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UPPER BOUNDS ON THE UNIFORM SPREADS OF THE SPORADIC SIMPLE GROUPS

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Abstract. A finite group G has uniform spread k if there exists a fixed conjugacy class C of elements in G with the property that for any k nontrivial elements $s_1, s_2, \ldots, s_k$ in G there exists $y \in C$ such that $G = \langle s_i, y \rangle$ for $i = 1, 2, \ldots, k$. Further, the exact uniform spread of G is the largest k such that G has the uniform spread k. In this paper we give upper bounds on the exact uniform spreads of thirteen sporadic simple groups.

1. Introduction

It is well-known that every finite simple group can be generated by two suitable elements [2, 17, 18]. In this case the group is called 2-generated. Binder showed that for any two non-trivial elements x_1 and x_2 of the symmetric group S_n there exists an element y such that $S_n = \langle x_1, y \rangle = \langle x_2, y \rangle$ [3]. From this Brenner and Wiegold made the following definition in [6].

Definition 1.1. *Let r be any positive integer. A finite non-abelian group G is said to have spread r, if for every set $S = \{s_1, s_2, \ldots, s_r\}$ of distinct non-trivial elements of G, there exists an element $y \in G$ such that $G = \langle s_i, y \rangle$ for $i = 1, 2, \ldots, r$. In this case y is called the mate of S. G has exact spread r if it has spread r but not $r + 1$ and this is denoted by $s(G) = r$.*

The stronger notion of uniform spread was introduced in [14].

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Definition 1.2. *A finite group G has uniform spread k if there exists a fixed conjugacy class C of elements in G with the property that for every set $S = \{s_1, s_2, \dots, s_k\}$ of distinct non-trivial elements of G there exists $y \in C$ such that $G = \langle s_i, y\rangle$ for $i = 1, 2, \dots, k$. Further, the exact uniform spread of G, denoted by $u(G)$, is the largest k such that G has the uniform spread k.*

The following lemma gives an equivalent definition of spread, was presented by Bradley and Moori in [5], such that is useful for computational purposes.

Lemma 1.3. [5, Lemma 1.1] *A finite non-abelian group G has spread r, if for every set $S = \{s_1, s_2, \dots, s_r\}$ of distinct elements of prime order in G, there exists an element $y \in G$ such that $G = \langle s_i, y\rangle$ for $i = 1, 2, \dots, r$.*

The following is a version of Lemma 1.3 adapted to uniform spreads.

Lemma 1.4. *A finite group G has uniform spread k if there exists a fixed conjugacy class C in G such that for every set $S = \{s_1, s_2, \dots, s_k\}$ of distinct elements of prime order in G, there exists an element $y \in C$ such that $G = \langle s_i, y\rangle$ for $i = 1, 2, \dots, k$.*

Proof. Let $S = \{s_1, s_2, \dots, s_k\}$ be any set of distinct non-trivial elements of G. Then there exist positive integers m_i for $i = 1, 2, \dots, k$ such that $s_i^{m_i} = x_i$ and order of x_i for $i = 1, 2, \dots, k$ is prime. Thus by assumption there exists $y \in C$ such that $G = \langle x_i, y\rangle = \langle s_i^{m_i}, y\rangle \subseteq \langle s_i, y\rangle \subseteq G$. Therefore $G = \langle s_i, y\rangle$ for all $i = 1, 2, \dots, k$. $\square$

Clearly $u(G) \leq s(G)$, and in general these numbers are distinct. The exact spread and uniform spread of finite simple groups has been studied in [3, 6, 7, 9, 15, 16]. The exact spread of only two sporadic simple groups have been determined. For the remaining twenty four sporadic simple groups, bounds of the exact spread are known [4, 5, 7, 11, 13, 21]. In [4, 21] it was proved that $s(M_{11}) = 3$ and Fairbairn has shown that $s(M_{23}) = 8064$ [12].

In this paper we give upper bounds on the exact uniform spreads of thirteen sporadic simple groups. First we give some lemmas that present upper bounds and then offer three algorithms to calculate these upper bounds of the exact uniform spread.

For information on the sporadic simple groups and their maximal subgroups we use Atlas [10]. All calculations were done with the aid of GAP [19] and Magma [8].

The rest of the paper is organized as follows: in Section 2, we give some preliminary results. Three algorithms to calculate upper bounds of the exact uniform spread are offered in Section 3, and in Section 4 we prove our Main Theorem.

2. Preliminaries

Let G be a finite group and let M be a maximal subgroup of G. As is standard we write M^G for the maximal subgroups of G that are conjugate to M and $\mathcal{M}$ for the set of all maximal subgroups

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of G. We write $cl(G)$ for the collection of all conjugacy classes of elements of G and $cl_p(G)$ for the collection of all conjugacy classes of elements of G of prime order. We write nX for a conjugacy class of elements in G of order n which $n>1$. For a conjugacy class nX we define $supp(nX) = \{M | M \in \mathcal{M} \text{ and } M \cap nX \neq \emptyset\}$.

