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## ON THE CONVERGENCE OF MAPPINGS WITH k-FINITE DISTORTION.

**Abstract.** We prove that a locally uniform limit of a sequence of homeomorphisms with finite  $k$ -distortion is also a mapping with finite  $k$ -distortion. We obtain also an estimation for the distortion coefficient of the limit mapping.

**Key words:** *mapping with  $k$ -finite distortion, distortion coefficient, passing to the limit, differential form.*

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**1. Introduction.** It is well known that the limit of a uniformly converging sequence of analytic functions is an analytic function. Reshetnyak generalized this result to mappings with bounded distortion: *the limit of a locally uniformly converging sequence of mappings with bounded distortion is a mapping with bounded distortion.*

**Definition 1.** [8] A mapping  $f : \Omega \rightarrow \mathbb{R}^n$  is called *mapping with bounded distortion* if  $f$  is continuous,  $f \in W_{n,\text{loc}}^1(\Omega)$ , the Jacobian  $J(x, f)$  does not change the sign in the domain  $\Omega$  and

$$|Df(x)|^n \leq K |J(x, f)| \quad \text{for almost all } x \in \Omega. \quad (1)$$

The smallest constant in this inequality is called the *distortion coefficient* of the mapping  $f$  and is denoted by the symbol  $K(f)$ . It is clear that

$$K(f) = \sup \left\{ \frac{|Df(x)|^n}{|J(x, f)|} : x \in \Omega, \quad J(x, f) \neq 0 \right\}.$$

Reshetnyak used the weak convergence of Jacobians to prove the following theorem on the limit of a sequence of mappings with bounded distortion.

**Theorem 1.** [8] Let  $f_m : \Omega \rightarrow \mathbb{R}^n$ ,  $m = 1, 2, \dots$ , be an arbitrary sequence of mappings with bounded distortion, locally converging in  $L_n(\Omega)$  to a mapping  $f_0 : \Omega \rightarrow \mathbb{R}^n$ . Assume that the sequence of distortion coefficients  $K(f_m)$ ,  $m = 1, 2, \dots$ , is bounded. Then the limit mapping  $f_0$  is a mapping with bounded distortion and the following inequality holds:

$$K(f_0) \leq \liminf_{k \rightarrow \infty} K(f_m). \quad (2)$$

We briefly outline the proof in the case of non-negative Jacobians. For a test function  $\varphi \in C_0^\infty(\Omega)$  we have

$$\begin{aligned} \int_{\Omega} |Df_0(x)|^n \varphi(x) dx &\leq \liminf_{m \rightarrow \infty} \int_{\Omega} |Df_m(x)|^n \varphi(x) dx \leq \\ &\leq K \liminf_{m \rightarrow \infty} \int_{\Omega} J(x, f_m) \varphi(x) dx = K \int_{\Omega} J(x, f_0) \varphi(x) dx. \end{aligned}$$

To justify the limit in the last equality, we apply the weak convergence of Jacobians. Consequently, for the limit mapping, the point-wise inequality  $|Df_0(x)|^n \leq KJ(x, f_0)$  holds a. e. in  $\Omega$ .

More recently, research has begun on mappings with finite distortion. They are a natural generalization of mappings with bounded distortion.

**Definition 2.** [7] Let a mapping  $f : \Omega \rightarrow \mathbb{R}^n$  belong to the Sobolev class  $W_{n, \text{loc}}^1(\Omega)$  and  $J(x, f) \geq 0$ . We define the pointwise distortion coefficient  $K(x, f)$  of the mapping  $f$  as a value

$$K(x, f) = \begin{cases} \frac{|Df(x)|^n}{|J(x, f)|} & \text{if } J(x, f) > 0, \\ 1 & \text{otherwise.} \end{cases}$$

The mapping  $f : \Omega \rightarrow \mathbb{R}^n$  is called mapping with finite distortion ( $f \in FD(\Omega)$ ) if

$$|Df(x)|^n \leq K(x, f)J(x, f) \text{ where } 1 \leq K(x, f) < \infty \text{ for almost all } x \in \Omega.$$

**Remark.** In other words, the condition of finite distortion is that the partial derivatives of the mapping  $f \in W_{n, \text{loc}}^1(\Omega)$  vanish a.e. on the set of zeros of the Jacobian  $J(x, f)$ .

