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Strong convergence theorems for two finite families of generalized asymptotically quasi-nonexpansive mappings with applications

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Abstract. In this paper, an implicit iteration process has been proposed for two finite families of generalized asymptotically quasi-nonexpansive mappings and establish some strong convergence theorems in the framework of convex metric spaces. Also, some applications of our result has been given. Our results extend and generalize several results from the current existing literature.

1. Introduction and Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers and $J = \{1, 2, \ldots, N\}$, the set of first N natural numbers. Denote by $F(\mathcal{T})$ the set of fixed points of $\mathcal{T}$ and by $\mathcal{F} := \Big(\cap_{j=1}^{N} \mathcal{F}(\mathcal{T}_j)\Big) \cap \Big(\cap_{j=1}^{N} \mathcal{F}(\mathcal{S}_j)\Big)$ the set of common fixed points of two finite families of mappings $\{\mathcal{T}_j : j \in J\}$ and $\{\mathcal{S}_j : j \in J\}$.

First, we recall some definitions.

Definition 1.1. A mapping $\mathcal{T} \colon X \to X$ is called:

(i) Nonexpansive if $d(\mathcal{T}x, \mathcal{T}y) \leq d(x, y)$, for all $x, y \in X$.

(ii) Quasi-nonexpansive if $\mathcal{F}(\mathcal{T}) \neq \emptyset$ and $d(\mathcal{T}x, p) \leq d(x, p)$, for each $x \in X$ and $p \in \mathcal{F}(\mathcal{T})$.

(iii) Asymptotically nonexpansive [6] if there exists a sequence $a_n \in [0, \infty)$ with $\lim_{n\to\infty} a_n = 0$ such that $d(\mathcal{T}^n x, \mathcal{T}^n y) \leq (1 + a_n) d(x, y)$, for all $x, y \in X$ and for each $n \in \mathbb{N}$.

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(iv) Asymptotically quasi-nonexpansive if $\mathcal{F}(\mathcal{T}) \neq \emptyset$ and there exists a sequence $a_n \in [0, \infty)$ with $\lim_{n\to\infty} a_n = 0$ such that $d(\mathcal{T}^n x, p) \leq (1 + a_n)d(x, p)$, for all $x \in X$, $p \in \mathcal{F}(\mathcal{T})$ and for each $n \in \mathbb{N}$.

(v) Generalized asymptotically nonexpansive [21] if there exist sequence $a_n \in [0, \infty)$ and $c_n \in [0, \infty)$ with $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} c_n = 0$ such that $d(\mathcal{T}^n x, \mathcal{T}^n y) \leq (1+a_n)d(x, y)+c_n$, for all $x, y \in X$ and for each $n \in \mathbb{N}$.

(vi) Generalized asymptotically quasi-nonexpansive [21] if there exist sequence $a_n \in [0, \infty)$ and $c_n \in [0, \infty)$ with $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} c_n = 0$ such that $d(\mathcal{T}^n x, p) \leq (1 + a_n)d(x, p) + c_n$, for all $x \in X$, $p \in \mathcal{F}(\mathcal{T})$ and for each $n \in \mathbb{N}$.

(vii) Uniformly L-Lipschitzian if there exists a constant $\mathcal{L} > 0$ such that $d(\mathcal{T}^n x, \mathcal{T}^n y) \leq \mathcal{L}\, d(x, y)$, for all $x, y \in X$ and for each $n \in \mathbb{N}$.

Remark 1.1. From the above definitions, it is clear that

- a nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping,
- a quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping,
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- a generalized quasi-nonexpansive mapping is a generalized asymptotically quasi-nonexpansive mapping.

However, the converse of each of above statements may not be true. A generalized asymptotically quasi-nonexpansive mapping is not an asymptotically quasi-nonexpansive mapping [21].

Definition 1.2. [23] Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times [0, 1]$ and $u \in X$ holds

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

A metric space X together with the convex structure W is called a *convex metric space.*

Definition 1.3. Let X be a convex metric space. A nonempty subset F of X is said to be convex if $W(x, y, \lambda) \in F$ whenever $(x, y, \lambda) \in F \times F \times [0, 1]$.

In 1982, Kirk [13] used the term "hyperbolic type spaces" for convex metric spaces, and studied iteration processes for nonexpansive mappings in this abstract setting. Later on, many authors discussed the existence of the fixed point and the convergence of the iterative process for various mappings in convex metric spaces (see, for example, [1, 3, 4, 8, 11, 12, 16, 18, 20, 23]).

Recently, Yildirim and Khan [26] extended Definition 1.2 as follows.

