UDC 78.01
doi:10.5937/ZbAkUm1801104M
Scientific review

Maria Mannone
University of Minnesota, USA (alumna)

Eri Kitamura
University of Minnesota, USA (alumna)

Jiawei Huang
Ryo Sugawara
Yoshifumi Kitamura
Research Institute of Electrical Communications
Tohoku University
Japan

Musical Combinatorics, Tonnetz, and the CubeHarmonic

Abstract: In this paper, we give an overview of some applications of combinatorics and permutations in music through the centuries. The concepts of permutation and tonnetz (spatial representation of voice leading and modulation) can be joined together in a physical device, the CubeHarmonic, a musical version of the Rubik's cube. We finally describe a prototype of the CubeHarmonic that uses the magnetic tracking technology developed at the Tohoku University.

Keywords: Group Theory, Rubik's Cube, Motion Tracking

1. Introduction

Given five different notes, how many „melodies" can we compose with them? The answer comes from mathematics. There are 5! = 5*4*3*2*1 = 120 different combinations of notes. Combinatorics (Hazewinkel, 2001) is a huge field of mathematical research, and its applications to music — to pitch, scales, combinations of several musical parameters — is another field of research.

Combinatorics is the study of finite and countable discrete mathematical structures. Applications of combinatorics to music stretch back to ancient Greece and India and have several interesting details, but a thorough historical account is not within the scope of this article. Combinatorics is relevant for several fields of mathematics, physics, and science in general. As in an incomplete list, we may cite graph theory, finite geometry, probability studies, combinatorics on words (Goldin-Meadow et al., 1995; Berstel and Karhumäki, 2003), statistics in physics, and genetics in biology (Hofacker et al., 1994). Let us consider some „real-world examples" to understand the importance of combinatorics in music. While considering musical parameters such as pitch and pitch classes, combinatorics helps answering questions on transformational strategies, such as answering the question: How to list the number of variations with a limited set of elements? Frequencies in acoustics are continuous, even in Western music, which often involves a continuous variation of pitches, such as a violin glissando, or a voice *portamento*. However, considering also discrete pitch is a useful strategy for learning and researching in music theory. In particular, the notion of pitch class is relevant in contemporary music composition and research, in the context of a tempered system. As an example, the note A = 440 Hz, and its higher octave at 880 Hz enter in the same pitch class of „A" that is, modulo one-octave shift.

Musical combinatorics is relevant in musical investigation, as some review articles describe (Nolan, 2000). In particular, during the scientific revolution of the seventeenth century, the ancient and classical music-mathematics relationship was highlighted (Nolan, 2000). The resources given by combinatorics studies do not involve only music: there are several applications in other arts. Furthermore, permutations were used in poetry, such as the combinatorial poems by Emmett Williams, and permutation strategies were employed by Italian futurist poet Tommaso Marinetti and by Austrian poet Gerhard Ruhm. In more recent times, we have the example of the French poet and writer Raymond Queneau, with „Cent milliards de poèmes" (Queneau, 1961). In this book, there are ten papers, each of them divided into fourteen rows/strips. The reader can rotate the horizontal strips as they were separated pages, choosing each time a different version of the poem. Finally, there are also applications of combinatorics to visual art and dance. We can think of Robert Morris' permutation and Deborah Hay's choreography.

The applications to music are several and varied; we will see some of them in Section 2. Some examples, from the German tradition, are Musikalische Würfelspiele (musical dice game) by Ramon Llull (Zweig, 2014), with the known realizations by Haydn and Mozart. These games allow for a music composition from random combinations of some pre-composed material.

