpoons y & \wedge && z \sqsubset x \sqsubset z & \Longrightarrow & x \rightleftharpoons y && (C5) \\
z \rightleftharpoons z' \wedge x \sqsubset z \sqsubset y & & \wedge && x \sqsubset z' \sqsubset y & \Longrightarrow & x \rightleftharpoons y && (C6) \\
& x \neq y & \wedge && \ell(x) = \ell(y) & \Longrightarrow & x \prec^{sym} y && (C7) \\
& x \neq y & \Longrightarrow && x \sqsubset y \vee y \sqsubset x \vee x \rightleftharpoons y &&&& (C8)
\end{aligned}
$$

The closure operation cannot be simplified and we leave $\mathsf{or2ir}_{\mathsf{con}} = \mathsf{or2ir}$.

Similar to the case of alphabets from Θ_{wdp} (see Lemma 1, Lemma 2, Lemma 3, and Theorem 1), we obtain $\mathsf{IR}_{\mathsf{con}} \subseteq \mathsf{IR}$, $\mathsf{IR}_{\mathsf{con}} \subseteq \mathsf{OR}_{\mathsf{con}}$ and $\mathsf{or2ir}_{\mathsf{con}}(\mathsf{OR}_{\mathsf{con}}) \subseteq \mathsf{IR}_{\mathsf{con}}$, which leads to

$$
\begin{array}{ccccc}
\mathsf{OR}_{\Theta_{\mathsf{con}}} & \subset & \mathsf{OR}_{\mathsf{con}} & \subset & \mathsf{OR} \\
\cup & & \cup & & \cup \\
\mathsf{IR}_{\Theta_{\mathsf{con}}} & \subset & \mathsf{IR}_{\mathsf{con}} & \subset & \mathsf{IR}
\end{array}
$$

Moreover, for every relational structure $or = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$,

$$
or \in \mathsf{IR}_{\mathsf{con}} \quad \Longleftrightarrow \quad (or \in \mathsf{IR} \wedge \forall x, y \in \Delta : x \neq y \Longleftrightarrow x \sqsubset y \vee y \sqsubset x \vee x \rightleftharpoons y)
$$

and

$$
\begin{aligned}
or \in \mathsf{sseq2or}_{\Theta_{\mathsf{con}}}(\mathsf{SSEQ}) \quad &\Longleftrightarrow \\
&(or \in \mathsf{sseq2or}_{\Theta}(\mathsf{SSEQ}) \wedge \forall x, y \in \Delta : x \sqsubset y \vee y \sqsubset x \vee x \rightleftharpoons y) \,,
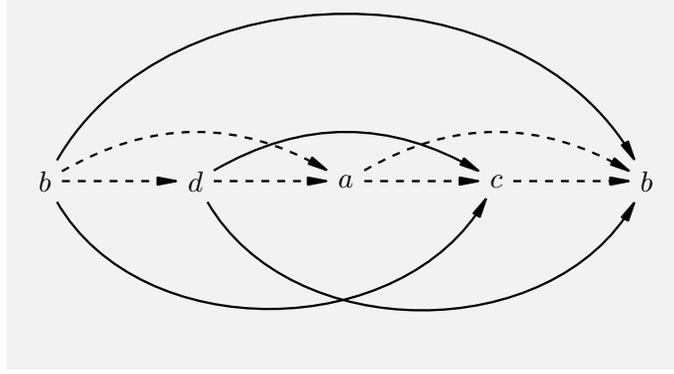\end{aligned}
$$

Figure 10: The invariant structure ir of $bdacb$ with the underlying occurrences ordered from left to right.

but

$$or \in \mathsf{IR}_{\Theta_{\mathsf{con}}} \implies (or \in \mathsf{IR}_\Theta \wedge \forall x, y \in \Delta : x \sqsubset y \vee y \sqsubset x \vee x \rightleftharpoons y) .$$