Definition 1.4. A mapping $W\colon X^3 \times [0,1]^3 \to X$ is said to be a convex structure on X, if it satisfies the following condition.
For any $(x, y, z; a, b, c) \in X^3 \times [0,1]^3$ with $a+b+c = 1$ and $u \in X$ holds

$$d(W(x, y, z; a, b, c), u) \le ad(x, u) + bd(y, u) + cd(z, u).$$

If (X, d) is a metric space with a convex structure W, then (X, d) is called a convex metric space.

Definition 1.5. Let (X, d) be a convex metric space. A nonempty subset E of X is said to be convex if $W(x, y, z; a, b, c) \in E$, for all $(x, y, z) \in E^3$, $(a, b, c) \in [0,1]^3$ with $a+b+c = 1$.

Remark 1.2. (i) $W(x, y, z; a, b, c) = ax + by + cz$, for all $x, y, z \in X$ and $a, b, c \in [0,1]$ with $a+b+c = 1$ represents a line segment joining the points $x, y, z \in X$, since it is a convex subset of X.

(ii) Since $W(x, y, z; a, b, c)$ is a convex subset of X, therefore

$$d(W(x, y, z; a, b, c), u) = \inf\Big\{d(v, u) : v \in W(x, y, z; a, b, c)\Big\},$$

that is, it represents a distance between u and the convex subset W of X.

Takahashi [23] has shown that open sphere $B(x, r) = \{y \in X : d(y, x) < r\}$ and closed sphere $B[x, r] = \{y \in X : d(y, x) \le r\}$ are convex. All normed spaces and their convex subsets are convex metric spaces. But there are many examples of convex metric spaces which are not embedded in any normed space (see [23]).

Remark 1.3. Every normed space is a special convex metric space with a convex structure $W(x, y, z; \alpha, \beta, \gamma) = \alpha\, x + \beta\, y + \gamma\, z$, for all $x, y, z \in X$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha+\beta+\gamma = 1$. In fact,

$$\begin{aligned} d(u, W(x, y, z; \alpha, \beta, \gamma)) &= \|u - (\alpha\, x + \beta\, y + \gamma\, z)\| \\ &\le \alpha\|u - x\| + \beta\|u - y\| + \gamma\|u - z\| \\ &= \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z),\ \forall\, u \in X. \end{aligned}$$

Implicit Iteration Process of [25]

In 2001, Xu and Ori [25] introduced the following implicit iteration process for common fixed points of a finite family of nonexpansive mappings $\{\mathcal{T}_i : i \in I\}$ in Hilbert spaces:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)\mathcal{T}_n x_n,\ n \in \mathbb{N}, \tag{1}$$

where $\mathcal{T}_n = \mathcal{T}_{n(mod\,N)}$ and $\{\alpha_n\}$ is a real sequence in $(0,1)$. They proved a weak convergence theorem using this process.

Implicit Iteration Process of [22]

In 2003, Sun [22] extended the process (1) to the following process for common fixed points of a finite family of asymptotically quasi-nonexpansive mappings $\{\mathcal{T}_i : i \in I\}$ in uniformly convex Banach spaces:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)\mathcal{T}_i^k x_n,\ n \in \mathbb{N}, \tag{2}$$

where $n = (k-1)N + i$, $i \in I$ and $\{\alpha_n\}$ is a real sequence in $(0,1)$.

Sun [22] studied the strong convergence of the process (2) for common fixed points of the mappings $\{\mathcal{T}_i : i \in I\}$, requiring only one member of the family to be semicompact. The results of Sun [22] generalized and extended the corresponding results of Xu and Ori [25].

In 2008, Khan et al. [7] studied the following n-step iterative processes for a finite family of mappings $\{\mathcal{T}_i : i = 1, 2, \ldots, k\}$. Let $x_1 \in K$ and the iterative sequence $\{x_n\}$ is defined as follows:

$$\begin{cases} x_{n+1} = (1-\alpha_{kn})x_n + \alpha_{kn}\mathcal{T}_k^n y_{(k-1)n}, \\ y_{(k-1)n} = (1-\alpha_{(k-1)n})x_n + \alpha_{(k-1)n}\mathcal{T}_{k-1}^n y_{(k-2)n}, \\ \quad\vdots \\ y_{2n} = (1-\alpha_{2n})x_n + \alpha_{2n}\mathcal{T}_2^n y_{1n}, \\ y_{1n} = (1-\alpha_{1n})x_n + \alpha_{1n}\mathcal{T}_1^n y_{0n},\ n \geq 1, \end{cases} \tag{3}$$

where $y_{0n} = x_n$ for all $n \in \mathbb{N} \cup \{0\}$ and $\alpha_{in} \in [0,1]$, $n \geq 1$ and $i \in \{1, 2, \ldots, k\}$.