For the first time, the essential properties of mappings with finite distortion were investigated in the paper [15] in the study of homeomorphisms inducing a bounded composition operator. The name "mapping with finite distortion" was proposed much later in the paper [7].

In [5] F. Gehring and T. Iwaniec showed that the limit of a weakly converging sequence of mappings with finite distortion is also a mapping with finite distortion, and they obtained also an estimation for the distortion coefficient of the limit mapping.

**Theorem 2.** [5] *Let  $f_m : \Omega \rightarrow \mathbb{R}^n$ ,  $m = 1, 2, \dots$ , be an arbitrary sequence of mappings with finite distortion converging weakly in  $W_{n,loc}^1(\Omega)$  to a mapping  $f_0 : \Omega \rightarrow \mathbb{R}^n$ . Assume that*

$$K(x, f_m) \leq M(x) < \infty, \quad m = 1, 2, \dots,$$

where  $\Omega \ni x \mapsto M(x) \in [1, \infty]$  is a measurable function. Then the limit mapping  $f_0$  is a mapping with finite distortion and the inequality

$$K(x, f_0) \leq M(x)$$

holds.

More precisely, in this paper existence of a subsequence  $f_{m_k}$  such that

$$K(x, f_0) \leq b * \lim_{k \rightarrow \infty} K(x, f_{m_k})$$

was shown. Here the limit is understood in the sense of so-called biting convergence.

**Definition 3.** [2] *Let  $h$  and  $h_k$ ,  $k \in \mathbb{N}$ , be Lebesgue measurable functions defined on a set  $E \subset \mathbb{R}^n$ . The sequence  $h_k$  is said to converge in the biting sense on  $E$  to  $h$  if there exists an increasing sequence  $E_\nu$  of measurable subsets of  $E$ ,*

$$\bigcup_{\nu} E_\nu = E,$$

such that

$$\lim_{k \rightarrow \infty} \int_{E_\nu} \varphi h_k dx = \int_{E_\nu} \varphi h dx$$

for any function  $\varphi \in L_\infty(E_\nu)$ .

In the paper [4] the limit of a sequence of homeomorphisms with finite distortion converging weakly in  $W_1^1$  was shown to have, also, a finite distortion under the condition that the limit mapping is a homeomorphism.

**Theorem 3.** [4] *Let  $\Omega, \Omega'$  be bounded domains in  $\mathbb{R}^n$ ,  $f_j : \Omega \rightarrow \Omega'$ ,  $j \in \mathbb{N}$ ,  $f : \Omega \rightarrow \Omega'$ , be homeomorphisms belonging to  $W_1^1(\Omega)$ , and  $f_j \rightarrow f$  weakly in  $W_1^1(\Omega)$ . Assume that*

$$|Df_j(x)|^n \leq K(x, f_j)J(x, f_j) \quad \text{for almost all } x \in \Omega,$$

where  $K(x, f_j) : \Omega \rightarrow [1, \infty)$  are Borel functions for all  $j$ , and the sequence  $K(x, f_j)$  converges in the biting sense to  $K(x)$  as  $j \rightarrow \infty$ . Then the limit mapping  $f$  is a mapping with finite distortion and the inequality  $K(x, f) \leq K(x)$  holds for almost all  $x \in \Omega$ .

In the paper [1] mappings with bounded  $(q, p)$ -distortion ( $n - 1 < q \leq p < \infty$ ) were defined and investigated; these mappings coincide with the class of mappings with bounded distortion if  $q = p = n$ .

In the paper [17] a locally uniform limit of a sequence of mappings with bounded  $(\theta, 1)$ -weighted  $(q, p)$ -distortion was shown to be, also, a mapping with a bounded  $(\theta, 1)$ -weighted  $(q, p)$ -distortion and an estimation similar to (2) was established. The proofs of the theorems like those in the article [17] and in this work are based on the method developed in [11] for extending Reshetnyak's result to Carnot groups.

