Journal of Algebraic Systems

Vol. 1, No. 1, (2013), pp 67-77

GENERALIZATIONS OF δ-LIFTING MODULES

Y. TALEBI* AND M. HOSSEINPOUR

Abstract. In this paper we introduce the notions of G_1^*L-module and G_2^*L-module which are two proper generalizations of δ-lifting modules. We give some characterizations and properties of these modules. We show that a G_2^*L-module decomposes into a semisimple submodule M_1 and a submodule M_2 of M such that every non-zero submodule of M_2 contains a non-zero δ-cosingular submodule.

1. Introduction

Throughout this article, all rings are associative with an identity, and all modules are unitary right $R-$modules. Let M be an R-module. By $N \leq M (N \leq_\oplus M)$ we mean that N is a submodule (direct summand) of M. A submodule N of a module M is called *essential* in M if for every nonzero submodule L of M, $N \cap L \neq 0$ (denoted by $N \leq_e M$) and A submodule N of a module M is called *small* in M if for every proper submodule L of M, $N + L \neq M$ (denoted by $N \ll M$). A module M is called *hollow* if every proper submodule of M is small in M. M is called a small module if there exists a module T such that $M \ll T$. If $N/K \ll M/K$, then K is called a *cosmall* submodule of N in M. A submodule N of M is called *coclosed* if N has no proper cosmall submodule. Recall that the *singular* submodule $Z(M)$ of a module M is the set of $m \in M$ with $mI = 0$ for some essential ideal I of R. If $Z(M) = M$ $(Z(M) = 0)$, then M is called a singular (*non-singular*) module. Let K, N be submodules of M. Following [14], as a

MSC(2010): Primary: 16D50; Secondary: 16D99

Keywords: δ-cosingular, non-δ-cosingular, G_1^*L-module, G_2^*L-module.

Received: 6 April 2013, Revised: 10 September 2013

*Corresponding author .

generalization of small submodules, N is called *δ-small* in M, if $M = N + K$ with M/K singular implies $M = K$ (denoted by $N \ll_\delta M$). K is a *δ-supplement* of N in M if and only if $M = N + K$ and $N \cap K \ll_\delta K$. Any R-module M is called *δ-supplemented* if every submodule of M has a δ-supplement in M. A module M is called a *lifting* module if, for every submodule A of M there exists a direct summand N of M with $N \subseteq A$ and $A/N \ll M/N$. A module M is called *δ-lifting* if, for every submodule A of M there exists a direct summand N of M with $N \subseteq A$ and $A/N \ll_\delta M/N$. Equivalently for any $A \leq M$, there exists a decomposition $M = N \oplus B$ such that $N \leq A$ and $A \cap B \ll_\delta B$. Let ρ be the class of all singular simple modules. For a module M, let $\delta(M) = Rej_M(\rho) = \bigcap\{N \subseteq M \mid M/N \in \rho\}$ be the reject in M of ρ. Dual to the notion of singular submodule of a module M, $Z(M)$ is defined by Talebi and Vanaja in [11], $\overline{Z}(M) = \bigcap\{Kerg \mid g : M \to N, N \text{ is a small module}\}$. If $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$), then M is called a *cosingular* (*non-cosingular*) module.

In [10], inspired by this definition Özcan defined the submodule $\overline{Z}_\delta(M)$ of M as $\overline{Z}_\delta(M) = \bigcap\{Kerg \mid g : M \to N, N \text{ is a } \delta\text{-small module}\}$. Clearly, $\overline{Z}_\delta(M) \subseteq \overline{Z}(M)$.

Any module M is called a *δ-cosingular* (*non-δ-cosingular*) module if $\overline{Z}_\delta(M) = 0$ ($\overline{Z}_\delta(M) = M$). Every cosingular module is δ-cosingular and every non-δ-cosingular module is non-cosingular.

In [13], Tribak and Orhan defined G_1L-modules and G_2L-modules and they investigated some properties of these modules and in [12], Talebi and Nematollahi defined C*-modules and studied some properties of such modules.