In 2010, Khan and Ahmed [8] considered the iteration process (3) in convex metric spaces as follows:

$$\begin{cases} x_{n+1} = W(\mathcal{T}_k^n y_{(k-1)n}, x_n; \alpha_{kn}), \\ y_{(k-1)n} = W(\mathcal{T}_{k-1}^n y_{(k-2)n}, x_n; \alpha_{(k-1)n}), \\ \quad\vdots \\ y_{2n} = W(\mathcal{T}_2^n y_{1n}, x_n; \alpha_{2n}), \\ y_{1n} = W(\mathcal{T}_1^n y_{0n}, x_n; \alpha_{1n}),\ n \geq 1, \end{cases} \tag{4}$$

where $y_{0n} = x_n$ for all $n \in \mathbb{N} \cup \{0\}$ and $\alpha_{in} \in [0,1]$, $n \geq 1$ and $i \in \{1, 2, \ldots, k\}$.

In 2010, Khan et al. [9] introduced an implicit iteration process for two finite families of nonexpansive mappings as follows:

Let $(E, \|.\|)$ be Banach space and $\mathcal{S}_i, \mathcal{T}_i \colon E \to E$, $(i \in I)$ be two families of nonexpansive mappings. For any given $x_0 \in E$, define an iteration process $\{x_n\}$ as

$$x_n = \alpha_n x_{n-1} + \beta_n \mathcal{S}_n x_n + \gamma_n \mathcal{T}_n x_n, \; n \in \mathbb{N}, \tag{5}$$

where $\mathcal{T}_n = \mathcal{T}_{n(modN)}$, $\mathcal{S}_n = \mathcal{S}_{n(modN)}$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$.

Recently, Yildirim and Khan [26] transformed iteration process (5) to the case of two families of asymptotically quasi-nonexpansive mappings in convex metric spaces as follows:

Let (X, d, W) be a convex metric space with convex structure W and $T_i, S_i \colon X \to X$ be two finite families of asymptotically quasi-nonexpansive mappings. For any given $x_0 \in X$, we define iteration process $\{x_n\}$ as follows.

$$\begin{aligned}
x_1 &= W(x_0, \mathcal{S}_1 x_1, \mathcal{T}_1 x_1; \alpha_1, \beta_1, \gamma_1), \\
x_2 &= W(x_1, \mathcal{S}_2 x_2, \mathcal{T}_2 x_2; \alpha_2, \beta_2, \gamma_2), \\
&\vdots \\
x_N &= W(x_{N-1}, \mathcal{S}_N x_N, \mathcal{T}_N x_N; \alpha_N, \beta_N, \gamma_N), \\
x_{N+1} &= W(x_N, \mathcal{S}_1^2 x_{N+1}, \mathcal{T}_1^2 x_{N+1}; \alpha_{N+1}, \beta_{N+1}, \gamma_{N+1}), \\
&\vdots \\
x_{2N} &= W(x_{2N-1}, \mathcal{S}_N^2 x_{2N}, \mathcal{T}_N^2 x_{2N}; \alpha_{2N}, \beta_{2N}, \gamma_{2N}), \\
x_{2N+1} &= W(x_{2N}, \mathcal{S}_1^3 x_{2N+1}, \mathcal{T}_1^3 x_{2N+1}; \alpha_{2N+1}, \beta_{2N+1}, \gamma_{2N+1}), \\
&\vdots
\end{aligned}$$

This iteration process can be rewritten in the following compact form:

$$x_n = W\Big(x_{n-1}, \mathcal{S}_i^k x_n, \mathcal{T}_i^k x_n; \alpha_n, \beta_n, \gamma_n\Big), \; n \in \mathbb{N}, \tag{6}$$

where $n = (k-1)N + i$, $i \in I$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and they established some strong convergence results which generalized some recent results from the literature (see, for example, [8, 9, 22, 24, 25]).

Notice that the iteration scheme (4) deals with one family and uses n-steps whereas (6) deals with two families and uses only one step. Hence our process is simpler than that used by [8] and is able to deal with two families at the same time.

Motivated and inspired by [22, 25, 26] and some others, we introduce and study the following iteration scheme:

Definition 1.6. Let (X, d, W) be a convex metric space with convex structure W, $\mathcal{T}_j, \mathcal{S}_j\colon X \to X$ be two finite families of generalized asymptotically quasi-nonexpansive mappings. For any given $x_0 \in X$, we define iteration process $\{x_n\}$ as follows.

