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ANALYSIS OF COMPLEX DECISION PROBLEMS BASED ON CUMULATIVE PROSPECT THEORY

Complex risky decision problems involve sequences of decisions and random events. The choice at a given stage depends on the decisions taken in the previous stages, as well as on the realizations of the random events that occurred earlier. In the analysis of such situations, decision trees are used, and the criterion for choosing the optimal decision is to maximize the expected monetary value. Unfortunately, this approach often does not reflect the actual choices of individual decision makers. In descriptive decision theory, the criterion of maximizing the expected monetary value is replaced by a subjective valuation that takes into account the relative outcomes and their probabilities. This paper presents a proposal to use the principles of cumulative prospect theory to analyse complex decision problems. The concept of a certainty equivalent is used to make it possible to compare risky and non-risky alternatives.

Keywords: *cumulative prospect theory, complex decision problem, decision tree*

1. Introduction

Complex decision problems often involve a sequence of decisions and the possible realizations of random events. In the analysis of such problems, decision trees are very useful and optimal decisions are selected based on the expected monetary value (EMV). This criterion is objective and rational but, as e.g., Birnbaum and Navarrete [1] show, decision makers sometimes do not act rationally and make decisions based on some subjective valuations.

The rules of prospect theory reflect the subjective valuation of outcomes and their probabilities but only in the valuation of simple alternatives. No procedure for the subjective valuation of alternatives in complex decision problems has yet been proposed.

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Thus the aim of this paper was to propose a procedure for the analysis of complex decision problems with risky alternatives based on cumulative prospect theory.

2. Decision trees

Decision trees are used to graphically represent available decisions, random factors and their consequences. They are normally drawn from left to right and consist of decision nodes (squares), chance nodes (circles), end nodes (triangles) and branches (arrows) leading from one node to another. All decision trees begin with a decision node called the root node. The branches emerging from the root node and other decision nodes represent the set of available decisions (alternatives). Only one of these alternatives can be selected. The realization of a decision may lead to a final outcome (end node, also called a leaf node), a random event (chance node) or another decision problem (decision node). The branches emerging from a chance node represent possible realizations of the corresponding random event and their probabilities. As a result of the realization of a random event, three types of situation can take place: the decision maker has to make another decision, another random event occurs or the final outcome is obtained.

The standard way of analysing a decision tree is called backward induction, which is carried out from the end nodes to the root node. It is assumed that the alternatives emerging from a node are disjoint, as are the realizations of a random event. At the chance nodes, the expected value of possible outcomes is assigned and at the decision nodes, the optimal decision is selected on the basis of the assumed decision criterion, which is usually the maximization of expected gains.

3. Cumulative prospect theory

Behavioural decision theory takes into account some psychological aspects of evaluating alternatives. As Kahneman and Tversky [6, 10] observed, decision makers compare possible outcomes y_i of a decision to a reference level y_{ref} and they express outcomes as gains and losses relative to this reference point. In cumulative prospect theory (CPT), a decision D which has an outcome from a discrete distribution can be defined as

$$\begin{aligned}
 D &= (x_1, p_1), \dots, (x_n, p_n), \dots, (x_N, p_N) \\
 x_1 &\leq \dots \leq x_n < 0 \leq x_{n+1} \leq \dots \leq x_N \\
 p_1 + \dots + p_n + p_{n+1} + \dots + p_N &= 1
 \end{aligned} \tag{1}$$

where $x_i = y_i - y_{\text{ref}}$ represents a gain ($x_i > 0$) or a loss ($x_i < 0$) relative to the reference point y_{ref} and p_i is the probability of obtaining the relative outcome x_i . The decision D related to the random event A corresponds to the fragment of a decision tree shown in Fig. 1.

