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Feedback Numbers of Goldberg Snark, Twisted Goldberg Snarks and Related Graphs

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Abstract. A subset of vertices of a graph *G* is called a feedback vertex set of *G* if its removal results in an acyclic subgraph. The minimum cardinality of a feedback vertex set is called the feedback number. In this paper, we determine the exact values of the feedback numbers of the Goldberg snarks G_n and its related graphs G_n^*, Twisted Goldberg Snarks TG_n and its related graphs TG_n^*. Let $f(n)$ denote the feedback numbers of these graphs, we prove that $f(n)=2n+1$, for $n\geq3$.

1 Introduction

Let $G = (V, E)$ be a graph or digraph without multiple edges, with vertex set $V(G)$ and edge set $E(G)$. A subset $F\subset V(G)$ is called a feedback vertex set if the subgraph G −F is acyclic, that is, if $G-F$ is a forest. The minimum cardinality of a feedback vertex set is called the feedback number (or decycling number proposed first by Beineke and Vandell [1]) of G. A feedback vertex set of this cardinality is called a minimum feedback vertex set.

Apart from its graph-theoretical interest, the minimum feedback vertex set problem has important application to several fields. For example, the problems are in operating systems to resource allocation mechanisms that prevent deadlocks [2], in artificial intelligence to the constraint satisfaction problem and Bayesian inference, in synchronous distributed systems to the study of monopolies and in optical networks to converters placement problem (see [3, 4]).

Determining the feedback number is quite difficult even for some elementary graphs. However, the problem has been studied for some special graphs and digraphs, such as hypercubes, meshes, toroids, butterflies, cube-connected cycles, directed split-stars (see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]). In fact, the minimum feedback set problem is known to be NP-hard for general graphs [14] and the best known approximation algorithm is one with an approximation ratio two [5].

Snarks are simple nontrivial connectes cubic graphs of chromatic index four [16]. The importance of the snar-ks does not only depend on the four colour theorem. Indeed, there are several important open problems such as the classical cycle double cover conjecture [18]. The smallest snark is the Petersen graph and it was the first discovered. In [17], it gave a survey of some results on some well known families of snarks, e.g. Flower snarks, Loupekine snarks and Goldberg snarks.

Goldberg Snarks is referred to Loupekine by Goldbe-rg himself. In fact, Goldberg Snarks are cartwheel snarks [19]. Both Goldberg Snarks and twisted Goldberg Snarks are graphs on $8n$ vertices with

$$V(G_n)=\{v_j^i \mid 0\leq i\leq n-1, 1\leq j\leq 8\}$$

• For every odd $n\geq3$, G_n is called the ***Goldberg Snark graph*** and adjacencies are defined as shown in Fig.1, where the vertex labels *i* are read modulo *n*.

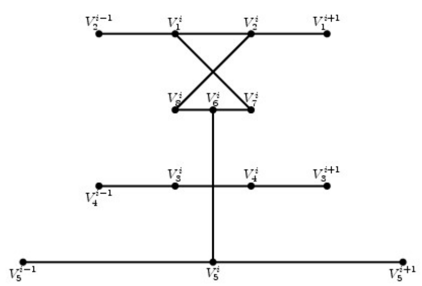

Figure 1. Used in the construction of Goldberg snark

• For every even $n\geq4$, G_n^* is called ***Related graphs of Goldberg Snark***.

• For every odd $n\geq3$, let TG_n be a graph obtained from G_n by replacing the edges $v_2^{n-1}v_1^0$ and $v_4^{n-1}v_3^0$ with $v_2^{n-1}v_3^0$ and $v_4^{n-1}v_1^0$, respectively. We denote TG_n as ***Twisted Goldberg Snark***.

• For every even $n\geq4$, TG_n^* is called ***Related graphs of Twisted Goldberg Snark***.

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The example of G_3, G_4^*, TG_3 and TG_4^* are shown in Fig.2.