$$\begin{aligned}
x_1 &= W(x_0, \mathcal{S}_1 x_1, \mathcal{T}_1 x_1; \alpha_1, \beta_1, \gamma_1),\\
x_2 &= W(x_1, \mathcal{S}_2 x_2, \mathcal{T}_2 x_2; \alpha_2, \beta_2, \gamma_2),\\
&\vdots\\
x_N &= W(x_{N-1}, \mathcal{S}_N x_N, \mathcal{T}_N x_N; \alpha_N, \beta_N, \gamma_N),\\
x_{N+1} &= W(x_N, \mathcal{S}_1^2 x_{N+1}, \mathcal{T}_1^2 x_{N+1}; \alpha_{N+1}, \beta_{N+1}, \gamma_{N+1}),\\
&\vdots\\
x_{2N} &= W(x_{2N-1}, \mathcal{S}_N^2 x_{2N}, \mathcal{T}_N^2 x_{2N}; \alpha_{2N}, \beta_{2N}, \gamma_{2N}),\\
x_{2N+1} &= W(x_{2N}, \mathcal{S}_1^3 x_{2N+1}, \mathcal{T}_1^3 x_{2N+1}; \alpha_{2N+1}, \beta_{2N+1}, \gamma_{2N+1}),\\
&\vdots
\end{aligned}$$

This iteration process can be rewritten in the following compact form:

$$x_n = W\Big(x_{n-1}, \mathcal{S}_i^k x_n, \mathcal{T}_i^k x_n; \alpha_n, \beta_n, \gamma_n\Big),\ n \in \mathbb{N}, \tag{7}$$

where $n = (k-1)N + j$, $j \in J$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$ and establish some strong convergence results in the setting of convex metric spaces.

Proposition 1.1. *Let X be a convex metric space and $\{\mathcal{T}_j : j \in J\}$ and $\{\mathcal{S}_j : j \in J\}$ be two finite families of generalized asymptotically quasi-nonexpansive mappings with $\mathcal{F} := \Big(\cap_{j=1}^N \mathcal{F}(\mathcal{T}_j)\Big) \cap \Big(\cap_{j=1}^N \mathcal{F}(\mathcal{S}_j)\Big) \neq \emptyset$. Then, there exists a point $p \in \mathcal{F}$ and sequences $\{w_n\}, \{g_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} w_n = 0 = \lim_{n\to\infty} g_n$ such that*

$$d(\mathcal{T}_j^n x, p) \leq (1 + w_n) d(x, p) + g_n \ \ \textit{and}\ \ d(\mathcal{S}_j^n x, p) \leq (1 + w_n) d(x, p) + g_n,$$

for all $x \in X$ and for each $j \in J$.

Proof. Since $\mathcal{T}_j, \mathcal{S}_j\colon X \to X$, $j \in J$ are generalized asymptotically quasi-nonexpansive mappings, therefore there exists a point $p \in \mathcal{F}$ and four sequences $\{u_n\}, \{v_n\}, \{c_n\}, \{d_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = 0$ such that $d(\mathcal{T}_j^n x, p) \leq (1 + u_n) d(x, p) + c_n$ and $d(\mathcal{S}_j^n x, p) \leq (1 + v_n) d(x, p) + d_n$ for all $x \in X$ and for each $j \in J$. Put $w_n = \sup\{u_n, v_n\}$ and $g_n = \sup\{c_n, d_n\}$ so that $d(\mathcal{T}_j^n x, p) \leq (1 + w_n) d(x, p) + g_n$

and $d(\mathcal{S}_j^n x, p) \leq (1+w_n)d(x,p) + g_n$ for all $x \in X$ and for each $j \in J$. This completes the proof. □

Lemma 1.1. *(See [14]) Let $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ be three sequences of nonnegative real numbers satisfying the following conditions:*

$$p_{n+1} \leq (1+q_n)p_n + r_n,\ n \geq 0, \quad \sum_{n=0}^{\infty} q_n < \infty, \quad \sum_{n=0}^{\infty} r_n < \infty.$$

Then:

(i) $\lim_{n\to\infty} p_n$ *exists.*
(ii) *In addition, if* $\liminf_{n\to\infty} p_n = 0$*, then* $\lim_{n\to\infty} p_n = 0$.

Remark 1.4. It is easy to verify that (ii) in Lemma 1.1 holds under the hypothesis $\limsup_{n\to\infty} p_n = 0$ as well. Therefore, the condition (ii) in Lemma 1.1 can be reformulated as follows:
(ii') If either $\liminf_{n\to\infty} p_n = 0$ or $\limsup_{n\to\infty} p_n = 0$, then $\lim_{n\to\infty} p_n = 0$.

2. Main Results

In this section, we prove some strong convergence results using iteration scheme (7) in the framework of convex metric spaces. First, we shall need the following lemma.