From Definition 1.2, it is clear that if y is a mate for the set $S = \{s_1, s_2, \ldots, s_k\}$, then no maximal subgroup contains both y and s_i for $i = 1, 2, \ldots, k$. We use this property in the following definitions.

Definition 2.1. *Let G be a finite group and let M be a maximal subgroup of G. Let S be a set of elements from a conjugacy class nX. When $M \cap nX \neq \emptyset$ we say that S supports M^G if $M' \cap S \neq \emptyset$ for every $M' \in M^G$. In this case we define $support_{nX}(M)$ for the size of the smallest S that supports M^G.*

From the Definition 2.1 it is clear that if $M \cap nX \neq \emptyset$ then $support_{nX}(M) \leq |M^G|$.

Consider a conjugacy class nY of elements in G. There exists a conjugacy class $mX \in cl_p(G)$ such that $supp(mX) \cap supp(nY) \neq \emptyset$. We use this fact in the following definition.

Definition 2.2. *Let G be a finite group and mX and nY be two conjugacy classes of elements of G such that $mX \in cl_p(G)$ and $supp(mX) \cap supp(nY) \neq \emptyset$. We define,*

$$support(mX, nY) = min\{support_{mX}(M) | M \in \mathcal{M} \text{ and } M \in supp(mX) \cap supp(nY)\}.$$

Lemma 2.3. *Let G be a finite group and mX and nY be two conjugacy classes of elements of G such that $mX \in cl_p(G)$ and $supp(mX) \cap supp(nY) \neq \emptyset$. Consider a set S of elements of the class mX. Let $u_{mX,nY}(G)$ be the largest integer such that any set S of this size has a mate from the class nY. Then we have,*

$$u(G) \leq max\{min\{u_{mX,nY}(G) | mX \in cl_p(G) \text{ and } supp(mX) \cap supp(nY) \neq \emptyset\} | nY \in cl(G)\}.$$

Proof. For a given conjugacy class nY consider $u_{mX,nY}(G)$ for each $mX \in cl_p(G)$ where $supp(mX) \cap supp(nY) \neq \emptyset$. The minimum of all these is then an upper bound on the size of a set that can have a mate from the class nY . The maximum of these values, when nY goes through all classes of G, is then an upper bound on $u(G)$. □

Lemma 2.4. *Let G be a finite group. Then,*

$$u(G) \leq max\{min\{support(mX, nY) | mX \in cl_p(G) \text{ and } supp(mX) \cap supp(nY) \neq \emptyset\} | nY \in cl(G)\} - 1.$$

Proof. Consider a conjugacy class nY of elements in G. There is a conjugacy class $cX \in cl_p(G)$ such that $support(cX, nY) = min\{support(mX, nY) | mX \in cl_p(G) \text{ and } supp(mX) \cap supp(nY) \neq \emptyset\}$. From the Definition 2.2 there exists a set S of elements from the class cX of size $support(cX, nY)$ such that has not any mate from the class nY. Therefore $u_{cX,nY}(G) \leq support(cX, nY) - 1$. Then the result is concluded from Lemma 2.3. □

Lemma 2.5. *Let M be a maximal subgroup of G and nX be a conjugacy class of elements of G such that $M \cap nX \neq \emptyset$. If each element of the class nX is inside a unique subgroup of class M^G then $support_{nX}(M) = |M^G|$.*

Proof. From the Definition 2.1 we have $support_{nX}(M) \leq |M^G|$. Let S be a set of elements from the class nX such that supports M^G. Let $|S| < |M^G|$. Each element of S is inside a unique subgroup of M^G. Therefore S has non-empty intersection with at most $|S|$ subgroups of M^G. Then S does not support M^G that is contradiction. □

In Table 1 bounds on the exact spread of thirteen sporadic simple groups are presented. The lower bounds were proved in [7] except for M_{23} whose lower bound was proved in [12]. The upper bounds were proved in [11] except for Fi_{22} and M_{23} whose upper bounds were proved in [4]. According to the fact that $u(G) \leq s(G)$ for any group G in general, upper bounds presented in Table 1 also are upper bounds on the exact uniform spread. The lower bounds given here are lower bounds on $s(G)$ and are therefore not necessarily lower bounds on $u(G)$.

Table 1. Bounds on the exact spread of thirteen sporadic simple groups.

G	Upper bound Lower bound	G	Upper bound Lower bound
M_{23}	8064 8064	Ly	1296826874 35049375
Ru	1252799 2880	Th	976841774 133997
$O'N$	2857238 3072	Fi_{23}	31670 911
Co_2	1024649 270	Co_1	46621574 3671
Fi_{22}	186 13	J_4	47766599363 1647124116
HN	74064374 8593	Fi'_{24}	7819305288794 269631216855
$\mathbb{M}$	5791748068511982636944259374 3385007637938037777290624		

In our Main Theorem we present upper bounds on the exact uniform spreads for thirteen sporadic simple groups. These results are presented in Table 2.

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TABLE 2. Upper bounds on the exact uniform spreads of twelve sporadic simple groups.