However, both Goldberg Snarks and twisted Goldberg Snarks can be obtained via the Loupekine construction from the Petersen graph [15].

In this paper, we determine the exact values of the feedback number of the Goldberg snarks G_n and its related graphs G_n^*, the twisted Goldberg snarks TG_n and its related graphs TG_n^*. Let $f(n)$ denote the feedback number of these graphs, we prove that $f(n)=2n+1$, for $n≥3$.

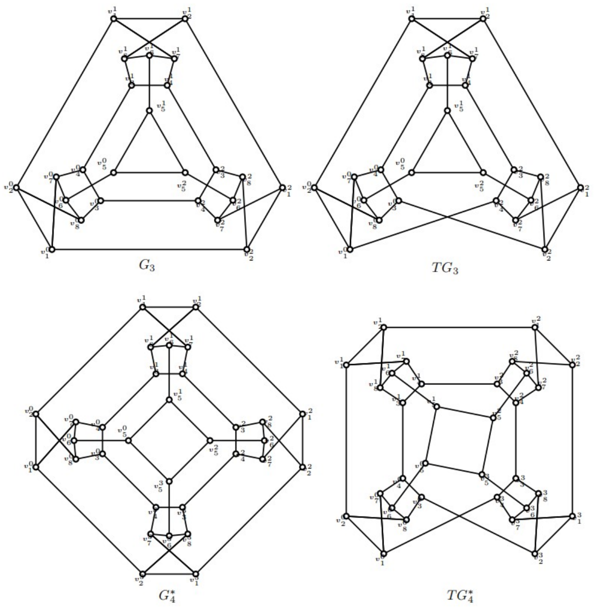

Figure 2. Graph G_3, G_4^*, TG_3 and TG_4^*

2 Acyclic Vertex Set of G_n and G_n^*

Let $n \bmod 2 = k$, then $n = 2m + k$, $m \geq 1$ *and* $0 \leq i \leq k$. Thus $k = 0$ and $k = 1$. Then, we discuss 2 cases as follows.

Case 1. If $n \bmod 2 = 1$ and $n \geq 3$

Let $V_a(G_n) = F_1^n \cup F_2^n$, where

$$F_1^n = \{v_2^0, v_3^0, v_8^0, v_6^0, v_7^0\}$$

$$F_2^n = \bigcup_{i=1}^{n-1} \{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$$

Case 2. If $n \bmod 2 = 0$ and $n \geq 4$.

Let $V_a(G_n^*) = F_1^n \cup F_2^n \cup F_3^n$, where

$$F_1^n = \{v_2^0, v_3^0, v_8^0, v_6^0, v_7^0\}$$

$$F_2^n = \{v_2^1, v_3^1, v_8^1, v_6^1, v_7^1, v_5^1\}$$

$$F_3^n = \bigcup_{i=2}^{n-1} \{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$$

Lemma 2.1. $G[V_a(G_n)]$ is acyclic for n mod 2=1 and $n≥3$.

Proof: Since $V_a(G_n) = F_1^n \cup F_2^n$, Then if we want to prove $G[V_a(G_n)]$ is acyclic, which is equivalent to verify that $G[F_1^n \cup F_1^n]$ is acyclic.

Firstly, we prove the $G[F_1^n \cup F_1^n]$ is a forest by induction on n.

We first prove the basic step for n=3.

Combining case 1 and the definition of G_n, we have

$$F_1^3 = \{v_2^0, v_3^0, v_8^0, v_6^0, v_7^0\}$$

$$F_2^3 = \{v_2^1, v_3^1, v_8^1, v_6^1, v_7^1, v_5^1, v_2^2, v_3^2, v_8^2, v_6^2, v_7^2, v_5^2\}$$

Obviously, the induced subgraph of vertex set F_1^3 and the induced subgraph of vertex set F_2^3 are acyclic graphs(see Fig.3 and Fig.4) and each graph of $G[F_1^3]$ and $G[F_2^3]$ is a tree.