In this paper, we defines G_1^*L-module and G_2^*L-module that are generalizations of G_1L-module and G_2L-module and we discuss more results which are different from the results of papers [12, 13]

A module M is called G_1L-module if, for every submodule N of M, there exists a direct summand K of M such that K is contained in N and the factor N/K is a small module. A module M is called G_2L-module (or C*-module), if for every submodule N of M, there exists a direct summand K of M such that K is contained in N and the factor N/K is a cosingular module.

A module M is called G_1^*L-module if, for every submodule N of M, there exists a direct summand K of M such that K is contained in N and the factor N/K is a δ-small module. A module M is called G_2^*L-module if, for every submodule N of M, there exists a direct summand K of M such that K is contained in N and the factor N/K is a

δ-cosingular module. It is easily seen that G_1^*L (G_2^*L)-modules are two generalizations of δ-lifting modules, we have the following hierarchy:
δ-lifting$\Rightarrow$ G_1^*L-module$\Rightarrow$ G_2^*L-module (The converse is not true. For example see Example 2.15(3)).

2. General Properties

Proposition 2.1. *For an R-module M the following statements are equivalent:*

(1) *M is G_1^*L (G_2^*L);*

(2) *For every submodule N of M there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2$ is δ-small (δ-cosingular) module;*

(3) *For every submodule N of M, N has a decomposition $N = N_1 \oplus N_2$ such that N_1 is a direct summand of M and N_2 is δ-small (δ-cosingular) module.*

Proof. It is obvious. □

Remark 2.2. The class of G_1^*L (G_2^*L)-modules is closed under submodules.

Proposition 2.3. *Let $M = M_1 \oplus M_2$ where M_1 is semisimple and M_2 is G_1^*L (G_2^*L). Then M is a G_1^*L (G_2^*L)-module.*

Proof. The proof is similar to [12, Theorem 2.10]. □

Proposition 2.4. *Let M be a G_2^*L- module. Then any homomorphic image of M is G_2^*L- module.*

Proof. Let $f : M \to N$ be an epimorphism and L a submodule of N. Then there is a submodule H of M such that $L \cong H/Kerf$. Since M is a G_2^*L- module, there are direct summands K, K' of M such that $M = K \oplus K'$, $K \leq H$ and that H/K is δ-cosingular. So $N \cong M/Kerf = (K/Kerf) \oplus (K' + Kerf)/Kerf$. Since $(H/Kerf)/(K/Kerf) \cong H/K$ is δ-cosingular, N is a G_2^*L- module. □

A module M is H-supplemented if for every submodule N of M there exists a direct summand D of M such that $(N + D)/N \ll M/N$ and $(N + D)/D \ll M/D$ (see [6]). We define a module to be H-δ-supplemented if for every submodule N of M there exists a direct summand D of M such that $(N + D)/N \ll_\delta M/N$ and $(N + D)/D \ll_\delta M/D$.

Proposition 2.5. *Let M be a non-δ-cosingular module. Then the following are equivalent:*

(1) *M is δ-lifting;*

(2) *M is H-δ-supplemented;*

(3) *M is G_1^*L.*

Proof. (1) $\Longrightarrow$ (2) This is easy.
(2) $\Longrightarrow$ (3) Let $N \leq M$. By assumption there exists a direct summand D of M such that $(N+D)/D \ll_\delta M/D$ and $(N+D)/N \ll_\delta M/N$. Since $(N+D)/N \cong D/(N \cap D)$. Hence $(N+D)/N$ is both non-δ-cosingular and δ-cosingular, and so $N+D = N$, therefore $D \leq N$ and $N/D \ll_\delta M/D$ and M is G_1^*L.
(3) $\Longrightarrow$ (1) This is easy. □

Theorem 2.6. *The following statements are equivalent for a ring R:*

(1) *Every right R-module satisfies G_2^*L;*

(2) *Every injective right R-module satisfies G_2^*L;*

(3) *Every right R-module is a direct sum of an injective module and a δ-cosingular module.*

Proof. The proof is similar to [12, Theorem 2.9]. □

Proposition 2.7. *If for every module M, $\overline{Z}_\delta(M)$ is a direct summand of M and every non-δ-cosingular module is injective, then every R-module is G_2^*L.*