In a more mathematical detail, combinatorics deals with the vast field of group theory. A „group" is a mathematical structure where the combination of two elements (of the group) gives a third element of the same group. Groups verify the properties of closure, associativity, identity, and invertibility (Rotman, 1991). In a permutation group, the elements are the permutations of a given set, and the operation is the composition of the permutations in the group. In music, a permutation group can be provided by the permutations of pitches, and the composition of their permutations. Let us consider the inversion operation: inverting the place of two notes of a melody still gives a melody. An exemplary device used to show group permutations in three dimensions is the Rubik's cube, invented as a teaching tool by a professor, Ernö Rubik. One more example of how music and combinatorics can be „combined" together, is given by the CubeHarmonic, a musical version of the Rubik's cube. In this article, we would like to create new math-music connections, creatively use a puzzle game, and, at the same time, connect two areas of music theory, combinatorics, and tonnetz.

In Section 2, we shortly review combinatorics in music. Then, in Section 3, we describe the concept of tonnetz, a visual-geometric representation of chord relationships and voice leading. We can join the two concepts, introducing combinatorics within a tonnetz, with the creation of a physical device that is the musical version of the Rubik's cube, called CubeHarmonic (see Section 4). We illustrate the CubeHarmonic as a theoretical idea first, and as a working prototype in Section 4.

2. Ars Combinatoria

How many variations can we get by permuting some notes, and which among them are good to hear? How do you manipulate the given musical material so that all combinations are 'right'? Music theory scholars and composers have been dealing with these questions for centuries. In his masterwork, Harmonie Universelle („Universal Harmony" published in Paris in 1636), the music theorist Marin Mersenne, a French priest, theologist, and defender of Galileo Galilei during the affair with the Inquisition, discussed the rejection of some possibilities due to aesthetic reasons (Nolan, 2000). Mersenne tabled permutations of 22 notes in the second book of Harmonie Universelle (Christensen, 2006).

Combinatorics is mostly applied to music theory but has a lot to offer regarding concrete strategies for composition. Some examples of combinatorics applied to music include musical dice games. A musical dice game is a tool to compose music without any previous knowledge, using random combinations of pre-composed musical fragments. Despite the randomness, all output results are „pleasant" to hear because the composer controls the initial choices. According to the scholar Lawrence Zbikowski, „In truth, chance played little part in the success of the music produced by such games. Instead, what was required of the compilers… [was] a little knowledge about how to put the game together and an understanding of the formal design of waltzes, etc. "(Zbikowski, 2002).

Mozart developed a famous dice game. It comprises of a table with 12 rows and 8 columns. The (i, j)-number on the table represents a musical measure. Throwing two dice the first time and adding them up, we get a number, a value of 'I', that is, the row number. In fact, the number of rows is given by adding on the two dice. Thus, the first measure will be the first (column j = 1) element in the i-th row. Throwing the dice again, we get the second number, and we take the element in the i'-th row of the 2nd column. So far, we got the first two measures of a new minuet. Keep going in the same manner and a sequence of 8 measures is obtained. If we have 16 columns on the table, we can compose a minuet of 16 measures. Summarizing, for each measure, we throw the dice, and we write down the corresponding musical measure. There are 11^8 possible combinations of eight measures each.

There are also examples of dice games composed by Haydn and Carl Philip Emmanuel Bach (Zweig, 2014), and the Athanasius Kircher's composition machine based on a matrix (Kircher, 1650). In the work of the Austrian-German theorist (and composer-violinist) Joseph Riepel, „Grundregeln zur Tonordnung insgemein" (Basic Rules for Tonal Order) of 1755, combinatorial-like procedure is meant to „stimulate musical imagination and transmit knowledge and skill in [the] manipulation of musical material" (Riepel, 1755; Nolan, 2000). In the same treatise, Riepel tabulates 120 permutations of five keys diatonically related to C (Christensen, 2006).

