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CLASSIFICATION OF LIE SUBALGEBRAS UP TO AN INNER AUTOMORPHISM

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Abstract. In this paper, a useful classification of all Lie subalgebras of a given Lie algebra up to an inner automorphism is presented. This method can be regarded as an important connection between differential geometry and algebra and has many applications in different fields of mathematics. After main results, we have applied this procedure for classifying the Lie subalgebras of some examples of Lie algebras.

1. Introduction

Lie algebras are one of the most applicable algebraic structures in mathematics, bacause of correspondency to Lie groups which plays a vast role in differential geometry, differential equations, physisc and etc. In this article we introduce a method based on the basic results in algebra for classifying subalgebras of a Lie algebra. This method could be applied to all finite-dimensional Lie algebras, by constructing an inner automorphism of a given Lie algebra, then, by an algorithmic method all subalgebras will be classified. This subalgebras which are called optimal system, are list of conjugacy inequivalent subalgebras with the property that any other subalgebras is conjugate to the precisely one subalgebra in the list.

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2. Lie Algebras

The vector space $\mathcal{G}$ over the field $\mathbb{R}$ is called a *Lie algebra* if it is equipted with a bilinear operation of commutator $(u, v) \mapsto [u, v]$, corresponding to the vectors $u, v \in \mathcal{G}$, whose bracket $[u, v]$ satisfies the axioms:

1) bilinearity: for any $u, v, w \in \mathcal{G}$ and $a, b \in \mathbb{R}$

$$[au + bv, w] = a[u, w] + b[v, w], \qquad [u, av + bw] = a[u, v] + b[u, w] \tag{2.1}$$

2) antisymmetry: for any $u, v \in \mathcal{G}$

$$[u, v] = -[v, u], \tag{2.2}$$

3) The Jocobi identity: for any $u, v, w \in \mathcal{G}$

$$[[u, v], w] + [[v, w], u] + [[w, u], v] = 0. \tag{2.3}$$

If the space $\mathcal{G}$ is infinite-dimensional, but is a Banach space with the norm $|u|$ of the vector $u \in \mathcal{G}$, then the Lie algebra $\mathcal{G}$ is called a Banach algebra of Lie if one more axiom is satisfied:

4) continuity: The is a number $\delta > 0$ such that for any $u, v \in \mathcal{G}$

$$|[u, v]| \leq \delta |u||u|. \tag{2.4}$$

Some examples of Lie algebras are the set of vectors of the three dimensional Euclidean space $\mathbb{R}^3$ where the operation of the commutator is taken as vector multiplication $[u, v] = u \times v$, and the set $\mathfrak{L}(X)$ of linear mapping $A : X \to X$ of some Banach space X into itself with the operation of commutator $[A, B] = A \circ B - B \circ A$.

The dimensionality of the Lie algebra $\mathcal{G}$ is the dimensionality of its vector space $\mathcal{G}$; if it has finite dimensionality r, then the Lie algebra $\mathcal{G}$ is called finite dimensional; if it is infinite dimensional, then the Lie algebra $\mathcal{G}$ is also called infinite dimensional.

2.1. Structural Constants.

The finite-dimensional Lie algebra $\mathcal{G}$ is usually given by indication of the vector basis $\{u_\alpha\}$ in the space $\mathcal{G}$. In this case, because of the (2.1), the bracket operation in $\mathcal{G}$ is fully defined by the table of commutators, that is, an $r \times r$ matrix in which the commutator $[u_\beta, u_\gamma]$ is placed at the intersection of the $\beta-$th row and $\gamma-$th column. With this, the equation for decomposition of the commutator of the basis vectors with respect to the basis is

$$[u_\beta, u_\gamma] = \sum_{\alpha=1}^{r} C^\alpha_{\beta\gamma} u_\alpha, \qquad (\beta, \gamma = 1, ..., r). \tag{2.5}$$

The constants $C^{\alpha}_{\beta\gamma}$ contained in (2.5), are called the *structural constants* of the Lie algebra $\mathcal{G}$ with respect to basis $\{u_\alpha\}$. As a result of (2.1),(2.2),(2.3), the following properties hold:

1) $(C^{\alpha}_{\beta\gamma})$ is a tensor of the third rank over the space $\mathcal{G}$,
2) structural constants are antisymmetric relative to the lower indices; $C^{\alpha}_{\beta\gamma} = -C^{\alpha}_{\gamma\beta}$,
3) the Jaccobi identity $C^{\alpha}_{\beta\sigma}C^{\sigma}_{\gamma\delta} + C^{\alpha}_{\gamma\sigma}C^{\sigma}_{\delta\beta} + C^{\alpha}_{\delta\sigma}C^{\sigma}_{\beta\gamma} = 0$.

2.2. **Homomorphisms.** Let $[\ ,\ ]$ be the symbol of the commutator in the Lie algebra $\mathcal{G}$ and $[\ ,\ ]'$ the symbol of the commutator in the Lie algebra $\mathcal{G}'$. The linear map $T : \mathcal{G} \to \mathcal{G}'$ is called a *homomorphism* of the Lie algebra $\mathcal{G}$ into the Lie algebra $\mathcal{G}'$ if for any $u, v \in \mathcal{G}$ we have

$$T([u, v]) = [T(u), T(v)]. \tag{2.6}$$

This homomorphism is called *isomorphism* if the map $T : \mathcal{G} \to T(\mathcal{G}) \subset \mathcal{G}'$ is invertible. One should distinguish the isomorphism into, when $T(\mathcal{G})$ is a subspace of $\mathcal{G}'$, and the isomorphism onto, when $T(\mathcal{G}) = \mathcal{G}'$. These two Lie algebras are isomorphic if there is an onto isomorphism. If $\mathcal{G} = \mathcal{G}'$, an onto isomorphism is called *automorphism*. If A_1 and A_2 are automorphisms of the Lie algebra $\mathcal{G}$, then their composition $A_1 \circ A_2$ is also an automorphism because

$$(A_1 \circ A_2)([u, v]) = A_1([A_2(u), A_2(v)]) = [(A_1 \circ A_2)(u), (A_1 \circ A_2)(v)].$$

Thus the set of all autopmorphisms of the given Lie algebra is a group, called the group of linear homomorphisms of the space $\mathcal{G}$. The group of all automorphisms of the Lie algebra $\mathcal{G}$ is called the *holomorph* of the Lie algebra and will be denoted by $\mathrm{Aut}(L)$.

In the finite-dimensional case, isomorphic Lie algebras have the same dimensionality. The criterion of isomorphism for finite-dimensional Lie algebras is stated in terms of structural constants in the following lemma, [6].

