Mathematica Moravica
Vol. **22**, No. **1** (2018), 31–41
doi:10.5937/MatMor1801031T

Well-posedness and asymptotic stability for the Lamé system with internal distributed delay

Noureddine Taouaf, Noureddine Amroun, Abbes Benaissa and Abderrahmane Beniani

Abstract. In this work, we consider the Lamé system in 3-dimension bounded domain with distributed delay term. We prove, under some appropriate assumptions, that this system is well-posed and stable. Furthermore, the asymptotic stability is given by using an appropriate Lyapunov functional.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Let us consider the following Lamé system with a distributed delay term:

$$\begin{cases} u''(x,t) - \Delta_e u(x,t) & \\ \qquad + \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s) u'(x,t-s)\mathrm{d}s + \mu_1 u'(x,t) = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \end{cases} \tag{1.1}$$

with initial conditions

$$\begin{cases} u(x) = u_0(x), \quad u'(x,0) = u_1(x), & \text{in } \Omega, \\ u'(x,-t) = f_0(x,-t), & \text{in } \Omega \times (0,\tau_2), \end{cases} \tag{1.2}$$

where (u_0, u_1, f_0) are given history and initial data. Here Δ denotes the Laplacian operator and Δ_e denotes the elasticity operator, which is the 3×3 matrix-valued differential operator defined by

$$\Delta_e u = \mu\Delta u + (\lambda+\mu)\nabla(\operatorname{div} u), \quad u = (u_1, u_2, u_3)^T$$

and μ and λ are the Lamé constants which satisfy the conditions

$$\mu > 0, \quad \lambda + \mu \geq 0. \tag{1.3}$$

Moreover, $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function and $\tau_1 < \tau_2$ are two positive constants.

2010 *Mathematics Subject Classification.* Primary: 35L05, 35L15, 35L70, 35B40.

Key words and phrases. Lamé system, delay terms, Lyapunov functions, decay rates.
Full paper. Received 29 September 2017, revised 2 January 2018, accepted 21 February 2018, available online 25 June 2018.

In the particular case $\lambda + \mu = 0$, $\Delta_e = \mu\Delta$ gives a vector Laplacian, that is, (1.1) describes the vector wave equation.

In recent years, the control of partial differential equations with time delay effects has become an active and attractive area of research see ([1, 7, 9, 14, 15, 16] and [21]), and the references therein. Recently, S. A. Messaoudi et al. [21] considered the following problem with a strong damping and a strong distributed delay:

$$(1.4)\qquad \begin{cases} u_{tt} - \Delta_x u(x,t) - \mu_1 \Delta u_t(x,t) & \\ \qquad - \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s)\Delta u_t(x,t-s)\mathrm{d}s = 0, & \text{in } \Omega \times (0,+\infty), \\ u = 0, & \text{on } \Gamma \times [0,+\infty), \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & \text{on } \Omega, \\ u_t(x,-t) = f_0(x,-t), & 0 < t \le \tau_2, \end{cases}$$

and under the assumption

$$(1.5)\qquad \mu_1 > \int_{\tau_1}^{\tau_2} \mu_2(s)\mathrm{d}s.$$

The authors proved that the solution is exponentially stable.

In [3], the authors considered the Bresse system in bounded domain with internal distributed delay

$$(1.6)\qquad \begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + lw + \psi)_x - Ehl(w_x - l\varphi) \\ \qquad + \mu_1\varphi_t + \mu_2\varphi_t(x,t-\tau_1) = 0, \\ \rho_2\psi_{tt} - El\psi_{xx} - Gh(\varphi_x - lw + \psi) \\ \qquad + \displaystyle\int_{\tau_1}^{\tau_2} \mu(s)\psi_t(x,t-s)\mathrm{d}s = 0, \\ \rho_1 w_{tt} - Eh(w_x - l\varphi)_x + lGh(\varphi_x + lw + \psi) \\ \qquad + \widetilde{\mu_1} w_t + \widetilde{\mu_2} w_t(x,t-\tau_2) = 0, \end{cases}$$

where $(x,t) \in]0,L[\times\mathbb{R}_+$, the authors proved, under suitable conditions, that the system is well-posed and its energy converges to zero when time goes to infinity. For Timoshenko-type system with thermoelasticity of second sound, in the presence of a distributed delay Apalara [1] considered the following system:

$$(1.7)\qquad \begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \mu\varphi_t & \\ \qquad + \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s)\varphi_t(x,t-s)\mathrm{d}s = 0, & \text{in } (0,1)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi xx + k(\varphi_x + \psi) + \gamma\theta_x = 0, & \text{in } (0,1)\times(0,\infty), \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0, & \text{in } (0,1)\times(0,\infty), \\ \tau q_t + Bq + \theta_x = 0, & \text{in } (0,1)\times(0,\infty), \end{cases}$$

and proved an exponential decay result under the assumption

$$\mu > \int_{\tau_1}^{\tau_2} \mu_2(s)\mathrm{d}s. \tag{1.8}$$

In [4], Beniani and al. considered the following Lamé system with time varing delay term:

$$\left\{\begin{array}{ll} u''(x,t) - \Delta_e u(x,t) + \mu_1 g_1(u'(x,t)) & \\ \qquad + \mu_2 g_2(u'(x,t-\tau(s)) = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \end{array}\right. \tag{1.9}$$

the authors proved, under suitable conditions, that energy converges to zero when time goes to infinity.

The paper is organized as follows. In Section 2, we prove the global existence and uniqueness of solutions of (1.1)-(1.2). In Section 3, we prove the stability results.

2. Well-posedness

In this section, we prove the existence and uniqueness of solutions of (1.1)-(1.2) using semigroup theory.

As in [20], we introduce the variable

$$z(x,\rho,t,s) = u'(x,t-\rho s), \quad (x,\rho,t,s) \in \Omega \times (0,1) \times (0,\infty) \times (\tau_1,\tau_2).$$

Then, it is easy to check that

$$s z_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0, \quad (x,\rho,t,s) \in \Omega \times (0,1) \times (0,\infty) \times (\tau_1,\tau_2). \tag{2.1}$$

Thus, system (1.1) becomes

$$\left\{\begin{array}{ll} u''(x,t) - \Delta_e u(x,t) + \mu_1 u'(x,t) & \\ \qquad + \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\mathrm{d}s = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ s z_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0, & \text{in } \Omega \times (0,1) \times (0,\infty) \times (\tau_1,\tau_2) \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & \text{in } \Omega \times (0,1) \times (\tau_1,\tau_2), \\ z(x,\rho,0,s) = f_0(x,-\rho s), & \text{in } \Omega \times (0,1) \times (\tau_1,\tau_2), \end{array}\right. \tag{2.2}$$

Next, we will formulate the system (1.1)-(1.2) in the following abstract linear first-order system:

$$\left\{\begin{array}{l} \mathcal{U}_t(t) = \mathcal{A}\mathcal{U}(t), \text{for } t > 0, \\ \mathcal{U}(0) = \mathcal{U}_0, \end{array}\right. \tag{2.3}$$

where $\mathcal{U} = (u, u_t, z)^T$, $\mathcal{U}_0 = (u_0, u_1, f_0)^T \in \mathcal{H}$

$$\mathcal{H} = H_0^1(\Omega)^3 \times (L^2(\Omega))^3 \times L^2((0,1), H)$$

We define the inner product in $\mathcal{H}$,

$$\langle V, \bar{V}\rangle_{\mathcal{H}} = \int_\Omega v\bar{v}\mathrm{d}x + \mu\int_\Omega \nabla u\nabla\bar{u}\mathrm{d}x + (\lambda+\mu)\int_\Omega \operatorname{div} u\cdot\operatorname{div}\bar{u}\mathrm{d}x \\ + \int_\Omega\int_{\tau_1}^{\tau_2} s\mu_2(s)\int_0^1 z(x,\rho,t,s)\bar{z}(x,\rho,t,s)\mathrm{d}\rho\mathrm{d}s\mathrm{d}x.$$

The operators $\mathcal{A}$ is linear and given by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} v \\ \Delta_e u(x,t) - \mu_1 v(x,t) - \int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,t,s)\mathrm{d}s \\ -\frac{1}{s}z_\rho(x,\rho,t,s) \end{pmatrix} \tag{2.4}$$

The domain $D(\mathcal{A})$ of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{V = (u,v,z)^T \in \mathcal{H}, \mathcal{A}V \in \mathcal{H}, z(,0) = v\right\}.$$

The well-posedness of problem (2.3) is ensured by the following theorem.