Let us make a short excursus on mathematics and combinatorics. Christensen (2006) cites several examples. We can refer to Printz (Phrynis Mytilenaeus oder Satirischer Componist, 1696), Heinichen (Der General-bass in der Composition, 1728), Mattheson (Der vollkommene Capellmeister, 1739), Kirnberger (Der allezeit fertige Menuetten- und Polonoisenkomponist, 1757), and Galeazzi (Ratner, 1970; Hook, 2007). Another connection between mathematics and music — also relevant for musical combinatorics — deals with the concept of pitch class jointly with a branch of mathematics, the modulo arithmetic. In modulo arithmetic, after a specific value (the „module"), numbers „wrap around."An example of this is the clock, with its 12-hour period. In music, an example of a module is the octave. Regarding pitch classes, we have „the same notes" modulo one octave. Thinking in terms of chromatic scale, we have the notes „modulo 12." This choice is possible in equal temperament, where all the half-tones are equal, and all the octaves are equal. According to Nolan (Nolan, 2009), modular arithmetic is about „the combined agency of modular arithmetic and equal temperament [that] enabled the formulation of theories of pitch structures based on algebraic methods and a recovery of pure speculation in music theory."The French music theorist (according to Christensen (2006), „with an evident mathematical training") Anatole Loquin investigated pitch classes and harmony, listing more than five hundred „harmonic effects" of five pitch classes (Loquin, 1873). He represented module as a circle with 12 points, with each point representing a note on the chromatic scale. This circular representation is used today by mathematical music theorists. In this circular representation, a triad is represented by a triangle, and the transposition of a chord of a half-tone can be represented higher or lower by rotating the image accordingly.

In more contemporary times, the contributions of Ernst Bacon and John Cage may be cited. Bacon deals with the extension of the traditional tonal language through the resources of new harmonies, combinatorics, and 12-pitch classes (Bacon, 1917). John Cage focuses on „the togetherness of differences," involving „differences in structure" (Perloff and Junkerman, 1994). However, Cage is not referring to a precise mathematical concept of classification, privileging instead a more 'poetical' way to see „disorganization" and to enjoy the different combinations of some existing material.

A more heavily-based mathematical study on music and combinatorics is the Redfield-Pólya Theorem (Redfield, 1927). Published by Redfield in 1927 first, and then by Pólya years later, it concerns the enumeration of discrete combinatorial objects depending on their „order." For example, depending on the number of nodes, we can enumerate how many graphs with that number of nodes there are. This theorem has been used to „enumerate musical objects," determining their equivalence classes — how many scales, chords… (Fripertinger, 1992). Musical applications of Pólya theorem appear in the works of both mathematicians and musicians (Jedrzejewski, 2006). A mathematically-detailed review of combinatorial techniques used by composers, music theorists, and mathematicians, is given by Hook (2007).

Finally, we can refer to algorithmic composition studies. The dice game suggests a sequence of pre-defined rules, and it constitutes a special example of algorithmic composition. See Nierhaus's work (Nierhaus, 2009) for a more detailed overview of algorithmic music.

3. Tonnetz

Christensen (2006) defines a tonnetz as „a potent two-dimensional image composed of a grid or lattice of parallel horizontal and vertical lines and nodes." *Tonnetz* is a German word meaning „tone-network," and it was first described by Leonhard Euler (1739) to represent tonal space. The tonnetz was rediscovered in the nineteenth century by Ernst Naumann, Arthur van Oettingen, and Hugo Riemann (Riemann, 1992). The tonnetz is a visual method for representing chords and voice leading relationships in the plane. Modern examples of the tonnetz include three-dimensional representation (Tymoczko, 2012), as well as animated renditions with movements in space and synchronized sounds (Baroin). David Lewin (1987) defines the tonnetz as a spatial metaphor for music theory. He describes intervals as displacements in the space of pitches (and pitch classes), before generalizing the concept of interval itself. Henry Klumpenhouer, a former student of David Lewin, defines transformations T (transposition) and I (inversion) to connect pitch classes. Given a graph, we can define other graphs with the same structure of transformations. In this case, we talk about isomorphic graphs, where the configuration of nodes-and-arrows is the same.

Networks with isomorphic graphs are called isographic (Klumpenhouwer, 1991).[1] The definition of space (a locus, as a set of data) and the transformations on it are relevant also for group theory. The tonnetz is also relevant in Tymoczko's and Hook's studies (Tymoczko, 2012; Hook, 2006).

