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Proof of Some Properties of Transfer Using Noncommutative Determinants *

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Abstract

A transfer is a group homomorphism from a group to an abelian quotient group of a subgroup of finite index. In this paper, we give a natural interpretation of the transfers in group theory in terms of noncommutative determinants.

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1 Introduction

A transfer is defined by Issai Schur (see [7]) as a group homomorphism from a group to an abelian quotient group of a subgroup of the group. In finite group theory, transfers play an important role in transfer theorems. Transfer theorems include, for example, Alperin's theorem (see [1], Theorem 4.2), Burnside's theorem (see [6], Hauptsatz 4.2.6), and Hall-Wielandt's theorem (see [5], Theorem 14.4.2).

On the other hand, Eduard Study defined the determinant of a quaternionic matrix (see [3]). The Study determinant uses a regular representation from $\mathrm{Mat}(n, \mathbb{H})$ to $\mathrm{Mat}(2n, \mathbb{C})$, where $\mathbb{H}$ is the quaternions. Similarly, we define a noncommutative determinant. It is similar to the Dieudonné determinant (see [2]).

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Tôru Umeda suggested that a transfer can be derived as a noncommutative determinant (see [8], Footnote 7). In this paper, we develop his ideas in order to explain the properties of the transfers by using noncommutative determinants. As a result, we give a natural interpretation of the transfers in group theory in terms of noncommutative determinants.

Let G be a group, H a subgroup of G of finite index, K a normal subgroup of H, and the quotient group H/K of K in H an abelian group. The transfer of G into H/K is a group homomorphism

$$V_{G \to H/K} : G \to H/K.$$

The definition of the transfer $V_{G \to H/K}$ uses the left (or right) coset representatives of H in G. We can show that a transfer has the following properties.

(1) A transfer is a group homomorphism from G to H/K (see [4], Theorem 3.1).

(2) A transfer is invariant under a change of coset representatives (see [4], Proposition 3.1).

(3) A transfer by left coset representatives equals a transfer by right coset representatives (see [4], Section 3.1).

Let R be a commutative ring with unity and RG the group algebra of G over R whose elements are all possible finite sums of the form

$$\sum_{g \in G} x_g g, x_g \in R.$$

The noncommutative determinant uses a left (or right) regular representation from RG to $\mathrm{Mat}(m, RH)$, where m is the index of H in G. Our main result is the following.

Theorem 1.1 *We can regard the transfer $V_{G \to H/K}$ as the noncommutative determinant* Det. *That is, we have*

$$\mathrm{Det}(g) = \mathrm{sgn}(g) V_{G \to H/K}(g) \quad (g \in G)$$

where the map

$$\mathrm{sgn} : G \to \{-1, 1\}$$

is a group homomorphism and 1 is the unit element of R. In addition, we can show that the above properties of the transfer (1), (2), and (3) by the following properties of the noncommutative determinant Det.

(1′) *The determinant is a multiplicative map from RG to $R(H/K)$.*

(2′) *The determinant is invariant under a change of a regular representation.*

(3′) *The determinant of any left regular representation equals the determinant of any right regular representation.*

2 Definition of the transfer

Here, we define the left and right transfer of G into H/K.

Let

$$G = t_1 H \cup t_2 H \cup \ldots \cup t_m H.$$

That is, we take a complete set $\{t_1, t_2, \ldots, t_m\}$ of left coset representatives of H in G. We define $\overline{g} = t_i$ for all $g \in t_i H$. The definition of the left transfer is the following.

Definition 2.1 (Left transfer; see Definition 3.3 of [4]) *We define the map*

$$V_{G \to H/K} : G \to H/K$$

by

$$V_{G \to H/K}(g) = \prod_{i=1}^{m} \left\{ \left(\overline{g t_i}\right)^{-1} g t_i \right\} K.$$

We call the map $V_{G \to H/K}$ *the left transfer of* G *into* H/K.

Next, we define the right transfer of G into H/K.

Let

$$G = Hu_1 \cup Hu_2 \cup \ldots \cup Hu_m.$$

That is, we take a complete set

$$\{u_1, u_2, \ldots, u_m\}$$

of right coset representatives of H in G. We define $\widetilde{g} = u_i$ for all $g \in Hu_i$. The definition of the right transfer is the following.