Lemma 2.1. *Let (X,d,W) be a convex metric space with convex structure W and $\mathcal{T}_j, \mathcal{S}_j \colon X \to X\,(j \in J)$ be two finite families of generalized asymptotically quasi-nonexpansive mappings with sequences $\{w_n\}, \{g_n\} \subset [0,\infty)$ as defined in Proposition 1.1. Suppose that $\mathcal{F} \neq \emptyset$ and that $x_0 \in X$, $\{\beta_n\} \subset (b, 1-b)$ for some $b \in (0, \frac{1}{2})$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} w_n < \infty$ and $\sum_{n=1}^{\infty} g_n < \infty$. Suppose that $\{x_n\}$ is as in (7). Then:*

(i) $\lim_{n\to\infty} d(x_n, p)$ *exists for all* $p \in \mathcal{F}$.
(ii) $\lim_{n\to\infty} d(x_n, \mathcal{F})$ *exists.*
(iii) *If* $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$*, where* $d(x, \mathcal{F}) = \inf\{d(x,y) : y \in \mathcal{F}\}$*, then* $\{x_n\}$ *is a Cauchy sequence.*

Proof. (i) Let $p \in \mathcal{F}$ and $n = (k-1)N + j$, $j \in J$. Then from (7), we have

$$\begin{aligned} d(x_n, p) &= d(W(x_{n-1}, \mathcal{S}_j^n x_n, \mathcal{T}_j^n x_n; \alpha_n, \beta_n, \gamma_n, p) \\ &\leq \alpha_n\, d(x_{n-1}, p) + \beta_n\, d(\mathcal{S}_j^n x_n, p) + \gamma_n\, d(\mathcal{T}_j^n x_n, p) \\ &\leq \alpha_n\, d(x_{n-1}, p) + \beta_n[(1+w_n)d(x_n,p) + g_n] \\ &\quad +\gamma_n[(1+w_n)d(x_n,p) + g_n] \\ &= \alpha_n\, d(x_{n-1}, p) + (\beta_n + \gamma_n + 2w_n)d(x_n,p) + (\beta_n+\gamma_n)g_n \\ &\leq \alpha_n\, d(x_{n-1}, p) + (\beta_n + \gamma_n + 2w_n)d(x_n,p) + 2g_n \\ &= \alpha_n\, d(x_{n-1}, p) + (\beta_n + \gamma_n + 2w_n)d(x_n,p) + t_n, \end{aligned} \tag{8}$$

where $t_n = 2g_n$ with $\sum_{n=1}^{\infty} t_n < \infty$. Since $\lim_{n\to\infty} \gamma_n = 0$, there exists a natural number n_1 such that $n > n_1$, $\gamma_n \le \frac{b}{2}$. Therefore

$$1 - \beta_n - \gamma_n \ge 1 - (1-b) - \frac{b}{2} = \frac{b}{2},$$

for $n > n_1$. Thus, from (8), we have

$$(1-\beta_n-\gamma_n)d(x_n,p) \quad \le \quad \alpha_n\, d(x_{n-1},p) + 2w_n d(x_n,p) + t_n,$$

so that

$$\begin{aligned} d(x_n,p) &\le \frac{\alpha_n}{1-\beta_n-\gamma_n} d(x_{n-1},p) + \frac{2w_n}{1-\beta_n-\gamma_n} d(x_n,p) \\ &\quad + \frac{t_n}{1-\beta_n-\gamma_n} \\ &\le d(x_{n-1},p) + \frac{4}{b} w_n d(x_n,p) + \frac{2}{b} t_n. \qquad (9) \end{aligned}$$

Since $\lim_{n\to\infty} w_n = 0$, there exists a natural number n_2 such that $n \ge n_2$ and

$$w_n \le \frac{b}{8}. \qquad (10)$$

From (9), we have

$$\Big(1 - \frac{4}{b} w_n\Big) d(x_n,p) \quad \le \quad d(x_{n-1},p) + \frac{2}{b} t_n.$$

That is,

$$d(x_n,p) \quad \le \quad \frac{b}{b-4w_n} d(x_{n-1},p) + \frac{2}{b-4w_n} t_n. \qquad (11)$$

Let

$$1 + \phi_n = \frac{b}{b-4w_n} = 1 + \frac{4w_n}{b-4w_n}.$$

But from (10), $2w_n \le \frac{b}{4}$, $b - 4w_n \ge b - \frac{b}{2} = \frac{b}{2}$ so that $\frac{1}{b-4w_n} \le \frac{2}{b}$ and so $\phi_n = \frac{4w_n}{b-4w_n} \le \frac{8}{b} w_n$.

Thus

$$\sum_{n=1}^{\infty} \phi_n = \sum_{n=1}^{\infty} \frac{8}{b} w_n < \infty.$$

Now by (11), we have

$$d(x_n,p) \quad \le \quad (1+\phi_n) d(x_{n-1},p) + \frac{4}{b} t_n. \qquad (12)$$

Since $\sum_{n=1}^{\infty} \phi_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, it follows from Lemma 1.1(i) that $\lim_{n\to\infty} d(x_n,p)$ exists.

(ii) Taking infimum over all $p \in \mathcal{F}$ in equation (12), we have that

$$d(x_n,\mathcal{F}) \quad \le \quad (1+\phi_n) d(x_{n-1},F) + \frac{4}{b} t_n.$$

Since $\sum_{n=1}^{\infty}\phi_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, it follows from Lemma 1.1(i) that $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists.