G	Upper bound
M_{12}	7
J_1	132
M_{22}	25
J_2	11
HS	32
J_3	458
M_{24}	32
McL	277
He	653
Suz	373
Co_3	1539
$\mathbb{B}$	3843461129719173164826623999999

Main Theorem. *If G is one of the groups M_{12}, J_1, M_{22}, J_2, HS, J_3, M_{24}, McL, He, Suz, Co_3 or $\mathbb{B}$, then the exact uniform spread $u(G)$ is bounded above by the numbers given in Table 2 and $u(M_{11}) = 3$.*

3. Three Algorithms

Let G be a finite group and M_i for $i \in I$ be some maximal subgroups of G, such that $(\cup_{i\in I} M_i) \cap nY \neq \emptyset$ which nY goes through all classes of G. According to Lemma 2.4 if we can determine $support_{mX}(M_i)$ for $i \in I$ and each $mX \in cl_p(G)$ that $M_i \in supp(mX)$, then

$$u(G) \leq max\{min\{support_{mX}(M_i) | mX \in cl_p(G)\, and\, M_i \in supp(mX)\} | i \in I\} - 1.$$

In this section for a maximal subgroup M and $mX \in cl_p(G)$ which $m \in supp(mX)$ we present three algorithms to compute upper bounds for $support_{mX}(M)$. The problem of computing upper bounds of $support_{mX}(M)$ can be interpreted in terms of intersection graphs.

Let $mX = \{x_1, x_2, \ldots, x_k\}$. Each $x \in mX$ is inside h conjugates of M. We denote by M_{x_i} the set of all conjugates of M containing x_i for $i = 1, 2, \ldots, k$. Each two M_{x_i} and M_{x_j} for $1 \leq i, j \leq k$, can be disjoint or have intersection of different size. It is clear that $\cup_{i=1}^{k} M_{x_i} = M^G$. Now we consider graph $\Gamma = (V, E)$ such that V, the set of vertices, is M_{x_i} for $i = 1, 2, \ldots, k$. Two vertices M_{x_i} and M_{x_j} are adjacent if $M_{x_i} \cap M_{x_j} \neq \emptyset$. The graph Γ is called the intersection graph of G defined by M and mX. Note that it is vertex transitive and is therefore regular. If $support_{mX}(M) = t$ then there exists a set I of size t such that $\cup_{i\in I} M_{x_i} = M^G$. Therefore determining $support_{mX}(M)$ is equivalent to

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finding a set S, consisting of vertices of Γ, of minimum size such that union of its members contains all conjugates of M.

In the following algorithms, let G be a finite group and mX and cY be two conjugacy classes of elements of G such that $mX \in cl_p(G)$. Also suppose M is a maximal subgroup of G that has intersection with mX and $n = |M^G|$. With Algorithm-1 we can obtain an upper bound for $support_{mX}(M)$.

Algorithm-1:

Input: group G, maximal subgroup M and conjugacy class $mX \in cl_p(G)$

Output: an upper bound for $support_{mX}(M)$

Step 1. Construct graph $\Gamma = (V, E)$ from G, M and mX. Set $min = n$.

Repeat for each vertex M_x of V:

Step 2. Find a maximum independent set S of Γ containing M_x. Then set $L = |S|$ and $N = \cup_{M_y \in S} M_y$. If $|N| = n$ then go to Step 4.

Step 3. Select a vertex M_z such that $|N \cap M_z|$ is of minimum size. Then set $N' := N \cup M_z$ and add one to L. If $|N'| = n$ then go to Step 4, else replace N with N' and go to Step 3.

Step 4. If $L < min$ then $min = L$, else continue.

End repeat.

Step 5. Return min.

If the maximal subgroup M contains elements of a class cY then min, the output of Algorithm-1, is an upper bound of $u_{mX,cY}(G)$. If each element of the class cY is in more than one conjugate of M then by a few changes in Algorithm-1 we can obtain better upper bound for $u_{mX,cY}(G)$.

Algorithm-2:

Input: group G, maximal subgroup M, conjugacy classes $mX \in cl_p(G)$ and cY

Output: an upper bound for $u_{mX,cY}(G)$

Step 1. Construct graph $\Gamma = (V, E)$ from G, M and mX. Set $min = n$.

Repeat for each vertex M_x of V:

Step 2. Find a maximum independent set S of Γ containing M_x. Then set $L = |S|$ and $N = \cup_{M_y \in S} M_y$. Also set T to be the union of elements of the class cY in conjugates of M contained in the set N. If $|N| = n$ or $|T| = |cY|$ then go to Step 4.

Step 3. Select a vertex M_z such that $|N \cap M_z|$ is of minimum size. Then set $N' := N \cup M_z$ and add one to L. Also add to T, elements of the class cY in conjugates of M that are in M_z. If $|N'| = n$ or $|T| = |cY|$ then go to Step 4, else replace N with N' and go to Step 3.

Step 4. If $L < min$ then $min = L$, else continue.

End repeat.