In this paper we extend the above-mentioned assertions to the class of mappings with  $k$ -finite distortion, which arise naturally in the problem of operating with differential forms of degree  $k$  (see [12]).

**2. Preliminaries.** Let  $U$  be a domain in  $\mathbb{R}^n$ . We consider the Banach space  $\mathcal{L}_p(U, \Lambda^k)$  of differential forms  $\omega$  of degree  $k$ ,  $k = 1, \dots, n$ , with measurable coefficients, which have the following finite norm:  $\|\omega\|_p = (\int_U |\omega|^p dx)^{1/p}$ .

A mapping  $f : U \rightarrow \mathbb{R}^n$  is said to be *approximate differentiable* at a point  $x \in U$  [3], if there exists a linear mapping  $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that

$$\lim_{r \rightarrow 0} \frac{|\{y \in B(x, r) : |f(y) - f(x) - L(y - x)| > \varepsilon\}|}{r^n} = 0 \quad (3)$$

for any  $\varepsilon > 0$ . Here the symbol  $|\cdot|$  denotes the Lebesgue measure. It is well known that the approximate differential is unique [3] if  $x$  is a density point. In what follows, it is denoted by the symbol  $\text{ap } Df(x)$ .

In our paper we consider the mappings belonging to some Sobolev class. Such mappings are unconditionally approximately differentiable.

Let  $\omega = \sum \omega_\beta dy^\beta$  be any  $k$ -form,  $k = 1, \dots, n$ , in  $W$  with continuous coefficients  $\omega_\beta : W \rightarrow \mathbb{R}$ , where the summation is over all  $k$ -dimensional ordered multi-indices  $\beta = (\beta_1, \dots, \beta_k)$ ,  $1 \leq \beta_1 < \dots < \beta_k \leq n$ , and  $dy^\beta = dy_{\beta_1} \wedge dy_{\beta_2} \wedge \dots \wedge dy_{\beta_k}$ . Let a mapping  $f = (f_1, \dots, f_n) : U \rightarrow W$  of Euclidean domains  $U, W \subset \mathbb{R}^n$  be approximately differentiable almost everywhere in  $U$ . We write the pull-back of the  $k$ -form  $\omega$  in the following way:

$$f^*\omega(x) = \sum_{\beta} \omega_{\beta}(f(x)) df_{\beta_1} \wedge df_{\beta_2} \wedge \dots \wedge df_{\beta_k} = \sum_{\alpha} \sum_{\beta} \omega_{\beta}(f(x)) M_{\alpha}^{\beta}(x) dx^{\alpha}.$$

In other words, it is a  $k$ -form with measurable coefficients, which are defined for almost all  $x \in U$  (here  $df_{\beta_k} = \sum_{i=1}^n \frac{\partial f_{\beta_k}}{\partial x_i} dx_i$  and the partial derivatives are understood in the approximate sense,  $M_{\alpha}^{\beta}(x)$  are  $(k \times k)$ -minors of the matrix  $\text{ap } Df(x) = \left( \frac{\partial f_j}{\partial x_i} \right)$ ,  $i, j = 1, \dots, n$ , with ordered lines and columns).

We recall that the approximate differential  $\text{ap } Df(x) : T_x U \rightarrow T_{f(x)} W$  is defined a.e. in  $U$ . It generates canonically the linear mapping  $\Lambda_k f(x) : \Lambda_k T_x U \rightarrow \Lambda_k T_{f(x)} W$  of the spaces of  $k$ -vectors, and the pullback operation  $f^*$  of  $k$ -forms. We denote the norm of the last linear mapping by the symbol  $|\Lambda^k f(x)|$ .

The minimal analytic and geometric properties of the mapping  $f$  were obtained in [12] for generating a bounded pullback operator

$$f^* : \mathcal{L}_p(W, \Lambda^k) \rightarrow \mathcal{L}_q(U, \Lambda^k), \quad 1 \leq q \leq p \leq \infty, \quad (4)$$

of differential forms of degree  $k = 1, \dots, n$ .