(iii) Note that, when $a > 0$, $1 + a \le e^a$. Thus from (12), we have

$$\begin{aligned}
d(x_{n+m}, p) &\le (1+\phi_{n+m})d(x_{n+m-1}, p) + \frac{4}{b}t_{n+m} \\
&\le e^{\phi_{n+m}} d(x_{n+m-1}, p) + \frac{4}{b}t_{n+m} \\
&\le e^{\phi_{n+m}}[e^{\phi_{n+m-1}} d(x_{n+m-2}, p) + \frac{4}{b}t_{n+m-1}] \\
&\quad +\frac{4}{b}t_{n+m} \\
&\le e^{\phi_{n+m}}.e^{\phi_{n+m-1}} d(x_{n+m-2}, p) + e^{\phi_{n+m}}\frac{4}{b}t_{n+m-1} \\
&\quad +\frac{4}{b}t_{n+m} \\
&\vdots \\
&\le \Big(e^{\sum_{l=0}^{m}\phi_{n+l}}\Big) d(x_n, p) + \frac{4}{b}\Big(e^{\sum_{l=0}^{m}\phi_{n+l}}\Big)\Big(\sum_{l=0}^{m} t_{n+l}\Big) \\
&= \mathcal{M} d(x_n, p) + \frac{4}{b}\mathcal{M}\Big(\sum_{l=0}^{m} t_{n+l}\Big),
\end{aligned}$$

for all $p \in \mathcal{F}$ and $n, m \in \mathbb{N}$ and $\mathcal{M} = e^{\sum_{l=0}^{\infty}\phi_l}$. That is,

$$d(x_{n+m}, p) \le \mathcal{M} d(x_n, p) + \frac{4}{b}\mathcal{M}\Big(\sum_{l=0}^{m} t_{n+l}\Big). \tag{13}$$

Now we use (13) to prove that $\{x_n\}$ is a Cauchy sequence. From the hypothesis $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$ and $\sum_{l=0}^{m} t_{n+l} < \infty$, for each $\varepsilon > 0$ there exists $n_3 \in \mathbb{N}$ such that

$$d(x_n, \mathcal{F}) < \frac{\varepsilon}{2(\mathcal{M}+1)} \ \forall\, n \ge n_3$$

and

$$\sum_{l=0}^{m} t_{n+l} < \frac{b\varepsilon}{8\mathcal{M}} \ \forall\, n \ge n_3. \tag{14}$$

Thus, there exists $z \in \mathcal{F}$ such that

$$d(x_n, z) < \frac{\varepsilon}{2(\mathcal{M}+1)} \ \forall\, n \ge n_3. \tag{15}$$

Using (14) and (15) in (13), we obtain

$$d(x_{n+m}, x_n) \le d(x_{n+m}, z) + d(x_n, z)$$

$$
\begin{aligned}
&\leq \mathcal{M}d(x_n,z)+\frac{4\mathcal{M}}{b}\Big(\sum_{l=0}^{m}t_{n+l}\Big)\\
&\quad +d(x_n,z)\\
&= (\mathcal{M}+1)d(x_n,z)+\frac{4\mathcal{M}}{b}\Big(\sum_{l=0}^{m}t_{n+l}\Big)\\
&< (\mathcal{M}+1).\Big(\frac{\varepsilon}{2(\mathcal{M}+1)}\Big)+\frac{4\mathcal{M}}{b}.\Big(\frac{b\varepsilon}{8\mathcal{M}}\Big)=\varepsilon,
\end{aligned}
$$

for all $n, m \geq n_3$. Thus $\{x_n\}$ is a Cauchy sequence. This completes the proof.

□

Theorem 2.1. *Let (X, d, W) be a convex metric space with convex structure W and $\mathcal{T}_j, \mathcal{S}_j \colon X \to X$ $(j \in J)$ be two finite families of generalized asymptotically quasi-nonexpansive mappings with sequences $\{w_n\}, \{g_n\} \subset [0,\infty)$ as defined in Proposition 1.1. Suppose that $\mathcal{F} \neq \emptyset$ and that $x_0 \in X$, $\{\beta_n\} \subset (b, 1-b)$ for some $b \in (0, \frac{1}{2})$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} w_n < \infty$ and $\sum_{n=1}^{\infty} g_n < \infty$. Suppose that $\{x_n\}$ is as in (7). Then*

(C_1) $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = \limsup_{n\to\infty} d(x_n, \mathcal{F}) = 0$ if $\{x_n\}$ converges to a unique point in $\mathcal{F}$.