Lemma 2.1. *If the structural constants of the Lie algebra $\mathcal{G}$ are equal to the corresponding structural constants of the same dimensional Lie algebra $\mathcal{G}'$, then these Lie algebras are isomorphic. Conversely, if these two Lie algebras are isomorphic, then there are basis in them with corresponding equal structural constants.*

2.3. **Subalgebras.** The vector space $\mathcal{H} \subset \mathcal{G}$ is called *subalgebra* of the Lie algebra $\mathcal{G}$, if it is closed under the commutator bracket of $\mathcal{G}$. In the case of an infinite-dimensional Banach algebra of Lie one distinguishes the closed subalgebra $\mathcal{H}$, where it is a closed subspace of the banach space $\mathcal{G}$.

The subalgebra $\mathcal{I} \subset \mathcal{G}$ is called an *ideal* of the Lie algebra $\mathcal{G}$ if $[\mathcal{I}, \mathcal{G}] \subset \mathcal{G}$, i.e.; for any $u \in \mathcal{G}$, $v \in \mathcal{I}$ we have $[u, v] \in \mathcal{I}$.

In any Lie algebra $\mathcal{G}$ one distinguishes two subalgebras (they are also ideals in $\mathcal{G}$): the *null subalgebra*, consisting of one null element of the space $\mathcal{G}$; and the Lie algebra $\mathcal{G}$ itself. All remaning subalgebras of $\mathcal{G}$ are called *proper subalgebras* and ideals.

Under the homomorphism of the Lie algebra $T : \mathcal{G} \to \mathcal{G}'$ its kernel is an ideal in $\mathcal{G}$. Since $T(v) = 0$ for $v \in$ker$T$, it follows that for any $u \in \mathcal{G}$ we have $T([u, v]) = [T(u), 0] = 0$, which means that $[u, v] \in$ker$T$.

The element $z \in \mathcal{G}$ is called *central* if $[z, u] = 0$ for any $u \in \mathcal{G}$. The union $\mathcal{Z}$ of all central elements is called the *center* of the Lie algebra $\mathcal{G}$. The center is an ideal in $\mathcal{G}$. We can construct an ideal in $\mathcal{G}$ called *derived algebra* of the Lie algebra $\mathcal{G}$.

Define

$$\mathcal{G}^{(1)} := [\mathcal{G}, \mathcal{G}] = \{[u, v] : u, v \in \mathcal{G}\}, \tag{2.7}$$

which is called first derived algebra and clearly is an ideal of $\mathcal{G}$. The derived algebra of $\mathcal{G}^{(1)}$, $\mathcal{G}^{(2)} := [\mathcal{G}^{(1)}, \mathcal{G}^{(1)}]$ is called the second derived algebra of $\mathcal{G}$. By induction one determines the $(k+1)-$th derived as a derived of the $k-$th derived: $\mathcal{G}^{(k+1)} := [\mathcal{G}^{(k)}, \mathcal{G}^{(k)}]$. In this way a series of commutants of the Lie algebra $\mathcal{G}$, appears:

$$\cdots \subset \mathcal{G}^{(n)} \subset \cdots \subset \mathcal{G}^{(2)} \subset \mathcal{G}^{(1)} \subset \mathcal{G}. \tag{2.8}$$

For a finite dimensional Lie algebra $\mathcal{G}$, its commutants series terminates either with a null ideal $\mathcal{G}^{(0)} = \{0\}$ or with an $\mathcal{G}^{(n)}$ such that $\mathcal{G}^{(n-1)} \neq \mathcal{G}^{(n)} = \mathcal{G}^{(n+1)}$, [1].

A Lie algebra $\mathcal{G}$ is called *solvable* if its series of commutants (2.8) terminates with a null ideal.

3. Associated Algebra

In this section we define a kind of automorphism which we use to classify subalgebras of a given Lie algebra.

3.1. **Algebra of Differentiation.** The linear map $d : \mathcal{G} \to \mathcal{G}$, satisfies the condition

$$d([u, v]) = [d(u), v] + [u, d(v)], \tag{3.1}$$

is called a *differentiation* of $\mathcal{G}$.

It is easy verify that if d_1 and d_2 are differentiation of a Lie algebra, then for any $a, b \in \mathbb{R}$ the linear map $ad_1 + bd_2$ is also a differentiation, [2]. Thus the set of all differentiations of a given Lie algebra generates a vector space, which is a subspace in $\mathfrak{L}(\mathcal{G})$, denoted by $d\mathcal{G}$. It is easy to check that this vector space is a subalgebra with commutator

$[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$, of the Lie algebra of all linear maps $\mathcal{G} \to \mathcal{G}$. This Lie algebra is called *Lie algebra of differentiation* of the Lie algebra $\mathcal{G}$.

For every $x \in \mathcal{G}$, define a linear map $\mathrm{ad}_x : \mathcal{G} \to \mathcal{G}$, by

$$\mathrm{ad}_x(u) := [u, x], \tag{3.2}$$

using (2.3), it shows that ad_x is a differentiation of the Lie algebra $\mathcal{G}$. This differentiation is called an *inner differentiation* of the Lie algebra $\mathcal{G}$. If $x, y \in \mathcal{G}$ and $a, b \in \mathbb{R}$ the equation

$$(a\mathrm{ad}(x) + b\mathrm{ad}(y))(u) = [u, ax + by] = \mathrm{ad}(ax + by)(u), \tag{3.3}$$

holds, thus $\mathfrak{D}$ is a vector space in $d\mathcal{G}$. Further, if $x \in \mathcal{G}$ and $d \in d\mathcal{G}$, then for any $u \in \mathcal{G}$

$$\begin{aligned} (d \circ \mathrm{ad}(x) - \mathrm{ad}(x) \circ d)(u) &= d([u, x]) - (d(u), x) = [u, d(x)] \quad (3.4) \\ &= \mathrm{ad}(d(x))(u). \end{aligned}$$

Hence $d \circ \mathrm{ad}(x) - \mathrm{ad}(x) \circ d = \mathrm{ad}(d(x)) \in \mathfrak{D}$. Thus we have:

Lemma 3.1. *The set of all inner differentiation of the Lie algebra $\mathcal{G}$ denoted by $\mathfrak{D}(\mathcal{G})$ is an ideal in $d\mathcal{G}$.*

The definition of the map (3.2) shows that there is a map $\mathrm{ad} : \mathcal{G} \to \mathfrak{D}(\mathcal{G})$, defined by $x \mapsto \mathrm{ad}_x$. The following lemma, [3], fixes the algebraic properties of this map.

