

Feedback Numbers of Möbius Ladders

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Abstract. A subset $F \subset V(G)$ is called a feedback vertex set if the subgraph $G - F$ is acyclic. The minimum cardinality of a feedback vertex set is called the feedback number of G , which is proposed by Beineke and Vandell [1]. In this paper, we consider a particular topology graph called Möbius ladders M_{2n} . We use $f(M_{2n})$ to denote the feedback number of M_{2n} . This paper proves that $f(M_{2n}) = \left\lceil \frac{n+1}{2} \right\rceil, n \geq 3$.

1 Introduction

It is well-known that the underlying topology of an interconnection network can be represented by a connected graph $G=(V,E)$, where V is the set of processors and E is the set of communication links in the network. In this paper, we use graphs and interconnection networks interchangeably.

Let $G=(V, E)$ be a graph or digraph without multiple edges, with vertex set $V(G)$ and edge set $E(G)$. A subset $F \subset V(G)$ is called a feedback vertex set if the subgraph $G-F$ is acyclic, that is, if $G-F$ is a forest. The minimum cardinality of a feedback vertex set is called the feedback number (or decycling number proposed first by Beineke and Vandell [1]) of G . A feedback vertex set of this cardinality is called a minimum feedback vertex set.

Apart from its graph-theoretical interest, the minimum feedback vertex set problem has important application to several fields. For example, the problems are in operating systems to resource allocation mechanisms that prevent deadlocks [2], in artificial intelligence to the constraint satisfaction problem and Bayesian inference, in synchronous distributed systems to the study of monopolies and in optical networks to converters placement problem (see [3, 4]).

Determining the feedback number is quite difficult even for some elementary graphs. However, the problem has been studied for some special graphs and digraphs, such as hypercubes, meshes, toroids, butterflies, cube-connected cycles, directed split-stars (see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]). In fact, the minimum feedback set problem is known to be NP-hard for general graphs [14] and the best known approximation algorithm is one with an approximation ratio two [5].

In this paper, we consider the feedback number of Möbius Ladders M_{2n} . Let $f(M_{2n})$ denote the feedback number of Möbius ladders. By using an approach to

construct an acyclic subgraph of M_{2n} , we proves that:

$$f(M_{2n}) = \left\lceil \frac{n+1}{2} \right\rceil, n \geq 3.$$

2 Preliminary

R. L.Guy and F. Harary consider the graph M_L (also called the Möbius ladder) in 1966, which we picture as a ladder with the ends joined after a twist, so that the graph is naturally embedded in a Möbius band [16].

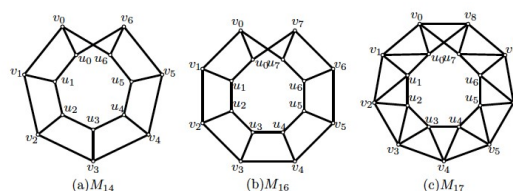


Figure 1. Examples of Möbius ladders

One particular molecular graph which is of interest is the molecular Möbius ladder, which was first synthesized by Walba et al. It was a significant achievement in chemistry because of its topologically interesting molecular structure. Simon proved that its associated embedded molecular graph was topologically chiral [17].

Although many authors only consider even L , but there exists odd L which be mentioned in ref. [18]. And as evidenced by the number of publications, there are many more established results for even L Möbius ladders than for odd. Obviously, the two cases (L even or odd) change the structure of these graphs quite radically:

Case1. When L is even, M_L is three-regular and Hamiltonian (See Fig.1(a)-(b)).

Case2. When L is odd, M_L is four-regular and therefore two-factorable (See Fig.1(c)).

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In this paper, we study only Case 1, and only even $L \geq 6$. Unless otherwise stated, assume that M_L means an even Möbius Ladder.

Definition 2.1. A Möbius Ladder is defined to be a graph on $2n(n \geq 3)$ vertices with

$$V(M_{2n}) = \{v_i, u_i : 0 \leq i \leq n-1\}$$

and

$$\begin{aligned} E(M_{2n}) = & \{v_i u_i : 0 \leq i \leq n-1\} \\ & \cup \{v_i v_{i+1}, u_i u_{i+1} : 0 \leq i \leq n-1\} \\ & \cup \{u_{n-1} v_0, v_{n-1} u_0\} \end{aligned}$$

Obviously, M_{2n} is three-regular and Hamiltonian.

3 Feedback vertex set of M_{2n}

Since $N = 2n$, let $n \bmod 2 = k$, then $n = 2m + k$, $m \geq 1$ and $k \in [0, 1]$.