Theorem 2.1. *Assume that*

$$\mu_1 > \int_{\tau_1}^{\tau_2}\mu_2(s)\mathrm{d}s. \tag{2.5}$$

Then, for any $\mathcal{U}_0 \in \mathcal{H}$, the system (2.3) *has a unique weak solution*

$$\mathcal{U} \in C(\mathbb{R}^+, \mathcal{H}).$$

Moreover, if $\mathcal{U} \in D(\mathcal{A})$, then the solution of (2.3) *satisfies* (*classical solution*)

$$\mathcal{U} \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C(\mathbb{R}^+, D(\mathcal{A})).$$

Proof. We prove that $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is a maximal monotone operator, that is, $\mathcal{A}$ is dissipative and $Id - \mathcal{A}$ is surjective. Indeed, a simple calculation implies that, for any $V = (u,v,z)^T \in D(\mathcal{A})$,

$$
\begin{aligned}
\langle \mathcal{A}V, V\rangle_{\mathcal{H}} = \mu \int_\Omega & \nabla v(x,t) \nabla u(x,t) \mathrm{d}x \\
& + (\lambda+\mu) \int_\Omega \mathrm{d}v(x,t) \cdot \mathrm{d}u(x,t) \mathrm{d}x \\
& + \int_\Omega \Big\{ \Delta_e u(x,t) - \mu_1 v(x,t) \\
& - \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s)\,\mathrm{d}s \Big\} v(x,t) \mathrm{d}x \\
& - \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) \int_0^1 z(x,\rho,t,s) z_\rho(x,\rho,t,s) \mathrm{d}\rho \mathrm{d}s \mathrm{d}x \\
= -\mu_1 & \int_\Omega v^2(x,t) \mathrm{d}x \\
& - \int_\Omega v(x,t) \Big(\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s) \mathrm{d}s \Big) \mathrm{d}x \\
& - \frac{1}{2} \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) \int_0^1 \frac{\partial}{\partial \rho} |z(x,\rho,t,s)|^2 \mathrm{d}\rho \mathrm{d}s \mathrm{d}x
\end{aligned} \tag{2.6}
$$

Using Young's inequality and taking into account that $z(.,0,.,.) = v$, we get

$$
\langle \mathcal{A}V, V\rangle_{\mathcal{H}} = -\Big(\mu_1 - \int_{\tau_1}^{\tau_2} \mu_2(s) \mathrm{d}s \Big) \int_\Omega v^2(x,t) \mathrm{d}x \tag{2.7}
$$

by virtue of (2.5). Therefore, $\mathcal{A}$ is dissipative. On the other hand, we prove that $Id - \mathcal{A}$ is surjective. Indeed, let $F = (f,g,h)^T \in \mathcal{H}$ we show that there exists $V = (u,v,z)^T \in D(\mathcal{A})$ satisfying

$$
(Id - \mathcal{A})V = F \tag{2.8}
$$

which is equivalent to

$$
\begin{cases}
u - v = f, \\
v - \Delta_e u + \mu_1 v + \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s) \mathrm{d}s = g, \\
s z(x,\rho,t,s) + z_\rho(x,\rho,t,s) = hs,
\end{cases} \tag{2.9}
$$

Using the equation in (2.9), we obtain

$$
z(x,t,\rho,s) = (u-f) e^{-\rho s} + e^{-\rho s} \int_0^\rho s h(x,\sigma) e^{\sigma s} \mathrm{d}\sigma.
$$

Replacing v by $u - f$ in the second equation of (2.9), we get

$$
Ku - \Delta_e u = G. \tag{2.10}
$$

where

$$
K = 1 + \mu_1 + \int_{\tau_1}^{\tau_2} e^{-s} \mu_2(s) \mathrm{d}s > 0 \tag{2.11}
$$

and

$$G = g + \Big(1 - \mu_1 - \int_{\tau_1}^{\tau_2} e^{-s}\mu_2(s)\mathrm{d}s\Big) f$$
$$+ \int_{\tau_1}^{\tau_2} se^{-s}\mu_2(s) \int_0^1 h(x,\sigma)\cdot e^{\sigma s}\mathrm{d}\sigma\mathrm{d}s.$$