Step 5. Return min.

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In the following algorithm let M_1 and M_2 be two maximal subgroups from different conjugacy classes such that have intersection with mX and cY and $|M_1^G| = n$. In Algorithm-3 by considering two conjugacy classes of maximal subgroups we can obtain better upper bound for $u_{mX,cY}(G)$.

Algorithm-3:

Input: group G, maximal subgroups M_1 and M_2, conjugacy classes $mX \in cl_p(G)$ and cY

Output: an upper bound for $u_{mX,cY}(G)$

Step 1. Construct graph $\Gamma = (V, E)$ from G, M_1 and mX. Set $min = n$.

Repeat for each vertex ${M_1}_x$ of V:

Step 2. Find a maximum independent set S of Γ containing ${M_1}_x$. Then set $L = |S|$ and $N = \cup_{{M_1}_y \in S} {M_1}_y$. Also set T to be the union of elements of the class cY in conjugates of M_1 contained in N and in conjugates of M_2 that contain element x. If $|N| = n$ or $|T| = |cY|$ then go to Step 4.

Step 3. Select a vertex ${M_1}_z$ such that $|N \cap {M_1}_z|$ is of minimum size. Then set $N' := N \cup {M_1}_z$ and add one to L. Also add to T, elements of the class cY in conjugates of M_1 that are in ${M_1}_z$ and in conjugates of M_2 that have element z. If $|N'| = n$ or $|T| = |cY|$ then go to Step 4, else replace N with N' and go to Step 3.

Step 4. If $L < min$ then $min = L$, else continue.

End repeat.

Step 5. Return min.

We can extend Algorithm-3 for any number of maximal subgroups from different conjugacy classes that intersect both mX and cY. Since Algorithm-2 and Algorithm-3 use large amounts of memory, we often use Algorithm-1 to find upper bounds in the proof of our Main Theorem. Therefore we give an implementation of Algorithm-1 in Appendix A.

4. Proof Of The Main Theorem

In this section we prove our Main Theorem. Therefore we consider the groups presented in Table 2, individually. For some sporadic simple groups for which we use Algorithm-3, the obtained bounds are given in some tables. In these tables upper bound for one of $u_{2A,nY}(G)$ or $u_{2B,nY}(G)$ or $u_{3A,nY}(G)$ for each conjugacy class nY, which is minimum, was computed and are presented. We use "-" in tables when corresponding $u_{mX,nY}(G)$ was not computed. In keeping with Atlas notation we will write $5AB$ to indicate the conjugacy classes $5A$ and $5B$ and similarly for other classes.

Our method does not work for sporadic simple groups presented in Table 1. Because for using algorithms of Section 3, we must construct some graphs. For a finite group G size of these graphs are depended on size of conjugacy classes of elements and conjugacy classes of maximal subgroups of G. Constructing these graphs for sporadic simple groups presented in Table 1 are not computationally possible yet.

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4.1. **Mathieu sporadic group M_{11}.** Bradley and Holmes in [4] proved that $s(M_{11})$ =3. They really just proved that any set of three elements has a mate from class $11A$, in other words $3 \leq u(M_{11})$ which combined with the fact that $u(M_{11}) \leq s(M_{11})$ tells us that $u(M_{11}) = 3$. Thus we have the following proposition.

Proposition 4.1. *$u(M_{11})$=3.*

4.2. **Mathieu sporadic group M_{12}.** Group M_{12} has eleven conjugacy classes of maximal subgroups and fifteen conjugacy classes of elements. Group M_{12} has two conjugacy classes of maximal subgroups isomorphic to M_{11}. These two conjugacy classes have intersection with all conjugacy classes of elements except $2A$, $3B$, $6A$ and $10A$. With Algorithm-1 we found $support_{2B}(M_{11})\leq 3$ for two conjugacy classes of M_{11}. For the remaining conjugacy classes $2A$, $3B$, $6A$ and $10A$ we have:
- With Algorithm-3 and conjugacy classes of two maximal subgroups M_{10}:2 we found $u_{2B,2A}(M_{12})<3$.
- With Algorithm-3 and conjugacy classes of maximal subgroups $L_2(11)$, $2\times S_5$ and $4^2{:}D_{12}$ we found $u_{2A,3B}(M_{12})<3$.
- With Algorithm-3 and conjugacy classes of maximal subgroups $L_2(11)$, $2\times S_5$, $4^2{:}D_{12}$ and $A_4\times S_3$ we found $u_{2A,6A}(M_{12})<8$.
- With Algorithm-3 and conjugacy classes of maximal subgroups M_{10}:2, M_{10}:2 and $2\times S_5$ we found $u_{2B,10A}(M_{12})<8$.

The results are presented in Table 3.