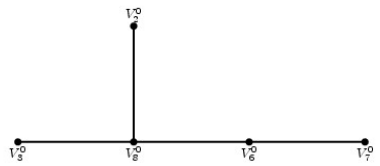

Figure 3. Subgraph of $G[F_1^3]$

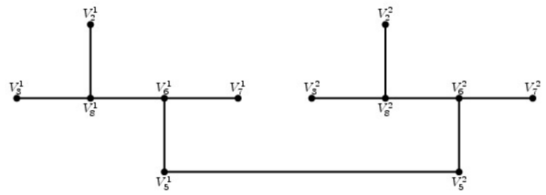

Figure 4. Subgraph of $G[F_2^3]$

Since $F_1^3 \cap F_2^3 = Ø$, then the induced subgraph of vertex set $F_1^3 \cup F_2^3$ is acyclic. That is, $G[F_1^3 \cup F_2^3]$ is a forest.

Suppose $G[F_1^n \cup F_2^n]$ is acyclic, we now prove that $G[F_1^{n+2} \cup F_2^{n+2}]$ is acyclic.

Since

$$F_1^{n+2} = \{v_2^0, v_3^0, v_8^0, v_6^0, v_7^0\}$$

$$F_2^{n+2} = \bigcup_{i=1}^{n+2-1} \{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$$

Then we have

$$F_1^{n+2} = F_1^n$$

$$F_2^{n+2} = \bigcup_{i=1}^{n-1}\{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\} \cup \bigcup_{i=n}^{n+1}\{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$$

$$= F_2^n \cup \bigcup_{i=n}^{n+1}\{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$$

For convenience, we denote T as vertex set $\bigcup_{i=n}^{n+1}\{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$, then we denote the induced subgraph of vertex set $\bigcup_{i=n}^{n+1}\{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$ as $G[T]$.

Obviously, $G[T] \cong G[F_2^3]$ (see Fig. 5), then $G[T]$ is a tree.

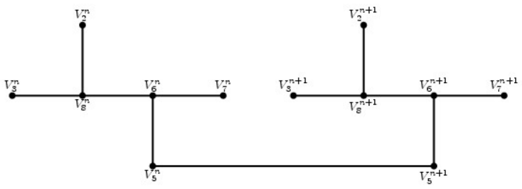

Figure 5. Subgraph of $G[T]$

Since $T \cap F_2^n = \emptyset$, then $G[T \cup F_2^n]$ is a forest.

Since $T \cap F_1^n = \emptyset$, then $G[T \cup F_1^n \cup F_2^n]$ is a forest. That is, $G[F_1^{n+2} \cup F_2^{n+2}]$ is acyclic.

Combining induction step with basic step, the $G[F_1^n \cup F_2^n]$ is a forest. Thus, $G[V_a(G_n)]$ is acyclic.

The lemma holds.

Lemma 2.2. $G[V_a(G_n^*)]$ is acyclic for $n \bmod 2 = 0$ and $n \geq 4$.

Proof: Since $V_a(G_n^*) = F_1^n \cup F_2^n \cup F_3^n$, Then if we want to prove $G[V_a(G_n^*)]$ is acyclic, which is equivalent to verify that $G[F_1^n \cup F_2^n \cup F_3^n]$ is acyclic.

Firstly, we prove the $G[F_1^n \cup F_2^n \cup F_3^n]$ is a forest by induction on *n*.

We first prove the basic step for *n*=4.

Combining case 2 and the definition of G_n^*, we have

$$F_1^4 = \{v_2^0, v_3^0, v_8^0, v_6^0, v_7^0\}$$

$$F_2^4 = \{v_2^1, v_3^1, v_8^1, v_6^1, v_7^1, v_5^1\}$$

$$F_3^4 = \{v_2^2, v_3^2, v_8^2, v_6^2, v_7^2, v_5^2, v_2^3, v_3^3, v_8^3, v_6^3, v_7^3, v_5^3\}$$

Obviously, $G[F_1^4]$, $G[F_2^4]$ and $G[F_3^4]$ are acyclic graphs, and each graph of $G[F_1^4], G[F_2^4]$ and $G[F_3^4]$ is a tree.