Proof. Let M be an R-module. We have $M = \overline{Z}_\delta(M) \oplus N$ for some submodule N of M. By [10, Proposition 2.5 (3)], N is δ-cosingular. Therefore $\overline{Z}_\delta(M) = \overline{Z}_\delta^2(M)$. Thus $\overline{Z}_\delta(M)$ is non-δ-cosingular. By hypothesis, $\overline{Z}_\delta(M)$ is injective. The result follows from Theorem 2.6. □

Let R be a ring. Recall that R is a *right δ-Harada ring (δ-H-ring for short)*, if every injective right R-module is δ-lifting. R is a right δ-H-ring if and only if every right R-module can be expressed as a direct sum of a δ-small R-module and an injective module. Also R is a *Quasi-Frobenius ring (QF-ring for short)*, if every injective module is projective if and only if every projective module is injective.

Corollary 2.8. *If R is QF-ring, Every right R-module satisfies G_2^*L.*

Proof. By [8, Corollary 2.11], R is a QF-ring if and only if every R-module is a direct sum of a projective module and a δ-small module. By definitions, every QF-ring is a right δ-H-ring. So see Theorem 2.6. □

Theorem 2.9. *Let M be an R-module, then:*

(1) *Let X be a submodule of M and D a direct summand of M. Assume that M/D is G_2^*L. If $X/(X \cap D)$ is non-δ-cosingular, then $D + X$ is a direct summand of M.*

(2) *If M is non-δ-cosingular and M/D is G_2^*L with D a direct summand of M, then $(D + X)/D$ is a direct summand of M/D for all direct summands X of M.*

Proof. (1) Let X be a submodule of M and D a direct summand of M. Consider the submodule $(X + D)/D \leq M/D$. Since M/D is G_2^*L, there exists a direct summand C/D of M/D such that $C/D \subseteq (X + D)/D$ and $(D + X)/C$ is δ-cosingular. On the other hand $(X + D)/D \cong X/(X \cap D)$ and so $(X + D)/D$ is non-δ-cosingular. Therefore since every homomorphic images of non-δ-cosingular is non-δ-cosingular ([10, Propositon 2.4]), $(D + X)/C$ is non-δ-cosingular. Hence $D + X = C$.

(2) Let M is non-δ-cosingular and M/D is G_2^*L with D a direct summand of M. Let X be a direct summand of M. Then $X/(X \cap D)$ is non-δ-cosingular by [10, Propositon 2.4]. By (1) $D + X$ is direct summand of M and hence $(X + D)/D$ is a direct summand of M/D. □

Recall that a module M has the *Summand Intersection Property*, (*SIP*) if the intersection of any two direct summands of M is again a direct summand (see [5]) and M has the *Summand Sum Property*, (*SSP*) if the sum of any two direct summands of M is again a direct summand (see [3]). Let M be any module. M is called a (D_3)-module if whenever M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, $M_1 \cap M_2$ is also a direct summand of M.

Proposition 2.10. *Every non-δ-cosingular G_2^*L module has the SSP.*

Proof. Let M be a non-δ-cosingular G_2^*L module. Let A and B be two direct summands of M. Let $M = A \oplus A' = B \oplus B'$ for some submodules A', B'. Note that A' and B' are G_2^*L modules. Since $M/A \cong A'$ and $M/B \cong B'$, $(A+B)/A$ is a direct summand of M/A and $(A + B)/B$ is a direct summand of M/B by Theorem 2.9(2). Hence $A + B$ is a direct summand of M. □

There exists modules having the SSP and be G_2^*L but not the SIP.

Example 2.11. Let F be a filed and R the upper triangular matrix ring $R = \begin{pmatrix} F & 0 \\ F & F \end{pmatrix}$.

For submodules $A = \begin{pmatrix} 0 & 0 \\ F & F \end{pmatrix}$ and $B = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$, $A \oplus (R/B)$ has the SSP by [3] and G_2^*L by [9]. But has not the SIP.

Lemma 2.12. *Assume that M is (D_3). If M has the SSP then M has the SIP.*

Proof. By [1, Lemma 19(2)]. □

Corollary 2.13. *Let M be non-δ-cosingular module with (D_3).Then we have:*
M is $G_2^*L \Longrightarrow M$ has SSP$\Longrightarrow M$ has SIP.