Let $V_a^1(M_{2n}) = F_{1,1}^n \cup F_{1,2}^n \subset V(M_{2n})$, then we discuss two cases as follows.

Case 1. If $n \bmod 2 = 1$ and $odd\ n \geq 3$, let

$$\begin{aligned} F_{1,1}^n &= \bigcup_{\substack{i=0, \\ i \bmod 2=0}}^{n-1} \{v_i\} \cup \bigcup_{\substack{i=0, \\ i \bmod 2=1}}^{n-2} \{u_i\} \\ F_{1,2}^n &= \bigcup_{\substack{i=0, \\ i \bmod 2=1}}^{n-2} \{v_i\} \end{aligned}$$

Case 2. If $n \bmod 2 = 0$ and $even\ n \geq 4$, let

$$\begin{aligned} F_{2,1}^n &= \bigcup_{\substack{i=0, \\ i \bmod 2=0}}^{n-1} \{v_i\} \cup \bigcup_{\substack{i=0, \\ i \bmod 2=1}}^{n-2} \{u_i\} \\ F_{2,2}^n &= \bigcup_{\substack{i=0, \\ i \bmod 2=1}}^{n-2} \{v_i\} \\ F_{2,3}^n &= \{u_{n-1}\} \end{aligned}$$

Definition 3.1. The open neighborhood and the closed neighborhood of a vertex $v \in V(M_{2n})$ are denoted by $N(v) = \{u \in V(M_{2n}) : vu \in E(M_{2n})\}$ and $N[v] = N(v) \cup \{v\}$, respectively. For a vertex set

$$S \subseteq V(M_{2n}), N(S) = \bigcup_{v \in S} N(v) \text{ and } N[S] = \bigcup_{v \in S} N[v].$$

Definition 3.2. For any vertex $u_j \in F_{i,2}^n$, $i \in [1, 2]$ and $j \in [0, n-1]$, denote $T_{1,j} = N[u_j]$, where $N[u_j]$ is the closed neighborhood of a vertex u_j and the induced subgraph of $T_{1,j}$ as $G[T_{1,j}]$.

In fact, for any j , $G[T_{1,j}]$ is isomorphic to $K_{1,3}$. That is, for any j , $G[T_{1,j}]$ is a tree, where vertex v_j is a root

vertex of tree $G[T_{1,j}]$ and vertex set $N(v_j)$ are leaf vertices of tree $G[T_{1,j}]$.

By the Definition 2.1, all the vertices in $F_{i,1}^n$ and $F_{i,2}^n$ are isolated vertex set for $i \in [1, 2]$, respectively. Then the induced subgraph of $F_{i,1}^n$ is acyclic and the induced sub-graph of $F_{i,2}^n$ is acyclic. By the Definition 3.2, we obtain $N(F_{i,2}^n) = F_{i,1}^n$.

Lemma 3.1. $G[V_a^1(M_{2n})]$ is acyclic for $n \bmod 2 = 1$ and $odd\ n \geq 3$.

Proof.
 Since

$$\begin{aligned} V_a^1(M_{2n}) &= F_{1,1}^n \cup F_{1,2}^n \\ &= N(F_{1,2}^n) \cup F_{1,2}^n \\ &= N[F_{1,2}^n] \end{aligned}$$

Then if we want to prove $G[V_a^1(M_{2n})]$ is acyclic, which is equivalent to verify that $G[N[F_{1,2}^n]]$ is acyclic.

Firstly, we prove that $G[N[F_{1,2}^n]]$ is a tree by induction on n .

We first prove the basic step for $n = 3$.

Since

$$F_{1,1}^3 = \{v_0, v_2\} \cup \{u_1\}, F_{1,2}^3 = \{v_1\}$$

And

$$\begin{aligned} T_{1,1}^3 &= N[v_1] = N\{v_1\} \cup \{v_1\} \\ &= \{v_0, v_2\} \cup \{v_1\} \\ &= F_{1,1}^3 \cup F_{1,2}^3 \\ &= N[F_{1,2}^3] \\ G[T_{1,1}^3] &\cong K_{1,3} \end{aligned}$$

Then $G[T_{1,1}^3]$ is a tree. That is, $G[F_{1,1}^3 \cup F_{1,2}^3]$ is a tree.

Suppose $G[F_{1,1}^n \cup F_{1,2}^n]$ is a tree, we now prove that $G[F_{1,2}^{n+2} \cup F_{1,2}^{n+2}]$ is a tree.