Since $F_1^4 \cap F_2^4 = \emptyset$, then the induced subgraph of vertex set $F_1^4 \cup F_2^4$ is acyclic.

Since $F_1^4 \cap F_3^4 = \emptyset$, then the induced subgraph of vertex set $F_1^4 \cup F_3^4$ is acyclic.

Since $F_2^4 \cap F_3^4 = \emptyset$, then the induced subgraph of vertex set $F_2^4 \cup F_3^4$ is acyclic.

Thus ,we have $F_1^4 \cap F_2^4 \cap F_3^4 = \emptyset$, then the induced subgraph of vertex set $F_1^4 \cup F_2^4 \cup F_3^4$ is acyclic.

That is, $G[F_1^4 \cup F_2^4 \cup F_3^4]$ is a forest. The vertices set is shown as Fig 6.

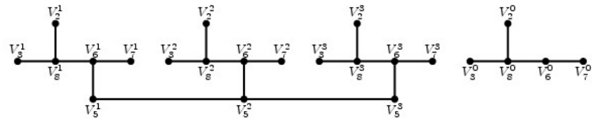

Figure 6. Subgraph of $G[F_1^4 \cup F_2^4 \cup F_2^4]$

Suppose $G[F_1^n \cup F_2^n \cup F_3^n]$ is a forest, we now prove that $G[F_1^{n+2} \cup F_2^{n+2} \cup F_3^{n+2}]$ is a forest.

Since

$$F_1^{n+2} = \{v_2^0, v_3^0, v_8^0, v_6^0, v_7^0\}$$

$$F_2^{n+2} = \{v_2^1, v_3^1, v_8^1, v_6^1, v_7^1, v_5^1\}$$

$$F_3^{n+2} = \bigcup_{i=1}^{n+2-1}\{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$$

Then, we have

$$F_1^{n+2} = F_1^n$$

$$F_2^{n+2} = F_2^n$$

$$F_3^{n+2} = \bigcup_{i=2}^{n-1}\{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\} \cup \bigcup_{i=n}^{n+1}\{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$$

$$= F_3^n \cup \bigcup_{i=n}^{n+1}\{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$$

For convenience, we also denote T as vertex set $\bigcup_{i=n}^{n+1}\{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$, then we denote the induced subgraph of vertex set $\bigcup_{i=n}^{n+1}\{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$ as $G[T]$.

Obviously, $G[T] \cong G[F_3^4]$, then $G[T]$ is a tree.

Since $T \cap F_3^n = \emptyset$,then $G[T \cup F_3^n]$ is a forest.

Since $T \cap F_1^n = \emptyset$,then $G[T \cup F_1^n \cup F_2^n]$ is a forest.

Since $T \cap F_2^n = \emptyset$,then $G[T \cup F_1^n \cup F_2^n \cup F_3^n]$ is a forest. That is, $G[F_1^{n+2} \cup F_2^{n+2} \cup F_3^{n+2}]$ is acyclic.

Combining induction step with basic step, the $G[F_1^n \cup F_2^n \cup F_3^n]$ is a forest. Thus, $G[V_a(G_n^*)]$ is acyclic.

The lemma holds.

3 Acyclic Vertex Set of TG_n and TG_n^*

Let $n \bmod 2 = k$, then $n = 2m + k$, $m \ge 1$ *and* $0 \le i \le k$. Thus $k = 0$ and $k = 1$. Then, we discuss 2 cases as follows.