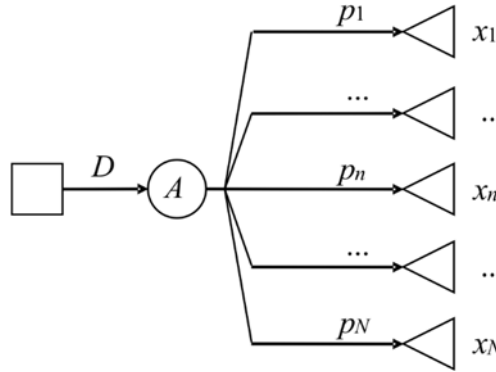


Fig. 1. Decision D related to the random event A as a fragment of a decision tree

Another behaviour of decision makers is that they subjectively evaluate relative outcomes, treating relative gains and relative losses in a different way. They exhibit risk aversion in the face of gains, but are risk seeking in the face of losses. Moreover, they exhibit loss aversion. Such behaviour can be expressed by an S-shaped value function of the following form [10]:

$$v(x) = \begin{cases} x^\alpha, & x \geq 0 \\ -\lambda(-x)^\beta, & x < 0 \end{cases} \quad (2)$$

The values of the parameters α and β were estimated to be 0.88 based on the research of Kahneman and Tversky [10]. The loss aversion ratio, $\lambda = 2.25$, models a decision maker’s perception, according to which “losses loom larger than gains” [6, p. 279].

Decision makers also subjectively evaluate probabilities. They overestimate small probabilities and underestimate moderate and large ones. Their observed behaviour can be modelled by a probability weighting function of the form

$$w(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}} \quad (3)$$

where $\gamma = 0.61$, if a probability concerns a relative gain and $\gamma = 0.69$ in the case of a relative loss.

The functions given in Eqs. (2) and (3) are not the only ones that have been applied. For other propositions of the value function and the probability weighting function which have the required properties see [3, 5, 8, 9, 11]. Originally, prospect theory introduced by Kahneman and Tversky [6] proposed a method of evaluating a decision with at most two possible non-zero relative outcomes. This theory turned out to be inconsistent with stochastic dominance. Cumulative prospect theory [10] overcame these shortcomings. It is consistent with stochastic dominance and can be used in the evaluation of decision with more than two possible outcomes. More information about the differences between prospect theory and cumulative prospect theory can be found in [4].

Based on the cumulative prospect theory, the valuation of a decision D (Eq. (1)) corresponding to a random event A is the sum of the valuation of the relative losses $CPT^-(A)$ and the valuation of the relative gains $CPT^+(A)$, see [10], i.e.

$$CPT(A) = CPT^-(A) + CPT^+(A) \quad (4)$$

The components $CPT^-(A)$ and $CPT^+(A)$ are calculated as follows:

$$CPT^-(A) = v(x_1)w(p_1) + \sum_{i=2}^n v(x_i) \left[w\left(\sum_{j=1}^i p_j\right) - w\left(\sum_{j=1}^{i-1} p_j\right) \right] \quad (5)$$

$$CPT^+(A) = \sum_{i=n+1}^{N-1} v(x_i) \left[w\left(\sum_{j=i}^N p_j\right) - w\left(\sum_{j=i+1}^N p_j\right) \right] + v(x_N)w(p_N) \quad (6)$$

If the possible outcomes of a decision involve only gains or only losses, Eq. (4) reduces to the evaluation of only gains or only losses, accordingly. When comparing two risky decisions on the basis of cumulative prospect theory the preferred one is that with the higher CPT value.

4. Analysis of a complex decision tree based on CPT

In the analysis of complex decision problems, decision makers often have to compare the random outcomes of decisions with a non-random final outcome. All evaluations should be expressed in the same units. There is no problem if the analysis is based on the expected monetary value, since the evaluation of each alternative is expressed in monetary units. However, if an evaluation is based on cumulative prospect theory, the valuation of a risky decision (with an uncertain outcome) is a number that is not ex-

pressed in monetary units and so cannot be directly compared to the deterministic monetary outcomes. Based on the theory of expected utility *the value of a certain prospect that yields the same utility as the expected utility of an uncertain prospect is called the certainty equivalent* [7]. In cumulative prospect theory, the certainty equivalent of a risky decision can be defined in a similar manner, as the deterministic monetary value whose valuation is the same as the CPT valuation of a risky decision. This means that

$$v(CE(A)) = CPT(A) \tag{7}$$

where $v(\cdot)$ is the value function given by Eq. (2) and thus

$$CE(A) = v^{-1}(CPT(A)) = \begin{cases} \alpha \sqrt[CPT(A)]{}, & CPT(A) \geq 0 \\ -\beta \sqrt[-CPT(A)]{}, & CPT(A) < 0 \end{cases} \tag{8}$$

A certainty equivalent is assigned to each alternative with a random outcome and such alternatives can then be compared with non-random final outcomes.