So, we multiply (2.10) by a test function $\varphi \in (H_0^1(\Omega))^3$ and we integrate by using Green's equality , obtaining the following variational formulation of (2.10):

$$a(u,\varphi) = L(\varphi) \quad \text{for } \varphi \in (H_0^1(\Omega))^3, \tag{2.12}$$

where

$$a(u,\varphi) = \int_\Omega (Ku\cdot\varphi + \mu\nabla u\cdot\nabla\varphi + (\lambda+\mu)\operatorname{div} u\cdot\operatorname{div}\varphi)\mathrm{d}x \tag{2.13}$$

and

$$L(\varphi) = \int_\Omega G\varphi\mathrm{d}x. \tag{2.14}$$

It is clear that a is a bilinear and continuous form on $(H_0^1(\Omega))^3\times(H_0^1(\Omega))^3$, and L is a linear and continuous form on $(H_0^1(\Omega))^3$. On the other hand, (1.3) and (2.11) imply that there exists a positive constant a_0 such that

$$a(u,u) \geq a_0\|u\|_{(H_0^1(\Omega))^3}, \quad \text{for each } v_1 \in (H_0^1(\Omega))^3,$$

which implies that a is coercive. Therefore, using the Lax-Milgram Theorem, we conclude that (2.12) has a unique solution $u \in (H_0^1(\Omega))^3$. By classical regularity arguments, we conclude that the solution u of (2.12) belongs into $(H^2(\Omega)\cap H_0^1(\Omega))^3$. Consequently, we deduce that (2.8) has a unique solution $V \in D(\mathcal{A})$. This proves that $Id - \mathcal{A}$ is surjective. Finally, (2.6) and (2.8) mean that $-\mathcal{A}$ is maximal monotone operator. Then, using Lummer-Phillips theorem (see [23]), we deduce that $\mathcal{A}$ is an infinitesimal generator of a linear C_0-semigroup on $\mathcal{H}$. □

3. Stability

In this section, we investigate the asymptotic behaviour of the solution of problem (2.3). In fact, using the energy method to produce a suitable Lyapunov functional, we define the energy associated with the solution of (1.1)-(1.2) by

$$\begin{aligned} E_u(t) = {} & \frac{1}{2}\int_\Omega \big(\mu|\nabla u|^2 + (\lambda+\mu)|\operatorname{div} u|^2 + |u'|^2\big)\mathrm{d}x \\ & + \frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,t,\rho,s)\mathrm{d}s\mathrm{d}\rho\mathrm{d}x. \end{aligned} \tag{3.1}$$

Theorem 3.1. *Assume that* (1.3) *and* (2.5) *hold. Then, for any* $\mathcal{U}_0 \in \mathcal{H}$*, there exist positive constants* δ_1 *and* δ_2*, such that the solution of* (2.3) *satisfies*

$$E(t) \leq \delta_2 e^{-\delta_1 t}, \quad \text{for } t \in \mathbb{R}^+. \tag{3.2}$$

We carry out the proof of Theorem 3.1. Firstly, we will estimate several Lemmas.

Lemma 3.2. *Suppose that* μ_1, μ_2 *satisfy* (2.5)*. Then energy functional satisfies, along the solution* u *of* (1.1)-(1.2)*,*

$$E'(t) \leq -\Big(\mu_1 - \int_{\tau_1}^{\tau_2} \mu_2(s)\mathrm{d}s\Big)\int_\Omega u'^2(x,t)\mathrm{d}x \leq 0 \tag{3.3}$$

Proof. A differentiation of $E(t)$ gives

$$\begin{aligned} E'(t) &= \int_\Omega \big(\mu\nabla u\nabla u' + (\lambda+\mu)\operatorname{div} u \operatorname{div} u' + u'u''\big)\mathrm{d}x \\ &\quad + \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z'(x,t,\rho,s)z(x,t,\rho,s)\mathrm{d}s\mathrm{d}\rho\mathrm{d}x. \end{aligned} \tag{3.4}$$