Lemma 3.2. *The map* $\mathrm{ad} : \mathcal{G} \to \mathfrak{D}(\mathcal{G})$ *is a homomorphism. The kernel of this homomorphism is the center $\mathcal{Z} \subset \mathcal{G}$.*

This homomorphism is called *natural homomorphism* of the Lie algebra $\mathcal{G}$ on its associated Lie algebra $\mathfrak{D}(\mathcal{G})$.

Suppose $\mathcal{G}$ be a finite Lie lagebra and its inner vector space be considered as a Banach space. The map $\mathrm{ad} : \mathcal{G} \to \mathfrak{D}(\mathcal{G})$ gives, a family $\mathfrak{E}$ of vector fields $e : \mathcal{G} \to \mathcal{G}$ according to the following rule: every vector field $e \in \mathfrak{E}$, define by the assignment of a vector $u \in \mathcal{G}$, is denoted by the symbol e_u, which acts by

$$e_u(x) = \mathrm{ad}_u(x) = [x, u]. \tag{3.5}$$

According to the bilinearity of the bracket $[\ ,\ ]$, (2.1), for any $a, b \in \mathbb{R}$ and $u, v \in \mathcal{G}$ we have $e_{au+bv} = ae_u + be_v$, thus $\mathfrak{E}$ is a vector space, i.e.; a subspace of all vector field on $\mathcal{G}$.

From definition (3.5), every vector field e_u is a linear map such as $\mathcal{G} \to \mathcal{G}$. an straight forward calculation shows that

$$[e_u, e_v] = e_{[u,v]}, \tag{3.6}$$

thus the space $\mathfrak{E}$ is a Lie algebra of vector fields. Also from (3.5) it follows that the map $\Gamma : \mathcal{G} \to \mathfrak{E}$, defined by $\Gamma(u) = e_u$, is a homomorphism of the Lie algebras.

3.2. **Inner Automorphism.** According to differential geometry, [5], every vector field e_u on $\mathcal{G}$ define a flow, which is a curve in $\mathcal{G}$ called one-parameter group obtained from the system of ordinary differential equation $x'_t = e_u(x')$, with the initial condition $x'(0) = 0$. Because of definition (3.5) this becomes the problem

$$x'_t = [x', u], \qquad x'(0) = x. \tag{3.7}$$

Since equation (3.7) is a linear relative to the unknown x, the solution of problem (3.7) must be the result of operation on the vector $x \in \mathcal{G}$ by some linear invertible map $\mathcal{G} \to \mathcal{G}$. This map will denoted by the symbol A_u^t, so the solution of the problem (3.7) is $x' = A_u^t(x)$. It is useful to note that this solution can be deduced by rewriting these equations directly from the transformation A_u^t as

$$\frac{d}{dt} A_u = \mathrm{ad}(u) \circ A_u, \qquad A_u(0) = \mathbb{I}, \tag{3.8}$$

using (3.5). Here the coefficient ad_u is a constant map (because of independency of t), and the exact solution is found to be

$$A_u^t = \exp(t\ \mathrm{ad}(u)). \tag{3.9}$$

From this, the exact expression for the solution of problem (3.7) follows from the generation map of one-parameter group of e_u, with the condition that $e_u = \mathrm{ad}_u \neq 0$,

$$x' = A_u^t(x) = \exp(t\ \mathrm{ad}_u)(x). \tag{3.10}$$

Equation (3.9) shows that the resulting transformations A_u^t of the space $\mathcal{G}$ are defined and invertible without any limitations on the values of the group parameter t and the determining vector $u \in \mathcal{G}$. A useful theorem, [6], shows that these transformations are automorphism.

Theorem 3.3. *The transformations (3.9) of the space $\mathcal{G}$ are automorphisms of the Lie algebra $\mathcal{G}$.*

This automorphism found by the foregoing method is called *inner automorphism* of the Lie algebra $\mathcal{G}$. The group of transformations of the space $\mathcal{G}$, generated by the transformations A_u^t which are one-parameter group, called *the group of inner automorphisms* of the Lie algebra $\mathcal{G}$, and will be denoted by $\mathrm{Int}(\mathcal{G})$.

To aid in the construction of an isomorphisms group, it should be noted that the Lie algebra $\mathfrak{E}$ is isomorphic to the quotient algebra $\mathcal{G}/\mathcal{Z}$. Let $\{u_\alpha\}$ be a basis in $\mathcal{G}$ and G_γ^1 be the one-parameter group generated by the vector field $e_{u'_\gamma}$. Since $e_{u_{\gamma'}} = [x, u_{\gamma'}] = 0$ because $u_{\gamma'} \in \mathcal{Z}$ for $\gamma' = 1, ..., s$, so the group G^1_γ consist of only one identity transformation. Thus, the group $G^1_{\gamma''}$ will be nontrivial only for $\gamma'' = s+1, ..., r$. Assume $x = \sum_\alpha x^\alpha u_\alpha$ and introduce the structural constants $C^\alpha_{\beta\gamma}$ of the Lie algebra $\mathcal{G}$, then, $[x, u_{\gamma''}]^\alpha = \sum_{\beta''} C^\alpha_{\beta''\gamma''} x^{\beta''}$. Consequently, for the group $G^1_{\gamma''}$, equation (3.7) in the basis $\{u_\alpha\}$ will assume the form of linear system of scalar equations with constants coefficients

$$x'^\alpha_t = \sum_{\beta'',\gamma''} C^\alpha_{\beta'',\gamma''} x'^{\beta''}, \qquad , x'^\alpha(0) = x^\alpha, \qquad (\alpha = 1, ..., r). \quad (3.11)$$

From the solution of the system (3.11) automorphisms $A^t_{\gamma''} \in G^1_{\gamma''}$ are given, in the same basis, by the matrix $(A^t_{\alpha\sigma})$, which acts by the equation

$$x'^\alpha = \sum_\sigma A^t_{\alpha\sigma} x^\sigma, \qquad (\alpha = 1, ..., r). \quad (3.12)$$

The set of inner automorphism $A \in \text{Int}(\mathcal{G})$ is obtained by the following rule: For every $\gamma'' = s + i$ $(i = 1, ..., r - s)$, the parameter of the group G^1_{s+i} is denoted by t_i and the composition of $r - s$ factors

$$A(t_1, t_2, ..., t_{r-s}) = A^{t_1}_{s+1} \circ A^{t_2}_{s+2} \circ \cdots \circ A^{t_{r-s}}_r, \quad (3.13)$$

is generated.