Table 3. An upper bound for $u(M_{12})$.

nY	$2AB, 3A, 4AB, 5A, 6B, 8AB, 11AB$	$3B$	$6A$	$10A$
$u_{2B,nY}(M_{12}) <$	3	-	-	8
$u_{2A,nY}(M_{12}) <$	-	3	8	-

For the sake of completeness, we give below some sets of elements that satisfy the conditions of Table 3. Consider the permutation representation of M_{12} on 12 points with standard generators from [1],

$$a = (1,4)(3,10)(5,11)(6,12)\,,\; b = (1,8,9)(2,3,4)(5,12,11)(6,10,7).$$

In Table 4 we give three elements from conjugacy class $2B$ that do not have a mate from conjugacy classes $2AB$, $3AB$, $4AB$, $5A$, $6B$, $8AB$ and $11AB$.

Table 4. Three elements of the class $2B$.

1	$(1,2)(3,7)(5,8)(6,12)$
2	$(1,2)(3,7)(4,10)(9,11)$
3	$(4,10)(5,8)(6,12)(9,11)$

In Table 5 we give eight elements from conjugacy class $2B$ that do not have a mate from conjugacy class $10A$.

TABLE 5. Eight elements of the class $2B$.

1	$(1,3)(2,7)(5,8)(9,11)$
2	$(1,6)(4,11)(5,12)(7,10)$
3	$(2,10)(3,6)(4,9)(8,12)$
4	$(2,4)(3,12)(5,6)(10,11)$
5	$(1,4)(3,11)(5,10)(7,8)$
6	$(1,9)(2,5)(3,4)(8,10)$
7	$(1,10)(2,6)(4,8)(11,12)$
8	$(1,12)(2,11)(3,5)(4,7)$

In Table 6 we give eight elements from conjugacy class $2A$ that do not have a mate from conjugacy class $6A$.

TABLE 6. Eight elements of the class $2A$.

1	$(1,2)(3,12)(4,6)(5,10)(7,8)(9,11)$
2	$(1,2)(3,7)(4,8)(5,10)(6,11)(9,12)$
3	$(1,2)(3,4)(5,10)(6,12)(7,9)(8,11)$
4	$(1,2)(3,11)(4,9)(5,10)(6,7)(8,12)$
5	$(1,2)(3,12)(4,10)(5,7)(6,11)(8,9)$
6	$(1,2)(3,7)(4,6)(5,11)(8,9)(10,12)$
7	$(1,2)(3,11)(4,5)(6,12)(7,10)(8,9)$
8	$(1,2)(3,4)(5,12)(6,7)(8,9)(10,11)$

Proposition 4.2. *$u(M_{12}) \leq 7$.*

Proof. Applying Lemma 2.3 to the information in Table 3 implies the result. □

The proofs of all subsequent propositions are similar to the above, so we omit them.

4.3. **Janko sporadic group J_1.** We know that J_1 has seven conjugacy classes of maximal subgroups and fifteen conjugacy classes of elements. Conjugacy classes of maximal subgroups $L_2(11)$, $2{\times}A_5$, 19:6 and $D_6{\times}D_{10}$ have intersection with all conjugacy classes of elements except $7A$. Conjugacy classes of maximal subgroups 2^3:7:3 and 7:6 have intersection with $7A$. We use Algorithm-1 to compute an upper bound for $support_{2A}(M)$ for $M{\in}\{L_2(11),\ 2{\times}A_5,\ 19{:}6,\ D_6{\times}D_{10}\}$ and Algorithm-3 to compute an upper bound for $u_{3A,7A}(J_1)$ with maximal subgroups 2^3:7:3 and 7:6. Computed bounds are

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$support_{2A}(L_2(11))\leq 33$, $support_{2A}(2\times A_5)\leq 91$, $support_{2A}(19{:}6)\leq 127$, $support_{2A}(D_6\times D_{10})\leq 133$ and $u_{3A,7A}(J_1)<125$.
The results are presented in Table 7.

TABLE 7. An upper bound for $u(J_1)$.

nY	$2A, 3A, 5AB, 6A, 11A$	$7A$	$10AB$	$15AB$	$19ABC$
$u_{2A,nY}(J1) <$	33	-	91	133	127
$u_{3A,nY}(J1) <$	-	125	-	-	-

Proposition 4.3. *$u(J_1) \leq 132$.*

4.4. **Mathieu sporadic group M_{22}.** Group M_{22} has eight conjugacy classes of maximal subgroups and twelve conjugacy classes of elements. Conjugacy classes of maximal subgroups $L_3(4)$, $2^4\times A_6$ and $L_2(11)$ have intersection with all conjugacy classes of elements. With Algorithm-1 we found $support_{2A}(L_3(4))\leq 5$, $support_{2A}(2^4\times A_6)\leq 7$ and $support_{2A}(L_2(11))\leq 26$, By Lemma 2.4, an upper bound of $u(M_{22})$ is 25.