(C_2) $\{x_n\}$ converges to a unique point in $\mathcal{F}$ if X is complete and either $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$ or $\limsup_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

Proof. (C_1) Let $q \in \mathcal{F}$. Since $\{x_n\}$ converges to p, $\lim_{n\to\infty} d(x_n, p) = 0$. So, for a given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, p) < \varepsilon \quad \forall\, n \geq n_0.$$

Taking the infimum over $p \in \mathcal{F}$, we obtain that

$$d(x_n, \mathcal{F}) < \varepsilon \quad \forall\, n \geq n_0.$$

This means $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$ so we obtain that $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = \limsup_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

(C_2) Suppose that X is complete and $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$ or $\limsup_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Then, we have from condition (ii) in Lemma 1.1 and Remark 1.4 that $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. From the completeness of X and Lemma 2.1, we get that $\lim_{n\to\infty} x_n$ exists and equals $v \in X$ (say). Moreover, since the set $\mathcal{F}$ of common fixed points of two finite families of mappings is closed, $v \in \mathcal{F}$ from $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. This shows that v is a common fixed point of $\{\mathcal{T}_j : j \in J\}$ and $\{\mathcal{S}_j : j \in J\}$. Hence $\{x_n\}$ converges to a unique point in $\mathcal{F}$. This completes the proof. □

3. Applications

As an application of Theorem 2.1, we establish some strong convergence results as follows.

Theorem 3.1. *Let (X, d, W) be a convex metric space with convex structure W and $\mathcal{T}_j, \mathcal{S}_j \colon X \to X$ $(j \in J)$ be two finite families of generalized asymptotically quasi-nonexpansive mappings with sequences $\{w_n\}, \{g_n\} \subset [0, \infty)$ as defined in Proposition 1.1. Suppose that $\mathcal{F} \neq \emptyset$ and that $x_0 \in X$, $\{\beta_n\} \subset (b, 1-b)$ for some $b \in (0, \frac{1}{2})$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} w_n < \infty$ and $\sum_{n=1}^{\infty} g_n < \infty$. Suppose that $\{x_n\}$ is as in (7). Assume that the following two conditions hold:*

(D_1) $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$;

(D_2) the sequence $\{y_n\}$ in X satisfying $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$ implies $\liminf_{n\to\infty} d(y_n, \mathcal{F}) = 0$ or $\limsup_{n\to\infty} d(y_n, \mathcal{F}) = 0$.

Then $\{x_n\}$ converges to a unique point in $\mathcal{F}$.

Proof. From conditions (D_1) and (D_2), we have that

$$\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0 \text{ or } \limsup_{n\to\infty} d(x_n, \mathcal{F}) = 0.$$

Therefore, we obtain from (C_2) in Theorem 2.1 that the sequence $\{x_n\}$ converges to a unique point in $\mathcal{F}$. This completes the proof. □

Theorem 3.2. *Let (X, d, W) be a convex metric space with convex structure W and $\mathcal{T}_j, \mathcal{S}_j \colon X \to X$ $(j \in J)$ be two finite families of generalized asymptotically quasi-nonexpansive mappings with sequences $\{w_n\}, \{g_n\} \subset [0, \infty)$ as defined in Proposition 1.1. Suppose that $\mathcal{F} \neq \emptyset$ and that $x_0 \in X$, $\{\beta_n\} \subset (b, 1-b)$ for some $b \in (0, \frac{1}{2})$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} w_n < \infty$ and $\sum_{n=1}^{\infty} g_n < \infty$. Suppose that $\{x_n\}$ is as in (7). Assume that $\lim_{n\to\infty} d(x_n, \mathcal{S}_j x_n) = \lim_{n\to\infty} d(x_n, \mathcal{T}_j x_n) = 0$ for all $j \in J$. If there exists an T_j or S_j, $j \in J$, which is semi-compact. Then the sequence $\{x_n\}$ converges to a point in $\mathcal{F}$.*

Proof. Without loss of generality, we can assume that $\mathcal{T}_1$ is semi-compact. From Lemma 2.1, we know that the sequence $\{x_n\}$ is bounded and by hypothesis of the theorem

$$\lim_{n\to\infty} d(x_n, \mathcal{S}_j x_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(x_n, \mathcal{T}_j x_n) = 0,$$

for all $j \in J$. Since $\mathcal{T}_1$ is semi-compact and $\lim_{n\to\infty} d(x_n, \mathcal{T}_1 x_n) = 0$, there exists a subsequence $\{x_{n_p}\}$ of $\{x_n\}$ such that $x_{n_p} \to q^* \in X$. Thus

$$d(q^*, \mathcal{T}_j q^*) = \lim_{p\to\infty} d(x_{n_p}, \mathcal{T}_j x_{n_p}) = 0$$

and

$$d(q^*, \mathcal{S}_j q^*) = \lim_{p\to\infty} d(x_{n_p}, \mathcal{S}_j x_{n_p}) = 0,$$

for all $j \in J$. Which implies that $q^* \in \mathcal{F}$ and so

$$\liminf_{n\to\infty} d(x_n, \mathcal{F}) \leq \liminf_{p\to\infty} d(x_{n_p}, \mathcal{F}) \leq \lim_{p\to\infty} d(x_{n_p}, q^*) = 0.$$