4. Optimal Subalgebraic System

The problem of enumeration of all subalgebras of a given finite-dimensional Lie algebra $\mathcal{G}$ is essential specially for the group analysis of differential equations [3, 6]. For this the group $\text{Int}(\mathcal{G})$ of the inner automorphisms can be considered as known. Since, under the action of an automorphism, every subalgebra transforms into a subalgebra of the same dimensionality, so the present problem can be solved correctly, up to transformations determined by the inner automorphisms.

The subalgebra $\mathcal{H}_1$ and $\mathcal{H}_2$ of the Lie algebra $\mathcal{G}$ are called *similar* if there is an inner autopmorphism $A \in \text{Int}(\mathcal{G})$ such that $A(\mathcal{H}_2) = \mathcal{H}_1$. By the relation of similarity, which is obviously a set theoretical indicator of equivalence, all subalgebras of the given Lie algebra $\mathcal{G}$ are decomposed into classes of similar algebras. The set of the representatives of similar algebra classes of given dimensionality s will be called the *optimal system (of order s)* and denoted by the symbol Θ_s. Thus, the solution of the stated problem for the finite-dimensional Lie algebra $\mathcal{G}$ must be tables of optimal system for every $s = 1, 2, ..., r - 1$.

The search for subalgebras of the given dimensionality $s > 1$ is reduced to an algebraic problem. Let $\{u_\alpha\}$ be a basis of the Lie algebra

$\mathcal{G}$ and $C^{\alpha}_{\beta\gamma}$ be its structural constants in this basis.It is obvious that the vectors $x_i = \sum_\alpha x_i^\alpha u_\alpha$ $(i = 1, ..., s < r)$ generate a basis of subalgebra $\mathcal{H}$ of dimensionality s if and only if the rank of the matrix (x_i^α) is equal to s and the relations $[x_i, x_j] = \sum_k \tilde{C}^k_{ij} x_k$, or

$$rank(x_i^\alpha) = s, \qquad C^{\alpha}_{\beta\gamma} x_i^\beta x_j^\gamma = \tilde{C}^k_{ij} x_k^\alpha \qquad (i, j = 1, ..., s; \alpha = 1, ..., r) \quad (4.1)$$

are satisfied, where $\tilde{C}^k_{ij} x_k$ are the structural constants of the subalgebra $\mathcal{H}$. Consequently, the problem is reduced to the solution of the algebraic equation (4.1) with unknown coefficients x_i^α and $\tilde{C}^k_{ij}$.

This difficulty algebraic problem has not yet been completely solved, in the sence of the formation of a difinite algorithm according to which the optimal system Θ_s can be uniquely constructed. The determination of optimal systems can be done, relatively easily, only for small dimensionality $s \leq 3$. Even in those cases the solution is accompolished by a semiprimitive method, which requires choices from a few possibilities at certain stages of the work, and the simplicity of the results obtained is not clear. Thus the theory for constructing optimal system of subalgebras needs to be developed further. There is a general result for the given problem that gives an important addition to the Levi theorem say, if $\mathcal{R}$ is the radical of a finite-dimensional Lie algebra $\mathcal{G}$, then there is a semisimple subalgebra $\mathcal{H} \subset \mathcal{G}$ such that $\mathcal{G} = \mathcal{H} \oplus \mathcal{R}$, (here $\mathcal{H}$ is called the Levi factor). For the sake of simplicity it is stated in a weak form.

Theorem 4.1. *Maltsev-Herish-Chandra* [3]. *If $\mathcal{G} = \mathcal{H} \oplus \mathcal{R}$, then for any semisimple subalgebra $\mathcal{H}_1 \subset \mathcal{G}$ there exist an automorphism $A \in \mathrm{Int}(\mathcal{G})$ such that $A(\mathcal{H}_1) = \mathcal{H}$.*

During the construction of an optimal system for a given Lie algebra $\mathcal{G}$ a special role is played by the center $\mathcal{Z}$. The point is that every vector $z \in \mathcal{Z}$ is invariant relative to any automorphism $A \in \mathrm{Int}(\mathcal{G})$. Indeed, as $[z, u] = 0$ for any $u \in \mathcal{G}$, so the unique solution of the problem (4.1) for the elements $z' = A_u^t(z)$, having the form $z'_t = [z', u],\ z'(0) = z$, is $z' = z$. Thus the central elements generate subalgebras (ideals), which cannot be changed by any automorphisms. They can be included as a direct term in any subalgebra of the Lie algebra $\mathcal{G}$. This means that if the optimal systems are known for the quotient algebra $\mathcal{G}/\mathcal{Z}$, then they can be considered as known for the entire Lie algebra $\mathcal{G}$. Cosequently, it is sufficient to describe methods for constructing optimal system only for Lie algebras with null centeres.

Let the basis $\{u_\alpha\}$ be fixed in a Lie algebra $\mathcal{G}$ with a null center. All inner automorphisms of this Lie algebra depends on $r-$parameters and is given by equation (3.13), in which it is necessary to assume $s = 0$.

The construction of one-dimensional subalgebras, Θ_1, begins with the selection of non-null vector $x = \sum_\alpha x^\alpha u_\alpha$ and its image $x' = A(x)$, obtained by the operation of the common automorphism $A = A(t_1, ..., t_r)$. If $(A_{\alpha\beta})$ is a matrix of the automorphism A in the basis $\{u_\alpha\}$, then the components of the vectors x' in the basis are given by an equation of the form (3.12)

$$x'^\alpha = \sum_\sigma A_{\alpha\sigma} x^\sigma, \qquad (\alpha = 1, ..., r). \tag{4.2}$$

The next step is the selection of values of the parameters $t_1, ..., t_r$, on which the automorphism A depends, to achieve the maximum possible simplification of the set $(x'^1, ..., x'^r)$ of components of the vector x'. This permits the choice of the simplest representative of each class of similar algebras to which the element x belongs. Usually, this means choosing the maximum possible number of null values for these components. Appearing here are the various alternative cases that give the classes of one-parameter subalgebras and, from them, the optimal system Θ_1.

