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Mutually Exclusive Nuances of Truth in Moisil Logic

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Abstract

Moisil logic, having as algebraic counterpart Łukasiewicz-Moisil algebras, provides an alternative way to reason about vague information based on the following principle: a many-valued event is characterized by a family of Boolean events. However, using the original definition of Łukasiewicz-Moisil algebra, the principle does not apply for subalgebras. In this paper we identify an alternative and equivalent definition for the n-valued Łukasiewicz-Moisil algebras, in which the determination principle is also saved for arbitrary subalgebras, which are characterized by a Boolean algebra and a family of Boolean ideals. As a consequence, we prove a duality result for the n-valued Łukasiewicz-Moisil algebras, starting from the dual space of their Boolean center. This leads us to a duality for MV$_n$-algebras, since are equivalent to a subclass of n-valued Łukasiewicz-Moisil algebras.

Keywords: Łukasiewicz-Moisil algebras, determination principle, duality, MV$_n$-algebras

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Introduction

The first systems of many-valued logic are the 3-valued and the n-valued *Łukasiewicz logic* introduced by J. Łukasiewicz in the 1920's, while the infinite valued Łukasiewicz logic was defined by J. Łukasiewicz and A. Tarski in 1930 [13, 14]. The investigation of the corresponding algebraic structures was a natural problem. The first who studied such an algebrization was Gr. C. Moisil who in 1941 introduced 3 and 4-valued *Łukasiewicz algebras* [15], and generalized later to the n-valued case [16] and the infinite case [17]. On the other hand, in 1958, C. C. Chang defined *MV-algebras* [2] as algebraic structures for Łukasiewicz logic. As an example of A. Rose showed in 1965 that for $n \geq 5$ the Łukasiewicz implication cannot be defined in an n-valued Łukasiewicz algebra, the structures introduced by Moisil are not appropriate algebraic counterpart for Łukasiewicz logic. In consequence, we are dealing with two different logical systems with different flavour: Łukasiewicz logic, from one side, having MV-algebras as algebraic counterpart, and *Moisil logic*, from another side, having Łukasiewicz algebras as corresponding algebras. Nowadays, we call Łukasiewicz algebras by *Łukasiewicz-Moisil algebras* and the standard monograph on these structures is [1].

The proper subclass of Łukasiewicz-Moisil algebras that correspond to n-valued Łukasiewicz logic, i.e. *proper* Łukasiewicz-Moisil algebras are characterized in [4]. Since MV_n-algebras [8] are the algebraic correspondent of the finite valued Łukasiewicz logic, proper Łukasiewicz-Moisil algebras and MV_n-algebras are categorical equivalent. The complex connections between Łukasiewicz-Moisil algebras and MV-algebras are deeply investigated in [9, 10, 11].

The main idea behind Moisil logic is that of *nuancing*: to a many-valued object we associate some Boolean objects, its *Boolean nuances*. We do not define an many-valued object by its Boolean nuances, but we characterize it through the nuances, we investigate its properties by reducing them to the study of some Boolean ones. Moisil logic is therefore derived from the classical logic by the idea of nuancing, mathematically expressed by a categorical adjunction. This is a general idea, that can be applied to any logical system, as pointed out in [7].

Moisil's *determination principle* plays a central role in the study of Łukasiewicz-Moisil algebras and Moisil logic. At algebraic level it gives an efficient method for obtaining important results, lifting properties of Boolean algebras to the level of Łukasiewicz-Moisil algebras (see Moisil's representation theorem, for example), while at logical level it gives an alternative way to reason about non-crisp objects, by evaluating some crisp ones.

The determination principle from the initial definition of Łukasiewicz-Moisil algebras does not hold in general for subalgebras. The initial definition for Łukasiewicz-Moisil algebras is given in terms of some lattice endomorphisms, called the *Chrysippian endomorphisms* by Moisil, i.e. the φ's. Using another family of unary operations, i.e. the J's introduced in [4], the determination principle for subalgebras can be proved [12], leading to the idea that the Boolean nuances of a subalgebra are Boolean ideals. Moreover, the alternative nuances J's are *mutually exclusive*, or simply *disjoint*.

In this paper we introduce an equivalent definition for Łukasiewicz-Moisil algebras using the J's and we further investigate the properties of these operations. We obtain a categorical equivalence that allow us to represent any Łukasiewicz-Moisil algebra as a Boolean algebra and a finite family of Boolean ideals. As a consequence, we develop a duality for Łukasiewicz-Moisil algebras starting from Boolean spaces and adding a family of open sets. As a corollary, we obtain a duality for MV_n-algebras.

The paper is organized as follows. In Section 1 we recall the basic definitions and properties on Łukasiewicz-Moisil algebras. In particular in 1.1 we present the adjunction between Łukasiewicz-Moisil algebras and Boolean algebra, the fundamental idea behind Moisil logic, while in 1.2 we recall the Stone-type duality for Łukasiewicz-Moisil algebras [3], which is developed starting from the dual space of a bounded distributive lattice. In Section 2 we introduce an alternative definition for Łukasiewicz-Moisil algebras using the J's. In Subsection 2.2 we prove the fundamental logic adjunction theorem via the J's, while in Subsection 2.3 we prove a categorical equivalence for Łukasiewicz-Moisil algebras. Sections 3 and 4 are devoted for Stone-type dualities for Łukasiewicz-Moisil algebras and MV-algebras, respectively, starting from the simple dual space of Boolean algebras.

1 Łukasiewicz-Moisil Algebras

We refer to [1] for all unexplained notions on the theory of Łukasiewicz-Moisil algebras. In the sequel n is a natural number and we use the notation $[n] := \{1, \ldots, n\}$.

Definition 1 *A* Łukasiewicz-Moisil algebra of order $n+1$ *(LM_{n+1}-algebra, for short) is a structure of the form*

$$(L, \vee, \wedge, ^*, \varphi_1, \ldots, \varphi_n, 0, 1)$$

such that $(L, \vee, \wedge, ^*, 0, 1)$ *is a De Morgan algebra, i.e. a bounded distributive lattice with a decreasing involution* * *satisfying the De Morgan property, and* $\varphi_1, \ldots, \varphi_n$[3] *are unary operations on* L *such that the following hold:*

(L1) $\varphi_i(x \vee y) = \varphi_i(x) \vee \varphi_i(y)$,

(L2) $\varphi_i(x) \vee \varphi_i(x)^* = 1$,

(L3) $\varphi_i \circ \varphi_j = \varphi_j$,

(L4) $\varphi_i(x^*) = \varphi_{n+1-i}(x)^*$,

(L5) if $i \leq j$ *then* $\varphi_i(x) \leq \varphi_j(x)$,

(L6) if $\varphi_i(x) = \varphi_i(y)$, *for all* $i \in [n]$, *then* $x = y$,

for any $i, j \in [n]$ *and* $x, y \in L$.