Representations of tonnetz start with the plane but are not limited to it. We have several examples of different geometries, such as the torus and the Möbius strip (Mazzola et al., 2016). If the Möbius strip represents relationships between the notes of a given tonality, a connection between different Möbius strips through the common grades represents a modulation. This is an example of geometrical studies on Western music theory. Some scholars look at the possible physiological bases of these geometries, analyzing how listeners' brain areas are activated (Janata et al., 2002).

Finally, some musical MIDI instruments based on the tonnetz have been invented: this is the case of „HarmonicTable"(http://c-thru-music.com/cgi/index.cgi). Manipulations of the tonnetz in combinatorial terms might be suggested in the so-called „slot-machine" transformations, but, at the best of our knowledge, no physical realizations of these have been made. A „slot-machine" transformation is a metaphor to indicate a permutation, more than a rotation, according to Alegant (2001), in a cross-partition. The cross-partition is a way to arrange pitch classes with each column having numbers that represent the notes of a chord. It is often used in Schoenberg twelve-tone music, as „a way to represent the pitch classes of an aggregate (or row) in a two-dimensional rectangular design" (Alegant and Mead, 2012). If we have the sequence:

0	3	6
1	4	7
2	5	8

and we permute the numbers in the first column, getting 1 2 0, the overall vertical content is the same, but the horizontal arrangement will change. The „harmonies" are the same, while the „melodies" change (Alegant and Mead, 2012):

1	3	6
2	4	7
0	5	8

The described process may remind us of a „slot-machine" rotation of the first column, which explains the origin of the name. If we see the rows and columns above as elements within a tonnetz, we may imagine the slot-machine transformations as combinatorial transformations within the tonnetz. Such a combinatorial effect requires one more dimension than the two of the plane. In Section 4, we introduce a tridimensional model that combines the tonnetz, group theory, and musical combinatorics.

1 Klumpenhouer networks, also known as K-nets, were developed well before the modern treatment of the *Tonnetz* in the music theory literature.

4. Tonnetz and Ars Combinatoria: the idea of the CubeHarmonic

How can we „combine“ combinatorics and tonnetz? A possibility is the application of the Rubik’s cube to music. The Rubik’s cube is the object of scientific investigation, and it constitutes an example of group transformations (Frey and Singmaster, 1982). The 4x4x4 cube (the „Revenge“) is analyzed in recent mathematical works (Weed, 2016).

The first example of CubeHarmonic starts from the 4x4x4 Rubik’s Cube (see Figure 1). Let us consider a face of the cube. Each little square (facet) is a note. A column (a slice) is a chord of four notes, and the sequence of the four chords is a well-defined cadence. On each face of the cube, there is a different cadence: two from classical harmony, one from ancient music (Landino’s cadence), and two from jazz music. However, we may also consider each little square (facet) as a complete chord. With a 3x3x3 cube, we can consider sequences of three chords with 3-part harmony. With a 2x2x2 cube, we have either sequences of bichords, or complete chords that involve each entire face. For example, we can have C-major chord in a face, with the single facets playing C E G C. Scrambling the cube provokes a mixing of the chord sequences first, and later of chords themselves. The initial idea was about playing each slide at the time, that is to say, playing one chord after the other.

The CubeHarmonic was first thought by Maria Mannone in 2013 during her studies of music and combinatorics at IRCAM. The idea was first published under the name of CubHarmonic (Mazzola et al. 2016), then modified into CubeHarmonic. A more general version of the CubeHarmonic can include different timbres, complete chords for each facet, and also rhythms. While the scope of the Rubik’s cube is solving the scrambled cube, the scope of the CubeHarmonic is in the scrambling itself, enjoying the variety of the musical combinations produced.