For the construction of the two-dimensional subalgebras, optimal system Θ_2, it is possible to assume, that one of the basis vectors of the two-dimensional subalgebra $\mathcal{H} \subset \mathcal{G}$ is taken from the system Θ_1. Let this vector be $x = \sum_\alpha x^\alpha u_\alpha$. Correspondingly, a vector $y = \sum_\alpha y^\alpha u_\alpha$ is chosen, so that the system of equations (4.1) is satisfied. Here that the system is

$$C^\alpha_{\beta\gamma} x^\beta y^\gamma = \lambda y^\alpha + \mu x^\alpha, \qquad (\alpha = 1, ..., r), \tag{4.3}$$

and it contains $r + 2$ unknown coefficients y^α, λ and μ. The set of solution for system (4.3) gives a two-dimensional algebra which contains the vector x. As the vectors x and y can be scaled by any nonzero multipliers, instead of the two-parameter set of system (4.3), with parameters λ and μ, it is enough to consider such systems in which $(\lambda, \mu) = (0, 0), (1, 0), (0, 1)$ and $(1, 1)$, Obviously the case $(1, 1)$ is reducible to the case $(1, 0)$ by the change $y \mapsto y + x$. In the case $(0, 1)$ there are no solutions. If $\mu = 0$, then (4.3) has the form of an eigenvalue problem for the linear map ad_x, with the matrix $(C^\alpha_{\beta\gamma} x^\beta)$, and can be written in the form $\mathrm{ad}_x(y) = \lambda y$. Its solution is reduced to finding the roots of the characteristic polynomial $\det(\lambda\mathbb{I} - \mathrm{ad}_x)$, which in the expand form is

$$\begin{aligned} \det(\lambda\mathbb{I} - \mathrm{ad}_x) &= \lambda^r - \tau_1(x)\lambda^{r-1} + \tau_2(x)\lambda^{r-2} - \cdots \\ &+ (-1)^\ell \tau_{r-1}(x)\lambda^\ell, \end{aligned} \tag{4.4}$$

where $\tau_{r-1}(x) \neq 0$ and $0 \leq \ell$. As $\mathrm{ad}_x(x) = 0$ and $x = 0$, it follows that $\ell > 0$.

Corresponding to every real non-zero root of polynomial (4.4) is one or more linearly independent solution of equation (4.3). (This is when $\mu = 0$. In the case $\ell > 1$ there are also one or more solutions corresponding to the root $\lambda = 0$. If $\ell = 1$, and there are no non-zero real roots, then the element x does not belong to any two-dimensional algebra of this kind with $\mu = 0$ in (4.3).) Consequently, it follows that such vectors $x \in \mathcal{H}$ can exist that do not belong to any two-dimensional Lie algebra $\mathcal{H}$.

The polynomial (4.4), considered for the variable $x \in \mathcal{G}$, is called *Killing's polynomial* (or the *characteristic polynomial*) of the Lie algebra $\mathcal{G}$. The maximal value of the number ℓ in the expression of killling's polynomial (4.4), obtained when the vector x ranges over the whole sapace $\mathcal{G}$, is called *the rank of the Lie algebra* $\mathcal{G}$. The elemernt $x \in \mathcal{G}$ for which the value ℓ in (4.4) is equal to the rank is called a *regular element* of $\mathcal{G}$.

After the construction of all two-dimensional subalgebras, for every $x \in \Theta_1$, they need to be simplified by the action of automorphisms in a manner analogous to the way it was done for the one-dimensional algebra. As the result, the optimal system Θ_2 is obtained.

The system Θ_3 can be developed by the method of expansion of two-dimensional subalgebras. For this take any two-dimensional sualgebra from Θ_2, with the basis $\{x_1, x_2\}$, and find a vector $y = \sum_\alpha y^\alpha u_\alpha$ such that the triple $\{x_1, x_2, y\}$ generates a basis of three-dimensional algebra. For that it is necessary and sufficient that the vector y satisfies the equations, following from the system (4.1),

$$\begin{aligned} C^\alpha_{\beta\gamma} x_1^\beta y^\gamma &= \lambda_1 y^\alpha + \mu_1 x_1^\alpha + \nu_1 x_2^\alpha, \\ C^\alpha_{\beta\gamma} x_2^\beta y^\gamma &= \lambda_2 y^\alpha + \mu_2 x_1^\alpha + \nu_2 x_2^\alpha, \qquad (\alpha = 1, ..., r), \end{aligned} \tag{4.5}$$

with $r + 6$ unknown coefficients $y^\alpha, \lambda_i, \mu_i, \nu_i$. Every solution of system (4.5) is linearly independent of $\{x_1, x_2\}$ and gives a three-dimensional subalgebra of $\mathcal{G}$.

A question says, are all three-dimensional subalgebras obtained by this porocess Obviously, this question is equivalent to: Do all three-dimensional subalgebras contain a two-dimensional sublagebra? More generally, the problem can be reduced to the question, are there three-dimensional real Lie algebras that do not contain two-dimensional subalgebras. The answer to the last question is affirmative, and the simple Lie algebra $\mathcal{SO}(3)$ of the simple Lie group SO(3), the rotation group in $\mathbb{R}^3$ is the example. Because if $\{u_1, u_2, u_3\}$ be a basis for $\mathcal{SO}(3)$ then , we have

$$[u_1, u_2] = u_3, \qquad [u_2, u_3] = u_1, \qquad [u_3, u_1] = u_2. \tag{4.6}$$

Lemma 4.2. *Any three-dimensional real Lie algebra not containing two-dimensional subalgebras is isomorphic to the Lie algebra of the rotations (4.6), [7].*

In the next section we will classify subalgebras of given some Lie algebras up to inner automorphism A.

5. Examples

In the first example we give a comprehensive illustration to classify subalgebras.

5.1. **Lie algebra of Special Euclidean Transformations in $\mathbb{R}^3$.**

Consider the Lie group $\mathrm{SE}(3) = \mathrm{SO}(3) \times \mathbb{R}^3$ of Special Euclidean transformations in $\mathbb{R}^3$ containing three-dimensional rotations and translations. Its Lie algebra is a six-dimensional vector space spanned by $\{u_1, ..., u_6\}$ with the commutator table

$$\begin{aligned} &[u_1, u_5] = -u_3, \qquad [u_1, u_6] = u_2, \qquad [u_2, u_4] = u_3, \\ &[u_2, u_6] = -u_1, \qquad [u_3, u_4] = -u_2, \qquad [u_3, u_5] = u_1, \\ &[u_4, u_5] = -u_6, \qquad [u_4, u_6] = u_5, \qquad [u_5, u_6] = -u_4. \end{aligned} \tag{5.1}$$

