

# Non-existence of solutions for a Timoshenko equations with weak dissipation

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ABSTRACT. In this paper, we consider the following Timoshenko equation

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2) \Delta u + u_t = |u|^{q-1} u$$

associated with initial and Dirichlet boundary conditions. We prove the non-existence of solutions with positive and negative initial energy.

## 1. INTRODUCTION

Let  $\Omega$  be a bounded domain with smooth boundary  $\partial\Omega$  in  $R^n$ . We study the following Timoshenko equation

$$(1) \quad \begin{cases} u_{tt} + \Delta^2 u - M(\|\nabla u\|^2) \Delta u + u_t = |u|^{q-1} u, & (x, t) \in \Omega \times (0, T), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \\ u(x, t) = \frac{\partial}{\partial \nu} u(x, t) = 0, & x \in \partial\Omega, \end{cases}$$

where  $q \geq 1$  is real number,  $\nu$  is the outer normal and  $M(s) = 1 + s^\gamma$ ,  $\gamma \geq 1$ .

This type equation arises beam theory [3]. Timoshenko [14], a pioneer in strength of materials, developed a theory in 1921 which is a modification of Euler's beam theory. The theory takes into account corrections for shear and rotatory inertia neglected in Euler's beam theory. The modified theory is called the "Timoshenko beam theory".

In the case of  $M(s) = 1$  and without fourth order term  $\Delta^2 u$ , the equation (1) can be written in the following form

$$(2) \quad u_{tt} - \Delta u + u_t = |u|^{q-1} u.$$

The existence and blow up in finite time of solutions for (2) were established in [6, 7, 8, 10, 15].

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In the case of  $M(s) = 0$  the equation (1) can be written in the following form

$$(3) \quad u_{tt} + \Delta^2 u + u_t = |u|^{q-1} u.$$

Messaoudi [11] studied the local existence and blow up of the solution to the equation (3). Wu and Tsai [16] obtained global existence and blow up of the solution of the problem (3). Later, Chen and Zhou [2] studied blow up of the solution of the problem (3) for positive initial energy.

The problem (1) was studied by Esquivel-Avila [4, 5], he proved blow up, unboundedness, convergence and global attractor. Pişkin [12] studied the local and global existence, asymptotic behavior and blow up. Later, Pişkin and Irkil [13] studied blow up of the solutions (1) with positive initial energy.

In this paper, we prove the nonexistence of solutions for the problem (1), with positive and negative initial energy.

This paper is organized as follows. In section 2, we present some lemmas and notations needed later of this paper. In section 3, nonexistence of the solution is discussed.

## 2. PRELIMINARIES

In this section, we shall give some assumptions and lemmas which will be used throughout this paper. Let  $\|\cdot\|$  and  $\|\cdot\|_p$  denote the usual  $L^2(\Omega)$  norm and  $L^p(\Omega)$  norm, respectively.

**Lemma 2.1** (Sobolev-Poincaré inequality [1]). *Let  $p$  be a number with  $2 \leq p < \infty$  ( $n = 1, 2$ ) or  $2 \leq p \leq \frac{2n}{n-2}$  ( $n \geq 3$ ), then there is a constant  $C_* = C_*(\Omega, p)$  such that*

$$\|u\|_p \leq C_* \|\nabla u\| \quad \text{for } u \in H_0^1(\Omega).$$

We define the energy function as follows

$$(4) \quad E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \left( \|\nabla u\|^2 + \|\Delta u\|^2 \right) + \frac{1}{2(\gamma+1)} \|\nabla u\|^{2(\gamma+1)} - \frac{1}{q+1} \|u\|_{q+1}^{q+1}.$$

**Lemma 2.2.**  *$E(t)$  is a nonincreasing function for  $t \geq 0$  and*

$$E'(t) = -\|u_t\|^2 \leq 0.$$

*Proof.* Multiplying the equation of (1) by  $u_t$  and integrating over  $\Omega$ , using integrating by parts, we get

$$E(t) - E(0) = - \int_0^t \|u_\tau\|^2 d\tau \quad \text{for } t \geq 0.$$

□

Next, we state the local existence theorem of problem (1), whose proof can be found in [12].

**Theorem 2.1** (Local existence). *Assume that  $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$  holds. Then there exists a unique solution  $u$  of (1) satisfying*

$$u \in C([0, T]; H_0^2(\Omega)), \quad u_t \in C([0, T]; L^2(\Omega)) \cap L^{p+1}(\Omega \times (0, T)).$$

Moreover, at least one of the following statements holds:

- (i)  $T = \infty$ ,
- (ii)  $\|u_t\|^2 + \|\Delta u\|^2 \rightarrow \infty$  as  $t \rightarrow T^-$ .