Property (L6) is called *the determination principle* and the system (L1)-(L6) is equivalent to (L1)-(L5), (L7) and (L8), where

(L7) $x \leq \varphi_n(x)$,

(L8) $x \wedge \varphi_i(x)^* \wedge \varphi_{i+1}(y) \leq y$, for any $i \in [n-1]$.

Therefore, the class of LM_{n+1}-algebras is equational.

Example 1 ([16]) *The* canonical LM_{n+1}*-algebra is the structure*

$$(L_{n+1}, \vee, \wedge, *, \varphi_1, \ldots, \varphi_n, 0, 1),$$

where $L_{n+1} := \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$, *the lattice order is the natural one,*

$$\frac{j}{n}^* := \frac{n-j}{n} \quad \text{and} \quad \varphi_i(\tfrac{j}{n}) := \begin{cases} 0, & \text{if } i+j < n+1, \\ 1, & \text{if } i+j \geq n+1 \end{cases},$$

for any $0 \leq j \leq n$ *and* $i \in [n]$.

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[3] These operations are called the *Chrysippian endomorphisms.*

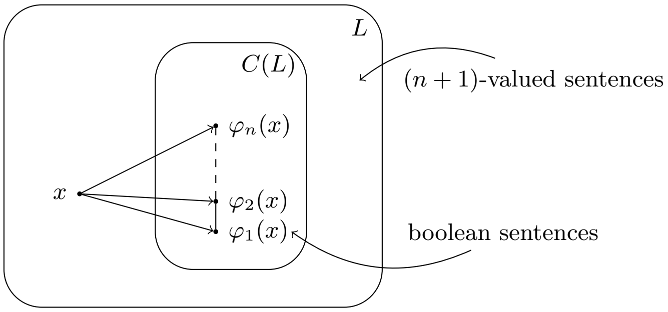

Figure 1: The determination principle for LM_{n+1}-algebras.

The canonical LM_2-algebra has only one Chrysippian endomophism which, by the determination principle, is forced to be a bijection, making the canonical LM_2-algebra a Boolean algebra. Therefore the "overloaded" notation L_2 for Boolean algebras and LM_2-algebra is consistent.

By *Moisil's representation theorem*, any LM_{n+1}-algebra is isomorphic to a subdirect product of LM_{n+1}-subalgebras of L_{n+1}.

Lemma 1 ([16]) *In any LM_{n+1}-algebra L, the following hold, for any $x, y \in L$ and any $i, j \in [n]$:*

(1) $\varphi_i(x \wedge y) = \varphi_i(x) \wedge \varphi_i(y)$,

(2) $\varphi_i(x) \wedge \varphi_i(x)^* = 0$,

(3) $\varphi_i(\varphi_j(x)^*) = \varphi_j(x)^*$,

(4) $x \leq y$ *iff* $\varphi_i(x) \leq \varphi_j(x)$,

(5) $\varphi_1(x) \leq x \leq \varphi_n(x)$.

For each LM_{n+1}-algebra L, we define its *Boolean center* $C(L)$ as the set of all complemented elements of L, i.e. $C(L) = \{x \in L \mid x \vee x^* = 1\}$. We can easily see that, for each $x \in L$, the following equivalences hold:

$$\begin{array}{lll} x \in C(L) & \text{iff} & \text{there exists } i \in [n] \text{ such that } \varphi_i(x) = x, \\ & \text{iff} & \text{for all } i \in [n], \varphi_i(x) = x, \\ & \text{iff} & \text{there exist } i \in [n] \text{ and } y \in L \text{ such that } \varphi_i(y) = x. \end{array}$$

Note that the determination principle can be represented as in Figure 1.

1.1 The Fundamental Logic Adjunction Theorem

The logic having as algebraic semantics LM_{n+1}-algebras is called nowadays *Moisil logic*. Moisil logic is derived from the classical logic by the *idea of nuancing*, mathematically expressed by a categorical adjunction, construction which is presented below.

Let $(B, \vee, \wedge, ^{-}, 0, 1)$ be a Boolean algebra. Let us consider the set

$$T(B) := \{(x_1, \ldots, x_n) \mid x_1 \leq \ldots \leq x_n\}.$$

On the set $T(B)$ we can define an LM_{n+1}-algebra structure as follows: the lattice operations, as well as 0 and 1, are defined componentwise, and for each $(x_1, \ldots, x_n) \in T(B)$ and $i \in [n]$, we consider

$$(x_1, \ldots, x_n)^* := (\overline{x_n}, \ldots, \overline{x_1}) \qquad \text{and} \qquad \varphi_i(x_1, \ldots, x_n) := (x_i, \ldots, x_i).$$

Remark that the LM_{n+1}-algebras L_{n+1} and $T(L_2)$ are isomorphic.

Let $\mathbf{LM}_{n+1}$ be the category of $(n+1)$-valued Łukasiewicz-Moisil algebras and **B**ool**A** be the category of Boolean algebras. Let

$$C : \mathbf{LM}_{n+1} \to \mathbf{B}\text{ool}\mathbf{A} \qquad \text{and} \qquad T : \mathbf{B}\text{ool}\mathbf{A} \to \mathbf{LM}_{n+1}$$

be two functors defined as follows: for each LM_{n+1}-algebra, $C(L)$ is the Boolean center of L, and for each LM_{n+1}-morphism $f : L \to L'$, $C(f) : C(L) \to C(L')$ is the restriction and co-restriction of f to $C(L)$ and $C(L')$; for each Boolean algebra B, $T(B)$ is the above LM_{n+1}-algebra, and for each Boolean morphism $g : B \to B'$, $T(g) : T(B) \to T(B')$ is defined by applying g on each component of any $u \in T(B)$.