The matrices of automorphisms $A^{t_i}, i = 1, ..., 6$ are:

$$A^{t_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & t_1 & 0 & 1 & 0 \\ 0 & -t_1 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.2}$$

$$A^{t_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -t_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ t_2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A^{t_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & t_3 & 0 & 1 & 0 & 0 \\ -t_3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A^{t_4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos t_4 & \sin t_4 & 0 & 0 & 0 \\ 0 & \sin t_4 & \cos t_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos t_4 & \sin t_4 \\ 0 & 0 & 0 & 0 & -\sin t_4 & \cos t_4 \end{pmatrix},$$

$$A^{t_5} = \begin{pmatrix} \cos t_5 & 0 & -\sin t_5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \sin t_5 & 0 & \cos t_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos t_5 & 0 & -\sin t_5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin t_5 & 0 & \cos t_5 \end{pmatrix},$$

$$A^{t_6} = \begin{pmatrix} \cos t_6 & \sin t_6 & 0 & 0 & 0 & 0 \\ -\sin t_6 & \cos t_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos t_6 & \sin t_6 & 0 \\ 0 & 0 & 0 & -\sin t_6 & \cos t_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let us apply the action of these matrices to the basis $\{u_1, ..., u_6\}$ of $\mathcal{SE}(3)$. Suppose $x = \sum_{\alpha=1}^{6} x^\alpha u_\alpha$ is an arbitrary element in $\mathcal{SE}(3)$ then,

$$\begin{aligned} A(t_6, ..., t_1)(x) \quad \mapsto & \qquad (5.3) \\ & (\cos t_5 \cos t_6 x^1 + \cos t_5 \sin t_6 x^2 - \sin t_5 x^3) u_1 + \cdots \end{aligned}$$

Now,we can simply x as follows:

If x^1, x^2 and $x^3 = 0$,then we can make the coefficients of $u_1, ..., u_5$ vanish, by $t_4 = -\arctan(x^5/x^6)$ and $t_5 = \arctan(x^4/x^6)$. Scaling x if necessary, we can assume that $x^6 = 1$. And u is reduced to the Case of A_1^1.

If x^2 and $x^3 = 0$ but $x^1 \neq 0$,then we can make the coefficients of u_2, u_3, u_5 and u_6 vanish, by $t_2 = -x^6/x^1$ and $t_3 = x^5/x^1$. Scaling u if necessary, we can assume that $x^1 = 1$. And u is reduced to the Case of A_1^2.

If x^1 and $x^3 = 0$ but $x^2 \neq 0$,then we can make the coefficients of u_1, u_2, u_3, and u_6 vanish, by $t_1 = x^6/x^2$ and $t_3 = -x^4/x^2$. Scaling u if necessary, we can assume that $x^2 = 1$. And u is reduced to the Case of A_1^3.

If x^1 and $x^2 = 0$ but $x^3 \neq 0$,then we can make the coefficients of u_1, u_2, u_4 and u_5 vanish, by $t_1 = -x^5/x^3$ and $t_2 = x^4/x^3$. Scaling u if necessary, we can assume that $x^3 = 1$. And u is reduced to the Case of A_1^4.

If x^1 and $x^2 \neq 0$,then we can make the coefficients of u_3, u_4 and u_5 vanish, by $t_1 = x^6/x^2$, $t_3 = x^5/x^1$ and $t_4 = -\arctan(x^3/x^2)$. Scaling u if necessary, we can assume that $x^1 = 1$. And u is reduced to the Case of A_1^5.

If $x^2 = 0$ but x^1 and $x^3 \neq 0$,then we can make the coefficients of u_2, u_5 and u_6 vanish, by $t_1 = -x^5/x^3$, $t_2 = -x^6/x^1$ and $t_4 = -\arctan(x^3/x^2)$. Scaling u if necessary, we can assume that $x^1 = 1$. And u is reduced to the Case of A_1^6.

If $x^1 = 0$ but x^2 and $x^3 \neq 0$,then we can make the coefficients of u_3, u_4 and u_5 vanish, by $t_1 = x^6/x^2$, and $t_2 = x^4/x^3$. Scaling u if necessary, we can assume that $t_2 = 1$. And u is reduced to the Case of A_1^7. See [4] for another comprehensive example.

Thus the one-dimensional optimal system Θ_1 of $\mathcal{SE}(3)$ is given by

$$\begin{array}{ll} A_1^1 : \langle u_6 \rangle, & A_1^2 : \langle u_1 + bu_4 \rangle, \\ A_1^3 : \langle u_2 + bu_5 \rangle, & A_1^4 : \langle u_3 + bu_6 \rangle, \\ A_1^5 : \langle u_1 + au_2 + bu_6 \rangle, & A_1^6 : \langle u_1 + au_3 + bu_4 \rangle, \\ A_1^7 : \langle u_2 + au_3 + bu_5 \rangle. & \end{array} \tag{5.4}$$

For finding Θ_2, let us assume that, $\mathcal{G} = \langle x_1, x_2 \rangle$ is a two-dimensional Lie subalgebra of $\mathcal{SE}(3)$.

Let $x_1 = u_6$ be as the Case of A_1^1, $x_2 = \sum_{i=1}^6 b_i u_i$, and $[x_1, x_2] = \lambda x_1 + \mu x_2$. Then, we have $x_2 = b_3 u_3 + b_6 u_6$ and $\lambda = \mu = 0$. By a suitable change of base of $\mathcal{G}$,we can assume that $x_2 = u_3$. Now $\mathcal{G}$ is reduced to the Case of A_2^2. We can not be used to further simplify this subalgebra,by A^{t_i}, $i = 1, ..., 6$ defined as (5.2).

Let $x_1 = u_1 + a_4 u_4$ be as the Case of A_1^2, $x_2 = \sum_{i=1}^6 b_i u_i$, and $[x_1, x_2] = \lambda x_1 + \mu x_2$. Then,we have $x_2 = b_3 u_3 + b_1 u$ and $\lambda = \mu = 0$. By a suitable change of base of $\mathcal{G}$, we can assume that $x_2 = u_3$. Now $\mathcal{G}$ is reduced to the Case of A_2^6. We can not be used to further simplify this subalgebra,by A^{t_i}, $i = 1, ..., 6$.

Let $x_1 = u_2 + a_5 u_5$ be as the Case of A_1^3, $x_2 = \sum_{i=1}^6 b_i u_i$, and $[x_1, x_2] = \lambda x_1 + \mu x_2$. Then,we have $x_2 = b_2 u_2 + b_5 u_5$ and $\lambda = \mu = 0$. By a suitable change of base of $\mathcal{G}$, we can assume that $x_1 = u_2$ and $x_2 = u_5$. Now, $\mathcal{G}$ is reduced to the Case of A_2^1. We can not be used to further simplify this subalgebra,by A^{t_i}, $i = 1, ..., 6$.

Let $x_1 = u_3 + a_6 u_6$ be as the Case of A_1^4, $x_2 = \sum_{i=1}^6 b_i u_i$, and $[x_1, x_2] = \lambda x_1 + \mu x_2$. Then,we have $x_2 = b_3 u_3 + b_6 u_6$ and $\lambda = \mu = 0$. By a suitable change of base of $\mathcal{G}$, we can assume that $x_1 = u_6$ and $x_2 = u_3$. Now, $\mathcal{G}$ is reduced to the Case of A_2^2.