Proposition 4.4. *$u(M_{22}) \leq 25$.*

4.5. **Hall-Janko sporadic group J_2.** We know that J_2 has nine conjugacy classes of maximal subgroups and twenty one conjugacy classes of elements. Conjugacy classes of maximal subgroups $U_3(3)$ and $3.PGL_2(9)$ have intersection with all conjugacy classes of elements except $5CD$, $6B$ and $10CD$. With Algorithm-1 we found $support_{2A}(U_3(3))\leq 5$ and $support_{2A}(3.PGL_2(9))\leq 9$. For the remaining conjugacy classes $5CD$, $6B$ and $10CD$ we have:
- With Algorithm-3 and conjugacy classes of maximal subgroups $2^{1+4}{:}A_5$ and $A_5\times D_{10}$ we found $u_{3A,5D}(J_2)<4$ and $u_{3A,5C}(J_2)<4$.
- With Algorithm-3 and conjugacy classes of maximal subgroups $2^{2+4}{:}(3\times S_3)$, $A_4\times A_5$, $L_3(2){:}2$ and $5^2{:}D_{12}$ we found $u_{2A,6B}(J_2)<12$.
- With Algorithm-3 and conjugacy classes of maximal subgroups $2^{1+4}{:}A_5$ and $A_5\times D_{10}$ we found $u_{3A,10D}(J_2)<12$ and $u_{3A,10C}(J_2)<12$.

The results are presented in Table 8

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Table 8. An upper bound for $u(J_2)$.

nY	$2A, 3AB, 4A, 6A, 7A, 8A, 12A$	$2B, 5AB, 10AB, 15AB$	$5CD$	$6B$	$10CD$
$u_{2A,nY}(J_2) <$	5	9	-	12	-
$u_{3A,nY}(J_2) <$	-	-	4	-	12

Proposition 4.5. *$u(J_2) \leq$11.*

Bradley and Holmes in [4] proved that $s(J_2)\leq 24$. Since the upper bounds of $s(J_2)$ and $u(J_2)$ differed by a significant margin we conjecture that $u(J_2)<s(J_2)$. Therefore we give below some sets of elements that satisfy the conditions of Table 8. Consider the permutation representation of J_2 on 100 points with standard generators a and b in [1]. Define $x = ab$ and $y = ab^{-1}$.

In Table 9 we give five elements from conjugacy class $2A$ that do not have a mate from conjugacy classes $2A$, $3AB$, $4A$, $6A$, $7A$, $8A$ and $12A$.

Table 9. Five elements of the class $2A$.

1	$by(yx)^4y^2((xy)^2yx)^2$
2	$by^2x^2((yx)^2y)^2x(xy)^2(yx^2)^2a$
3	$x(xy)^3(xy^2x)^4yx^2a$
4	$b^{-1}x(yxy)^2x(xy)^4x(xy)^2yx^2yx$
5	$x^{-3}b((yx)^2y)^2yx^2ya$

In Table 10 we give nine elements from conjugacy class $2A$ that do not have a mate from conjugacy classes $2B$, $5AB$, $10AB$ and $15AB$.

Table 10. Nine elements of the class $2A$.

1	$((xy)^3x^2y)^2yx(xy)^2a$
2	$((ba)^2x^{-1})^2(b^{-1}xa)^2b((yx)^2y)^2x^2y^2xa$
3	$b^{-1}xy^2x(xy)^4x^2yx^2$
4	$b(x(xy)^2)^2x^2yx^2$
5	$b(xy)^2x(yx^2y)^4xyx^2$
6	$(yx^2y)^2xy((xy)^2x)^2a$
7	$b^{-1}(yx)^2(yxy)^2((yx)^2x)^2y(yx^2)^2a$
8	$by^2xy(y(xyx)^2yx)^2y^2x$
9	$b(xy)^2y((xy)^2yx)^2(xy^2)^2xyx^2$

In Table 11 we give twelve elements from conjugacy class $2A$ that do not have a mate from conjugacy classes $6B$. In Table 12 we give four elements from conjugacy class $3A$ that do not have

Table 11. Twelve elements of the class $2A$.

1	$(b^{-1}((xy)^2x)^2ya)^2$
2	$(bya)^2(ba)^2b^{-1}((yx)^2y^2x)^2xy$
3	$(xy(yx)^2)^2y(yx)^4y^2xa$
4	$byxy(xy^2x)^4(yx)^2ya$
5	$b^{-1}(yx)^4y^2xy(yx^2y^2x)^2yxa$
6	$b^{-1}(yx)^2xy((yx)^2y^2x)^2x^2$
7	$byx(xy^2)^2x(yx(xy)^2)^2x(xy)^2a$
8	$(ba)^2x^{-2}b(xy^2xy)^2(xy)^3$
9	$((bya)^2ba)^3$
10	$b^{-1}x(xy)^5x^3$
11	$bx(xy^2xy)^2xy^2x^2y$
12	$bx(yx^2y)^2xy((xy)^2x)^2y^2$

Table 12. Four elements of the class $3A$.

1	$(xy)^2yx(xy)^3x^2y^2a$
2	$y(yx)^4xy^2x^2yxa$
3	$(yxy)^2(xy)^2(yx^2)^2a$
4	$b^{-1}(yx)^2((yx)^2y)^2yxy^2$

a mate from conjugacy classes $5CD$.