We say that an approximately differentiable mapping  $f : U \rightarrow W$  has  $k$ -finite distortion,  $1 \leq k \leq n$ , (shortly  $f \in \mathcal{CD}^k(U; W)$ ) if  $\text{rank ap } Df(x) < k$  almost everywhere on a set  $Z$ . (Hereinafter  $Z = \{x \in U : \det \text{ap } Df(x) = 0\}$ .) For  $k = 1$  ( $k = n - 1$ ) and  $f \in W_{1, \text{loc}}^1(U)$  ( $f \in W_{n-1, \text{loc}}^1(U)$ ), this notion is well-known in literature: it is just the class of Sobolev mappings with finite distortion (codistortion), which is characterized by the property:  $\text{ap } Df(x) = 0$  ( $\text{adj } Df(x) = 0$ ) almost everywhere on  $Z$  (see [10], [14] for the second notion).

Besides of the property of  $k$ -finite distortion we consider mappings with a certain behavior of some characteristics of the distortion containing in

itself the ratio  $\frac{|\Lambda^k f(x)|^q}{|J(x,f)|}$ , where  $J(x,f) = \det \operatorname{ap} Df(x)$  [12]: operator (4) is bounded if and only if the mapping  $f \in \mathcal{CD}^k(U; W)$  and the distortion function  $\mathbb{W} \ni y \mapsto H_{k,q}(y) =$

$$= \begin{cases} \left( \sum_{x \in f^{-1}(y) \setminus (\Sigma \cup Z)} \frac{|\Lambda^k f(x)|^q}{|J(x,f)|} \right)^{\frac{1}{q}} & \text{if } f^{-1}(y) \setminus (\Sigma \cup Z) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

belongs to  $L_{\varkappa}(W)$  where  $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$  if  $q < p$ , and  $\varkappa = \infty$  ( $\varkappa = q$ ) if  $q = p$  ( $p = \infty$ ). Moreover, the norm of the operator  $f^*$  is comparable with the value  $\|H_{k,q}(\cdot) | L_{\varkappa}(W)\|$ :

$$\alpha_{q,p} \|H_{k,q}(\cdot) | L_{\varkappa}(\Omega)\| \leq \|f^*\| \leq \|H_{k,q}(\cdot) | L_{\varkappa}(\Omega)\|$$

where  $\alpha_{q,p}$  is some constant.

Hereinafter  $\Sigma \subset U$  is a set of measure zero outside of which the mapping  $f$  has the Luzin property  $\mathcal{N}$ .

For homeomorphic mappings one can use a simpler characteristic.

**Corollary 1.** [12] *Let  $f : U \rightarrow W$  be an approximate differentiable homeomorphism. The operator  $f^* : \mathcal{L}_p(W, \Lambda^k) \rightarrow \mathcal{L}_q(U, \Lambda^k)$ ,  $1 \leq q \leq p \leq \infty$ ,  $k = 1, \dots, n$ , is bounded if and only if the following conditions are satisfied:*

- 1)  $f : U \rightarrow W$  has the  $k$ -finite distortion;
- 2) the function  $K_{k,p}(x, f) = \begin{cases} \frac{|\Lambda^k f(x)|}{|J(x,f)|^{1/p}} & \text{if } J(x,f) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$  belongs to  $L_{\varkappa}(U)$ , where  $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$  if  $q < p$ , and  $\varkappa = \infty$  ( $\varkappa = q$ ) if  $q = p$  ( $p = \infty$ ).

In this case, the norm of the operator  $f^*$  is comparable with

$$\|K_{k,p}(\cdot, f) | L_{\varkappa}(U)\| : \alpha_{q,p} \|K_{k,p}(\cdot, f) | L_{\varkappa}(\Omega)\| \leq \|f^*\| \leq \|K_{k,p}(\cdot, f) | L_{\varkappa}(\Omega)\|,$$

where  $\alpha_{q,p}$  is some constant.

### 3. Main results.

**Definition 4.** [12] *An approximately differentiable homeomorphism  $f : U \rightarrow W$  belongs to the class  $\mathcal{CD}_{q,p}^k(U; W)$  if the following conditions hold*

- 1)  $f \in \mathcal{CD}^k(U; W)$ ;

2)  $K_{k,p}(\cdot, f) \in L_{\varkappa}(U)$  where  $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$ ,  $1 \leq q \leq p < \infty$ .

