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The measurement and control of interest rate risk

Abstract

This paper develops a general measure of interest rate risk in an infinite factor model. We show that interest rate risk measures developed by Fong and Vasicek (M^2), by Bowden (*Direction X*), and by Nawalkha and Chambers ($|M|$) are special cases derived by placing different restrictions on term structure changes. Futher, we extend Bowden's measurement to the case of an arbitrary time horizon. This extension has theoretical advantages when long-term interest rate volatility is small and has computational advantages inherent in positive quadratic forms.

Keywords: immunization, duration, interest rate risk, term structure of interest rates.

Introduction

Traditional immunization theory (Redington, 1952; Fisher and Weil, 1971) assumes that a single factor determines the evolution of interest rates. Fong and Vasicek (1984) examined immunization risk in an infinite factor model. They considered the effect of an arbitrary change in the instantaneous forward rates, subject to a restriction on the maximum slope of the forward rate curve. They found that the classically immunized portfolio has a negative lower bound that depends upon the magnitude of the rate change and a portfolio dispersion measured, which they call M-squared (M^2).

Fong and Vasicek's work has received considerable attention. Two promising variations on the same theme have appeared. Barber and Copper (1998) and Bowden (1997) obtained a lower bound based upon a restriction on the root-mean square of the interest rate innovation. Nawalkha and Chambers (1996) obtained a lower bound based upon a restriction on the maximum forward rate change.

The contribution of this paper is to show how the ideas are related, and how they may all be developed off the same platform. In the process, we generalize Fong and Vasicek's M^2 risk measure, and recast and extend Bowden's risk measure. In fact, both Nawalkha and Chamber's risk measure and Bowden's are seen to be special members of a large family of possible risk measures.

1. Interest rate risk

We will examine the interest rate risk of an account A consisting of a schedule of future cash flows. Let $a(t)$ denote the cumulative cash flow into account A from time 0 to a future time $t \geq 0$. Although cash flows typically are discrete, we will allow cash to continuously flow in or out of A. A discrete cash flow c at time t_0 means the cumulative cash flow will have discrete jump at time t_0 of c. For any two successive times, t_1 and t_2, $\mu([t_1, t_2])$ will be the

measure of net cash flows credited into A from time t_1 to t_2. Any cash flows occurring precisely at t_1 and t_2 are also included[1].

Next, let $P(t_1, t_2)$ be a function that provides the value at time t_1 of a dollar promised at time t_2. If $t = 0$ is the present, $P(0, t_2)$ would be the present value of a dollar promised at t_2, while $P(T, t_2)$ is the value at horizon date T of a dollar promised at t_2. Now, let $V(T)$ denote the value of A at time T, based upon the current term structure (spot yield curve) observed at time 0. In other words, $V(T)$ is the future value of the portfolio at the horizon T assuming that term structure does not change. Our goal is to examine how the value at the horizon date changes when the term structure shifts in some manner.

The future value of account A under the current term structure is determined by the following Stieltjes integral:

$$V(T) = \int P(T,t) d\mu_a(t). \qquad (1)$$

If the set of cash flows is a discrete set, say $C_1,..., C_N$ occurring at times $t_1,..., t_N$, then (1) can be expressed in the usual form as:

$$V(T) = \sum_{i=1}^{N} C_i P(T, t_i).$$

If the cash flow stream is continuous, then $d\mu_\alpha(t) = \alpha'(t)dt$ and

$$V(T) = \int P(T,t) \alpha'(t) dt.$$

In this case $\alpha'(t)$ is usually called the continuous cash flow stream. The advantages of the Stieltjes integral are: (1) the integrals can easily be manipulated using well-known formulas from the calculus, and (2) the integral form of the present value allows us to examine both the continuous and discrete (and mixtures of both) cash flow streams at the same time[2].

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[1] Technically, we are assuming that α be of finite variation in finite intervals of time so that μ_α denotes the corresponding Lebesgue-Stieltjes measure (cf. Hewitt and Stromberg, 1965).

[2] Bowden (1997), for example, separately derives his interest rate risk measure for continuous and discrete cash flow streams.