Proposition 2.14. *Let M be G_2^*L such that $Soc(M) \neq 0$. Then for every minimal submodule N of M, either N is δ-cosingular or $N \leq_\oplus M$.*

Proof. Let $N \leq M$ be minimal. Since M is a G_2^*L-module, N contains a direct summand K of M such that N/K is δ-cosingular. Since N is minimal, $K = 0$ or $K = N$. If $K = 0$, N is δ-cosingular and if $K = N$, N is a direct summand of M. □

Example 2.15. (1) Let R be a commutative domain which is not a field. Harada proved [4, Theorem 2] that the module R_R is small. Therefore R_R is a G_1^*L-module.

(2) Let R be a right semisimple ring and M be a nonzero right R-module. Then M is nonsingular and semisimple. Every submodule of M(even M itself) is δ-small in M. So M is δ-lifting and G_1^*L-module.

(3) Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. Then M is a G_1^*L-module. On the other hand, by [7, Example 2.8], M is not δ-lifting.

Proposition 2.16. *Every δ-cosingular module (and so every δ-small module) is G_2^*L.*

We have the following implications:

$$\begin{array}{ccccc} small & \Longrightarrow & \delta - small & \Longrightarrow & G_1^*L \\ \Downarrow & & \Downarrow & & \Downarrow \\ cosingular & \Longrightarrow & \delta - cosingular & \Longrightarrow & G_2^*L \end{array}$$

Lemma 2.17. *Every non-δ-cosingular submodule of a G_2^*L-module M is a direct summand of M.*

Proof. Let $N \leq M$ be a non-δ-cosingular submodule. By assumption, N contains a direct summand K of M such that N/K is δ-cosingular. Since N is non-δ-cosingular, N/K is non-δ-cosingular. Hence $N = K$ is a direct summand of M. □

Proposition 2.18. *Let M be a G_2^*L-module. Then $\overline{Z}_\delta^2(M) = \overline{Z}_\delta(\overline{Z}_\delta(M))$ is non-δ-cosingular and it is a direct summand of M.*

Proof. Since $\overline{Z}_\delta(M)$ is a submodule of M and M is a G_2^*L-module, there exists a decomposition $M = K \oplus K'$ such that $\overline{Z}_\delta(M)/K$ is δ-cosingular. This gives $\overline{Z}_\delta^2(M) + K = K$. Thus $\overline{Z}_\delta^2(M) \subseteq K$. But $\overline{Z}_\delta^2(M) = \overline{Z}_\delta^2(K) \oplus \overline{Z}_\delta^2(K')$. Then $\overline{Z}_\delta^2(M) = \overline{Z}_\delta^2(K)$. Since $\overline{Z}_\delta(M)/\overline{Z}_\delta^2(M)$ is δ-cosingular, so is $K/\overline{Z}_\delta^2(M)$. Thus $\overline{Z}_\delta(K/\overline{Z}_\delta^2(K) = 0$. It follows that $\overline{Z}_\delta(K) + \overline{Z}_\delta^2(K) = \overline{Z}_\delta^2(K)$. Therefore $\overline{Z}_\delta(K) = \overline{Z}_\delta^2(K) = \overline{Z}_\delta^2(M)$. So $\overline{Z}_\delta^2(M)$ is non-δ-cosingular and by Lemma 2.17 is a direct summand of M. □

Example 2.19. (1) Let M be the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$, where p is a prime. M is a G_2^*L-module since it is a hollow module.

(2) By Proposition 2.3, the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}/q\mathbb{Z}$, where p and q are primes is a G_2^*L-module

(3) For every module M, the factor module $M/\overline{Z}_\delta(M)$ is a G_2^*L-module since it is δ-cosingular.

(4) R is semiperfect if and only if the right (left) R-module R is lifting [9, Corollary 4.42]. Hence every semiperfect ring R, as a right(left) R-module is G_2^*L.

3. The Main Results

In this section we consider some important properties of G_2^*L-module. We show that a G_2^*L-module decomposes into a semisimple submodule M_1 and a submodule M_2 of M such that every non-zero submodule of M_2 contains a non-zero δ-cosingular submodule. Let R be any ring. Let M be a module. We denote the sum of all δ-cosingular submodules of M by $Soc_\delta(M)$.