**Theorem 4.** Let  $f_m \in \mathcal{CD}_{q,p}^k(U; W)$ ,  $m \in \mathbb{N}$ , be a sequence of homeomorphisms of the Sobolev class  $W_{l,\text{loc}}^1(U)$  with  $k < l$ ,  $q \leq l/k$ ,  $1 < q \leq p < \infty$ . Suppose that the sequence  $f_m$  is locally bounded in  $W_l^1(U)$ , and locally uniformly converges to a homeomorphism  $f : U \rightarrow W$  as  $m \rightarrow \infty$ . Assume also that there exists a sequence of functions  $U \ni x \mapsto M_m(x)$ , belonging to  $L_{\varkappa}(U)$ , that is bounded in  $L_{\varkappa}(U)$ ,  $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$ , for which the inequality

$$K_{k,p}(x, f_m) \leq M_m(x) \text{ for almost all } x \in U \quad (5)$$

is true.

Then there exists a function  $U \ni x \mapsto M(x)$  of  $L_{\varkappa}(U)$  such that some subsequence

- (i) in the case  $1 < q < p < \infty$ : of functions  $\{M_m(x)^{\varkappa}\}_{m \in \mathbb{N}}$  converges in the biting sense to  $M(x)^{\varkappa}$ ;
- (ii) in the case  $1 < q = p < \infty$ : of numbers  $\{\|M_m \mid L_{\infty}(U)\|\}_{m \in \mathbb{N}}$  converges to  $M = \varliminf_{m \rightarrow \infty} \|M_m \mid L_{\infty}(U)\|$ ;

the limit mapping  $f$  belongs to  $\mathcal{CD}_{q,p}^k(U; W)$  and  $K_{k,p}(\cdot, f) \in L_{\varkappa}(U)$ , where  $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$ .

Moreover, the inequalities

$$\begin{cases} K_{k,p}(x, f) \leq M(x) & \text{in the case } q < p, \\ K_{k,p}(x, f) \leq M & \text{in the case } q = p, \end{cases}$$

hold for almost all  $x \in U$ .

In the proof we use some arguments from the paper [17], where Theorem 4 is proved for  $k = 1$ .

**Proof.** It follows from the conditions of the theorem that  $f \in W_{l,\text{loc}}^1(U)$ . First, we show that the limit mapping  $f$  belongs to  $\mathcal{CD}^k(U; W)$ . For doing this, we show that the mapping  $f$  induces a bounded operator  $f^* : \mathcal{L}_p(W, \Lambda^k) \rightarrow \mathcal{L}_q(U, \Lambda^k)$ ,  $1 < q \leq p < \infty$ . Since every mapping  $f_m \in \mathcal{CD}_{q,p}^k(U; W)$ , it follows from Corollary 1 that the homeomorphism  $f_m : U \rightarrow W$  induces the bounded operator  $f_m^* : \mathcal{L}_p(W, \Lambda^k) \rightarrow \mathcal{L}_q(U, \Lambda^k)$ ,  $1 < q \leq p < \infty$ ,  $m \in \mathbb{N}$ . Moreover, the norms of the operators  $f_m^*$  are totally bounded

$$\|f_m^*\| \leq \|K_{k,p}(\cdot, f_m) \mid L_{\varkappa}(U)\| \leq \|M_m(\cdot) \mid L_{\varkappa}(U)\| \leq \tilde{M} < \infty.$$

Take a  $k$ -form  $\omega \in \mathcal{L}_p(W, \Lambda^k) \cap \mathcal{C}(W, \Lambda^k)$  and set  $\sigma_m = f_m^*(\omega)$ . Since  $\|f_m^*\| \leq \tilde{M}$ , the sequence of forms  $\sigma_m$  is bounded in  $\mathcal{L}_q(U, \Lambda^k)$ . Therefore, we can extract a weakly converging subsequence. We assume that the sequence  $\sigma_m$  converges weakly in  $\mathcal{L}_q(U, \Lambda^k)$  to a form  $\sigma_0$ . The weak convergence of forms means that coefficients of the forms  $\sigma_m$  converge weakly in  $L_q(U)$  to the corresponding coefficients of the form  $\sigma_0$ . Since the sequence  $\sigma_m$  converges weakly in  $\mathcal{L}_q(U, \Lambda^k)$  to  $\sigma_0$  as  $m \rightarrow \infty$ , we have