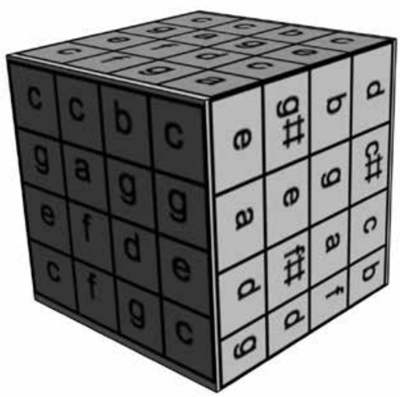

Figure 1

The very first prototype (developed by Mannone) used a giant cube with sound modules (of greeting cards) on each facet; see Figure 2 (left: the cube with big sound modules; right: a detail of a small sound module covered with paper). This allowed the playing of each facet separately, of each slice, and the selection of random melodies selecting different paths on the cube. However, there are still open problems, such as an adequate substitution of batteries, and the reduction of the size of the cube (depending on the minimal size of the loudspeaker and electric circuit on each facet).

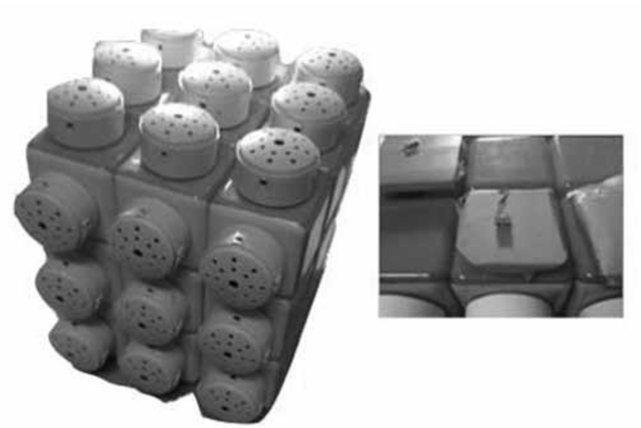

Figure 2

A technical prototype has been recently developed in collaboration with the ICD team of the Tohoku University, headed by Prof. Yoshifumi Kitamura, see Figure 3. The prototype uses a novel magnetic tracking technology (Huang et al., 2015, 2016). The system generates a magnetic field to drive multiple, tiny, and wireless LC coils and detects the resonant magnetic flux to compute their 3D location and rotation. When the LC coils are attached to a Rubik's cube, the movements of the facets can be tracked in real-time. Thus, it is possible to visualize the Rubik's cube in a virtual environment as well as to convert the facet combination into sounds. Instead of choosing which facet or which slide to play, it has been chosen to play all the notes at the same time. To avoid any chaotic effects, the variety of chords was limited first to two chords (a chord per each group of three faces, with several doublings), and then to three chords (a chord for each group of two facets). Moreover, to improve the idea of „motion" and „twisting," different loudness levels, accordingly to the position of the facets, were set (with the simple choice of the horizontal distance concerning the screen). The farther the notes are from the screen, the louder they play. In this way, we can have a Doppler-like effect of „approaching"pieces of the cube during a 45-degree rotation forward of the right (or left) side. Even if we do not change the pitch but only the loudness of the notes corresponding to the moved slice, we perceive a pitch change. The pieces play with different levels of „depth" due to their different intensity: the closer pieces play louder, as well as the approaching pieces. In recent updates of this prototype, simultaneous playing of all faces can be substituted by a gradual transition from one slice to another according to twisting. IM3D technology has also enabled to map into sound the overall position of the cube within the horizontal plane: in this way, the user/performer can change overall pitch (obtaining glissando) and overall loudness, allowing expressive playing.