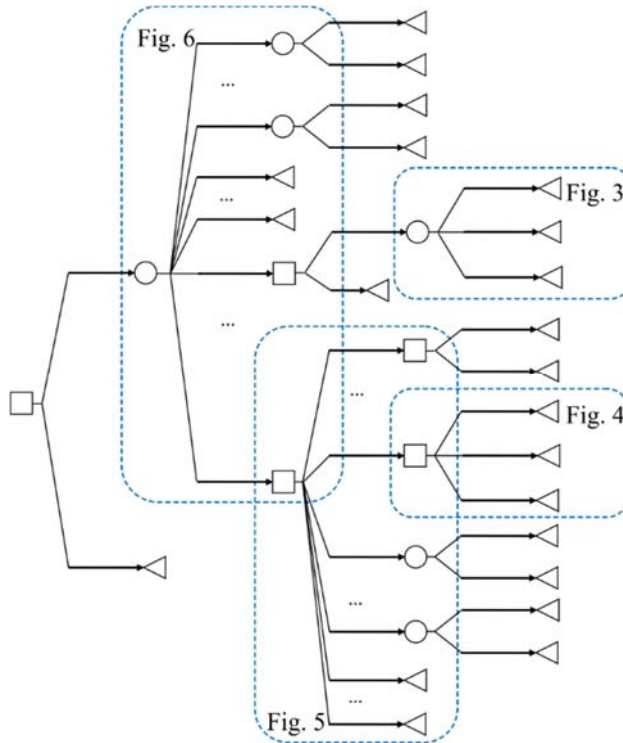


Fig. 2. Example of a complex decision tree

In complex decision trees, there are many decision nodes and chance nodes. After any decision node, other decision nodes, chance nodes, and leaves may occur. Similarly, after any chance node, other chance nodes, leaves, or decision nodes may also occur. Only after a leaf node (end node) can no other node occur. An example of a complex decision tree where alternatives have random outcomes is presented in Fig. 2. For the sake of clarity, the names of the nodes and branches, as well as the probabilities and outcomes corresponding to the end nodes are omitted. On the other hand, some fragments of the tree are highlighted. These fragments are referred to in the description of the procedure for analysing decision problems based on cumulative prospect theory.

The analysis of a complex decision tree starts from chance nodes and decision nodes whose branches lead only to end nodes. These nodes are assigned values. Then the preceding nodes are analysed in sequence up to the root. The following procedure is descriptive insofar as there is no indexing of chance, decision or end nodes, or of final outcomes and probabilities. The symbols and indices relate to the fragment of the tree shown in the corresponding figure. In the following procedure, it is assumed that the decision maker wants to maximize the certainty equivalent of the outcome:

1. All of the chance nodes whose branches lead only to end nodes (leaves) are considered. A suitable part of such a decision tree is shown in Fig. 3.

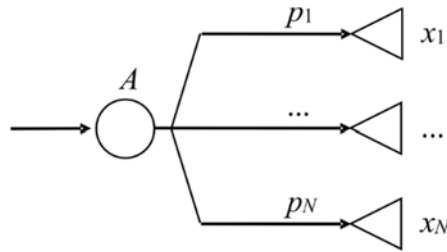


Fig. 3. Chance node whose branches lead only to end nodes

For each such chance node, the $CPT(A)$ is calculated (Eqs. (4)–(6)) and then its certainty equivalent (Eq. (8)) is assigned to that chance node.

2. All of the decision nodes whose branches lead only to end nodes (leaves) are now considered. Figure 4 shows such a decision node.