$$\begin{aligned} \|\sigma_0 \mid L_q(U)\| &\leq \varliminf_{m \rightarrow \infty} \|\sigma_m \mid L_q(U)\| = \varliminf_{m \rightarrow \infty} \|f_m^* \omega \mid L_q(U)\| \leq \\ &\leq \varliminf_{k \rightarrow \infty} \|f_m^*\| \cdot \|\omega \mid L_p(U)\| \leq \tilde{M} \cdot \|\omega \mid L_p(U)\|. \quad (6) \end{aligned}$$

The following lemma is proved in the book [8, Chapter 2, §4].

**Lemma 1.** [8] *Suppose  $U$  is an open subset in  $\mathbb{R}^n$ , and suppose that  $\varphi_m = (\varphi_{m1}, \varphi_{m2}, \dots, \varphi_{mk})$ ,  $1 \leq k \leq n$ ,  $m = 1, 2, \dots$ , is a sequence of vector-functions of  $W_{l, \text{loc}}^1(U)$ ,  $k \leq l$ , locally bounded in  $W_l^1(U)$ . Assume that, as  $m \rightarrow \infty$ , the functions  $\varphi_m$  converge in  $L_{1, \text{loc}}$  to a vector function  $\varphi_0 = (\varphi_{01}, \varphi_{02}, \dots, \varphi_{0k})$ , and set  $\omega_m = d\varphi_{m1} \wedge d\varphi_{m2} \wedge \dots \wedge d\varphi_{mk}$ . Then the sequence of forms  $\omega_m$  weakly converges in  $L_{l/k, \text{loc}}(U)$  to a form  $\omega_0$ <sup>1</sup>.*

Since the homeomorphisms  $f_m$  locally uniformly converge to  $f$  and the form  $\omega$  has continuous coefficients, the functions  $\omega_\beta(f_m(x))$  converge locally uniformly to  $\omega_\beta(f(x))$  as  $m \rightarrow \infty$ . Lemma 1 implies that the minors of the matrices  $Df_m$  converge weakly in  $L_{l/k, \text{loc}}(U)$  to minors of the matrix  $Df$ . Therefore, the forms  $\sigma_m$  converge weakly in  $L_{l/k, \text{loc}}(U)$  to  $f^*(\omega)$ .

It is not hard to see that both limits  $\sigma_0$  and  $f^*(\omega)$  coincide:  $\sigma_0 = f^*(\omega)$ . In view of (6) the mapping  $f$  induces a bounded operator  $f^* : \mathcal{L}_p(W, \Lambda^k) \rightarrow \mathcal{L}_q(U, \Lambda^k)$ ,  $1 < q \leq p < \infty$ . By Corollary 1,  $f \in \mathcal{CD}^k(U; W)$ .

First, we consider the case  $q < p$ . The following lemma is valid.

**Lemma 2.** [2] *Every sequence of mappings  $h_m$ ,  $m = 1, 2, \dots$ , that is bounded in  $L_1(U)$ , contains a subsequence, converging in the biting sense to some function  $h \in L_1(U)$ .*

This lemma implies existence of a function  $U \ni x \mapsto M(x)$  of  $L_{\mathcal{X}}(U)$  such that some subsequence of the function  $h_m = M_m(x)$ <sup>2</sup> converges

<sup>1</sup>It means that the sequence of forms  $\omega_m$  converges weakly in  $L_{l/k}(D)$  to a form  $\omega_0$  on every subdomain  $D \Subset U$ .



in the biting sense to  $h = M(x)^\varkappa$ . We assume that the given sequence  $M_m(x)^\varkappa$  converges in the biting sense to the function  $M(x)^\varkappa$  (the set of the Definition 3 is denoted by  $E_\nu$ ).

Now we estimate the distortion coefficient of the limit mapping  $f$ . For this, we consider estimates on a set  $E_\nu$ . Let  $Z_m$  be the set of zeros of the Jacobian of the mapping  $f_m$ . Since the rank of the matrix  $Df_m$  on the set  $Z_m$  is less than  $k$ , it follows that all  $k$ -th-order minors are equal to zero on the set  $Z_m$ .