Theorem 1 (The Fundamental Logic Adjunction Theorem[16]) *The above functors C and T satisfy the following:*

(1) C is faithful and T is full and faithful,

(2) C is a left adjoint of T, where the unit and the counit are given by

$$\eta_L : L \to TC(L),\ \eta_L(x)(i) = \varphi_i(x), \text{ for any } x \in L \text{ and any } i \in [n],$$
$$\epsilon_B : CT(B) \to B,\ \epsilon_B(u) = u(1), \text{ for all } u \in CT(B).$$

(3) η_L is an LM_{n+1} embedding and ϵ_B is a Boolean isomorphism.

1.2 Stone Duality using the Chrysippian Endomorphisms

Two categories $\mathbf{C}$ and $\mathbf{D}$ are *dually equivalent* if there exists a pair of contravariant functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ such that both FG and GF are naturally isomorphic with the corresponding identity functors, i.e. for each object C in $\mathbf{C}$ and D in $\mathbf{D}$ there are isomorphisms $\eta_C\colon GF(C) \to C$ and $\kappa_D\colon FG(D) \to D$ such that

$$\begin{tikzcd} GF(C_1) \arrow[r, "GF(f)"] \arrow[d, "\eta_{C_1}"'] & GF(C_2) \arrow[d, "\eta_{C_2}"] \\ C_1 \arrow[r, "f"'] & C_2 \end{tikzcd} \qquad \begin{tikzcd} FG(D_1) \arrow[r, "FG(g)"] \arrow[d, "\kappa_{D_1}"'] & FG(D_2) \arrow[d, "\kappa_{D_2}"] \\ D_1 \arrow[r, "g"'] & D_2 \end{tikzcd}$$

for each $f\colon C_1 \to C_2$ in $\mathbf{C}$ and $g\colon D_1 \to D_2$ in $\mathbf{D}$.

A *Stone space* is a topological space X such that: *(i)* X is a compact T_0 space, *(ii)* the set KX of compact open subsets of X is closed with respect to finite intersections and unions, and is a basis for the topology of X, and *(iii)* if $\mathcal{C} \subseteq KX$ is closed with respect to finite intersections and $F \subseteq X$ is a closed set such that $F \cap Y \neq \emptyset$, for every $Y \in \mathcal{C}$, then $F \cap \bigcap_{Y \in \mathcal{C}} Y \neq \emptyset$. A map $f : X \to Y$ between two Stone spaces X and Y is called *strongly continuous* if $f^{-1}(A) \in KX$, for every $A \in KX$. We denote by $\mathbf{B}$ool$\mathbf{S}$ the category of Stone spaces and strongly continuous functions. The category of bounded distributive lattices is dually equivalent with the category $\mathbf{B}$ool$\mathbf{S}$.

A *Stone space with involution* is a couple (X, h) such that: *(i)* X is a Stone space, and *(ii)* $h : X \to X$ is a function satisfying $h^2 = id_X$ and $X \setminus h(A) \in KX$, for every $A \in KX$. We denote by $\mathbf{S}$t(I) the category of Stone spaces with involution, where the morphisms between (X, g) and (X', g') are the arrows $f : X \to X'$ in $\mathbf{B}$ool$\mathbf{S}$ such that $g' \circ f = f \circ g$. The category of De Morgan algebras is dually equivalent with the category $\mathbf{S}$t(I).

An $(n+1)$*-valued Łukasiewicz-Moisil space* (an LM_{n+1}*-space*, for short) is a tuple $(X, h, h_1, \ldots, h_n)$ such that

i) (X, h) is a Stone space with involution,

ii) $\{h_i : X \to X\}_{i \in [n]}$ is a family of functions satisfying the conditions:

  (1) h_i is strongly continuous,

  (2) $X \setminus h_i^{-1}(A) \in KX$,

  (3) $h_i \circ h_j = h_i$,

(4) if $i \leq j$ then $h_i^{-1}(A) \subseteq h_j^{-1}(A)$,

(5) if $h_i^{-1}(A) = h_i^{-1}(B)$, for all $i \in [n]$, then $A = B$,

for any $i, j \in [n]$ and any $A, B \in KX$,

iii) $h \circ h_i = h_{n+1-i}$ and $h_i \circ h = h_i$, for any $i \in [n]$,

A morphism between two LM_{n+1}-spaces $(Xh, h_1, \ldots, h_n)$ and $(X', h', h'_1, \ldots, h'_n)$ is an arrow $f : (X, h) \to (X', h')$ in $\mathbf{S}\mathrm{t}(\mathrm{I})$ such that $f \circ g_i = g'_i \circ f$, for any $i \in [n]$. We denote by $\mathbf{S}\mathrm{t}(\mathrm{LM_{n+1}})$ the category of LM_{n+1}-spaces.

Theorem 2 ([3]) $\mathbf{LM}_{n+1}$ *and* $\mathbf{S}\mathrm{t}(\mathrm{LM_{n+1}})$ *are dually equivalent.*

2 Mutually Exclusive Nuances of Truth

Using the original definition of LM_{n+1}-algebras (Definition 1), the determination principle holds just for ideals and not for arbitrary subalgebras. For example, if $n \geq 4$ and $K_{n+1} = \{0, \frac{1}{n}, \frac{n-1}{n}, 1\}$, then K_{n+1} and L_{n+1} are distinct LM_{n+1}-algebras such that $\varphi_i(K_{n+1}) = \varphi_i(L_{n+1}) = \{0, 1\}$ for all $i \in [n]$.

In this section we propose an alternative definition for LM_{n+1}-algebras which save the determination principle also for subalgebras.

2.1 Alternative Definition

Definition 2 *An LM_{n+1}-algebra is a structure of the form*

$$(L, \vee, \wedge, ^*, \mathrm{J}_1, \ldots, \mathrm{J}_n, 0, 1)$$

such that $(L, \vee, \wedge, ^*, 0, 1)$ is a De Morgan algebra and $\mathrm{J}_1, \ldots, \mathrm{J}_n$ are unary operations on L such that the following hold:

(J1) $\bigvee_{k=n-i+1}^{n} \mathrm{J}_k(x \vee y) = \bigvee_{k=n-i+1}^{n} (\mathrm{J}_k(x) \vee \mathrm{J}_k(y))$,

(J2) $\mathrm{J}_i(x) \vee \mathrm{J}_i(x)^* = 1$,

(J3) $\mathrm{J}_k(\mathrm{J}_i(x)) = 0$ *and* $\mathrm{J}_n(\mathrm{J}_i(x)) = \mathrm{J}_i(x)$,

(J4) $\mathrm{J}_k(x^*) = \mathrm{J}_{n-k}(x)$ *and* $\mathrm{J}_n(x^*) = \bigwedge_{i=1}^{n} \mathrm{J}_i(x)^*$,

(J5) $\mathrm{J}_l(x) \leq (\mathrm{J}_1(x) \vee \ldots \vee \mathrm{J}_{l-1}(x))^*$,

(J6) if $\mathrm{J}_i(x) = \mathrm{J}_i(y)$, *for all* $i \in [n]$, *then* $x = y$,

for any $i, j \in [n]$, $k \in [n-1]$, $1 < l \leq n$ *and* $x, y \in L$.