Using (2.2) and integrating by parts, we get

$$\begin{aligned} E'(t) &= -\mu_1\int_\Omega u'^2(x,t)\mathrm{d}x - \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,t,1,s)u'(x,t)\mathrm{d}s\mathrm{d}x \\ &\quad - \frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\frac{\partial}{\partial\rho}\Big(z^2(x,t,\rho,s)\Big)\mathrm{d}s\mathrm{d}\rho\mathrm{d}x \\ &= -\mu_1\int_\Omega u'^2(x,t)\mathrm{d}x - \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,t,1,s)u'(x,t)\mathrm{d}s\mathrm{d}x \\ &\quad - \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,t,1,s)\mathrm{d}s\mathrm{d}x \\ &\quad + \frac{1}{2}\Big(\int_{\tau_1}^{\tau_2}|\mu_2(s)\mathrm{d}s\Big)\int_\Omega u'^2(x,t)\mathrm{d}x. \end{aligned} \tag{3.5}$$

Young's inequality leads to the desired estimate. □

Lemma 3.3. *The functional*

$$\phi(t) = \int_\Omega u\cdot u'\mathrm{d}x, \quad \text{for } t \in \mathbb{R}^+ \tag{3.6}$$

satisfies, along the solution u *of* (1.1)-(1.2)

$$\begin{aligned} \phi'(t) &\leq c\int_\Omega |u'|^2\mathrm{d}x - (\mu - c)\int_\Omega|\nabla u|^2\mathrm{d}x - (\lambda+\mu)\int_\Omega|\operatorname{div} u|^2\mathrm{d}x \\ &\quad + c\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,t,1,s)\mathrm{d}s\mathrm{d}x, \end{aligned} \tag{3.7}$$

for a positive constant c.

Proof. By differentiating (3.6) and using (2.2), yields

$$\phi'(t) = \int_\Omega |u'|^2 dx - \mu \int_\Omega |\nabla u|^2 \mathrm{d}x - (\lambda+\mu)\int_\Omega |\operatorname{div} u|^2 \mathrm{d}x - \mu_1 \int_\Omega uu' \mathrm{d}x - \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| uz(x,t,1,s) \mathrm{d}s\mathrm{d}x. \tag{3.8}$$

By using Young's inequality, we obtain

$$\begin{aligned}\phi'(t) \le {} & \Big(\frac{\mu_1^2}{2}+1\Big)\int_\Omega |u'|^2 \mathrm{d}x - \mu \int_\Omega |\nabla u|^2 \mathrm{d}x \\ & - (\lambda+\mu)\int_\Omega |\operatorname{div} u|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega u^2(x,t)\mathrm{d}x \\ & + \frac{1}{2}\Big(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\mathrm{d}s\Big)\int_\Omega u^2(x,t)\mathrm{d}x \\ & + \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,t,1,s)\mathrm{d}s\mathrm{d}x.\end{aligned} \tag{3.9}$$

Then, Poincaré's inequality leads to the desired estimate. □

Lemma 3.4. *The functional*

$$I(t) = \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_2(s)| z^2(x,t,\rho,s)\mathrm{d}s\mathrm{d}\rho\mathrm{d}x, \quad \textit{for } t \in \mathbb{R}^+ \tag{3.10}$$

satisfy

$$\begin{aligned} I'(t) \le {} & -e^{-\tau_2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,t,1,s)\mathrm{d}s\mathrm{d}x \\ & + \Big(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\mathrm{d}s\Big)\int_\Omega u'^2(x,t)\mathrm{d}x \\ & - e^{-\tau_2}\int_\Omega \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,t,\rho,s)\, d\mathrm{d}s\mathrm{d}\rho\mathrm{d}x.\end{aligned} \tag{3.11}$$

Proof. Using (2.1), the derivative of I entails

$$\begin{aligned} I'(t) &= 2\int_\Omega \int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_2(s)| z'(x,t,\rho,s) z(x,t,\rho,s)\mathrm{d}s\mathrm{d}\rho\mathrm{d}x \\ &= -\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| e^{-s\rho}\frac{\partial}{\partial\rho}\Big(z^2(x,t,\rho,s)\Big)\mathrm{d}s\mathrm{d}\rho\mathrm{d}x \\ &= -\int_\Omega \int_{\tau_1}^{\tau_2} se^{-s}|\mu_2(s)| z^2(x,t,1,s)\mathrm{d}s\mathrm{d}x \\ &\quad + \Big(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\Big)\int_\Omega u'^2(x,t)\mathrm{d}x \\ &\quad - \int_\Omega \int_{\tau_1}^{\tau_2} s|\mu_2(s)| \int_0^1 e^{-s\rho} z^2(x,t,\rho,s)\mathrm{d}\rho\mathrm{d}s\mathrm{d}x,\end{aligned} \tag{3.12}$$

and the desired estimate follows immediately. □

Now, we prove our main stability results (3.2).