Applying the Hölder inequality, and taking into account that  $\frac{q}{\varkappa} + \frac{q}{p} = 1$  on each intersection  $E_\nu \cap B(x_0, r)$ , where  $x_0 \in E_\nu$ ,  $B(x_0, r) \Subset U$ , in view of (5) we have

$$\begin{aligned}
 & \int_{E_\nu \cap B(x_0, r)} |\Lambda^k f_m(x)|^q dx = \int_{(E_\nu \cap B(x_0, r)) \setminus Z_m} \frac{|\Lambda^k f_m(x)|^q}{|J(x, f_m)|^{\frac{q}{p}}} |J(x, f_m)|^{\frac{q}{p}} dx \leq \\
 & \leq \left( \int_{(E_\nu \cap B(x_0, r)) \setminus Z_m} \frac{|\Lambda^k f_m(x)|^{q \frac{\varkappa}{q}}}{|J(x, f_m)|^{\frac{q \varkappa}{q}}} dx \right)^{\frac{q}{\varkappa}} \left( \int_{E_\nu \cap B(x_0, r)} |J(x, f_m)|^{\frac{q}{p} \frac{p}{q}} dx \right)^{\frac{q}{p}} = \\
 & = \left( \int_{E_\nu \cap B(x_0, r)} (K_{k,p}(f_m))^\varkappa(x) dx \right)^{\frac{q}{\varkappa}} \left( \int_{E_\nu \cap B(x_0, r)} |J(x, f_m)| dx \right)^{\frac{q}{p}} \leq \\
 & \leq \left( \int_{E_\nu \cap B(x_0, r)} M_m^\varkappa(x) dx \right)^{\frac{q}{\varkappa}} \left( \int_{B(x_0, r)} |J(x, f_m)| dx \right)^{\frac{q}{p}}. \quad (7)
 \end{aligned}$$

Elements of the matrix  $\Lambda^k(f_m)(x)$  are the  $k$ -th-order minors of  $Df_m(x)$ . In view of Lemma 1, they converge weakly in  $L_{l/k, \text{loc}}(U)$  to elements of the matrix  $\Lambda^k(f)(x)$ . Since  $q \leq l/k$ ,  $\Lambda^k(f_m)(x)$  converges weakly in  $L_{q, \text{loc}}(U)$  to  $\Lambda^k(f)(x)$ . Since the norm is semicontinuous in the Banach space  $L_q$ , the left-hand side of the inequality can be estimated as

$$\int_{E_\nu \cap B(x_0, r)} |\Lambda^k f(x)|^q dx \leq \liminf_{m \rightarrow \infty} \int_{E_\nu \cap B(x_0, r)} |\Lambda^k f_m(x)|^q dx.$$

We have also  $\int_{B(x_0, r)} |J(x, f_m)| dx \leq |f_m(B(x_0, r))|$ .

Since  $|f(B(x_0, r))| < \infty$  and the mapping  $f$  is a homeomorphism, the images  $f(S(x_0, r))$  of the spheres  $S(x_0, r)$  do not intersect under different

$r$ . It follows that the  $n$ -measure of the image of any sphere is zero for almost all  $r$ :  $|f(S(x_0, r))| = 0$ . We fix  $r$  so that

$$|f(S(x_0, r))| = 0$$

and surround the image of the sphere  $f(S(x_0, r))$  by an  $\varepsilon$ -neighborhood  $U_\varepsilon$ . Since the mappings  $f_m$  converge locally uniformly to the mapping  $f$ , it follows that, starting from a number  $m_0$ , the images of the spheres  $f_m(S(x_0, r))$ ,  $m \geq m_0$ , are contained in this  $\varepsilon$ -neighborhood. It is clear that  $|U_\varepsilon| \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , and hence  $|f_m(B(x_0, r))| \rightarrow |f(B(x_0, r))|$  as  $m \rightarrow \infty$ .