In Table 13 we give twelve elements from conjugacy class $3A$ that do not have a mate from conjugacy classes $10CD$.

Table 13. Twelve elements of the class $3A$.

1	$b^{-1}(yx)^3y(yxy^2x)^2x^2$
2	$y(yx)^4xy^2x^2yxa$
3	$yxy(xy^2x)^3yxya$
4	$(y(yx)^2)^2$
5	$((bya)^2x^{-1})^2$
6	$((ba)^2x^{-1})^2by(yx)^3xy^2$
7	$x((yx)^2y)^2yx^2y^2xya$
8	$y(xyx)^2y^2xy(xyx)^2a$
9	$x^{-2}(ba)^2b^{-1}x(yxy)^2x^2yx^2$
10	$y((xy)^2x)^2xy^2x^2yxa$
11	$x(yx^2y)^2(xy)^3ya$
12	$(xyx)^2(yxy)^2x(xy)^2a$

4.6. **Higman-Sims sporadic group HS.** We know that HS has twelve conjugacy classes of maximal subgroups and twenty four conjugacy classes of elements. Conjugacy classes of maximal subgroups M_{22}, $U_3(5){:}2$ and S_8 have intersection with all conjugacy classes of elements. We found $support_{2A}(S_8)\leq 33$, $support_{2A}(U_3(5){:}2)\leq 11$ and $support_{2A}(M_{22})\leq 5$ with Algorithm-1. By Lemma 2.4, an upper bound of $u(HS)$ is 32. So, we have the following proposition.

Proposition 4.6. *$u(HS)\leq 32$.*

4.7. **Janko sporadic group J_3.** Group J_3 has nine conjugacy classes of maximal subgroups and twenty one conjugacy classes of elements. Conjugacy classes of maximal subgroups $L_2(16){:}2$, $L_2(19)$, and $(3\times A_6){:}2$ have intersection with all conjugacy classes of elements. With Algorithm-1 we found $support_{2A}(L_2(16){:}2)\leq 83$, $support_{2A}(L_2(19))\leq 375$ and $support_{2A}(3\times A_6){:}2)\leq 459$. By Lemma 2.4, an upper bound of $u(J_3)$ is 458. Therefore the following proposition is concluded.

Proposition 4.7. *$u(J_3)\leq 458$.*

4.8. **Mathieu sporadic group M_{24}.** Group M_{24} has nine conjugacy classes of maximal subgroups and twenty six conjugacy classes of elements. Conjugacy classes of maximal subgroups M_{23}, $M_{22}{:}2$, $2^4{:}A_8$, $M_{12}{:}2$ and $L_3(4){:}S_3$ have intersection with all conjugacy classes of elements. By Algorithm-1 $support_{2A}(M_{23})\leq 3$, $support_{2A}(M_{22}{:}2)\leq 13$, $support_{2A}(2^4{:}A_8)\leq 16$, $support_{2A}(M_{12}:2)\leq 27$ and $support_{2A}(L_3(4):S_3)\leq 33$ were calculated. By Lemma 2.4, an upper bound of $u(M_{24})$ is 32. So, we have the following proposition.

Proposition 4.8. *$u(M_{24})\leq 32$.*

4.9. **McLaughlin sporadic group McL.** We know that McL has twelve conjugacy classes of maximal subgroups and twenty four conjugacy classes of elements. Conjugacy classes of maximal subgroups $U_4(3)$, M_{22}, $U_3(5)$, $L_3(4){:}2$ and $2.A_8$ have intersection with all conjugacy classes of elements. With Algorithm-1 we found $support_{2A}(U_4(3))\leq 13$, $support_{2A}(M_{22})\leq 46$, $support_{2A}(U_3(5))\leq 107$, $support_{2A}(L_3(4):2)\leq 160$ and $support_{2A}(2.A_8)\leq 278$. By Lemma 2.4, an upper bound of $u(McL)$ is 277. Therefore the following proposition is concluded.

Proposition 4.9. *$u(McL)\leq 277$.*

4.10. **Held sporadic group He.** We know that He has eleven conjugacy classes of maximal subgroups and thirty three conjugacy classes of elements. Conjugacy classes of maximal subgroups $S_4(4){:}2$, $2^2.L_3(4).S_3$, $2^6{:}3.S_6$, $2^{1+6}.L_3(2)$ and $3.S_7$ have intersection with all conjugacy classes of elements. With Algorithm-1 we found $support_{2A}(S_4(4){:}2)\leq 23$, $support_{2A}(2^2.L_3(4).S_3)\leq 46$, $support_{2A}(2^6{:}3.S_6)\leq 147$, $support_{2A}(2^{1+6}.L_3(2))\leq 654$ and $support_{2A}(3.S_7)\leq 642$. By Lemma 2.4, an upper bound of $u(He)$ is 653. Therefore the following proposition is concluded.