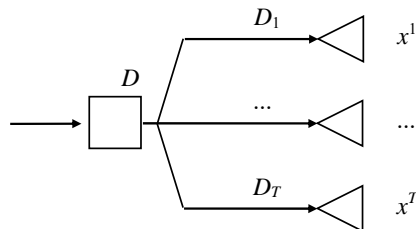


Fig. 4. Decision node whose branches lead only to end nodes

The final outcome of the best alternative is assigned to each such decision node:

$$V(D) = \max_{t \in \{1, \dots, T\}} \{x^t\} \quad (9)$$

where T is the number of available alternatives and x^1, \dots, x^T are the final outcomes.

3. Other decision nodes whose branches lead to other decision nodes and/or chance nodes and/or end nodes are now taken into account, as long as the succeeding nodes have all been assigned a value. An example of such a decision node is shown in Fig. 5.

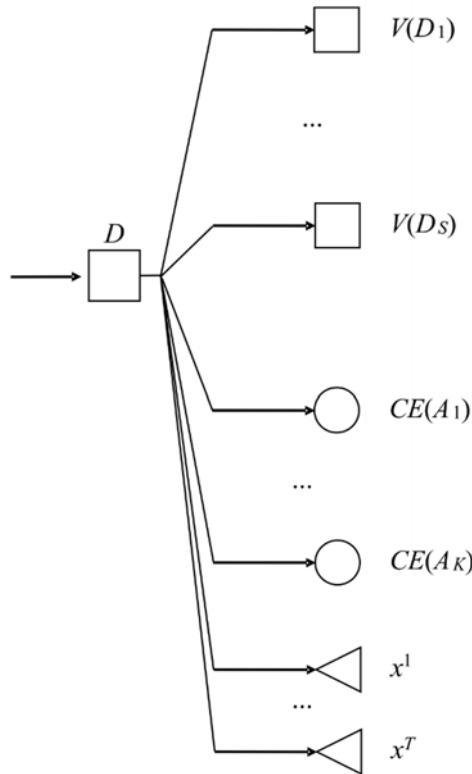


Fig. 5. Decision node

The value of the best alternative is assigned to each decision node:

$$V(D) = \max \{V(D_1), \dots, V(D_S), CE(A_1), \dots, CE(A_K), x^1, \dots, x^T\} \quad (10)$$

where: S is the number of alternatives leading to other decision nodes, K is the number of alternatives leading to chance nodes, $V(D_1), \dots, V(D_S)$ are the values assigned earlier

to the decision nodes D_1, \dots, D_S , $CE(A_1), \dots, CE(A_K)$ are certainty equivalents assigned earlier to chance nodes, x^1, \dots, x^T are final outcomes assigned to end nodes.

4. Other chance nodes whose branches lead to other chance nodes and/or decision nodes and/or end nodes are now taken into account, as long as the succeeding nodes have all been assigned a value. Figure 6 shows an example of such a chance node.

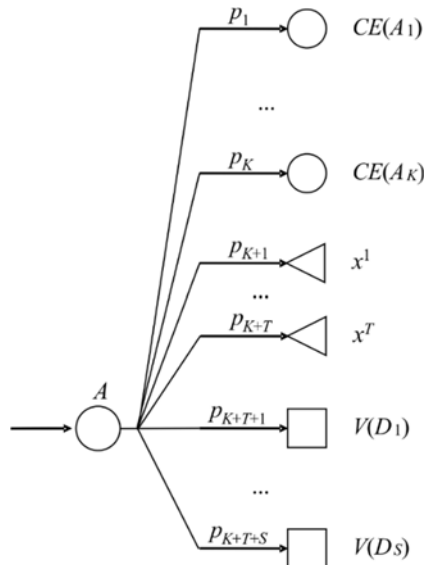


Fig. 6. Chance node

For each such chance node A , $CPT(A)$ is calculated by first sorting the set $CE(A_1), \dots, CE(A_K), x^1, \dots, x^T, V(D_1), \dots, V(D_S)$ in ascending order. Then a certainty equivalent $CE(A)$ can be assigned to a given chance node.