Definition 2.2 (Right transfer; see Definition 3.3 of [4]) *We define the map*

$$\widetilde{V}_{G \to H/K} : G \to H/K$$

by

$$\widetilde{V}_{G \to H/K}(g) = \prod_{i=1}^{m} \left\{ u_i g \left(\widetilde{u_i g}\right)^{-1} \right\} K.$$

We call the map $\widetilde{V}_{G \to H/K}$ *the right transfer of* G *into* H/K.

The definitions of the left and right transfers use the coset representatives of H in G. But, we can show that the left and right transfers are invariant under a change of coset representatives. Furthermore, we can show that a transfer is a group homomorphism from G to H/K and a transfer by left coset representatives equals a transfer by right coset representatives.

3 Definition of the noncommutative determinant

Here, we define the noncommutative determinant.

First, we define the left regular representation of RG. We take a complete set
$$T = \{t_1, t_2, \dots, t_m\}$$
of left coset representatives of H in G. Then, for all $\alpha \in RG$, there exists a unique $L_T(\alpha) \in \mathrm{Mat}(m, RH)$ such that

$$\alpha(t_1 \quad t_2 \quad \dots \quad t_m) = (t_1 \quad t_2 \quad \dots \quad t_m) L_T(\alpha),$$

where we regard $\alpha(t_1 \dots t_m)$ as scalar multiplication $(\alpha t_1 \dots \alpha t_m)$. The R-algebra homomorphism

$$L_T : RG \ni \alpha \mapsto L_T(\alpha) \in \mathrm{Mat}(m, RH)$$

is called the left regular representation with respect to T.

Let
$$T' = \{t'_1, t'_2, \dots, t'_m\}$$
be an another complete set of left coset representatives of H in G. Then, there exists $P \in \mathrm{Mat}(m, RH)$ such that $L_T = P^{-1} L_{T'} P$.

Example 3.1 *Let* $G = \mathbb{Z}/2\mathbb{Z} = \{\overline{0}, \overline{1}\}$, $H = \{\overline{0}\}$, *and* $\alpha = x\overline{0} + y\overline{1} \in RG$. *Then, we have*

$$\alpha(\overline{0} \quad \overline{1}) = (\overline{0} \quad \overline{1}) \begin{bmatrix} x\overline{0} & y\overline{0} \\ y\overline{0} & x\overline{0} \end{bmatrix}.$$

To obtain an expression for L_T, we define the indicator function $\dot{\chi}$ by

$$\dot{\chi}(g) = \begin{cases} 1 & g \in H, \\ 0 & g \notin H \end{cases}$$

for all $g \in G$.

Lemma 3.2 *Let*
$$\alpha = \sum_{g \in G} x_g g.$$
Then, we have
$$L_T(\alpha)_{ij} = \sum_{g \in G} \dot{\chi}\left(t_i^{-1} g t_j\right) x_g t_i^{-1} g t_j.$$

Proof — We have

$$\begin{aligned}
&\begin{pmatrix} t_1 & t_2 & \ldots & t_m \end{pmatrix} \left(\sum_{g \in G} \dot{\chi}\left(t_i^{-1} g t_j\right) x_g t_i^{-1} g t_j \right)_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant m} \\
&= \begin{pmatrix} \sum_{i=1}^{m} \sum_{g \in G} \dot{\chi}(t_i^{-1} g t_1) x_g g t_1 & \ldots & \sum_{i=1}^{m} \sum_{g \in G} \dot{\chi}(t_i^{-1} g t_m) x_g g t_m \end{pmatrix} \\
&= \left(\sum_{g \in G} x_g g \right) \begin{pmatrix} t_1 & t_2 & \ldots & t_m \end{pmatrix}.
\end{aligned}$$

This completes the proof. □

From Lemma 3.2, we have

$$\begin{aligned}
L_T(g)_{ij} &= \dot{\chi}\left(t_i^{-1} g t_j\right) t_i^{-1} g t_j \\
&= \begin{cases} t_i^{-1} g t_j & t_i^{-1} g t_j \in H, \\ 0 & t_i^{-1} g t_j \notin H \end{cases}.
\end{aligned}$$

From $t_i^{-1} g t_j \in H$ if and only if $\overline{g t_j} = t_i$, we have

$$L_T(g)_{ij} = \begin{cases} \left(\overline{g t_j}\right)^{-1} g t_j & t_i^{-1} g t_j \in H, \\ 0 & t_i^{-1} g t_j \notin H. \end{cases}$$

As for the definition of the noncommutative determinant, let

$$\psi : \mathrm{Mat}(m, RH) \to \mathrm{Mat}\left(m, R\left(H/K\right)\right)$$

be an R-linear map such that

$$\psi\left(h E_{ij}\right) = (hK) E_{ij}$$

for all $h \in H$ and $1 \leqslant i, j \leqslant m$, where E_{ij} is the matrix with 1 in the $(i\ j)$ entry and 0 otherwise. Obviously, ψ is an R-algebra homomorphism. The definition of the noncommutative determinant is the following.