### 3. NON-EXISTENCE OF SOLUTIONS

In this section, we deal with the blow up of the solution for the problem (1). Let us begin by stating the following two lemmas, which will be used later.

**Lemma 3.1.** *Let us have  $\delta > 0$  and let  $B(t) \in C^2(0, \infty)$  be a nonnegative function satisfying*

$$B''(t) - 4(\delta + 1)B'(t) + 4(\delta + 1)B(t) \geq 0.$$

*If  $B'(0) > r_2 B(0) + K_0$ , with  $r_2 = 2(\delta + 1) - 2\sqrt{(\delta + 1)\delta}$ , then  $B'(t) > K_0$  for  $t > 0$ , where  $K_0$  is a constant.*

*Proof.* See [9]. □

**Lemma 3.2.** *If  $H(t)$  is a nonincreasing function on  $[t_0, \infty)$  and satisfies the differential inequality*

$$[H'(t)]^2 \geq a + b[H(t)]^{2+\frac{1}{\delta}}, \quad \text{for } t \geq t_0,$$

*where  $a > 0$ ,  $b \in \mathbb{R}$ , then there exists a finite time  $T^*$  such that*

$$\lim_{t \rightarrow T^{*-}} H(t) = 0.$$

*Upper bounds for  $T^*$  are estimated as follows:*

- (i) *If  $b < 0$  and  $H(t_0) < \min\{1, \sqrt{-\frac{a}{b}}\}$ , then*

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{-\frac{a}{b}}}{\sqrt{-\frac{a}{b}} - H(t_0)}.$$

- (ii) *If  $b = 0$ , then*

$$T^* \leq t_0 + \frac{H(t_0)}{H'(t_0)}.$$

- (iii) *If  $b > 0$ , then*

$$T^* \leq \frac{H(t_0)}{\sqrt{a}} \quad \text{or} \quad T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \left[ 1 - (1 + cH(t_0))^{-\frac{1}{2\delta}} \right],$$

$$\text{where } c = \left(\frac{a}{b}\right)^{2+\frac{1}{\delta}}.$$

*Proof.* See [9]. □

**Definition 3.1.** A solution  $u$  of (1) is called blow up if there exists a finite time  $T^*$  such that

$$(5) \quad \lim_{t \rightarrow T^{*-}} \left[ \int_{\Omega} u^2 dx + \int_0^t \int_{\Omega} u^2 dx d\tau \right] = \infty.$$

Let

$$(6) \quad a(t) = \int_{\Omega} u^2 dx + \int_0^t \int_{\Omega} u^2 dx d\tau, \text{ for } t \geq 0.$$

**Lemma 3.3.** Assume  $\frac{\gamma}{2} \leq \delta \leq \frac{q-1}{4}$ , and that  $\gamma \geq 0$ , then we have

$$(7) \quad a''(t) \geq 4(\delta + 1) \int_{\Omega} u_t^2 dx - 4(2\delta + 1)E(0) + 4(2\delta + 1) \int_0^t \|u_{\tau}\|^2 d\tau.$$

*Proof.* By differentiating (6) with respect to  $t$ , we have

$$(8) \quad a'(t) = 2 \int_{\Omega} uu_t dx + \|u\|^2,$$

$$(9) \quad \begin{aligned} a''(t) &= 2 \int_{\Omega} u_t^2 dx + 2 \int_{\Omega} uu_{tt} dx + 2 \int_{\Omega} uu_t dx \\ &= 2 \left( \|u_t\|^2 + \|u\|_{\frac{q+1}{q}}^{q+1} \right) - 2 \left( \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} + \|\Delta u\|^2 \right). \end{aligned}$$

Then from (4) and (9), we have

$$\begin{aligned} a''(t) &= 4(\delta + 1) \int_{\Omega} u_t^2 dx - 4(2\delta + 1)E(0) \\ &\quad + 4\delta \left( \|\nabla u\|^2 + \|\Delta u\|^2 \right) + \left( \frac{4\delta + 2}{\gamma + 1} - 2 \right) \|\nabla u\|^{2(\gamma+1)} \\ &\quad + \left( 2 - \frac{4(2\delta + 1)}{q + 1} \right) \|u\|_{\frac{q+1}{q}}^{q+1} + 4(2\delta + 1) \int_0^t \|u_{\tau}\|_{\frac{p+1}{p}}^{p+1} d\tau. \end{aligned}$$

Since  $\frac{\gamma}{2} \leq \delta \leq \frac{q-1}{4}$ , we obtain (7).  $\square$

**Lemma 3.4.** Assume  $\frac{\gamma}{2} \leq \delta \leq \frac{q-1}{4}$ ,  $\gamma \geq 0$  and one of the following statements are satisfied

- (i)  $E(0) < 0$  and  $\int_{\Omega} u_0 u_1 dx > 0$ ,
- (ii)  $E(0) = 0$  and  $\int_{\Omega} u_0 u_1 dx > 0$ ,
- (iii)  $E(0) > 0$  and

$$(10) \quad a'(0) > r_2 \left[ a(0) + \frac{K_1}{4(\delta + 1)} \right] + \|u_0\|^2$$

holds.