Again, as in the case of Θ_{ssi}, the last implication in the reverse direction does not hold. This can be seen as follows. Consider the step alphabet $\theta = \langle \Sigma, \mathsf{sim}, \mathsf{seq} \rangle$ with $\Sigma = \{a, b, c, d\}$, $\mathsf{sim} = (\Sigma \times \Sigma) \setminus id_\Sigma$, and $\mathsf{seq} = \{(a,b), (b,a), (d,a), (a,c), (c,b), (b,d)\}$. Let $ir \in \mathsf{IR}_\Theta$ be the invariant structure over θ of the step sequence $bdacb$. Then ir has domain $\Delta = \{a_1, b_1, b_2, c_1, d_1\}$ with $\ell(x_i) = x$ (see Figure 10).

Suppose that $\theta' \in \Theta$ is such that ir is an invariant structure over θ'. Note that events b_1 and a_1 as well as a_1 and b_2 are related by $\sqsubset$. On the other hand, they cannot all be present in the same step (there are two instances of action b). Hence, by (1), $(a,b) \notin \mathsf{rig}_{\theta'}$ and $(a,b) \notin \mathsf{inl}_{\theta'}$, as in both situations we would observe $b_1 \rightleftharpoons a_1 \rightleftharpoons b_2$. Similarly, $(a,b) \notin \mathsf{sim}_{\theta'}$, as in this situation we would observe $b_1 \rightleftharpoons a_1$ or $a_1 \sqsubset b_1$. Finally, $(a,b) \notin \mathsf{wdp}_{\theta'}$ and $(a,b) \notin \mathsf{sse}_{\theta'}$ as in those situations we would observe $b_1 \rightleftharpoons a_1$ or $a_1 \rightleftharpoons b_2$. As a result, we get that $(a,b) \in \mathsf{con}_{\theta'}$ and $\theta' \notin \Theta_{\mathsf{con}}$.

7 Conclusion

In this paper we have proposed a new classification of step alphabets. As the distinctions between the different subclasses are more detailed in this

paper, this has resulted in an extension of the classification in [7] from eight to sixteen classes. The identification of the various classes is based on a partition of the basic relations between actions into six meaningful relations and then assuming that some of these do not occur (are empty). The eight new subclasses have been divided into three families and we have described some prominent additional features of their associated order structures.

The research in this paper yields a comparison of (families of) subclasses of step alphabets in terms of behaviour (potential relations between events) and properties of their associated dependence and invariant structures. This comparative study is however not yet complete and should still be extended to all sixteen classes by identifying a characteristic property for each of them. We anticipate that more investigations in this direction will lead to more insight in their practical potential. After all, also in the case of the four classes investigated in [7], some well known and useful trace models of concurrent systems could be identified.

Another aspect touched upon in this paper, but not yet entirely exploited, is related to the synthesis of step alphabets (see, e.g., [6, 16]). Properties of invariant structures that can be realised by particular step alphabets might be used in the procedure of revealing the concurrent structure of a process or system under investigation. Requiring membership of a specific class of the step alphabet to be synthesised, would then be a possible design choice.

Acknowledgements

References

[1] V.Diekert and G.Rozenberg (eds.). *The Book of Traces*. World Scientific, Singapore, 1995. doi:10.1142/9789814261456.

[2] R. Janicki, J. Kleijn, M. Koutny, and Ł. Mikulski. Causal Structures for General Concurrent Behaviours. *CEUR Workshop Proceedings*, 1032, pages 193–205, 2013.

[3] R. Janicki, J. Kleijn, M. Koutny, and Ł. Mikulski. Characterising Concurrent Histories. *Fundamenta Informaticae*, 139(1), pages 21–42, 2015. `doi:10.3233/FI-2015-1224`.

[4] R. Janicki, J. Kleijn, M. Koutny, and Ł. Mikulski. Step Traces. *Acta Informatica*, 53(1), pages 35–65, 2016. `doi:10.1007/s00236-015-0244-z`.