Taking into account that  $M_m(x)^\varkappa$  converge in the biting sense to  $M(x)^\varkappa$ , we pass to the lower limit in (7) as  $m \rightarrow \infty$ . We get

$$\int_{B(x_0, r)} |\Lambda^k f(x)|^q \chi_{E_\nu}(x) dx \leq \left( \int_{B(x_0, r)} M^\varkappa(x) \chi_{E_\nu}(x) dx \right)^{\frac{q}{\varkappa}} |f(B(x_0, r))|^{\frac{q}{p}}.$$

Dividing both sides of this inequality by the measure of the ball  $B(x_0, r)$ , we obtain the following inequality

$$\begin{aligned} \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} |\Lambda^k f(x)|^q \chi_{E_\nu}(x) dx &\leq \\ &\leq \left( \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} M^\varkappa(x) \chi_{E_\nu}(x) dx \right)^{\frac{q}{\varkappa}} \left( \frac{|f(B(x_0, r))|}{|B(x_0, r)|} \right)^{\frac{q}{p}}. \end{aligned} \quad (8)$$

Since the homeomorphism  $f$  is Sobolev differentiable, then by [13, Section 2.3, formula (2.5)] we have

$$\frac{|f(B(x_0, r))|}{|B(x_0, r)|} \rightarrow |J(x_0, f)| \text{ as } r \rightarrow 0 \text{ for almost all } x_0 \in E_\nu.$$

Hence, by the Lebesgue differentiability theorem, letting  $r$  go to 0 we obtain that

$$|\Lambda^k f(x)|^q \leq M^q(x) |J(x, f)|^{\frac{q}{p}} \text{ for almost all } x \in E_\nu. \quad (9)$$

As  $U = \bigcup_\nu E_\nu$ , the point-wise inequality (9) holds in  $U$  almost everywhere.

In the case  $1 < q = p < \infty$  we can assume that a sequence of numbers  $\{\|M_m \mid L_\infty(U)\|\}_{m \in \mathbb{N}}$  converges to  $M = \varliminf_{m \rightarrow \infty} \|M_m \mid L_\infty(U)\| \in \mathbb{R}$ . In

this case, instead of (7), for any  $\varepsilon > 0$  there exists  $m_1$  such that for all  $m \geq m_1$  we have

$$\int_{B(x_0, r)} |\Lambda^k f_m(x)|^q dx \leq (M + \varepsilon) \left( \int_{B(x_0, r)} |J(x, f_m)| dx \right).$$

Further, proceeding as in the case  $q < p$ , we obtain the estimation

$$K_{k,p}(x, f) \leq (M + \varepsilon) \text{ for almost all } x \in U.$$

Since  $\varepsilon > 0$  is an arbitrary number, we get the desired estimation.

By the Corollary 1, we have proved that the limit mapping  $f$  belongs to  $\mathcal{CD}_{q,p}^k(U; W)$ .  $\square$

As a straightforward consequence of Theorem 4 we get the following

**Corollary.** [16] *Let  $f_m \in \mathcal{CD}_{q,p}^k(U; W)$ ,  $m \in \mathbb{N}$ , be a sequence of homeomorphisms of the Sobolev class  $W_{l,\text{loc}}^1(U)$  with  $k < l$ ,  $q \leq l/k$ ,  $1 < q \leq p < \infty$ . Suppose that the sequence  $f_m$  is locally bounded in  $W_l^1(U)$ , and converge locally uniformly to a homeomorphism  $f : U \rightarrow W$  as  $m \rightarrow \infty$ . Assume also that there exists a function  $M(x) \in L_\infty(U)$ ,  $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$ , such that*

$$K_{k,p}(\cdot, f_m)(x) \leq M(x) \text{ for all } m \in \mathbb{N}$$

*in  $U$  almost everywhere. Then the limit mapping  $f$  belongs to  $\mathcal{CD}_{q,p}^k(U; W)$  and  $K_{k,p}(\cdot, f) \in L_\infty(U)$ , where  $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$ .*

*Moreover, the inequality  $K_{k,p}(x, f) \leq M(x)$  holds almost everywhere.*

**Remark.** There exists a misprint in the paper [16]: in the statement of the main result the condition  $q \leq l/k$  is missing.

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