We assume there exists an instantaneous *forward rate function r* that explains the time value of money:

$$\log P(t_1, t_2) = -\int_{t_1}^{t_2} r(u)\,du . \qquad (2)$$

The forward rates are determined from the current term structure. Since our goal here is to analyze how the future value of *A* might change, we write *V(r, T)* to indicate the dependence of *V* on the particular array of bond prices given by *r*. In fact, let us consider the transition from one pricing regime *r* to another *r + h*. Here *h* is the function, whose value at *t* is simply the difference in regimes as it pertains to the time *t* forward rate. If λ is a parameter that ranges in value from 0 to 1, then *V(r + λh, T)* ranges from the value of *A* under the old regime to the new. Under the assumption that *h* is "small" in an appropriate sense, we can safely use the derivative:

$$D_h V(r,T) = \left| \frac{\partial}{\partial \lambda} V(r + \lambda h, T) \right|_{\lambda = 0} \qquad (3)$$

to estimate the change *V(r, T)* to *V(r + h, T)*. In fact, this derivative, when $h \equiv -1$ or 1 and *V(r, T)* is scaled to 1, is the standard duration. Following Fong and Vasicek, Nawalkha and Chambers, and Bowden, we relax the requirement that *h* is known explicitly and consider what then can be said regarding the interest rate risk of *A*. Even with *h* unknown, we will follow Fabozzi (1996) and refer to (3) as the *dollar duration* of account *A*. Note that the immunization condition requires that the dollar duration equals zero.

Incorporating the forward rate (2) and differentiating under the integral sign, we can formulate the dollar duration as:

$$- \int \int_{T}^{t} h(u)\,du\, P(T, t)\,d\mu_\alpha(t) .$$

Furthermore, by letting $\beta(T, t)$ denote the *T* value of all cash flows attributed to account *A* up to and including time *t* and by integrating by parts, we may express this same dollar duration as[1]:

$$D_h V(r,t) = \int h(t)\beta(T,t)\,dt - V(T)\int_{T}^{\infty} h(t)\,dt . \qquad (4)$$

2. A Fong-Vasicek-type inequality

In this section, we derive the analog of the Fong and Vasicek (1984) inequality appropriate to our context. In our notation, their restriction on changes

in the forward rate curve means that the magnitude of the slope of the change in forward rate curve $|h'(t)|$ is bounded by positive number *K*: $|h'(t)| < K$ for all *t*. It will help our exposition to assume that the horizon *T* is fixed and to write $\beta(t) = \beta(T, t)$ and

$$d\mu_\beta = P(T, \cdot)\,d\mu_\alpha .$$

Integrating the right hand side of (4) by parts, leads to the expression:

$$- \int_{a}^{T} h'(t)\left(\int_{a}^{t} t - s\,d\mu_\beta(s) \right) dt +$$

$$+ \int_{T}^{b} h'(t)\left(\int_{T}^{t} s - T\,d\mu_\beta(s) + (t-T)(\beta(b) - \beta(t)) \right) dt +$$

$$+ h(T)\int_{a}^{T} T - s\,d\mu_\beta(s) - h(b)\int_{T}^{b} s - T\,d\mu_\beta(s) .$$

Now assume that $|h'(t)| < K$ for some fixed constant *K* and all *t*. Furthermore, assume that $\mu_\beta > 0$. Under these assumptions, the expression we have just obtained will always dominate the quantity

$$- K\int_{a}^{b} (s-T)^2\,d\mu_\beta(s) + K\int_{T}^{b} (s-T)(b-s)\,d\mu_\beta(s) +$$

$$+ h(T)\int_{a}^{T} T - s\,d\mu_\beta(s) - h(b)\int_{T}^{b} s - T\,d\mu_\beta(s) .$$

Unfortunately, this expression still depends on *h*. However, if we assume that account *A* is immunized to time horizon *T* in the sense that

$$\int t - T\,d\mu_\beta(t) = 0,$$

then, by virtue of the mean value theorem, *h* drops out and

$$D_h V(r,T) > -\frac{3}{2} K\int_{a}^{b} (s-T)^2\,d\mu_\beta(s) .$$

Finally, by using $-h$ as well as *h* and letting the limits of integration include all cash flows, we obtain

$$|D_h V(r,T)| < \frac{3}{2}\|h'\|_\infty \int (s-T)^2 P(T,s)\,d\mu_\alpha(s) . \qquad (5)$$

The integral in this expression evaluates to Fong and Vasicek's M^2 whenever cash flows are finite in number and discrete. The extra factor of 3 results from the linearization implicit in differentiation and the resultant neglect of the convexity of present value of positive cash flows. Nevertheless the use of this inequality in controlling interest rate is exactly the same as detailed in (Fong and Vasicek, 1984): *If one knows a bound for h' or, better, a confidence interval for that bound, then the interest rate risk inherent in using account A to fund a single liability at time T is given by M^2.*

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[1] It is necessary to place some restriction on *h;* say, that *h* be integrable.