It follows from Theorem 2.1 that $\{x_n\}$ converges strongly to a point in $\mathcal{F}$. This completes the proof. □

Theorem 3.3. *Let (X, d, W) be a convex metric space with convex structure W and $\mathcal{T}_j, \mathcal{S}_j\colon X \to X\ (j \in J)$ be two finite families of generalized asymptotically quasi-nonexpansive mappings with sequences $\{w_n\}, \{g_n\} \subset [0, \infty)$ as defined in Proposition 1.1 satisfying condition $\lim_{n\to\infty} d(x_n, \mathcal{S}_j x_n) = \lim_{n\to\infty} d(x_n, \mathcal{T}_j x_n) = 0$, for all $j \in J$. Suppose that $\mathcal{F} \neq \emptyset$ and that $x_0 \in X$, $\{\beta_n\} \subset (b, 1-b)$ for some $b \in (0, \frac{1}{2})$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} w_n < \infty$ and $\sum_{n=1}^{\infty} g_n < \infty$. Suppose that $\{x_n\}$ is as in (7). If either of the following condition is true, then the sequence $\{x_n\}$ defined by (7) converges to a unique point in $\mathcal{F}$.*

(E_1) If there exist constants $\mathcal{K}_1, \mathcal{K}_2 > 0$ such that either $d(x_n, \mathcal{T}_j x_n) \geq \mathcal{K}_1 d(x_n, \mathcal{F})$ or $d(x_n, \mathcal{S}_j x_n) \geq \mathcal{K}_2 d(x_n, \mathcal{F})$ for all $n \in \mathbb{N}$ and $j \in J$.

(E_2) There exists a function $f\colon [0, \infty) \to [0, \infty)$ which is right continuous at 0, $f(0) = 0$ and $f(d(x_n, \mathcal{T}_j x_n)) \geq d(x_n, \mathcal{F})$ or $f(d(x_n, \mathcal{S}_j x_n)) \geq d(x_n, \mathcal{F})$ for all $n \in \mathbb{N}$ and $j \in J$.

Proof. First suppose that (E_1) holds. Then, in both the cases, we obtain

$$\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0.$$

Thus, $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$ or $\limsup_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Hence by Theorem 2.1, the sequence $\{x_n\}$ must converges strongly to a point in $\mathcal{F}$.

Next, assume that (E_2) holds. Then either

$$\lim_{n\to\infty} d(x_n, \mathcal{F}) \leq \lim_{n\to\infty} f(d(x_n, \mathcal{T}_j x_n)) = f(\lim_{n\to\infty} d(x_n, \mathcal{T}_j x_n) = f(0) = 0$$

or

$$\lim_{n\to\infty} d(x_n, \mathcal{F}) \leq \lim_{n\to\infty} f(d(x_n, \mathcal{S}_j x_n)) = f(\lim_{n\to\infty} d(x_n, \mathcal{S}_j x_n) = f(0) = 0.$$

Again in both the cases, $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Thus, $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$ or $\limsup_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Hence by Theorem 2.1, the sequence $\{x_n\}$ converges to a point in $\mathcal{F}$. This completes the proof. □

Now, we give an example in support of our result: take two mappings $T_1 = T_2 = \cdots = T_N = T$ and $S_1 = S_2 = \cdots = S_N = S$ as follows:

Example 3.1. Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$. For each $x \in X$, define two mappings T, $S\colon X \to X$ by

$$T(x) = \begin{cases} \frac{x}{2} \sin\frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

and

$$S(x) = \begin{cases} \frac{x}{5}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then T and S are asymptotically quasi-nonexpansive mappings with sequences $\{w_n\} = \{\frac{1}{n^2}\}$ and $\{g_n\} = \{\frac{1}{n^3}\}$ for all $n \in \mathbb{N}$ and hence are generalized asymptotically quasi-nonexpansive mappings by Remark 1.1. Also

$F(S) = \{0\}$ is the unique fixed point of S and $F(T) = \{0\}$ is the unique fixed point of T, that is, $F = F(S) \cap F(T) = \{0\}$ is the unique common fixed point of S and T.

4. Concluding remarks

In this paper, we proposed and study an implicit iteration process for two finite families of generalized asymptotically quasi-nonexpansive mappings in convex metric spaces and establish some strong convergence results. Also, we give some applications of our result in the setting of convex metric spaces. The results presented in this paper are extensions and improvements of several corresponding results from the current existing literature (see, for example, [8, 9, 17, 19, 22, 24, 25, 26] and many others).

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