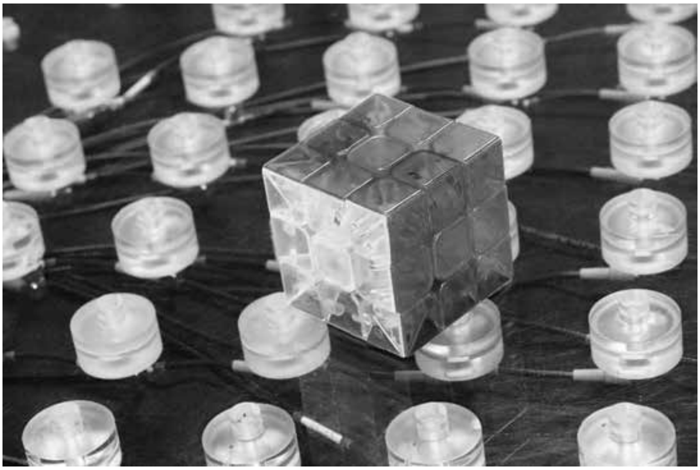

Figure 3

Independently from this work, a paper describing the Rubik's cube in a musical setup recently appeared (Polfreman and Oliver, 2017). However, only one face per time is involved in the musical production, while the discussed prototypes use all faces.

Theoretical and practical applications of the CubeHarmonic will involve music pedagogy, math pedagogy, as well as musical creativity in the fields of composition and improvisation. For example, we can challenge composition students to compose a song based on a sequence of chords obtained by random rotations of the cube. Another use can involve music theory: if we set the chords on the cube to match specific sequences, students can play and investigate voice leading patterns. The study of the useful initial configuration of chords on the cube and of the minimum amount of rotations to create harmonic variety can be the topic of theoretical research and musical practice.

Other applications may deal with the development of creativity in music, and the fruition of the Cube's game by people with disabilities. Let us see some potential applications in more detail.

Pedagogy of mathematics. Professor Rubik probably invented the 3x3x3 cube to teach group theory and transformations. With the CubeHarmonic, we add an auditory channel to the visual one, allowing for a better understanding, and a more enjoyable approach to mathematical topics through sound examples.

Pedagogy of music, especially music theory. The study of chords (at least 3-notes), bichords, chord sequences, consonances, and dissonances may profit from a tangible tool. Users may either play around with random combinations or try to get precise and pleasant chords out of the cub e. A great part of the outcome is „determined“ by the starting material (pitch, timbre) to associate with each little piece of the cube. Notes and chords can be carefully chosen to allow „pleasant combinations“ also if the cube is completely scrambled.

Pedagogy of math & music. Access to math & music topics usually requires a double training, which needs time. A game can help overcome this difficulty on the one side, and on the other, it can complete and accompany the theoretical and practical learning of students of both disciplines. A mathematical exercise can have a musical solution, or a sequence of mathematical passages can be compared step-by-step with musical transformations.

Research in music theory, in particular, musical combinatorics. There is a huge tradition in this field. A tangible device can embody years of abstract research, but allowing a more effective popularization of ideas.

Research in musical perception and movement. The link between movements and sound is highlighted if we change musical parameters in real time with the twisting: for example, adjusting the loudness of a piece if it is moving toward or away from us. The association would be the most intuitive as possible, modeled upon simple principles of musical (and sound) perception. Moreover, playing with the CubeHarmonic would itself lead to new research.

People with disabilities: 1. Visually-impaired people may enjoy the game if they can hear a note instead of seeing a color, for example. They can thus play the classic Rubik’s cube puzzle solving it through the use of sounds as a guide. 2. Physical therapy of the hands may also benefit from the sound channel to check if the movements are performed in the correct way.

Musical creativity and composition. Professional composers, amateurs, or even children can enjoy different musical combinations. The choice of different timbre (and even the addition of some rhythm), with the possibility of saving the sound produced, may constitute a source of material for composers.

No prior knowledge/practice of the solving techniques of the Rubik’s cube is required. The CubeHarmonic is mainly enjoyed while scrambling, not by solving it as a puzzle.

Fun! Rubik's cube was conceived as a pedagogy tool, but it has become one of the most sold and loved toys. The reason is that it is fun. The musical version of the cube adds the layer of sounds, with some creative and entertainment applications we have briefly summarized.