Since

$$\begin{aligned} F_{1,1}^{n+2} &= \bigcup_{\substack{i=0, \\ i \bmod 2=0}}^{n+2-1} \{v_i\} \cup \bigcup_{\substack{i=0, \\ i \bmod 2=1}}^{n+2-2} \{u_i\} \\ &= \bigcup_{\substack{i=0, \\ i \bmod 2=0}}^{n-1} \{v_i\} \cup \bigcup_{\substack{i=n, \\ i \bmod 2=0}}^{n+1} \{v_i\} \cup \bigcup_{\substack{i=0, \\ i \bmod 2=1}}^{n-2} \{u_i\} \cup \bigcup_{\substack{i=n-1, \\ i \bmod 2=1}}^n \{u_i\} \\ &= F_{1,1}^n \cup \bigcup_{\substack{i=n, \\ i \bmod 2=0}}^{n+1} \{v_i\} \cup \bigcup_{\substack{i=n-1, \\ i \bmod 2=1}}^n \{u_i\} \\ &= F_{1,1}^n \cup \{v_{n+1}\} \cup \{u_n\} \end{aligned}$$

and

$$F_{1,2}^{n+1} = \bigcup_{\substack{i=0, \\ i \bmod 2=1}}^{n+2-2} \{v_i\} = \bigcup_{\substack{i=0, \\ i \bmod 2=1}}^{n-2} \{v_i\} \cup \bigcup_{\substack{i=n-1, \\ i \bmod 2=1}}^n \{v_i\} \\ = F_{1,2}^n \cup \{v_n\}$$

Then

$$F_{1,1}^{n+2} \cup F_{1,2}^{n+2} = F_{1,1}^n \cup F_{1,2}^n \cup \{v_{n+1}, u_n, v_n\} \\ = F_{1,1}^n \cup F_{1,2}^n \cup T_{1,n}^{n+2}$$

Since $G[T_{1,n}^{n+2}] \cong K_{1,3}$, then $G[T_{1,n}^{n+2}]$ is a tree.

By the induction hypothesis, $G[F_{1,1}^n \cup F_{1,2}^n]$ is a tree.

Tree $G[T_{1,n}^{n+2}]$ and tree $G[F_{1,1}^n \cup F_{1,2}^n]$ just have one common leaf vertex v_{n-1} , then $G[F_{1,1}^n \cup F_{1,2}^n \cup T_{1,n}^{n+2}]$ is a tree.

Combining induction step with basic step,

$G[F_{1,2}^n \cup F_{1,2}^n]$ is a tree.

Thus, $G[V_a^1(M_{2n})] = G[F_{1,1}^n \cup F_{1,2}^n]$ is acyclic.

The lemma holds.

Lemma 3.2. $G[V_a^2(M_{2n})]$ is acyclic for $n \bmod 2 = 0$ and even $n \geq 4$.

Proof.

Since

$$V_a^2(M_{2n}) = F_{2,1}^n \cup F_{2,2}^n \cup F_{2,3}^n \\ = N(F_{2,2}^n) \cup F_{2,2}^n \cup \{u_{n-1}\} \\ = N[F_{2,2}^n] \cup \{u_{n-1}\}$$

Then if we want to prove $G[V_a^2(M_{2n})]$ is acyclic, which is equivalent to verify that $G[N[F_{2,2}^n] \cup \{u_{n-1}\}]$ is acyclic.

Firstly, we prove that $G[N[F_{2,2}^n] \cup \{u_{n-1}\}]$ is a tree by induction on n .

We first prove the basic step for $n = 4$.

Since

$$F_{2,1}^4 = \{v_0, v_2\} \cup \{u_1\}, F_{2,2}^4 = \{v_1\}, F_{2,3}^4 = \{u_3\}$$

And

$$T_{2,1}^4 = N[v_1] = N(v_1) \cup \{v_1\} = \{v_0, v_2, u_1\} \cup \{v_1\} \\ = F_{2,1}^4 \cup F_{2,2}^4 = N[F_{2,2}^4]$$

Since $G[T_{2,1}^4] \cong K_{1,3}$, then $G[T_{2,1}^4]$ is a tree. That is, $G[F_{2,1}^4 \cup F_{2,2}^4]$ is a tree.

Suppose $G[F_{1,2}^n \cup F_{1,2}^n]$ is a tree, we now prove that

$G[F_{1,2}^{n+2} \cup F_{1,2}^{n+2}]$ is a tree.