Note that for any $i, j \in [n]$ such that $i \neq j$, $\mathrm{J}_i(x)$ and $\mathrm{J}_j(x)$ are *mutually exclusive* (or *disjoint*, for short), i.e. $\mathrm{J}_i(x) \wedge \mathrm{J}_j(x) = 0$, for any $x \in L$. Indeed, assuming that $i < j$ and using (J5) and (J2), we have

$$\mathrm{J}_i(x) \wedge \mathrm{J}_j(x) = \mathrm{J}_i(x) \wedge \mathrm{J}_j(x) \wedge \mathrm{J}_{j-1}(x)^* \wedge \ldots \wedge \mathrm{J}_1(x)^* = 0.$$

While any of the operations φ_i from Definition 1 is a lattice endomophism and preserves arbitrary suprema and infima, whenever they exist, the operations $\mathrm{J}_1, \ldots, \mathrm{J}_{n-1}$ are not lattice endomorphisms, as can be seen from condition (J1).

Theorem 3 *Definitions 1 and 2 for* LM_{n+1}*-algebras are equivalent.*

Proof: Let $(L, \vee, \wedge, ^*, \varphi_1, \ldots, \varphi_n, 0, 1)$ be a structure as in Definition 1. As in [12], we define the unary operations

$$\mathrm{J}_n(x) := \varphi_1(x) \text{ and } \mathrm{J}_i(x) := \varphi_{n-i+1}(x) \wedge \varphi_{n-i}(x)^*, \text{ for } i \in [n-1], \tag{1}$$

for any $x \in L$. Conditions (J2), (J3) and (J6) are already proved in [12]. Notice that, for any $i, k \in [n-1]$, we have

$$\begin{aligned}
\bigvee_{j=i}^{k} \mathrm{J}_j(x) &= [\varphi_{n-i+1}(x) \wedge \varphi_{n-i}(x)^*] \vee [\varphi_{n-i}(x) \wedge \varphi_{n-i-1}(x)^*] \vee \bigvee_{j=i+2}^{k} \mathrm{J}_j(x) \\
&= [\varphi_{n-i+1}(x) \wedge \varphi_{n-i-1}(x)^*] \vee [\varphi_{n-i-1}(x) \wedge \varphi_{n-i-2}(x)^*] \vee \bigvee_{j=i+3}^{k} \mathrm{J}_j(x) \\
&= [\varphi_{n-i+1}(x) \wedge (\varphi_{n-i+1}(x) \vee \varphi_{n-i-2}(x)^*) \wedge \varphi_{n-i-2}(x)^*] \vee \bigvee_{j=i+3}^{k} \mathrm{J}_j(x) \\
&= [\varphi_{n-i+1}(x) \wedge \varphi_{n-i-2}(x)^*] \vee \bigvee_{j=i+3}^{k} \mathrm{J}_j(x) = \ldots \\
&= \varphi_{n-i+1}(x) \wedge \varphi_{n-k}(x)^*
\end{aligned}$$

Therefore, for any $i, k \in [n-1]$, we also have

$$\bigwedge_{j=i}^{k} \mathrm{J}_j(x)^* = \varphi_{n-i+1}(x)^* \vee \varphi_{n-k}(x).$$

Using the above equality, (J5) is obtained as follows, for any $1 < l \leq n$:

$$\begin{aligned}
\mathrm{J}_l(x) \wedge \bigwedge_{j=1}^{l-1} \mathrm{J}_j(x)^* &= \varphi_{n-l+1}(x) \wedge \varphi_{n-l}(x)^* \wedge (\varphi_n(x)^* \vee \varphi_{n-l+1}(x)) \\
&= \varphi_{n-l+1}(x) \wedge \varphi_{n-l}(x)^* = \mathrm{J}_l(x)
\end{aligned}$$

The first part of condition (J4) follows from [12], while for the second part we have

$$\begin{aligned}
\bigwedge_{j=1}^{n} \mathrm{J}_j(x)^* &= \bigwedge_{j=1}^{n-1} \mathrm{J}_j(x)^* \wedge \mathrm{J}_n(x)^* = (\varphi_n(x)^* \vee \varphi_1(x)) \wedge \varphi_1(x)^* \\
&= \varphi_n(x)^* \wedge \varphi_1(x)^* = \varphi_n(x)^* = \varphi_1(x^*) = \mathrm{J}_n(x^*)
\end{aligned}$$

Now let us show condition (J1) by induction over i. For $i = 1$, the conclusion follows from (L1). Assume that (J1) holds for i and let us prove it for $i + 1$. First notice that, for any $2 \leq i \leq n$, we have $\varphi_i(x)^* \vee \mathrm{J}_{n-i+1}(x) = \varphi_{i-1}(x)^*$. We have the following chain of equalities:

$$\begin{aligned}
&\bigvee_{k=n-(i+1)+1}^{n} \mathrm{J}_k(x \vee y) = \mathrm{J}_{n-i}(x \vee y) \vee \bigvee_{k=n-i+1}^{n} \mathrm{J}_k(x \vee y) \\
= \; & [\mathrm{J}_{n-i}(x) \wedge \varphi_i(y)^*] \vee [\mathrm{J}_{n-i}(y) \wedge \varphi_i(x)^*] \vee \bigvee_{k=n-i+1}^{n} (\mathrm{J}_k(x) \vee \mathrm{J}_k(y)) \\
= \; & [\mathrm{J}_{n-i}(x) \wedge \varphi_i(y)^*] \vee [\mathrm{J}_{n-i}(y) \wedge \varphi_i(x)^*] \vee [\mathrm{J}_{n-i+1}(x) \vee \mathrm{J}_{n-i+1}(y)] \vee \\
& \bigvee_{k=n-i+2}^{n} (\mathrm{J}_k(x) \vee \mathrm{J}_k(y)) \\
= \; & [(\mathrm{J}_{n-i}(x) \vee \mathrm{J}_{n-i+1}(y)) \wedge \varphi_{i-1}(y)^*] \vee [(\mathrm{J}_{n-i}(y) \vee \mathrm{J}_{n-i+1}(x)) \wedge \varphi_{i-1}(x)^*] \vee \\
& \bigvee_{k=n-i+2}^{n} (\mathrm{J}_k(x) \vee \mathrm{J}_k(y)) \\
= \; & \ldots \\
= \; & [(\mathrm{J}_{n-i}(x) \vee \mathrm{J}_{n-i+1}(y) \vee \ldots \vee \mathrm{J}_{n-1}(y)) \wedge \varphi_1(y)^*] \vee \\
& [(\mathrm{J}_{n-i}(y) \vee \mathrm{J}_{n-i+1}(x) \vee \ldots \vee \mathrm{J}_{n-1}(x)) \wedge \varphi_1(x)^*] \vee \mathrm{J}_n(x) \vee \mathrm{J}_n(y) \\
= \; & [\mathrm{J}_{n-i}(x) \vee \ldots \vee \mathrm{J}_n(x)] \vee [\mathrm{J}_{n-i}(y) \vee \ldots \vee \mathrm{J}_n(y)] \\
= \; & \bigvee_{k=n-(i+1)+1}^{n} (\mathrm{J}_k(x) \vee \mathrm{J}_k(y)).
\end{aligned}$$

Conversely, let $(L, \vee, \wedge, ^*, \mathrm{J}_1, \ldots, \mathrm{J}_n, 0, 1)$ be a structure as in Definition 2. Again as in [12], we define

$$\varphi_i(x) := \bigvee_{k=n-i+1}^{n} \mathrm{J}_k(x), \tag{2}$$

for any $i \in [n]$ and $x \in L$. Condition (L5) follows immediately from the definition of φ_i's, (L1) follows directly from (J1), while (L6) follows from (J6) noticing that $\mathrm{J}_i(x) = \varphi_{n-i+1}(x) \wedge \varphi_{n-i}(x)^*$, for any $i \in [n-1]$, and $\mathrm{J}_n(x) = \varphi_1(x)$. Using (J2), condition (L2) is obtained as follows:

$$\begin{aligned} \varphi_i(x) \vee \varphi_i(x)^* &= \bigvee_{k=n-i+1}^{n} \mathrm{J}_k(x) \vee \bigwedge_{j=n-i+1}^{n} \mathrm{J}_j(x)^* \\ &= \bigwedge_{j=n-i+1}^{n} (\bigvee_{k=n-i+1}^{n} \mathrm{J}_k(x) \vee \mathrm{J}_j(x)^*) = 1 \end{aligned}$$

Since for any $j \neq k$ and any $x \in L$, $\mathrm{J}_j(x)$ and $\mathrm{J}_k(x)$ are disjoint, it follows that $\mathrm{J}_j(x) \vee \mathrm{J}_k(x)^* = \mathrm{J}_k(x)^*$. Therefore, using (J2) and (J4), we get (L4):

$$\begin{aligned} \varphi_i(x^*) &= \bigvee_{k=n-i+1}^{n} \mathrm{J}_k(x^*) = \bigvee_{k=n-i+1}^{n-1} \mathrm{J}_k(x^*) \vee \mathrm{J}_n(x^*) \\ &= \bigvee_{k=n-i+1}^{n-1} \mathrm{J}_{n-k}(x) \vee \mathrm{J}_n(x^*) = \bigvee_{j=1}^{i-1} \mathrm{J}_j(x) \vee \bigwedge_{k=1}^{n} \mathrm{J}_k(x)^* \\ &= \bigwedge_{k=1}^{n} (\bigvee_{j=1}^{i-1} \mathrm{J}_j(x) \vee \mathrm{J}_k(x)^*) = \bigwedge_{k=i}^{n} (\bigvee_{j=1}^{i-1} \mathrm{J}_j(x) \vee \mathrm{J}_k(x)^*) \\ &= \bigwedge_{k=i}^{n} \mathrm{J}_k(x)^* = (\bigvee_{k=i}^{n} \mathrm{J}_k(x))^* = \varphi_{n-i+1}(x)^* \end{aligned}$$

Finally, let us prove condition (L3) by induction over j. For $j = 1$, using (J3) we have

$$\varphi_i(\varphi_1(x)) = \bigvee_{k=n-i+1}^{n} \mathrm{J}_k(\mathrm{J}_n(x)) = \mathrm{J}_n(x) = \varphi_1(x).$$

Assume that (L3) holds for j and let us prove it for $j+1$. Noticing that $\varphi_{j+1}(x) = \varphi_j(x) \vee \mathrm{J}_{n-j}(x)$ and using (J1) and (J3), we have

$$\begin{aligned}
\varphi_i(\varphi_{j+1}(x)) &= \bigvee_{k=n-i+1}^{n} \mathrm{J}_k(\varphi_{j+1}(x)) = \bigvee_{k=n-i+1}^{n} \mathrm{J}_k(\varphi_j(x) \vee \mathrm{J}_{n-j}(x)) \\
&= \bigvee_{k=n-i+1}^{n} \mathrm{J}_k(\varphi_j(x)) \vee \bigvee_{k=n-i+1}^{n} \mathrm{J}_k(J_{n-j}(x)) \\
&= \varphi_j(x) \vee \mathrm{J}_{n-j}(x) = \varphi_{j+1}(x).
\end{aligned}$$

□

2.2 The Fundamental Logic Adjunction Theorem via J's

The fundamental Theorem 1 which allows one to transfer properties from the category of Boolean algebras to the category of LM_{n+1}-algebras can be equivalently stated using the alternative definition.