Case 1. If $n \bmod 2 = 1$ and $n \ge 3$

Let $V_a(TG_n) = F_1^n \cup F_2^n$, where

$$F_1^n = \{v_2^0, v_3^0, v_8^0, v_6^0, v_7^0, v_2^{n-1}, v_3^{n-1}, v_8^{n-1}, v_6^{n-1}, v_7^{n-1}, v_5^{n-1}\}$$

$$F_2^n = \bigcup_{i=1}^{n-2} \{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$$

Case 2. If $n \bmod 2 = 0$, and $n \ge 4$

Let $V_a(TG_n^*) = F_1^n \cup F_2^n$, where

$$F_1^n = \{v_2^0, v_3^0, v_8^0, v_6^0, v_7^0, v_2^{n-1}, v_3^{n-1}, v_8^{n-1}, v_6^{n-1}, v_7^{n-1}, v_5^{n-1}\}$$

$$F_2^n = \bigcup_{i=1}^{n-2} \{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$$

Lemma 3.1. $G[V_a(TG_n)]$ is acyclic for $n \bmod 2 = 1$ and $n \ge 3$.

Proof: Since $V_a(TG_n) = F_1^n \cup F_2^n$, Then if we want to prove $G[V_a(TG_n)]$ is acyclic, which is equivalent to verify that $G[F_1^n \cup F_2^n]$ is acyclic.

Firstly, we prove the $G[F_1^n \cup F_2^n]$ is a tree by induction on *n*.

We first prove the basic step for *n*=3.

Combining case 1 and the definition of TG_n, we have

$$F_1^3 = \{v_2^0, v_3^0, v_8^0, v_6^0, v_7^0, v_2^2, v_3^2, v_8^2, v_6^2, v_7^2, v_5^2\}$$
$$F_2^3 = \{v_2^1, v_3^1, v_8^1, v_6^1, v_7^1, v_5^1\}$$

Obviously, by the definition of TG_n, we have that $G[F_1^3]$ and $G[F_2^3]$ are acyclic graphs, and each graph of $G[F_1^3]$ and $G[F_2^3]$ is a tree.

Since $F_1^3 \cap F_2^3 = \emptyset$, then the induced subgraph of vertex set $F_1^3 \cup F_2^3$ is acyclic. That is, $G[F_1^3 \cup F_2^3]$ is a tree. The vertices set is shown as Fig 7.

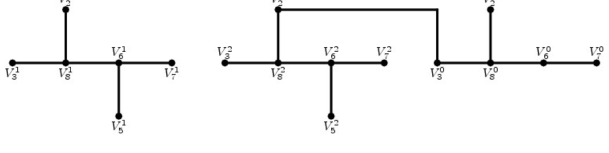

Figure 7. Subgraph of $G[F_1^3 \cup F_2^3]$

Suppose $G[F_1^n \cup F_2^n]$ is a tree, we now prove that $G[F_1^{n+2} \cup F_2^{n+2}]$ is a tree.

Since

$$F_1^{n+2} = \{v_2^0, v_3^0, v_8^0, v_6^0, v_7^0, v_2^{n+1}, v_3^{n+1}, v_8^{n+1}, v_6^{n+1}, v_7^{n+1}, v_5^{n+1}\}$$

$$\begin{aligned} F_2^{n+2} &= \bigcup_{i=1}^{n+2-2} \{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\} \\ &= \bigcup_{i=1}^{n-2} \{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\} \cup \bigcup_{i=n-1}^{n} \{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\} \\ &= F_2^n \cup \bigcup_{i=n-1}^{n} \{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\} \end{aligned}$$

Then we have

$$F_1^{n+2} \cong F_1^n$$

$$F_2^{n+2} = F_2^n \cup \bigcup_{i=n-1}^{n} \{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$$

For convenience, we denote T as vertex set $\bigcup_{i=n-1}^{n} \{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$, then we denote the induced subgraph of vertex set $\bigcup_{i=n-1}^{n} \{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$ as $G[T]$.

Obviously, $G[T] \cong G[F_2^3]$, then $G[T]$ is a tree.