5. Conclusion

To sum up, we made an overview of some applications of combinatorics in music, of the use of tonnetz graphical representations, and finally we introduced the CubeHarmonic, describing some current prototypes. The project of CubeHarmonic is very flexible, and there may be several potential applications. The starting point in the development of the CubeHarmonic was music theory and mathematical theory of music. It can help students, researchers, artists, and everybody to develop creativity and hopefully have fun.

REFERENCES:

1. Alegant, Brian. 2001. *Cross-Partitions as Harmony and Voice Leading in Twelve-Tone Music*. Music Theory Spectrum, 23 (1), pp. 1-40.
2. Alegant, Brian, and Mead, Andrew. 2012. *Having the Last Word: Schoenberg and the Ultimate Cadenza*. Music Theory Spectrum, 34 (2), pp. 107-136.
3. Bacon, Ernst. 1917. *Our Musical Idiom*. Reprinted from The Monist of October, 1917. The Open Court Publishing Company.
4. Baroin, Gilles (et al.). *Original Musical Spaces based on Graph Theory*. Retrieved from http://planetes.info, and http://www.mathemusic4d.net.
5. Berstel, Jean, and Karhumäki, Juhani. 2003. *Combinatorics on Words – A Tutorial*. Bulletin-European Association for Theoretical Computer Sciences EATCS, 79, p. 178-228.
6. Christensen, Thomas (eds.). 2006. *The Cambridge History of Western Music Theory*. Cambridge University Press.
7. Euler, Leonhard. 1739. *Tentamen novae theoriae musicae ex certissismis harmoniae principiis dilucide expositae.* Saint Petersburg Academy, p. 147.
8. Frey, Alexander Jr., and Singmaster, David. 1982. *Handbook of the Cubik Math*, Onslow Publishers.
9. Fripertinger, Harald. 1992. *Enumeration in Musical Theory*. Beiträge zur Elektronischen Musik 1, 1992
10. Goldin-Meadow, Susan, Mylander, Carolyn, and Butcher, Cynthia. 1995. *The resilience of combinatorial structure at the word level: morphology in self-styled gesture system*. Cognition 56, pp. 195-262.
11. Hashi, Shuichiro, Toyoda, Masaru, Yabukami, Shin, Ishiyama, Kazushi, Okazaki, Yasuo, and Arai, Ken Ichi. 2006. *Wireless Magnetic Motion Capture System for Multi-Marker Detection*. IEEE Transactions on Magnetics. 42 (10).