Proposition 3.1. *Let M be any G_2^*L-module. Then the module $M/Soc_\delta(M)$ is semisimple.*

Proof. Let $N/Soc_{\delta}(M)$ be a submodule of $M/Soc_{\delta}(M)$. Then there exist submodules K and K' of M such that $M = K \oplus K'$, $K \leq N$ and N/K is δ-cosingular. Hence $N = K \oplus (N \cap K')$ and $N \cap K'$ is δ-cosingular. Thus $N \cap K' \subseteq Soc_{\delta}(M)$, and we deduce that $M/Soc_{\delta}(M) = (N/Soc_{\delta}(M)) \oplus [(K' + Soc_{\delta}(M))/Soc_{\delta}(M)]$. That is, $N/Soc_{\delta}(M)$ is a direct summand of $M/Soc_{\delta}(M)$. So $M/Soc_{\delta}(M)$ is semisimple. □

Corollary 3.2. *Let R be a ring such that every simple R-module is δ-cosingular and M any G_2^*L-module. Then $Soc_{\delta}(M)$ is an essential submodule of M.*

Proof. Let N be any submodule of M such that $N \cap Soc_{\delta}(M) = 0$. So N can be embedded in $M/Soc_{\delta}(M)$. By Proposition 3.1, N is semisimple, so that, by hypothesis, $N \subseteq Soc_{\delta}(M)$. Hence $N = 0$. Thus $Soc_{\delta}(M)$ is an essential submodule of M. □

Lemma 3.3. *Let M be a G_2^*L-module and N be any submodule of M. Then N contains a non-zero δ-cosingular submodule or N is a semisimple direct summand of M.*

Proof. Suppose that N does not contain a δ-cosingular. Let P be any submodule of N. By Proposition 2.1, $P = K \oplus L$ for some direct summand K of M and δ-cosingular submodule L of M. But $L = 0$, and hence, $P = K$. By [2, Theorem 9.6], N is a semisimple direct summand of M. □

Proposition 3.4. *Let M be a G_2^*L-module. Then there exist a semisimple submodule M_1 and a submodule M_2 of M such that $M = M_1 \oplus M_2$ and every non-zero submodule of M_2 contains a non-zero δ-cosingular submodule.*

Proof. Let $\mathcal{A} = \{N \leq M$ such that N does not contain a non-zero δ-cosingular submodule$\}$. By Zorn's Lemma, $\mathcal{A}$ contains a maximal element M_1. By Lemma 3.3, M_1 is a semisimple direct summand of M. So there exists a submodule M_2 such that $M = M_1 \oplus M_2$. Let N be a non-zero submodule of M_2. Then $M_1 \oplus N$ contains a non-zero δ-cosingular submodule K, by the choice of M_1. Note that $K \cap M_1$ is a δ-cosingular submodule and hence $K \cap M_1 = 0$. Thus K can be embedded in N and hence N contains a non-zero δ-cosingular submodule. □

An internal direct sum $\oplus_{i \in I} X_i$ of submodules of a module M is called a local summand of M if, given any finite subset F of the index set I, the direct sum $\oplus_{i \in F} X_i$ is a direct summand of M.

Theorem 3.5. *Every non-δ-cosingular G_2^*L module is a direct sum of indecomposable modules.*

Proof. Let M be a non-δ-cosingular G_2^*L module and $X = \oplus_{i\in I}X_i$ a local summand of M. Since each X_i is a direct summand of M, and $X_i = \overline{Z}_\delta(X_i) \leq \overline{Z}_\delta(X)$. Then $\overline{Z}_\delta(X) = \overline{Z}_\delta(\oplus_{i\in I}X_i) = \oplus_{i\in I}\overline{Z}_\delta(X_i) = \oplus_{i\in I}X_i = X$. So X is non-δ-cosingular. It follows that $X \leq_\oplus M$. Hence every local summand is summand. Therefore by [9, Theorem 2.17] , M is a direct sum of indecomposable modules. □

Recall that an R-module M is an *extending* module if for every submodule A of M there exists a direct summand B of M such that $A \leq_e B$.