Proof of Theorem 3.1. Let

$$L(t) = NE(t) + \epsilon\phi(t) + I(t), \tag{3.13}$$

where N and ϵ are positive constants that will be fixed later. Taking the derivative of $L(t)$ with respect to t and making use of (3.3), (3.6) and (3.11), we obtain

$$\begin{aligned} L'(t) \leq & -\Big\{\Big(\mu_1 - \int_{\tau_1}^{\tau_2} \mu_2(s)\mathrm{d}s\Big)N - c\epsilon - \int_{\tau_1}^{\tau_2} \mu_2(s)\mathrm{d}s\Big\} \int_\Omega |u'|^2\mathrm{d}x \\ & - (\lambda+\mu)\int_\Omega |\operatorname{div} u|^2\mathrm{d}x - (\mu - c)\epsilon \int_\Omega |\nabla u|^2\mathrm{d}x \\ & - (e^{-\tau_2} - c\epsilon)\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,t,1,s)\mathrm{d}s\mathrm{d}x \\ & - e^{-\tau_2}\int_\Omega \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,t,\rho,s)\mathrm{d}s\mathrm{d}\rho\mathrm{d}x. \end{aligned} \tag{3.14}$$

At this point, we choose our constants in (3.14), carefully, such that all the coefficients in (3.14) will be negative. It suffices to choose ϵ so small such that

$$e^{-\tau_2} - c\epsilon > 0,$$

then pick N large enough such that

$$\Big(\mu_1 - \int_{\tau_1}^{\tau_2} \mu_2(s)\mathrm{d}s\Big)N - c\epsilon - \int_{\tau_1}^{\tau_2} \mu_2(s)\mathrm{d}s > 0.$$

Consequently, recalling (3.1), we deduce that there exist also $\eta_2 > 0$, such that

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t} \leq -\eta_2 E(t), \quad \text{for } t \geq 0. \tag{3.15}$$

On the other hand, it is not hard to see that from (3.13) and for N large enough, there exist two positive constants β_1 and β_2 such that

$$\beta_1 \; E(t) \leq L(t) \leq \beta_2 E(t), \quad \text{for } t \geq 0. \tag{3.16}$$

Combining (3.15) and (3.15), we deduce that there exists $\Lambda > 0$ for which the estimate

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t} \leq -\Lambda L(t), \quad \forall t \geq 0, \tag{3.17}$$

holds. Integrating (3.15) over $(0, t)$ and using (3.15) once again, then (3.2) holds. Then, the proof is complete. □

References

[1] T. A. Apalara, *Well-posedness and exponential stability for a linear damped Timoshenko system with second sound thermoelasticity and internal distributed delay*, Electron. J. Differ. Equations, Vol. 254 (2014), pp. 1–15.

[2] M. Aassila, *Strong asymptotic stability of isotropic elasticity systems with internal damping*, Acta Sci. Math. (Szeged), 64 (1998), pp. 103–108.

[3] L. Bouzettouta, S. Zitouni, K. Zennir and A. Guesmia, *Stability of Bresse system with internal distributed delay*, Journal of Mathematical and Computational Science, 7 (1) (2017), pp. 92–118.

[4] A. Beniani, Kh. Zennir and A. Benaissa, *Stability For The Lamé System With A Time Varying Delay Term In A Nonlinear Internal Feedback*, Clifford Analysis, Clifford Algebras And Their Applications, 5 (4) (2016), pp. 287–298.

[5] A. Bchatnia and M. Daoulatli, *Behavior of the energy for Lamé systems in bounded domains with nonlinear damping and external force*, Electron. J. Dif. Equa, 1 (2013), pp. 1–17.