Since

$$F_{1,1}^{n+2} = \bigcup_{\substack{i=0, \\ i \bmod 2=0}}^{n+2-1} \{v_i\} \cup \bigcup_{\substack{i=0, \\ i \bmod 2=1}}^{n+2-2} \{u_i\} \\ = \bigcup_{\substack{i=0, \\ i \bmod 2=0}}^{n-1} \{v_i\} \cup \bigcup_{\substack{i=n, \\ i \bmod 2=0}}^{n+1} \{v_i\} \cup \bigcup_{\substack{i=0, \\ i \bmod 2=1}}^{n-2} \{u_i\} \cup \bigcup_{\substack{i=n-1, \\ i \bmod 2=1}}^n \{u_i\} \\ = F_{1,1}^n \cup \bigcup_{\substack{i=n, \\ i \bmod 2=0}}^{n+1} \{v_i\} \cup \bigcup_{\substack{i=n-1, \\ i \bmod 2=1}}^n \{u_i\} \\ = F_{1,1}^n \cup \{v_{n+1}\} \cup \{u_n\}$$

and

$$F_{1,2}^{n+1} = \bigcup_{\substack{i=0, \\ i \bmod 2=1}}^{n+2-2} \{v_i\} \\ = \bigcup_{\substack{i=0, \\ i \bmod 2=1}}^{n-2} \{v_i\} \cup \bigcup_{\substack{i=n-1, \\ i \bmod 2=1}}^n \{v_i\} \\ = F_{1,2}^n \cup \{v_n\}$$

Then

$$F_{1,1}^{n+2} \cup F_{1,2}^{n+2} \\ = F_{1,1}^n \cup F_{1,2}^n \cup \{v_{n+1}, u_n, v_n\} \\ = F_{1,1}^n \cup F_{1,2}^n \cup T_{1,n}^{n+2}$$

Since $G[T_{1,n}^{n+2}] \cong K_{1,3}$, then $G[T_{1,n}^{n+2}]$ is a tree.

By the induction hypothesis, $G[F_{1,1}^n \cup F_{1,2}^n]$ is a tree.

Tree $G[T_{1,n}^{n+2}]$ and tree $G[F_{1,1}^n \cup F_{1,2}^n]$ just have one common leaf vertex v_{n-1} , then $G[F_{1,1}^n \cup F_{1,2}^n \cup T_{1,n}^{n+2}]$ is a tree.

Combining induction step with basic step, $G[F_{1,2}^n \cup F_{1,2}^n]$ is a tree.

Thus, $G[V_a^1(M_{2n})] = G[F_{1,1}^n \cup F_{1,2}^n]$ is acyclic.

The lemma holds.

By Lemma 3.1 and Lemma 3.2, we obtain the Lemma 3.3 as follows.

Lemma 3.3. $V(M_{2n}) \setminus V_a^i(M_{2n})$ is The feedback vertex set of M_{2n} for $i \in [1, 2]$.

4 Feedback Number of M_{2n}

By reference [1], Beineke and Vandell prove a lower bound of general graphs $G(V, E)$:

$$f(G) \geq \left\lceil \frac{|E| - |V| + 1}{\Delta - 1} \right\rceil$$

Lemma 4.1. Denote $f(M_{2n})$ as the feedback number of M_{2n} , the lower bound of the feedback number is

$$f(M_{2n}) \geq \left\lceil \frac{n+1}{2} \right\rceil$$

Proof.

Since $|V(M_{2n})|=2n$, $|E(M_{2n})|=3n$ and M_{2n} is 3-regular graph. Then, we obtain a lower bound as follows:

$$f(M_{2n}) \geq \left\lceil \frac{|E|-|V|+1}{\Delta-1} \right\rceil = \left\lceil \frac{3n-2n+1}{2} \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil$$

Lemma 4.2. Denote $f(M_{2n})$ as the feedback number of M_{2n} , the upper bound of the feedback number is

$$f(M_{2n}) \leq \left\lfloor \frac{n+1}{2} \right\rfloor$$

Proof. By Lemma 3.3, we have

$$f(M_{2n}) \leq V(M_{2n}) \setminus V_a^i(M_{2n})$$

Then

$$f(M_{2n}) \leq \begin{cases} 2n - (n + \frac{n-1}{2}) & \text{for odd } n \\ 2n - [(n-1) + (\frac{n}{2}-1) + 1] & \text{for even } n \end{cases} \\ = \left\lfloor \frac{n+1}{2} \right\rfloor$$

By the Lemma 4.1 and Lemma 4.2, we obtain the feedback number of M_{2n} as follows.

Theorem1. For $n \geq 3$, the feedback number of M_{2n} is $f(M_{2n}) = \left\lfloor \frac{n+1}{2} \right\rfloor$

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