Since $T \cap F_2^n = \emptyset$, then $G[T \cup F_2^n]$ is a tree.

Since $T \cap F_1^n = \emptyset$, then $G[T \cup F_1^n \cup F_2^n]$ is a tree. That is, $G[F_1^{n+2} \cup F_2^{n+2}]$ is acyclic.

Combining induction step with basic step, the $G[F_1^n \cup F_2^n]$ is a tree. Thus, $G[V_a(TG_n)]$ is acyclic.

The lemma holds.

Lemma 3.2. $G[V_a(TG_n^*)]$ is acyclic for $n \bmod 2 = 0$ and $n \ge 4$.

Proof: Since $V_a(TG_n^*) = F_1^n \cup F_2^n$, Then if we want to prove $G[V_a(TG_n^*)]$ is acyclic, which is equivalent to verify that $G[F_1^n \cup F_2^n]$ is acyclic.

Firstly, we prove the $G[F_1^n \cup F_2^n]$ is a tree by induction on *n*.

We first prove the basic step for *n*=4.

Combining case 2 and the definition of TG_n^*, we have

$$F_1^4 = \{v_2^0, v_3^0, v_8^0, v_6^0, v_7^0, v_2^3, v_3^3, v_8^3, v_6^3, v_7^3, v_5^3\}$$
$$F_2^4 = \{v_2^1, v_3^1, v_8^1, v_6^1, v_7^1, v_5^1, v_2^2, v_3^2, v_8^2, v_6^2, v_7^2, v_5^2\}$$

Obviously, $G[F_1^4]$ and $G[F_2^4]$ are acyclic graph, and each graph of $G[F_1^4]$ and $G[F_2^4]$ is a tree. Since $F_1^4 \cap F_2^4 = \emptyset$, then the induced subgraph of vertex set $F_1^4 \cup F_2^4$ is acyclic. That is, $G[F_1^4 \cup F_2^4]$ is a tree. The vertices set is shown as Fig 8.

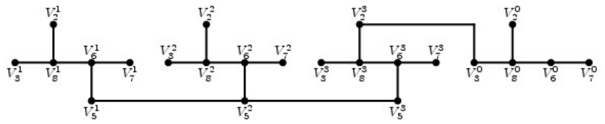

Figure 8. Subgraph of $G[F_1^4 \cup F_2^4]$

Suppose $G[F_1^n \cup F_2^n]$ is a tree, we now prove that $G[F_1^{n+2} \cup F_2^{n+2}]$ is a tree.

Since

$$F_1^{n+2} = \{v_2^0, v_3^0, v_8^0, v_6^0, v_7^0, v_2^{n-1}\} \cup \{v_3^{n+1}, v_8^{n+1}, v_6^{n+1}, v_7^{n+1}, v_5^{n+1}\}$$

$$F_2^{n+2} = \bigcup_{i=1}^{n+2-2} \{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$$

Then we have

$$F_1^{n+2} \cong F_1^n$$

$$F_2^{n+2} = F_2^n \cup \bigcup_{i=n-1}^{n} \{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$$

For convenience, we also denote T as vertex set $\bigcup_{i=n-1}^{n} \{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$, then we denote the induced subgraph of vertex set $\bigcup_{i=n-1}^{n} \{v_2^i, v_3^i, v_8^i, v_6^i, v_7^i, v_5^i\}$ as $G[T]$.

Obviously, $G[T] \cong G[F_2^4]$, then $G[T]$ is a tree.

Since $T \cap F_2^n = \emptyset$, then $G[T \cup F_2^n]$ is a tree.

Since $T \cap F_1^n = \emptyset$, then $G[T \cup F_1^n \cup F_2^n]$ is a tree. That is, $G[F_1^{n+2} \cup F_2^{n+2}]$ is acyclic.