Let $x_1 = u_1 + a_2 u_2 + a_4 u_4$ be as the Case of A_1^5, $x_2 = \sum_{i=1}^6 b_i u_i$, and $[x_1, x_2] = \lambda x_1 + \mu x_2$. Then,we have $x_2 = b_1 u_1 + (b_2/a_2) x_1$ and $\lambda = \mu = 0$. By a suitable change of base of $\mathcal{G}$, we can assume that

$x_1 = u_1$ and $x_2 = u_2 + au_4$. Now, $\mathcal{G}$ is reduced to the Case of A_2^3. We can not be used to further simplify this subalgebra,by A^{t_i}, $i = 1, ..., 6$.

Let $x_1 = u_2 + a_3u_3 + a_4u_4$ be as the Case of A_1^6, $x_2 = \sum_{i=1}^{6} b_iu_i$, and $[x_1, x_2] = \lambda x_1 + \mu x_2$. Then,we have $x_2 = b_1u_1 + (b_3/a_3)x_1$ and $\lambda = \mu = 0$. By a suitable change of base of $\mathcal{G}$,we can assume that $x_1 = u_1$ and $x_2 = u_3 + au_4$. Now, $\mathcal{G}$ is reduced to the Case of A_2^4. We can not be used to further simplify this subalgebra,by A^{t_i}, $i = 1, ..., 6$.

Let $x_1 = u_2 + a_3u_3 + a_5u_5$ be as the Case of A_1^7, $x_2 = \sum_{i=1}^{6} b_iu_i$, and $[x_1, x_2] = \lambda x_1 + \mu x_2$. Then,we have $x_2 = b_2u_2 + (b_3/a_3)x_1$ and $\lambda = \mu = 0$. By a suitable change of base of $\mathcal{G}$,we can assume that $x_1 = u_2$ and $x_2 = u_3 + au_5$. Now, $\mathcal{G}$ is reduced to the Case of A_2^5. We can not be used to further simplify this subalgebra,by A^{t_i}, $i = 1, ..., 6$.

Thus, a two-dimensional optimal system of $\mathcal{SE}(3)$ is given by

$$\begin{array}{lll} A_2^1 : \langle u_2, u_5\rangle, & A_2^2 : \langle u_3, u_6\rangle, & A_2^3 : \langle u_1, u_2 + au_4\rangle, \\ A_2^4 : \langle u_1, u_3 + au_4, \rangle & A_2^5 : \langle u_2, u_3 + au_5, \rangle & A_2^6 : \langle u_3, u_1 + au_4\rangle. \end{array} \quad (5.5)$$

For finding Θ_3, let us assume that, $\mathcal{G} = \langle x_1, x_2, x_3\rangle$ is a 3-dimensional Lie subalgebra of $\mathcal{SE}(3)$. Let $x_1 = u_2$ and $x_2 = u_5$ are as the Case of A_2^1, $x_3 = \sum_{i=1}^{6} b_iu_i$, $[x_1, x_3] = \lambda_1x_1 + \mu_1x_2 + \nu_1x_3$ and $[x_2, x_3] = \lambda_2x_1 + \mu_2x_2 + \nu_2x_3$. Then, we have $x_3 = b_2x_1 + b_5x_2$. By a suitable change of base of $\mathcal{G}$,we can assume that $x_3 = 0$, and $\mathcal{G}$ is not a three-dimensional subalgebra. Thus, in this case we have not any three-dimensional subalgebra.

The Cases A_2^i, $i = 2, 3, 4, 5$ are similar. Thus, in these cases we have not any three-dimensional subalgebra.

Let $x_1 = u_3$ and $x_2 = u_1 + au_4$ are as the Case of A_2^6, $x_3 = \sum_{i=1}^{6} b_iu_i$, $[x_1, x_3] = \lambda_1x_1 + \mu_1x_2 + \nu_1x_3$ and $[x_2, x_3] = \lambda_2x_1 + \mu_2x_2 + \nu_2x_3$. Then,we have $x_3 = (b_1/\nu_1).\big(-\nu_1x_1 + \nu_2x_2 - au_2\big)$. By a suitable change of base of $\mathcal{G}$, we can assume that $x_1 = u_1 + au_4$, $x_2 = u_2$ and $x_3 = u_3$. Now, $\mathcal{G}$ is reduced to the Case of A_3^1 and A_3^2. We can not be used to further simplify this subalgebra, by A^{t_i}, $i = 1, ..., 6$ defined as (5.2). Thus, a three-dimensional optimal system of $\mathcal{SE}(3)$ is given by

$$A_3^1 : \langle u_1 + au_4, u_2, u_3\rangle, \qquad A_3^2 : \langle u_4, u_5, u_6\rangle. \quad (5.6)$$

For finding Θ_4, let us assume that, $\mathcal{G} = \langle u_1 + au_4, u_2, u_3, x_1\rangle$ be a Lie subalgebra of $\mathcal{SE}(3)$, where $x = \sum_{i=1}^{6} b_i\, x_i$. Then,we have $x_1 = \sum_{i=1}^{4} b_iu_i$. By a suitable change of base of $\mathcal{G}$, we can assume that $x_1 = u_4$. Now, $\mathcal{G}$ is reduced to the Case of A_4. We can not be used to further simplify this subalgebra, by A^{t_i}, $i = 1, ..., 6$ defined as (5.2). Thus, a four-dimensional optimal system of $\mathcal{SE}(3)$ is given by

$$A_4 : \langle u_1, u_2, u_3, u_4\rangle. \quad (5.7)$$

$\mathcal{SE}(3)$ has not any five-dimensional Lie subalgebra. Because if $\mathcal{G} = \langle u_1, u_2, u_3, u_4, x_1\rangle$ be a Lie subalgebra of $\mathcal{SE}(3)$, where $x_1 = \sum_{i=1}^{6} b_i u_i$. Then, we have $x_1 = \sum_{i=1}^{4} b_i u_i$, and $\mathcal{G} = \langle u_1, u_2, u_3, u_4\rangle$ is not a five-dimensional subalgebra.

5.2. **Subalgebra Classification of All Two-dimensional Lie Algebras.** As mentioned in [5] there are two inequivalent two-dimensional Lie algebras.

i) $\mathcal{G}_{2,1}\ :\ [u_1, u_2] = 0$; this algebra is isomorphic to $\mathbb{R}^2$ and it is an Abelian Lie algebra.

ii) $\mathcal{G}_{2,2}\ :\ [u_1, u_2] = u_2$; this algebra is isomorphic to the Lie algebra of Affine transformations of order one and it is a non-Abelian Lie algebra.