[5] R. Janicki, J. Kleijn, M. Koutny, and Ł. Mikulski. Alphabets of Acyclic Invariant Structures. *Fundamenta Informaticae*, 154(1–4), pages 207–224, 2017. `doi:10.3233/FI-2017-1562`.

[6] R. Janicki, J. Kleijn, M. Koutny, and Ł. Mikulski. Invariant Structures and Dependence Relations. *Fundamenta Informaticae*, 155(1-2), pages 1–29, 2017. `doi:10.3233/FI-2017-1574`.

[7] R. Janicki, J. Kleijn, M. Koutny, and Ł. Mikulski. Classifying Invariant Structures of Step Traces. *Journal of Computer and System Sciences*, In Press, available online 2017. `doi:10.1016/j.jcss.2017.05.002`.

[8] R. Janicki and M. Koutny. Structure of Concurrency. *Theoretical Computer Science* 112(1), pages 5–52, 1993. `doi:10.1016/0304-3975(93)90238-O`.

[9] R.Janicki and M.Koutny. Semantics of Inhibitor Nets. *Inf. Comput.* 123(1), pages 1–16, 1995. `doi:10.1006/inco.1995.1153`.

[10] A. Laarman. Stubborn Transaction Reduction. In: A.Dutle, C.A.Muñoz, and A.Narkawicz, editors, *NASA Formal Methods Symposium*, Lecture Notes in Computer Science, vol. 10811, pages 290–298. Springer, Berlin, 2018. `doi:10.1007/978-3-319-77935-5_20`.

[11] D.T.M. Le. On Three Alternative Characterizations of Combined Traces. *Fundamenta Informaticae* 113(3), pages 265–293, 2011. `doi:10.3233/FI-2011-609`.

[12] A. W. Mazurkiewicz. Concurrent Program Schemes and Their Interpretations. DAIMI Rep. PB 78, Aarhus University (1977). `doi:10.7146/dpb.v6i78.7691`.

[13] A. W. Mazurkiewicz. Trace Theory. In: W. Brauer, W. Reisig, and G. Rozenberg, editors, *Petri Nets: Central Models and Their Properties, Advances in Petri Nets 1986, Part II*, Lecture Notes in Computer Science, vol. 255, pages 278–324. Springer, Berlin, 1987. `doi:10.1007/3-540-17906-2_30`

[14] A.W. Mazurkiewicz. Basic Notions of Trace Theory. In: J.W. de Bakker, W.P. de Roever, and G. Rozenberg, editors, *REX Workshop* Lecture Notes in Computer Science, vol. 354, pages 285–363. Springer, Berlin, 1988. `doi:10.1007/BFb0013025`.

[15] Ł. Mikulski. Algebraic Structure of Combined Traces. *Logical Methods in Computer Science* 9(3:8), 2013. `doi:10.2168/LMCS-9(3:8)2013`.

[16] A. Muscholl. Automated Synthesis of Distributed Controllers. In: *Automata, Languages, and Programming - 42nd International Colloquium, ICALP*, Lecture Notes in Computer Science 9135, pages 11–27, Springer, 2015. `doi:10.1007/978-3-662-47666-6_2`.

[17] A. Nagy and A. Arif. Trajectories and Traces on Non-Traditional Regular Tessellations of the Plane. In: *Combinatorial Image Analysis*, Lecture Notes in Computer Science 10256, pages 16–29, Springer, 2017. `doi:10.1007/978-3-319-59108-7_2`.

[18] L. Paulevé. Goal-Oriented Reduction of Automata Networks. In: International Conference on Computational Methods in Systems Biology, pages 252–272, 2016. `doi:10.1007/978-3-319-45177-0_16`.

[19] V.R.Pratt. Modeling Concurrency With Partial Orders. *Int. J. Parallel Program* 15(1), pages 33 – 71, 1986. `doi:10.1007/BF01379149`.

[20] E. Szpilrajn. Sur l'extension de l'ordre partiel. *Fundam. Math.* 16, 386–389, 1930. `doi:10.4064/fm-16-1-386-389`.