5. Steps 3 and 4 are repeated till all nodes have values assigned to them.

6. The solution to the complex decision tree is read starting from the root node. At each decision node, the best decision is kept, while the others alternatives as well as the parts of the tree following these alternatives, are discarded. The value assigned to the root node expresses the subjective monetary valuation of the best combination of decisions selected, while taking into account random events at chance nodes.

The following example illustrates the application of this procedure.

Example 1. How can the number of tourists coming to a city be increased?²

² This example is inspired by the model for deciding how a city should be promoted, published on <https://github.com/SilverDecisions/SilverDecisions/wiki/Gallery#city-promotion-decision-model>

A community considers two alternative strategies: a traditional one and applying to become a European Capital of Culture. The traditional alternative involves commercials, leaflets, advertising gadgets, etc. It would cost \$ 100 000 and cause an increase in the number of tourists by 7% (\$ 110 000 additional income) with probability 0.80 or by 10% (\$ 145 000 additional income) with probability 0.20. The second alternative requires investments that would cost \$ 250 000. Obtaining the title of European Capital of Culture is not guaranteed. The chance of success is estimated at 40%. Should the community fail, the number of tourists will increase by only 5% (\$ 100 000 additional income). In the case that the city becomes a European Capital of Culture, the number of tourists increases by 30% (\$ 300 000 additional income) with probability 0.25 or by 60% (\$ 1 000 000 additional income) with probability 0.75, depending on other random factors.

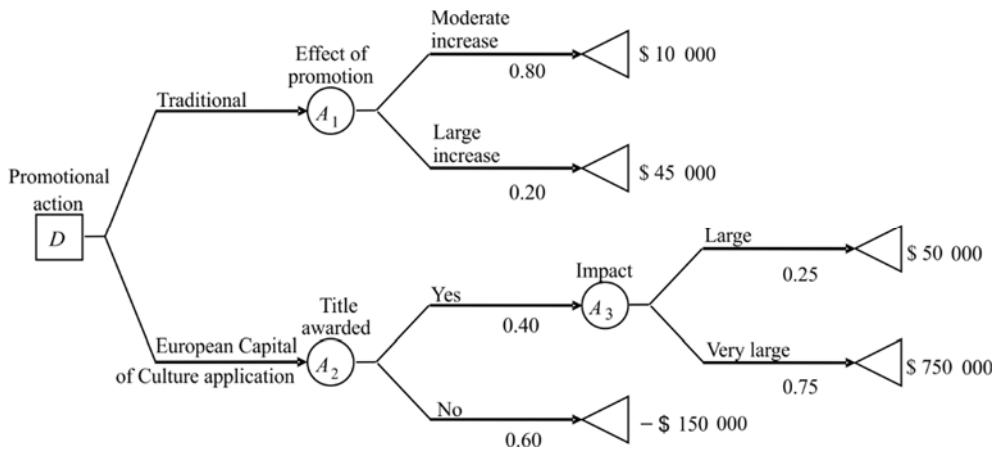


Fig. 7. Decision tree for city promotion problem

Figure 7 presents the decision tree for the city promotion decision problem. The final outcomes include the costs of each promotional action. Inclusion of the costs of action are necessary, since their omission means a change in the reference point, which usually causes a change in the valuation of an alternative [2]. In this decision problem, only one decision has to be taken concerning the way of promoting the city. The complexity of the problem lies in two chance nodes following one after the other (A_2 and A_3). Based on cumulative prospect theory, such a pair of chance nodes cannot be combined together because the probability weighting function is not linear and the combination of the transformations of two probabilities does not equal the transformation of the combination of these two probabilities.