According to this, $\Theta_i, i = 1, 2$ are easily determined and do not need any illustraion.

5.3. **Subalgebra Classification of All Three-dimensional Solvable Lie Algebras.** The classification of set of three-dimensional solvable Lie algebras was done by Bassarab and et. al in [7]. In this section we classify all subalgebras by using described method. But the classification is just given in some tables without using the expanded method such as the last example.

1) $\mathcal{G}_{3,1}\ :\ [u_1, u_2] = u_1, \qquad [u_1, u_3] = u_2, \qquad [u_2, u_3] = u_3,$

$$\begin{aligned} \Theta_1 &: \langle u_1\rangle, \qquad \langle u_2\rangle, \qquad \langle u_3\rangle, \\ \Theta_2 &: \langle u_1, au_2 + bu_3\rangle, \qquad \langle u_2, u_3\rangle. \end{aligned}$$

2) $\mathcal{G}_{3,2}\ :\ [u_1, u_2] = u_2,$

$$\begin{aligned} \Theta_1 &: \langle u_1\rangle, \qquad \langle u_2\rangle, \qquad \langle u_3\rangle, \\ \Theta_2 &: \langle u_1, au_2 + bu_3\rangle, \qquad \langle u_2, u_3\rangle. \end{aligned}$$

3) $\mathcal{G}_{3,3}\ :\ [u_2, u_3] = u_1,$

$$\begin{aligned} \Theta_1 &: \langle u_1\rangle, \qquad \langle au_2 + bu_3\rangle, \\ \Theta_2 &: \langle u_1, au_2 + bu_3\rangle. \end{aligned}$$

4) $\mathcal{G}_{3,4}\ :\ [u_1, u_3] = u_1, \qquad [u_2, u_3] = u_1 + u_2,$

$$\begin{aligned} \Theta_1 &: \langle u_3\rangle, \qquad \langle au_2 + bu_3\rangle, \\ \Theta_2 &: \langle u_1, au_2 + bu_3\rangle, \qquad \langle u_2, u_3\rangle. \end{aligned}$$

5) $\mathcal{G}_{3,5}\ :\ [u_1, u_3] = u_1, \qquad [u_2, u_3] = u_2,$

$$\begin{aligned} \Theta_1 &: \langle u_3\rangle, \qquad \langle au_2 + bu_3\rangle, \\ \Theta_2 &: \langle u_1, au_2 + bu_3\rangle, \qquad \langle u_2, u_3\rangle. \end{aligned}$$

6) $\mathcal{G}_{3,6}$: $[u_1, u_3] = u_1, \qquad [u_2, u_3] = -u_2,$

$$\begin{aligned} \Theta_1 &: \langle u_3 \rangle, \qquad \langle au_2 + bu_3 \rangle, \\ \Theta_2 &: \langle u_1, au_2 + bu_3 \rangle, \qquad \langle u_2, u_3 \rangle. \end{aligned}$$

7) $\mathcal{G}_{3,7}$: $[u_1, u_3] = u_1, \qquad [u_2, u_3] = qu_2 \qquad (0 < |q| < 1),$

$$\begin{aligned} \Theta_1 &: \langle u_3 \rangle, \qquad \langle au_2 + bu_3 \rangle, \\ \Theta_2 &: \langle u_1, au_2 + bu_3 \rangle, \qquad \langle u_2, u_3 \rangle. \end{aligned}$$

8) $\mathcal{G}_{3,8}$: $[u_1, u_3] = -u_2, \qquad [u_2, u_3] = u_1,$

$$\begin{aligned} \Theta_1 &: \langle u_3 \rangle, \qquad \langle au_2 + bu_3 \rangle, \\ \Theta_2 &: \langle u_1, u_2 \rangle, \qquad \langle u_2, u_3 \rangle. \end{aligned}$$

9) $\mathcal{G}_{3,9}$: $[u_1, u_3] = qu_1 - u_2, \qquad [u_2, u_3] = u_1 + qu_2,$

$$\begin{aligned} \Theta_1 &: \langle u_3 \rangle, \qquad \langle au_2 + bu_3 \rangle, \\ \Theta_2 &: \langle u_1, u_2 \rangle. \end{aligned}$$

5.4. **Subalgebra Classification of Some Four-dimensional Solvable Lie Algebras.** Among the four-dimensional Lie algebras there are ten decomposible and ten non-decomposible solvable Lie algebra, [7]. In this section the classification subalgebras of some of them is given.

1) $\mathcal{G}_{4,1}$: $[u_2, u_4] = u_1, \qquad [u_3, u_4] = u_2,$

$$\begin{aligned} \Theta_1 &: \langle au_1 + bu_2 \rangle, \qquad \langle au_1 + bu_3 \rangle, \qquad \langle u_4 \rangle, \\ \Theta_2 &: \langle u_1, au_2 + bu_3 + cu_4 \rangle, \qquad \langle u_2, u_3 \rangle, \\ \Theta_3 &: \langle u_1, u_2, au_3 + bu_4 \rangle. \end{aligned}$$

2) $\mathcal{G}_{4,2}$: $[u_1, u_4] = qu_1, \qquad [u_3, u_4] = u_2 + u_3 \qquad (q \neq 0),$

$$\begin{aligned} \Theta_1 &: \langle au_1 + bu_2 + cu_3 \rangle, \qquad \langle u_2 \rangle, \qquad \langle u_4 \rangle, \\ \Theta_2 &: \langle au_1 + bu_2, cu_3 + du_4 \rangle, \\ \Theta_3 &: \langle u_1, u_2, u_3 \rangle, \qquad \langle u_2, u_3, u_4 \rangle. \end{aligned}$$

3) $\mathcal{G}_{4,3}$: $[u_1, u_4] = u_1, \qquad [u_3, u_4] = u_2,$

$$\begin{aligned} \Theta_1 &: \langle au_1 + bu_2 + cu_3 \rangle, \qquad \langle u_2 \rangle, \qquad \langle u_4 \rangle, \\ \Theta_2 &: \langle u_1, au_2 + bu_3 + cu_4 \rangle, \qquad \langle u_2, au_3 + bu_4 \rangle, \\ \Theta_3 &: \langle u_1, u_2, au_3 + bu_4 \rangle. \end{aligned}$$

6. Conclusion

In this paper we give a method for classifying Lie subalgebras of a given Lie algebra. By increasing the dimension of the Lie algebra, claculations increases. In this situation we can use some calculation softwares such as **Maple** and **Mathematica** through an applicable programming with some packages.

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