Definition 3.3 *We define the map* $\mathrm{Det} : RG \to R\left(H/K\right)$ *by*

$$\mathrm{Det} = \det \circ \psi \circ L_T.$$

Since there is P such that $L_T = P^{-1} L_{T'} P$, we have

$$\mathrm{Det} = \det \circ \psi \circ L_T = \det \circ \psi \circ L_{T'}.$$

Thus, the determinant is invariant under a change of left regular representations, so the determinant Det is well-defined. If K is the commutator subgroup of H, the determinant is similar to the Dieudonné determinant.

Obviously, the map Det is a homomorphism. That is,

$$\mathrm{Det}(\alpha\beta) = \mathrm{Det}(\alpha)\mathrm{Det}(\beta)$$

for all $\alpha, \beta \in RG$. Therefore, we obtain properties (1′) and (2′).

Remark 3.4 *In general, that $\alpha \in RG$ is invertible is not equivalent to that*

$$\mathrm{Det}(\alpha) \in R(H/K)$$

is invertible. For example, let $R = \mathbb{C}$, $\mathbb{Z}/2\mathbb{Z} = \{\overline{0}, \overline{1}\}$ be the group of order two, S_3 be the symmetric group of degree three, $G = \mathbb{Z}/2\mathbb{Z} \times S_3$, $H = S_3$, and $K = [H, H]$ the commutator subgroup of H. Then

$$\alpha = \left(\overline{0}, e\right) + \left(\overline{0}, (123)\right) + \left(\overline{0}, (132)\right)$$

is not invertible, where e is the unit element of H. But, $\mathrm{Det}(\alpha) = 9K$ *is invertible.*

4 Proof of the properties

Here, we prove the transfer properties by using the noncommutative determinant's properties.

For all $g \in G$ and for all $t \in T$, there exists a unique $t_j \in T$ such that

$$t_i^{-1} g t_j \in H.$$

Therefore, there exists $\mathrm{sgn}(g) \in \{-1, 1\}$ such that

$$\mathrm{Det}(g) = \det\left(\psi\left(L_T(g)\right)\right)$$

$$= \mathrm{sgn}(g) \prod_{i=1}^{m} \left\{ \left(\overline{g t_i}\right)^{-1} g t_i \right\} K = \mathrm{sgn}(g) V_{G \to H/K}(g).$$

Thus, we have

$$\mathrm{sgn}(gh) V_{G \to H/K}(gh) = \mathrm{Det}(gh) = \mathrm{Det}(g)\mathrm{Det}(h)$$

$$= \mathrm{sgn}(g)\mathrm{sgn}(h)V_{G\to H/K}(g)V_{G\to H/K}(h).$$

Hence, we obtain

$$\mathrm{sgn}(gh) = \mathrm{sgn}(g)\mathrm{sgn}(h),$$
$$V_{G\to H/K}(gh) = V_{G\to H/K}(g)V_{G\to H/K}(h).$$

Therefore, from property (1′) that Det is a homomorphism, the left transfer

$$V_{G\to H/K}$$

is a group homomorphism (assuming, that is, $R = \mathbb{F}_2$, and we do not consider the signature).

Next, we show that the left transfer is invariant under a change of coset representatives by using property (2′) that the determinant is invariant under a change of regular representations. That is, we show that

$$\prod_{i=1}^{m} \left\{ \left(\overline{gt_i}\right)^{-1} gt_i \right\} K = \prod_{i=1}^{m} \left\{ \left(\overline{\overline{gt_i'}}\right)^{-1} gt_i' \right\} K$$

where we define $\overline{\overline{g}} = t_i'$ for all $g \in t_i'H$.