Then  $a'(t) > \|u_0\|^2$  for  $t > t^*$ , where  $t_0 = t^*$  is given by (11) in case (i) and  $t_0 = 0$  in cases (ii) and (iii).

Where  $K_1$  and  $t^*$  are defined in (15) and (11), respectively.

*Proof.* (i) If  $E(0) < 0$ , then from (7), we have

$$a'(t) \geq 2 \int_{\Omega} u_0 u_1 \, dx + \|u_0\|^2 - 4(2\delta + 1)E(0)t, \quad t \geq 0.$$

Thus we get  $a'(t) > \|u_0\|^2$ , for  $t > t^*$ , where

$$(11) \quad t^* = \max \left\{ \frac{a'(0) - \|u_0\|^2}{4(2\delta + 1)E(0)}, 0 \right\}.$$

(ii) If  $E(0) = 0$  and  $\int_{\Omega} u_0 u_1 \, dx > 0$ , then  $a''(t) \geq 0$ , for  $t \geq 0$ . We have  $a'(t) > \|u_0\|^2$ ,  $t \geq 0$ .

(iii) If  $E(0) > 0$ , we first note that

$$(12) \quad 2 \int_0^t \int_{\Omega} u u_t \, dx \, d\tau = \|u\|^2 - \|u_0\|^2.$$

By Hölder inequality and Young inequality, we get

$$(13) \quad \|u\|^2 \leq \|u_0\|^2 + \int_0^t \|u\|^2 \, dt + \int_0^t \|u_t\|^2 \, dt.$$

By Hölder inequality, Young inequality and (13), we have

$$(14) \quad a'(t) \leq a(t) + \|u_0\|^2 + \int_{\Omega} u_t^2 \, dx + \int_0^t \|u_t\|^2 \, d\tau.$$

Hence, by (7) and (14), we obtain

$$a''(t) - 4(\delta + 1)a'(t) + 4(\delta + 1)a(t) + K_1 \geq 0,$$

where

$$(15) \quad K_1 = 4(2\delta + 1)E(0) + 4(\delta + 1) \int_{\Omega} u_0^2 \, dx.$$

Let

$$b(t) = a(t) + \frac{K_1}{4(\delta + 1)}, \quad t > 0.$$

Then  $b(t)$  satisfies Lemma 3.1. Consequently, we get from (10)  $a'(t) > \|u_0\|^2$ ,  $t > 0$ , where  $r_2$  is given in Lemma 3.1.  $\square$

**Theorem 3.1.** *Assume  $\frac{\gamma}{2} \leq \delta \leq \frac{q-1}{4}$ ,  $\gamma \geq 0$  and one of the following statements are satisfied*

- (i)  $E(0) < 0$  and  $\int_{\Omega} u_0 u_1 \, dx > 0$ ,
- (ii)  $E(0) = 0$  and  $\int_{\Omega} u_0 u_1 \, dx > 0$ ,
- (iii)  $0 < E(0) < \frac{(a'(t_0) - \|u_0\|^2)^2}{8[a(t_0) + (T_1 - t_0)\|u_0\|^2]}$  and (10) holds.

Then the solution  $u$  blow up in finite time  $T^*$  in the case of (5).

In case (i)

$$T^* \leq t_0 - \frac{H(t_0)}{H'(t_0)}.$$

Furthermore, if  $H(t_0) < \min\{1, \sqrt{-\frac{a}{b}}\}$ , we have

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{-\frac{a}{b}}}{\sqrt{-\frac{a}{b}} - H(t_0)},$$

where

$$(16) \quad a = \delta^2 H^{2+\frac{2}{\delta}}(t_0) \left[ \left( a'(t_0) - \|u_0\|^2 \right)^2 - 8E(0) H^{-\frac{1}{\delta}}(t_0) \right] > 0,$$

$$(17) \quad b = 8\delta^2 E(0).$$

In case (ii)

$$T^* \leq t_0 - \frac{H(t_0)}{H'(t_0)}.$$

In case (iii)

$$T^* \leq \frac{H(t_0)}{\sqrt{a}}$$

or

$$T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \left( \frac{a}{b} \right)^{2+\frac{1}{\delta}} \frac{\delta}{\sqrt{a}} \left\{ 1 - \left[ 1 + \left( \frac{a}{b} \right)^{2+\frac{1}{\delta}} H(t_0) \right]^{-\frac{1}{2\delta}} \right\}$$

where  $a$  and  $b$  are given (16), (17).