Proposition 4.10. *$u(He)\leq 653$.*

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4.11. **Suzuki sporadic group** Suz**.** Group Suz has seventeen conjugacy classes of maximal subgroups and forty three conjugacy classes of elements. Conjugacy classes of maximal subgroups $G_2(4)$, $3.U_4(3){:}2$, $U_5(2)$, $2^{1+6}.U_4(2)$ and $J_2{:}2$ have intersection with all conjugacy classes of elements of Suz. By Algorithm-1 we found $support_{3A}(G_2(4)) \leq 11$, $support_{2A}(3.U_4(3){:}2) \leq 150$, $support_{2A}(U_5(2)) \leq 69$, $support_{3A}(2^{1+6}.U_4(2)) \leq 204$ and $support_{3A}(J_2{:}2) \leq 374$. By Lemma 2.4, an upper bound of $u(Suz)$ is 373. So, we have the following proposition.

Proposition 4.11. *$u(Suz) \leq 373$.*

4.12. **Conway sporadic group** Co_3**.** We know that Co_3 has fourteen conjugacy classes of maximal subgroups and forty two conjugacy classes of elements. Conjugacy classes of maximal subgroups $McL{:}2$, HS, M_{23}, $3^5{:}(2\times M_{11})$ and $U_3(5){:}S_3$ have intersection with all conjugacy classes of elements of Co_3. By Algorithm-1 we found $support_{2A}(McL{:}2) \leq 12$, $support_{2A}(HS) \leq 79$, $support_{2A}(M_{23}) \leq 160$, $support_{2A}(3^5{:}(2\times M_{11})) \leq 456$ and $support_{2A}(U_3(5){:}S_3) \leq 1540$. By Lemma 2.4, an upper bound of $u(Co_3)$ is 1539. Therefore the following proposition is concluded.

Proposition 4.12. *$u(Co_3) \leq 1539$.*

4.13. **Baby Monster sporadic group** $\mathbb{B}$**.** Group $\mathbb{B}$ has thirty conjugacy classes of maximal subgroups [20] and one hundred eighty four conjugacy classes of elements. Let M is a maximal subgroup of $\mathbb{B}$ and $nX \in cl_p(\mathbb{B})$. It is clear that if $M \cap nX \neq \emptyset$ then $support_{nX}(M) \leq |M^{\mathbb{B}}|$. Therefore by Lemma 2.4, $u(\mathbb{B})$ is less than the index of maximal subgroup of minimum order of $\mathbb{B}$. Maximal subgroup 47:23 of $\mathbb{B}$ has minimum order among all maximal subgroups. This subgroup has index 3843461129719173164826624000000. Then the following proposition is concluded.

Proposition 4.13. *$u(\mathbb{B}) \leq 3843461129719173164826623999999$.*

We are now ready to prove our Main Theorem.

Main Theorem. *If G is one of the groups M_{12}, J_1, M_{22}, J_2, HS, J_3, M_{24}, McL, He, Suz, Co_3 or $\mathbb{B}$, then the exact uniform spread $u(G)$ is bounded above by the numbers given in Table 2 and $u(M_{11}) = 3$.*

Proof. The proof follows from Propositions 4.1 to 4.13. □

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Appendix A

In the following we present a Magma [8] implementation of Algorithm-1. This code computes an upper bound of $support_{2A}(3.PGL_2(9))$ in the group J_2. By choosing proper values for "f", "m" and "size_mX" and loading groups presented in Table 2 one can check the results of Section 4. For groups J_3, McL, He, Suz and Co_3, Magma can not compute maximal subgroups. In these cases we loaded the group and its maximal subgroup M from [1] and omitted the lines "mG:=MaximalSubgroups(G)" and "M:=mG[f]` subgroup" from code.

In the following code "f" is the number corresponding to subgroup M in maximal subgroups of G in Magma ordering. "m" is order of an element of conjugacy class mX. "size_mX" is size of conjugacy class mX.

We used Magma version 2.10 [8].

```
f:=6;
m:=2;
size_mX:=315;
load j2;
mG:=MaximalSubgroups(G);
M:=mG[f]`subgroup;
G1:=CosetImage(G,M);
cG1:=ConjugacyClasses(G1);
n:=Index(G,M);
for i in [1..#cG1] do
  if cG1[i,1] eq m and cG1[i,2] eq size_mX then
   gX:=cG1[i,3];
   break;
  end if;
end for;
v:=Fix(gX)^G1;
v:=Set(v);
v:=SetToIndexedSet(v);
min:=n;
bound:=#Fix(gX);
for k in [1..#v] do
  best:=bound;
  t:=0;
  L:=1;
  N:={};
  range:=[1..#v];
  Exclude(~range,k);
  N:=N join v[k];
  while t ne n do
```

```
    for i in range do
      r:=#(v[i] meet N);
      if r le best then
        best :=r;
        temp:=i;
      end if;
    end for;
    N:=N join v[temp];
    L:=L+1;
    Exclude(~range,temp);
    if L gt min then
      break;
    end if;
    best:=bound;
    t:=#N;
  end while;
  if L lt min then
    min:=L;
    print ''Temporary upper bound is ='', min;
  end if;
end for;
print ''Upper bound is='', min;
```

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