Proposition 3.6. *Let M be an extending module. Then M is G_2^*L if and only if every submodule of M is a direct sum of an extending module and a δ-cosingular module.*

Proof. Suppose that M be G_2^*L. Let $N \leq M$. Then $N = N_1 \oplus N_2$ where $N_1 \leq_\oplus M$ and N_2 is δ-cosingular. It follows that N_1 is extending. Conversely, Suppose that every submodule of M is a direct sum of an extending module and a δ-cosingular module. Let L be any submodule of M. Then $L = L_1 \oplus L_2$ for some extending module L_1 and δ-cosingular module L_2. Since L_1 is extending, there exists a direct summand K of M such that $L_1 \leq_e K$. It follows that $K \cap L_2 = 0$ and $L = K \oplus L_2$. Hence M is G_2^*L. □

It is well known that there are $\mathbb{Z}$-modules which are not extending, for example $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. But by Example 2.15, M is G_2^*L.

Proposition 3.7. *The following are equivalent for a ring R.*

(1) *Every right R-module is G_2^*L;*

(2) *Every extending right R-module is G_2^*L;*

(3) *Every quasi-injective right R-module is G_2^*L;*

(4) *Every injective right R-module is G_2^*L;*

(5) *Every right R-module is a direct sum of an extending module and a δ-cosingular module;*

(6) *Every right R-module is a direct sum of an injective module and a δ-cosingular module.*

Proof. (1) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (6) By Theorem 2.6. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) Clear. (2) $\Longleftrightarrow$ (5) By Proposition 3.6. □

Acknowledgments

This research partially is supported by the Research center in Algebraic Hyperstructure and Fuzzy Mathematics University of Mazandaran, Babolsar, Iran.

References

1. M. Alkan and A. Harmanci, On summand sum and summand intersection property of modules, *Turkish J. Math.*, **26** (2002), 131-147.
2. F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Graduate Texts in Mathematics, Vol. 13, Springer-Verlag, New York, 1992.
3. J. L. Garcia, Properties of direct summands of modules, *Comm. Algebra*, **17**(1) (1989), 73-92.
4. M. Harada, On small submodules in the total quotient ring of a commutative ring, *Rev. Un. Mat. Argentina*, **28** (1977), 99-102.
5. J. Hausen, Modules with the summand intersection property, *Comm. Algebra*, **17**(1) (1989), 135-148.
6. D. Keskin, M. J. Nematollahi and Y. Talebi, On H-supplemented modules, *Algebra Colloq*, **18**(Spec 1) (2011), 915–924.
7. M. T. Kosan, δ-lifting and δ-supplemented modules, *Algebra Colloq*, **14**(1) (2007), 53–60.
8. M. T. Kosan and A. C. Özcan, δ-M-Small and δ-Harada modules, *Comm. Algebra.*, **36** (2008), 423-433.
9. S. M. Mohamed and B. J. Müller, *Continuous and Discrete Modules*, London Math. Soc. Lecture Notes Series 147, Cambridge, University Press, 1990.
10. A. C. Özcan, The torsion theory cogenerated by δ-M-small modules and GCO-modules, *Comm. Algebra.*, **35** (2007), 623-633.
11. Y. Talebi and N. Vanaja, The torsion theory cogenerated by M-small modules, *Comm. Algebra.*, **30**(3) (2002), 1449-1460.
12. Y. Talebi and M. J. Nematollahi, Modules with C^*-condition, *Taiwanese Journal of Mathematics*, **13**(5) (2009), 1451–1456.
13. N. Orhan and R. Tribak, Two generalizations of lifting modules, *Int. J. Algebra*, **3**(13) (2009), 599–612.
14. Y. Zhou, Generalizations of perfect, semiperfect, and semiregular rings, *Algebra Colloq*, **7**(3) (2000), 305–318.

Yahya Talebi

Department of Mathematics, Faculty of Mathematical Sciences

University of Mazandaran, Babolsar, Iran

Email: talebi@umz.ac.ir

Mehrab Hosseinpour

Department of Mathematics, Faculty of Mathematical Sciences

University of Mazandaran, Babolsar, Iran

Email: m.hpour@umz.ac.ir