12. Hazewinkel, Michiel, ed. 2001. *Combinatorial Analysis*, from *Encyclopedia of Mathematics*, Springer.
13. Hofacker, Ivo, Schuster, Peter, and Stadler, Peter. 1994. *Combinatorics of RNA secondary structures.* Discrete Applied Mathematics, 88 (1-3), pp. 207-237.
14. Hook, Julian. 2006. *Exploring Musical Space.* Science, 313 (5783), pp. 49-50.
15. Hook, Julian. 2007. *Why Are There Twenty-Nine Tetrachords? A Tutorial on Combinatorics and Enumeration in Music Theory*. Music Theory Online.
16. Huang, Jiawei, Mori, Tsuyoshi, Takashima, Kazuki, Hashi, Shuichiro, and Kitamura, Yoshifumi. 2015. *IM6D: Magnetic Tracking System with 6-DOF Passive Markers for Dexterous 3D Interaction and Motion.* ACM Trans. Graph. 34 (6), Article 217.
17. Huang, Jiawei, Mori, Tsuyoshi, Takashima, Kazuki, Hashi, Shuichiro, and Kitamura, Yoshifumi. 2016. *6-DOF Computation and Marker Design for Magnetic 3D Dexterous Motion-Tracking System.* In: Proceedings of the 22nd ACM Conference on Virtual Reality Software and Technology. ACM, 2016. p. 211-217.
18. Janata, Petr, Birk, Jeffrey, Van Horn, John, Leman, Marc, Tillmann, Barbara, and Bharucha, Jamshed. 2002. The Cortical Topography of Tonal Structures Underlying Western Music. Science, 13, 298 (5601), pp. 2167-2170.
19. Jedrzejewski, Franck. 2006. *Mathematical Theory of Music*. Editions Delatour France.
20. Johnson, Tom, and Jedrzejewski, Franck. 2014. *Looking at Numbers.* Springer.
21. Kircher, Athanasius. 1650. *Musurgia Universalis.*
22. Klumpenhouer, Henry. 1991. *Aspects of Row Structure and Harmony in Martino's Impromptu Number 6*. Perspectives of New Music, 29 (2), pp. 318-354.
23. Lewin, David. 1987. *Generalized Musical Intervals and Transformations*. Oxford University Press (Reprint 2011).
24. Loquin, Anatole. 1873. *Tableau de tous les effets harmoniques de un à cinq.*
25. Mannone, Maria. 2017. *Musical Gestures between Scores and Acoustics: A Creative Application to Orchestra.* Ph.D. dissertation, University of Minnesota, USA.
26. Mazzola, Guerino, Mannone, Maria, and Pang, Yan. 2016. *Cool Math for Hot Music.* Springer
27. Nierhaus, Gerhard. 2009. *Algorithmic Composition: Paradigms of Automated Music Generation.* Springer.
28. Nolan, Catherine. 2000. *On Musical Space and Combinatorics: Historical and Conceptual Perspectives in Music Theory*. Bridges.
29. Perloff, Marjorie, and Junkerman, Charles. 1994. *John Cage: Composed in America.* University of Chicago Press.
30. Polfreman, Richard, and Oliver, Benjamin. 2017. *Rubik's Cube, Music's Cube*. Proceedings of NIME Conference.
31. Ratner, Leonard. 1970. *Ars combinatoria: Chance and Choice in Eighteenth-Century Music.* In *Studies in Eighteenth-Century Music*, ed. H. C. Robbins Landon and Roger E. Chapman. Oxford University Press, pp. 343–63.
32. Redfield, J. Howard. 1927. *The Theory of Group-Reduced Distributions*. American Journal of Mathematics, 49 (3), pp. 433-455.
33. Riepel, Joseph. 1755. *Grundregeln zur Tonordnung Insgemein.* Nabu Press (Reprint 2011).
34. Riemann, Hugo [1914-15]. 1992. *Ideas for a Study 'On the Imagination of Tone.'* Translated by Robert Wason and Elizabeth West Marvin. Journal of Music Theory 36 (1), pp. 81-117.
35. Rotman, Joseph. 1991. *An Introduction to the Theory of Groups*. Springer.

36. Tymoczko, Dmitri. 2012. *The Generalized Tonnetz.* Journal of Music Theory, 56 (1).
37. Queneau, Raymond. 1961. *Cent mille milliards de poèmes.* Gallimard.
38. Weed, Jared. 2016. *Sub-Optimal Multi-Phase Path Planning: A Method for Solving Rubik's Revenge.* ArXiv.
39. Zbikowski, Lawrence. 2002. *Conceptualizing Music: Cognitive Structure, Theory, and Analysis*. Oxford University Press, pp. 142-143.
40. Zweig, Janet. 2014. *Ars Combinatoria*. Art Journal, 56 (3), pp. 20-29.

Muzička kombinatorika, *Tonnetz*, i *CubeHarmonic*

Apstrakt: U radu je dat pregled nekih primena kombinatorike i permutacije u muzici kroz vekove. Koncepti permutacije i *Tonnetza* (prostornog prikaza vođenja glasova i modulacije) mogu biti spojeni u jednu fizičku napravu – *CubeHarmonic* (harmonsku kocku), muzičku verziju Rubikove kocke. Konačno, opisujemo prototip *CubeHarmonic* u kome je korišćena tehnologija magnetskog praćenja, razvijena na Tohoku univerzitetu.

Kljlučne reči: grupna terapija, Rubikova kocka, praćenje pokreta