[6] A. Bchatnia and A. Guesmia, *well-posedness and asymptotic stability for the Lamé system with infinite memories in a bounded domain*, Math. Cont. And Related Fields, 4 (2014), pp. 451–463.

[7] A. Benaissa and A. Benguessoum, *Global existence and energy decay of solutions to a viscoelastic wave equation with a delay term in the non-linear internal feedback* Int. J. Dynamical Systems and Differential Equations, 5 (1) (2014), pp. 1–26.

[8] A. Benaissa and S. Mokeddem, *Global existence and energy decay of solutions to the Cauchy problem for a wave equation with a weakly nonlinear dissipation*, Abstr. Appl. Anal, 11 (2004), pp. 935–955.

[9] S. Berrimi and S. A. Messaoudi, *Existence and decay of solutions of a viscoelastic equation with a nonlinear source.*, Nonl. Anal, 64 (2006), pp. 2314–2331.

[10] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. A. Soriano, *Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping*, E. J. Differ. Equ., 44 (2002), pp. 1–14.

[11] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. Ferreira, *Existence and uniform decay for a non-linear viscoelastic equation with strong damping*, Math. Methods Appl. Sci, 24 (2001), pp. 1043-1053.

[12] M. M. Cavalcanti, H. P. Oquendo, *Frictional versus viscoelastic damping in a semilinear wave equation*, SIAM J. Control Optim, 42 (4) (2003), pp. 1310–1324.

[13] C. M. Dafermos, *Asymptotic stability in viscoelasticity*, Arch. Ration. Mech. Anal, 37 (1970), pp. 297–308.

[14] E. Fridman, S. Nicaise, J. Valein, *Stabilization of second order evolution equations with unbounded feedback with time-dependent delay*, SIAM J. Control Optim, 8 (2010), pp. 5028–5052.

[15] M. Kirane, B. Said-Houari, *Existence and asymptotic stability of a viscoelastic wave equation with a delay*, Z. Angew. Math. Phys, 62 (2011), pp. 1065–1082.

[16] A. Guesmia, *Well-posedness and exponential stability of an abstract evolution equation with infinite memory and time delay*, Journal of Mathematical Control and Information, (2013), pp. 1–20.

[17] M. Kafini, N-e Tatar, *A decay result to a viscoelastic in* $\mathbb{R}^n$ *with an oscillating kernel*, Journal of Mathematical Physics, 51 (7) (2010), Article ID 073506.

[18] B. Said-Houari, S. A. Messaoudi, A. Guesmia, *General decay of solutions of a nonlinear system of viscoelastic wave equations*, Nonlinear Differ. Equ. Appl, 18 (2011), pp. 659–684.

[19] A. Guesmia, S. Messaoudi, *A general decay result for a viscoelastic equation in the presence of past and infinite history memories*, Nonlinear Analysis, 13 (2012), pp. 476–485.

[20] S. Nicaise, C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks*, SIAM J. Control Optim, 45 (2006), pp. 1561–1585.

[21] S. A. Messaoudi, A. Fareh, N. Doudi, *Well posedness and exponential stability in a wave equation with a strong damping and a strong delay*, Journal of Mathematical Physics, 57 (11) (2016), Article ID 111501.

[22] B. Said-Houari, *A stability result for a Timoshenko system with past history and a delay term in the internal feedback*, Dynam. Systems Appl., 20 (2011), pp. 327–354.

[23] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, New York: Springer, 1983.

Noureddine Taouaf
Laboratory ACEDP
University of Djillali Liabes – B.P. 89
Sidi Bel Abbes 22000
Algeria
E-mail address: noureddine.taouaf@univ-sba.dz

Noureddine Amroun
Laboratory ACEDP
University of Djillali Liabes – B.P. 89
Sidi Bel Abbes 22000
Algeria
E-mail address: amroun@univ-sba.dz

Abbes Benaissa
Laboratory ACEDP
University of Djillali Liabes – B.P. 89
Sidi Bel Abbes 22000
Algeria
E-mail address: benaissa_abbes@yahoo.com

Abderrahmane Beniani
Laboratory ACEDP
Center University of Belhadj Bouchaib – B.P. 284 RP
Ain Temouchent 46000
Algeria
E-mail address: a.beniani@yahoo.fr