3. Bowden-type inequalities

Let us now return to the expression for dollar duration (4). Just as we used $\beta(T, t)$ to represent the time T value of all cash flows attributed to account A up to and including time t, now let $\overline{\beta}(T,t)$ denote the time T value of all cash flows attributed thereafter, $\beta(T,t) + \overline{\beta}(T,t) = V(T)$. Thus, for example, if T is taken before any cash flow, then dollar duration is compactly expressed

$$-\int h(t)\overline{\beta}(T,t)dt.$$

Continuing with this assumption, we have, by Holder's inequality:

$$\left|D_h V(r,T)\right| < \left(\int \left|h(t)\right|^p dt\right)^{\frac{1}{p}}\left(\int \left|\overline{\beta}(T,t)\right|^q dt\right)^{\frac{1}{q}}, \quad (6)$$

whenever p and q are nonnegative extended real numbers, whose reciprocals total to one.

In particular, if we specialize to the case $p = 2$, then (6) becomes the Cauchy inequality,

$$\left|D_h V(r,T)\right| < \left\|h\right\|_2 \left(\int \left|\overline{\beta}(T,t)\right|^2 dt\right)^{\frac{1}{2}}.$$

Bowden has observed that since Cauchy's inequality becomes an equality when $h = \overline{\beta}(T,\cdot)$, that $\overline{\beta}(T,\cdot)$ represents the "direction" of interest rate shift to which the T-value of account A is most sensitive (Bowden's (1997), "Direction X").

4. Inequalities for arbitrary time horizon

The logic of Fong and Vasicek's derivation of their inequality underlines the importance of being able to measure interest rate risk relative to horizons other than the present. This suggests a generalization of Bowden's inequalities. If we place no restriction on T and work directly from (4), then (6) generalizes to

$$\left|D_h V(r,T)\right| < \left(\int \left|h(t)\right|^p dt\right)^{\frac{1}{p}}$$

$$\left[\left(\int_{-\infty}^{T} \left|\beta(T,t)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_{T}^{\infty} \left|\overline{\beta}(T,t)\right|^q dt\right)^{\frac{1}{q}}\right]. \quad (7)$$

Now, if we take the extreme case $q = 1$ in (7) we obtain[1]

$$\left|D_h V(r,T)\right| < \left\|h\right\|_\infty \int \left|s - T\right| P(T,s) d\left|\mu_\alpha\right|(s), \quad (8)$$

an inequality quite similar to the Fong and Vasicek inequality, except that the second moment M^2 is replaced by the first moment of T-valued cash flows and the maximum of h' is replaced by the maximum of h itself. This inequality generalizes that obtained by Nawalkha and Chambers in (1996).

In fact, the manager of account A who feels that interest rate risk arises from term structure shifts that are "close" to parallel might consider this rough and ready strategy to control the interest rate risk resulting from funding a single liability with a variety of instruments with different maturities: *Minimize both the horizon "duration" (8) and the horizon "convexity" (5). The importance of minimizing "duration" increases with the magnitude of the interest shift expected while the importance of minimizing "convexity" increases with the magnitude of the "twist" in the term structure anticipated.*

Bowden's version of assumption (7) has computational tractability. In fact, if we take $p = 2$ in (7), we may express it as

$$\left|D_h V(r,T)\right| <$$

$$< \left\|h\right\|_2 \left(\int\int \left(\left|\frac{u+v}{2} - T\right| - \left|\frac{u-v}{2}\right|\right) + P(T,u)P(T,v)d\mu_\alpha(u),d\mu_\alpha(v)\right)^{\frac{1}{2}}$$

Here we have used the notation x_+ as shorthand for $max(0, x)$. This inequality shows that, with $\left\|h\right\|_2$ fixed, the risk of the portfolio is given by a positive quadratic form evaluated at the portfolio vector. Such forms have several attractive properties. Here we mention the minimum value of such a form constrained to a linear manifold may be found by solving the appropriate linear equations using Lagrange multiplier (Fleming, 1977).

Conclusion

We show that interest rate risk measures developed by Fong and Vasicek, by Bowden, and by Nawalkha and Chambers are special cases derived by placing different restrictions on term structure changes. Futher, we extend Bowden's measurement to the case of an arbitrary time horizon. This extension has theoretical advantages when long term interest rate volatility is small and has computational advantages inherent in positive quadratic forms.

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[1] The measure $\left|\mu_\alpha\right|$ is defined in the same way as μ_α except that the *signed* net cash flows for each date are replaced by the *absolute value* of the net cash flows. In particular, $\left|\mu_\alpha\right| = \mu_\alpha$ whenever all cash flows are positive.

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