Let $(B, \vee, \wedge, ^{-}, 0, 1)$ be a Boolean algebra. For any family of elements $\{y_1, \ldots, y_n\}$ from B we have

$$y_i \wedge y_j = 0,\ i \neq j \quad \text{iff} \quad y_i \leq \overline{y_1} \wedge \ldots \wedge \overline{y_{i-1}},\ 1 < i \leq n,$$

We call a family of elements $\{y_1, \ldots, y_n\}$ with the above property *disjoint.* Let us define the set

$$J(B) := \{(y_1, \ldots, y_n) \in B^n \mid \{y_1, \ldots, y_n\} \text{ disjoint family of elements } \}.$$

Lemma 2 *There exists a bijective correspondence between the sets $J(B)$ and $T(B)$.*

Proof: It is straightforward to check that the functions

$$\begin{aligned}
f : J(B) \to T(B), &\qquad f(y_1, \ldots, y_n) = (y_1, y_1 \vee y_2, \ldots, y_1 \vee \ldots \vee y_n), \\
g : T(B) \to J(B), &\qquad g(x_1, \ldots, x_n) = (x_1, x_2 \wedge \overline{x_1}, \ldots, x_n \wedge \overline{x_{n-1}})
\end{aligned}$$

define a bijective correspondence between the sets $J(B)$ and $T(B)$. □

As a consequence of Lemma 2, $J(B)$ can be endowed with a structure of an LM_{n+1}-algebra $(J(B), \vee, \wedge, ^{*}, \mathrm{J}_1, \ldots, \mathrm{J}_n, 0, 1)$. Consider the LM_{n+1}-structure on $T(B)$ defined in Subsection 1.1, $(T(B), \vee, \wedge, ^{*}, \varphi_1, \ldots, \varphi_n, 0, 1)$.

Using 1, the J's on the set $J(B)$ are defined as follows:

$$\begin{aligned}
\mathrm{J}_i(y_1,\ldots,y_n) &:= g(\varphi_{n-i+1}(f(y_1,\ldots,y_n))\wedge\varphi_{n-i}(f(y_1,\ldots,y_n))^*)\\
\mathrm{J}_n(y_1,\ldots,y_n) &:= g(\varphi_1(f(y_1,\ldots,y_n)))
\end{aligned}$$

for any $i\in[n-1]$. By simple computation, we obtain that

$$\mathrm{J}_i(y_1,\ldots,y_n)=(y_{n-i+1},0,\ldots,0) \text{ and } \mathrm{J}_n(y_1,\ldots,y_n)=(y_1,0,\ldots,0),$$

for $i\in[n-1]$. In a similar way we get:

$$\begin{aligned}
(y_1,\ldots,y_n)^* &= (\bigwedge_{i=1}^{n} y_i^*, y_n^*,\ldots,y_2^*)\\
(y_1,\ldots,y_n)\vee(z_1,\ldots,z_n) &= (w_1,\ldots,w_n)z_{i-1})^*\wedge\cdots\wedge(y_1\vee z_1)^* \text{ for } i>1
\end{aligned}$$

where $w_1=y_1\vee z_1$, $w_i=(y_i\vee z_i)\wedge(y_{i-1}\vee z_{i-1})^*\wedge\cdots\wedge(y_1\vee z_1)^*$, for $i>1$.

Remark 1 *The Fundamental Logic Adjunction Theorem from Subsection 1.1 can also be expressed in terms of the functor J defined in the obvious way.*

2.3 A Categorical Equivalence for LM_{n+1}-Algebras

One can see that the algebras $K_{n+1}=\{0,\frac{1}{n},\frac{n-1}{n},1\}$ and L_{n+1} are distinguished using the J's instead of the φ's: if $n\geq 4$ then $J_i(K_{n+1})=\{0\}\neq\{0,1\}=J_i(L_{n+1})$ for all $i\in\{2,\ldots,n-2\}$.

In this section we show that LM_{n+1}-algebras are equivalent with a category whose elements are Boolean algebras endowed with a particular set of Boolean ideals. This categorical equivalence is a powerful tool for working with LM_{n+1}-algebras, as exemplified in the next section.

We shall call a finite set $\{I_1,\ldots,I_{n-1}\}$ of ideals on a Boolean algebra B with the property $I_i=I_{n-i}$, for any $i\in[n-1]$, an *n symmetric sequence of Boolean ideals.* Consider the category $\mathbf{Bool}\mathbf{I}_{n+1}$ in which objects are tuples of the form

$$(B,I_{n-1},\ldots,I_1),$$

where B is a Boolean algebra and $\{I_1,\ldots,I_{n-1}\}$ is an n symmetric sequence of Boolean ideals on B, and arrows are of the form $g:(B,I_{n-1},\ldots,I_1)\to(B',I'_{n-1},\ldots,I'_1)$, where $g:B\to B'$ is a Boolean morphism and $g(I_i)\subseteq I'_i$, for any $i\in[n-1]$.

Let us define the functor

$$\Lambda : \mathbf{LM}_{n+1} \to \mathbf{Bool}\mathbf{I}_{n+1}$$

as follows: for any LM_{n+1}-algebra $(L, \vee, \wedge, ^*, \mathrm{J}_1, \ldots, \mathrm{J}_n, 0, 1)$, set

$$\Lambda(L) = (C(L), J_{n-1}(L), \ldots, J_1(L)),$$

while for any LM_{n+1}-morphism $f : L \to L'$, set $\Lambda(f) : \Lambda(L) \to \Lambda(L')$ to be the co-restriction of f to $C(L)$ and $C(L')$. The fact that $\Lambda(L)$ is an object in $\mathbf{Bool}\mathbf{I}_{n+1}$ follows from [12, Proposition 5.2].

Moreover, let us define the functor

$$\Sigma : \mathbf{Bool}\mathbf{I}_{n+1} \to \mathbf{LM}_{n+1}$$

as follows: for any object $(B, I_{n-1}, \ldots, I_1)$ in $\mathbf{Bool}\mathbf{I}_{n+1}$, set

$$\Sigma(B) = \{(y_1, \ldots, y_n) \in B^n \mid y_i \in I_{n-i+1}, \{y_1, \ldots, y_n\} \text{ disjoint elements}\},$$

and for each arrow $g : (B, I_{n-1}, \ldots, I_1) \to (B', I'_{n-1}, \ldots, I'_1)$ in $\mathbf{Bool}\mathbf{I}_{n+1}$, $\Sigma(g)$ is defined by applying g on each component of any $u \in \Sigma(B)$. Applying [12, Proposition 5.3] for the LM_{n+1}-algebra $J(B)$ defined in Subsection 2.2, we obtain that $\Sigma(B)$ is an LM_{n+1}-algebra.