*Proof.* Let

$$(18) \quad H(t) = \left[ a(t) + (T_1 - t) \|u_0\|^2 \right]^{-\delta}, \text{ for } t \in [0, T_1],$$

where  $T_1 > 0$  is a certain constant which will be specified later. Then we get

$$\begin{aligned} H'(t) &= -\delta \left[ a(t) + (T_1 - t) \|u_0\|^2 \right]^{-\delta-1} \left[ a'(t) - \|u_0\|^2 \right] \\ &= -\delta H^{1+\frac{1}{\delta}}(t) \left[ a'(t) - \|u_0\|^2 \right], \end{aligned}$$

$$\begin{aligned} H''(t) &= -\delta H^{1+\frac{2}{\delta}}(t) a''(t) \left[ a(t) + (T_1 - t) \|u_0\|^2 \right] \\ &\quad + \delta H^{1+\frac{2}{\delta}}(t) (1 + \delta) \left[ a'(t) - \|u_0\|^2 \right]^2 \end{aligned}$$

and

$$(19) \quad H''(t) = -\delta H^{1+\frac{2}{\delta}}(t) V(t),$$

where

$$(20) \quad V(t) = a''(t) \left[ a(t) + (T_1 - t) \|u_0\|^2 \right] - (1 + \delta) \left[ a'(t) - \|u_0\|^2 \right]^2.$$

For simplicity of calculation, we define

$$\begin{aligned} P_u &= \int_{\Omega} u^2 \, dx, & R_u &= \int_{\Omega} u_t^2 \, dx, \\ Q_u &= \int_0^t \|u\|^2 \, dt, & S_u &= \int_0^t \|u_t\|^2 \, dt. \end{aligned}$$

From (8), (12) and Hölder inequality, we get

$$\begin{aligned} (21) \quad a'(t) &= 2 \int_{\Omega} uu_t \, dx + \|u_0\|^2 + 2 \int_0^t \int_{\Omega} uu_t \, dx \, dt \\ &\leq 2 \left( \sqrt{R_u P_u} + \sqrt{Q_u S_u} \right) + \|u_0\|^2. \end{aligned}$$

If case (i) or (ii) holds, by (7) we have

$$(22) \quad a''(t) \geq (-4 - 8\delta) E(0) + 4(1 + \delta)(R_u + S_u).$$

Thus, from (20)-(22) and (18), we obtain

$$\begin{aligned} V(t) &\geq [(-4 - 8\delta) E(0) + 4(1 + \delta)(R_u + S_u)] H^{-\frac{1}{8}}(t) \\ &\quad - 4(1 + \delta) \left( \sqrt{R_u P_u} + \sqrt{Q_u S_u} \right)^2. \end{aligned}$$

From (6),

$$\begin{aligned} a(t) &= \int_{\Omega} u^2 \, dx + \int_0^t \int_{\Omega} u^2 \, dx \, ds \\ &= P_u + Q_u \end{aligned}$$

and (18), we get

$$V(t) \geq (-4 - 8\delta) E(0) H^{-\frac{1}{8}}(t) + 4(1 + \delta) \left[ (R_u + S_u)(T_1 - t) \|u_0\|^2 + \Theta(t) \right],$$

where

$$\Theta(t) = (R_u + S_u)(P_u + Q_u) - \left( \sqrt{R_u P_u} + \sqrt{Q_u S_u} \right)^2.$$

By the Schwarz inequality, and  $\Theta(t)$  being nonnegative, we have

$$(23) \quad V(t) \geq (-4 - 8\delta) E(0) H^{-\frac{1}{8}}(t), \quad t \geq t_0.$$

Therefore, by (19) and (23), we get

$$(24) \quad H''(t) \leq 4\delta(1 + 2\delta) E(0) H^{1+\frac{1}{8}}(t), \quad t \geq t_0.$$

By Lemma 3.3, we know that  $H'(t) < 0$ , for  $t \geq t_0$ . Multiplying (24) by  $H'(t)$  and integrating it from  $t_0$  to  $t$ , we get

$$H'^2(t) \geq a + bH^{2+\frac{1}{8}}(t),$$

for  $t \geq t_0$ , where  $a, b$  are defined in (16) and (17) respectively.

If case (iii) holds, by the steps of case (i), we get  $a > 0$  if and only if

$$E(0) < \frac{\left(a'(t_0) - \|u_0\|^2\right)^2}{8 \left[a(t_0) + (T_1 - t_0) \|u_0\|^2\right]}.$$

Then by Lemma 3.2, there exists a finite time  $T^*$  such that  $\lim_{t \rightarrow T^{*-}} H(t) = 0$  and upper bound of  $T^*$  is estimated according to the sign of  $E(0)$ . This means that (5) holds.  $\square$

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