In step 1, the values of two chance nodes, A_1 and A_3 , are calculated. Due to the fact that the outcomes of A_1 and A_3 involve only gains, Eq. (4) is simplified to value only gains.

$$\begin{aligned} \text{CPT}(A_1) &= \text{CPT}^+(A_1) \\ &= v(10\,000) [w(1) - w(0.20)] + v(45\,000) w(0.20) = 5691.79 \\ \text{CE}(A_1) &= \sqrt[0.88]{5691.79} = \$ 18\,506.66 \end{aligned}$$

$$\begin{aligned} \text{CPT}(A_3) &= \text{CPT}^+(A_3) \\ &= v(50\,000) [w(1) - w(0.75)] + v(750\,000) w(0.75) = 89\,956.03 \\ \text{CE}(A_3) &= \sqrt[0.88]{89\,956.03} = \$ 426\,167.10 \end{aligned}$$

Step 2 is skipped, because there is no decision node whose branches lead only to end nodes. Step 3 is also omitted, because the calculations for decision node D cannot be done until the certainty equivalent for chance node A_2 is obtained, which is done in step 4. In the valuations for chance node A_2 , the certainty equivalent for chance node A_3 is used.

$$\begin{aligned} \text{CPT}(A_2) &= \text{CPT}^-(A_2) + \text{CPT}^+(A_2) \\ &= v(-150\,000) w(0.60) + v(426\,167.10) w(0.40) = -8549.98 \\ \text{CE}(A_2) &= \sqrt[0.88]{-(-8549.98) / 2.25} = -\$ 11\,693.24 \end{aligned}$$

Now the valuation for decision node D can be obtained (as described in step 3). The certainty equivalent of the traditional alternative is 18 506.66, while the certainty equivalent of the other alternative is -11 693.24. The best alternative for the community, based on this behavioural evaluation, is to promote the city in a traditional way.

If the community evaluates both alternatives based on the expected monetary value, then the values assigned to the chance nodes are as follows:

$$\begin{aligned} \text{EMV}(A_1) &= 0.80 \times 10\,000 + 0.20 \times 45\,000 = \$ 17\,000 \\ \text{EMV}(A_3) &= 0.25 \times 50\,000 + 0.75 \times 750\,000 = \$ 575\,000 \\ \text{EMV}(A_2) &= 0.40 \times 575\,000 + 0.60 \times (-150\,000) = \$ 140\,000. \end{aligned}$$

The higher expected monetary value corresponds to applying to become a European Capital of Culture, thus this decision should be taken on the basis of the EMV criterion.

The criterion based on CPT valuation indicates the opposite alternative to the one based on EMV, because CPT places a lot of weight on possible losses (due to the loss aversion parameter) and even the possibility of a very high income cannot compensate for the fear of losing.

5. Example of using other criteria

The procedure described in Section 4 can also be used with other criteria in which some transformation of outcomes are applied, e.g., the theory of expected utility. The concept of certainty equivalent is useful in the analysis of complex decision trees based on such approaches. For example, if the decision maker's utility from the relative outcome is described by an exponential function of the form

$$u(x) = -e^{-\gamma x} \quad (11)$$

then he is risk averse if $\gamma > 0$. The expected utility of decision D (given by Eq. (1)) corresponding to a random event A is as follows:

$$U(A) = \sum_{i=1}^N p_i u(x_i) \quad (12)$$

The certainty equivalent for a decision with a random outcome based on the exponential utility function is given by

$$CE_U(A) = -\frac{\ln(-U(A))}{\gamma} \quad (13)$$

Using Equations (11)–(13) and $\gamma = 1000$, the certainty equivalents assigned to the chance nodes in Example 1 are given by: $CE_U(A_1) = \$ 10\,223.14$, $CE_U(A_3) = \$ 513\,86.29$ and $CE_U(A_2) = -\$ 149\,489.17$. This means that the community should choose the traditional way of promotion.

6. Conclusions

The optimal decisions chosen on the basis of a behavioural approach often differ from optimal decisions based on the objective criterion of maximizing the expected monetary value. The approach proposed in the procedure uses the subjective valuations

of relative outcomes and their probabilities and allows us to analyse complex decision problems in accordance with the decision maker's attitude towards risk (based on an S-shaped value function), losses (loss aversion parameter) and probabilities (overestimating low and underestimating moderate and high probabilities). The concept of a certainty equivalent allows us to compare random and non-random outcomes based on cumulative prospect theory, as well as other criteria that non-linearly transform outcomes or probabilities.

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