The functors Λ and Σ yield a categorical equivalence:

Theorem 4 *The categories* $\mathbf{LM}_{n+1}$ *and* $\mathbf{Bool}\mathbf{I}_{n+1}$ *are equivalent.*

3 Duality for LM_{n+1}-Algebras using Boolean Spaces

The duality presented in Subsection 1.2 uses the dual spaces of bounded distributive lattices. As an application of the categorical equivalence obtained in Subsection 2.3, we develop a duality for LM_{n+1}-algebras starting from the Stone duality for Boolean algebras.

Let us recall the Stone duality for Boolean algebras. Let $\mathbf{Bool}\mathbf{A}$ be the category of Boolean algebras and Boolean homomorphisms and $\mathbf{Bool}\mathbf{S}$ the category of Boolean spaces (i.e. topological spaces that are Hausdorff and compact, and have a basis of clopen subsets) and continuous maps. The category $\mathbf{Bool}\mathbf{S}$ and $\mathbf{Bool}\mathbf{A}$ are dually equivalent via the functors

$$S^a : \mathbf{Bool}\mathbf{A} \to \mathbf{Bool}\mathbf{S} \qquad \text{and} \qquad S^t : \mathbf{Bool}\mathbf{S} \to \mathbf{Bool}\mathbf{A}.$$

The Boolean space $S^a(B)$ of a Boolean algebra B is the topological space whose underlying set is the collection $X := Ult(B)$ of ultrafilters of B, and whose topology has a basis consisting of all sets of the form

$$N_b = \{U \in X \mid b \in U\},$$

for any $b \in B$. For every Boolean homomorphism $h : A \to B$, $S^a(f) : S^a(B) \to S^a(A)$ is defined as $S^a(f)(u) = h^{-1}(u)$, for every $u \in S^a(B)$. Conversely, if X is a Boolean space, we consider $S^t(X) = co(X)$, the set of all clopen subsets of X, and for every continuous map $f : X \to Y$, $\varphi = S^t(f) : co(Y) \to co(X)$, $\varphi(N) = f^{-1}(N)$, for every $N \in co(Y)$.

Note that for any $a, b \in B$, $N_{a\vee b} = N_a \cup N_b$, $N_{a\wedge b} = N_a \cap N_b$ and

$$\text{if } a \leq b \text{ then } N_a \subseteq N_b$$

Via the Stone duality, for every ideal of a Boolean algebra we can associate an open set: if I is an ideal of a Boolean algebra B, consider the open set in $S^a(B)$

$$N_I = \bigcup\{N_a \mid a \in I\} = \{U \in X \mid U \cap I \neq \emptyset\}. \tag{3}$$

For every $b \in B$, we have $b \in I$ iff $N_b \subseteq N_I$. Conversely, for every open set of a Boolean space we can associate a Boolean ideal: if O is an open subset of a Boolean space X, consider the ideal in $S^t(X)$

$$I_O = \{b \in S^t(X) \mid N_b \subseteq O\}. \tag{4}$$

In the following we provide a Stone-type duality for LM_{n+1}-algebras using Boolean spaces. As we have seen in Subsection 2.3 we can represent any LM_{n+1}-algebra L as a Boolean algebra endowed with an n symmetric sequence of Boolean ideals on it,

$$(C(L), J_{n-1}(L), \ldots, J_1(L)),$$

the categories $\mathbf{LM}_{n+1}$ and $\mathbf{Bool}\mathbf{I}_{n+1}$ being equivalent. Therefore we shall construct a Stone-type duality for the category $\mathbf{Bool}\mathbf{I}_{n+1}$.

Definition 3 *A* Boolean space with n symmetric open sets *is a tuple*

$$(X, O_1, \ldots, O_{n-1}),$$

where X is a Boolean space and $O_1, \ldots, O_{n-1}$ are open sets in X such that $O_i = O_{n-i}$, for any $i \in [n-1]$.

Denote by $\mathbf{B}\mathrm{ool}\mathbf{SO}_n$ the category of Boolean spaces with n symmetric open sets with continuous maps $f : (X, O_1, \ldots, O_{n-1}) \to (Y, U_1, \ldots, U_{n-1})$ such that $f^{-1}(U_i) \subseteq O_i$, for any $i \in [n-1]$.

Theorem 5 *The categories* $\mathbf{B}\mathrm{ool}\mathbf{I}_{n+1}$ *and* $\mathbf{B}\mathrm{ool}\mathbf{SO}_n$ *are dually equivalent.*

Proof: We let define the functors

$$\Theta^a : \mathbf{B}\mathrm{ool}\mathbf{I}_{n+1} \to \mathbf{B}\mathrm{ool}\mathbf{SO}_n \qquad \text{and} \qquad \Theta^t : \mathbf{B}\mathrm{ool}\mathbf{SO}_n \to \mathbf{B}\mathrm{ool}\mathbf{I}_{n+1}.$$

For every object $(B, I_{n-1}, \ldots, I_1)$ in $\mathbf{B}\mathrm{ool}\mathbf{I}_{n+1}$, define using (3),

$$\Theta^a(B, I_{n-1}, \ldots, I_1) = (S^a(B), N_{I_1}, \ldots, N_{I_{n-1}}),$$

and for every arrow $g : (B, I_{n-1}, \ldots, I_1) \to (B', I'_{n-1}, \ldots, I'_1)$ in $\mathbf{B}\mathrm{ool}\mathbf{I}_{n+1}$, consider $\Theta^a(g) = S^a(g)$, i.e. $\Theta^a(g)(u) = g^{-1}(u)$, for any $u \in S^a(B')$. It is easy to check that $\Theta^a(g)^{-1}(N_{I_i}) \subseteq N'_{I'_i}$, for every $i \in [n-1]$, and therefore $\Theta^a(g)$ is an arrow in $\mathbf{B}\mathrm{ool}\mathbf{SO}_n$. Conversely, for every Boolean space with n symmetric open sets $(X, O_1, \ldots, O_{n-1})$, define using (4)

$$\Theta^t(X, O_1, \ldots, O_{n-1}) = (S^t(X), I_{O_{n-1}}, \ldots, I_{O_1}),$$

and for every arrow $f : (X, O_1, \ldots, O_{n-1}) \to (Y, U_1, \ldots, U_{n-1})$, define $\Theta^t(f) = S^t(f)$. Since $\Theta^t(f)(I_{U_i}) \subseteq I_{O_i}$, for any $i \in [n-1]$, $\Theta^t(f)$ is an arrow in $\mathbf{B}\mathrm{ool}\mathbf{I}_{n+1}$. □

4 Duality for MV$_n$-Algebras using Boolean Spaces

In this section we specialize the duality developed in Section 3 in order to obtain a duality for the n-valued MV-algebras (MV_n-algebras), which are categorically equivalent to a proper subclass of LM_n-algebras.