From property (2′), there exists $\mathrm{sgn}'(g) \in \{-1, 1\}$ such that

$$\prod_{i=1}^{m} \left\{ \left(\overline{gt_i}\right)^{-1} gt_i \right\} K = \mathrm{sgn}(g)\mathrm{Det}(g)$$
$$= \mathrm{sgn}(g)\mathrm{sgn}'(g) \prod_{i=1}^{m} \left\{ \left(\overline{\overline{gt_i'}}\right)^{-1} gt_i' \right\} K.$$

Therefore, we have $\mathrm{sgn}(g)\mathrm{sgn}'(g) = 1$ and

$$\prod_{i=1}^{m} \left\{ \left(\overline{gt_i}\right)^{-1} gt_i \right\} K = \prod_{i=1}^{m} \left\{ \left(\overline{\overline{gt_i'}}\right)^{-1} gt_i' \right\} K.$$

Hence, the left transfer is invariant under a change of coset representatives.

Now let us prove property (3) that

$$V_{G\to H/K} = \widetilde{V}_{G\to H/K}$$

from property (3′) that any left regular representation is equivalent to any right regular representation.

Let

$$G = Hu_1 \cup Hu_2 \cup \ldots \cup Hu_m.$$

That is, we take a complete set

$$U = \{u_1, u_2, \ldots, u_m\}$$

of right coset representatives of H in G. Then, for all $\alpha \in RG$, there exists

$$R_U(\alpha) \in \mathrm{Mat}(m, RH)$$

such that

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} \alpha = R_U(\alpha) \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}.$$

The R-algebra homomorphism

$$R_U : RG \ni \alpha \mapsto R_U(\alpha) \in \mathrm{Mat}(m, RH)$$

is called the right regular representation.

The same as the left transfer, we can show that the following lemma.

Lemma 4.1 *Let* $\alpha = \sum_{g \in G} x_g g$. *Then, we have*

$$R_U(\alpha)_{ij} = \sum_{g \in G} \dot{\chi}(u_i g u_j^{-1}) x_g u_i g u_j^{-1}.$$

Therefore, there exists $\widetilde{\mathrm{sgn}}(g) \in \{-1, 1\}$ such that

$$(\det \circ \psi \circ R_U)(g) = \widetilde{\mathrm{sgn}}(g) \widetilde{V}_{G \to H/K}(g)$$

and $\widetilde{V}_{G \to H/K}$ is invariant under a change of coset representatives of H in G. We have properties (1) and (2).

Since T is a complete set of left coset representatives of H in G, we can take a complete set of

$$T^{-1} = \{t_1^{-1}, t_2^{-1}, \ldots, t_m^{-1}\}$$

of right coset representatives of H in G. Therefore,

$$\begin{aligned} R_{T^{-1}}(\alpha)_{ij} &= \sum_{g \in G} \dot{\chi}\left(t_i^{-1} g (t_j^{-1})^{-1}\right) x_g t_i^{-1} g (t_j^{-1})^{-1} \\ &= L_T(\alpha)_{ij}. \end{aligned}$$

We obtain property (3′). As a result,

$$(\det \circ \psi \circ R_U)(g) = (\det \circ \psi \circ L_T)(g).$$

Therefore, we have

$$\widetilde{\mathrm{sgn}}(g) = \mathrm{sgn}(g),$$
$$\widetilde{V}_{G \to H/K} = V_{G \to H/K}.$$

We obtain property (3). This completes the proof of Theorem 1.1.

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REFERENCES

[1] J.L. Alperin: "Sylow intersections and fusion", *J. Algebra* 6 (1967), 222–241.

[2] E. Artin: "Geometric algebra", *Interscience*, New York (1957).

[3] H. Aslaksen: "Quaternionic determinants", *Math. Intelligencer* 18 (1996), 57–65.

[4] D.C. Mayer: "Artin transfer patterns on descendant trees of finite p-groups", *Adv. Pure Math.* 6 (2016), 66–104.

[5] M. Hall: "The Theory of Groups", *Macmillan*, New York (1959).

[6] B. Huppert: "Endliche Gruppen I", *Springer*, Berlin (1967).

[7] I. Schur: "Neuer Beweis eines Satzes über endliche Gruppen", *Sitzber. Akad. Wiss. Berlin* (1902), 1013–1019.

[8] T. Umeda: "On some variants of induced representations", *Proc. Symp. Repr. Theory* (2012), 7–17.

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