Combining induction step with basic step, the $G[F_1^n \cup F_2^n]$ is a tree. Thus, $G[V_a(TG_n^*)]$ is acyclic.

The lemma holds.

4 Feedback Number of graphs

By the Lemma 2.1 and Lemma 2.2, we obtain

Lemma 4.1. $V(G_n) \setminus V_a(G_n)$ is feedback vertex set of G_n .

Lemma 4.2. $V(G_n^*) \setminus V_a(G_n^*)$ is feedback vertex set of G_n^* .

By the Lemma 3.1 and Lemma 3.2, we obtain

Lemma 4.3. $V(TG_n) \setminus V_a(TG_n)$ is feedback vertex set of TG_n .

Lemma 4.4. $V(TG_n^*) \setminus V_a(TG_n^*)$ is feedback vertex set of TG_n^* .

For convenience, we denote $f(n)$ as the feedback numbers of Goldberg Snark, Twist Goldberg Snark and their related graphs.

Lemma 4.5. For $n \geq 3$, the upper bound of feedback number is

$$f(n) \leq 2n+1$$

Proof: By Lemma 4.1, we have the feedback number of $V(G_n)$ as follows.

$$\begin{aligned} &|V(G_n) \setminus V_a(G_n)| \\ &\leq |V(G_n) \setminus (F_1^n \cup F_2^n)| \\ &= |V(G_n)| - |(F_1^n \cup F_2^n)| \\ &= 8n - [5 + 6(n-1)] \\ &= 2n+1 \end{aligned}$$

By Lemma 4.2, we have the feedback number of $V(G_n^*)$ as follows.

$$\begin{aligned} &|V(G_n^*) \setminus V_a(G_n^*)| \\ &\leq |V(G_n^*) \setminus (F_1^n \cup F_2^n \cup F_3^n)| \\ &= |V(G_n^*)| - |(F_1^n \cup F_2^n \cup F_3^n)| \\ &= 8n - [5 + 6 + 6(n-2)] \\ &= 2n+1 \end{aligned}$$

By Lemma 4.3, we have the feedback number of $V(TG_n)$ as follows.

$$\begin{aligned} &|V(TG_n) \setminus V_a(TG_n)| \\ &\leq |V(TG_n) \setminus (F_1^n \cup F_2^n)| \\ &= |V(TG_n)| - |(F_1^n \cup F_2^n)| \\ &= 8n - [11 + 6(n-2)] \\ &= 2n+1 \end{aligned}$$

By Lemma 4.4, we have the feedback number of $V(TG_n)$ as follows.

$$\begin{aligned} &|V(TG_n^*) \setminus V_a(TG_n^*)| \\ &\leq |V(TG_n^*) \setminus (F_1^n \cup F_2^n)| \\ &= |V(TG_n^*)| - |(F_1^n \cup F_2^n)| \\ &= 8n - [11 + 6(n-2)] \\ &= 2n+1 \end{aligned}$$

Then, we have the upper bound of feedback number of is $f(n) \leq 2n+1$

Lemma 4.6. For $n \geq 3$, the lower bound of feedback number is

$$f(n) \geq 2n+1$$

Proof: By reference [20], Beineke and Vandell prove a lower bound of general graphs $G(V,E)$:

$$f(G) \geq \left\lceil \frac{|E|-|V|+1}{\Delta-1} \right\rceil$$

Since $|V|=8n$, $|E|=12n$ and $\Delta=3$. Thus, we obtain a lower bound of these graphs as follows:

$$f(n) \geq \left\lceil \frac{|E|-|V|+1}{\Delta-1} \right\rceil = \left\lceil \frac{12n-8n+1}{3-1} \right\rceil = \left\lceil \frac{4n+1}{2} \right\rceil = \lceil 2n+1 \rceil$$

By the Lemma 4.5 and Lemma 4.6, we obtain the feedback number of these graphs as follows.

Theorem1. For $n \geq 3$, the feedback number of these graphs is $f(n)=2n+1$.

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