An *MV-algebra* is an algebraic structure $(A, \oplus, ^*, 0)$ of type $(2, 1, 0)$ such that $(A, \oplus, 0)$ is an abelian monoid $(x^*)^* = x$ and $x \oplus (y \oplus z) = (x \oplus y) \oplus z$. MV-algebras were defined by [2] and they stay to Łukasiewicz ∞-valued logic as Boolean algebras stay to classical logic. We refer to [5] for an introduction in the theory of MV-algebras.

If A is an MV-algebra and $x, y \in A$ we set $x \odot y = (x^* \oplus y^*)^*$ and $1 = 0^*$. For any natural number n we define

$$0x = 0, \qquad x^0 = 1,$$
$$(n+1)x = (nx) \oplus x, \quad x^{n+1} = (x^n) \odot x.$$

The structures corresponding to the $n+1$-valued Łukasiewicz logic were defined in [8] under the name of MV_n-algebras and they satisfy the following additional properties for any $x \in A$ and $1 < j < n$ such that j does not divide n:

$$(n+1)x = nx, \qquad [(jx) \odot (x^* \oplus ((j1)x)^*)]^n = 0.$$

In [4] the *proper* LM_{n+1}-algebras are defined and they are those LM_{n+1}-algebras adequate for the $n+1$-valued Łukasiewicz logic. Moreover, LM_{n+1}-algebras are term equivalent with MV_{n+1}-algebras [9, 10]. As in [12], we call these structures *-proper*, in order to avoid the confusion with the usual terminology from universal algebra. The Ł-proper LM_{n+1}-algebras are a full subcategory of $\mathbf{LM}_{n+1}$.

Remark 2 *In [6, Section 3] the category* $\mathbf{MV}_{n+1}$ *is proved to be equivalent with a category* $\mathbf{BM}_{n+1}$ *whose objects are pairs* (B, R) *such that* B *is a Boolean algebra and* $R \subset B^n$ *satisfies some particular properties; the morphisms form* (B_1, R_1) *to* (B_2, R_2) *in* $\mathbf{BM}_{n+1}$ *is a morphism of Boolean algebras* $f : B_1 \to B_2$ *such that* $(x_1, \ldots, x_n) \in B_1$ *implies* $(f(x_1), \ldots, f(x_n)) \in B_2$*. Moreover, it is proved that an object* (B, R) *can be characterized by a sequence* $I_1(R), \ldots, I_{n-1}(R)$ *of Boolean ideals such that* $J_i(L) \cap J_{i-k}(L) \subseteq J_i(L)$ *for* $2 \leq i \leq n-2$ *and* $j < i$*. In [12] this result is stated in the context of* LM_{n+1}*-algebras.*

Lemma 3 *For an* LM_{n+1}*-algebra* L *the following are equivalent:*

(a) L *is Ł-proper,*

(b) $J_i(L) \cap J_k(L) \subseteq J_{n-i+k-1}(L)$*, for any* $3 \leq i \leq n-2$*,* $1 \leq k \leq n-4$*,* $k < i$*.*

Proof: It is straightforward by [12, Proposition 5.11]. □

Denote by $\mathbf{MV}_{n+1}$ the category of MV$_{n+1}$-algebras and by $\mathbf{BoolIMV}_{n+1}$ the full subcategory of $\mathbf{BoolI}_{n+1}$ whose objects are tuples of the form

$$(B, I_{n-1}, \ldots, I_1) \text{ such that } I_i \cap I_k \subseteq I_{n-i+k-1},$$
$$\text{for any } 3 \leq i \leq n-2,\ 1 \leq k \leq n-4,\ k < i.$$

From Theorem 4 and Lemma 3 we immediately infer the following result, which was proved directly in [6, Section 3].

Corollary 1 *The categories* $\mathbf{MV}_{n+1}$ *and* $\mathbf{BoolIMV}_{n+1}$ *are equivalent.*

The duality result for MV_{n+1}-algebras is now straightforward. Denote by $\mathbf{BoolSOMV}_n$ the full subcategory of category $\mathbf{B}\text{ool}\mathbf{SO}_n$ whose objects are Boolean spaces with n symmetric open sets

$$(X, O_1, \ldots, O_{n-1}) \text{ such that } O_i \cap O_k \subseteq O_{n-i+k-1}, \\ \text{for any } 3 \leq i \leq n-2,\ 1 \leq k \leq n-4,\ k < i.$$

Theorem 6 *The categories* $\mathbf{MV}_{n+1}$ *and* $\mathbf{BoolSOMV}_n$ *are dually equivalent.*

Proof: It is a direct consequence of Theorem 5 and Corollary 1.

□

5 Conclusion

Nuances of truth provide an alternative and robust way to reason about vague information: a many-valued object is uniquely determined by some Boolean objects, its nuances. However, a many-valued object cannot be recovered only from its Boolean nuances. This idea is mathematically expressed by a categorical adjunction between Boolean algebras and Łukasiewicz-Moisil algebras.

Since the initial nuances of truth proposed by Moisil do not allow us to distinguish the subalgebras, in this paper we explore a different family of unary operators that take Boolean values and satisfy the determination principle, so they may act as truth nuances as well. Previously defined by Cignoli in [4] these operations were used merely as a technical ingredient. As consequence, the theory of LM_{n+1}-algebras posses two families of nuances: the φ's used in the original definition and the J's defined in [4] and further explored in this paper. The φ's have the advantage of being lattice homomorphisms, while the J's are not. However, the J's are mutually exclusive and they were used to prove a determination principle for subalgebras in [12].

Using the J's as truth nuances we characterized any LM_{n+1}-algebra as a sequence of Boolean ideals. This result led us to a new Stone-type duality for Łukasiewicz-Moisil algebras, which can be seen as a generalization of the Stone duality for Boolean algebras. Moreover this duality provides a direct and simple way to characterize the subclass of MV_{n+1}-algebras, structures that correspond to the $(n+1)$-valued Łukasiewicz logic.

As consequence, the theory of Łukasiewicz-Moisil algebras offers various and powerful tools for analyzing uncertainty, its distinguished feature being the determination principle. In the future we plan to investigate these results from logical perspective.

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