

# Generalized $C_\beta^\psi$ – rational contraction and fixed point theorem with application to second order differential equation

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ABSTRACT. In this article, generalized  $C_\beta^\psi$ - rational contraction is defined and the existence and uniqueness of fixed points for self map in partially ordered metric spaces are discussed. As an application, we apply our result to find existence and uniqueness of solutions of second order differential equations with boundary conditions.

## 1. INTRODUCTION

From last 15 years, several authors have studied and derived various fixed point results for many contractions in partially ordered sets. Ran and Reurings [1] derived a fixed point result on partially ordered sets in which contractive condition assumed to be hold on comparable elements. After that, author in [9, 10] deduced some results to get fixed point for monotone, non-decreasing operator with partially ordered relation on a set  $Y$  without using the continuity of maps. They also discussed few applications of their main findings and gave existence as well as uniqueness theorem ordinary differential equation of first order and first degree with restricted boundary conditions. Number of results after that have been investigated to establish fixed point in partially ordered metric spaces (for more detail see [2, 4, 7, 8, 11, 12, 13, 15, 18, 19, 21, 22]).

In 1975, Jaggi [23] and Das and Gupta [24] derived some fixed point results for rational type contraction. There exist several results in the literature for self and pair of maps satisfying rational expression in different spaces [20, 25].

In 2007, Suzuki [16] introduced the weaker  $C$ - contractive condition and proved some fixed point theorems. The existence as well as uniqueness of fixed point of such types of operator have also been extensively studied in [3, 17].

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**Definition 1.1.** [16] Let  $(Y, d)$  be a metric space. Then a map  $f$  on  $Y$  is said to satisfies the  $C$ - condition if, for all  $u, v \in Y$ ,

$$\frac{1}{2}d(u, fu) \leq d(u, v) \quad \text{implies} \quad d(fu, fv) \leq d(u, v).$$

We begin with the following definition and lemmas which are useful in proving our result.

**Definition 1.2.** [14] Let  $\Psi$  denote the class of function  $\psi : [0, \infty) \rightarrow [0, \infty)$  (called altering distance function), which satisfies the following assumptions:

- ( $\Psi$ 1.)  $\psi$  is non-decreasing and continuous,
- ( $\Psi$ 2.)  $\psi(\omega) = 0$  if and only if  $\omega = 0$ .

**Lemma 1.1.** [5] Let  $\pi : [0, \infty) \rightarrow [0, \infty)$  is a continuous function. If  $\psi$  is an altering distance function satisfying condition  $\psi(\omega) > \pi(\omega)$  for all  $\omega > 0$ , then  $\pi(0) = 0$ .

**Lemma 1.2.** [6] Let  $(Y, d)$  be a metric space. Let  $\{u_n\}$  be a sequence in  $Y$  such that

$$\lim_{n \rightarrow \infty} d(u_n, u_{n+1}) = 0.$$

If  $\{u_n\}$  is not a Cauchy sequence in  $Y$  then there exist an  $\epsilon > 0$  and sequences of positive integers  $(m_k)$  and  $(n_k)$  with  $m_k > n_k > k$  such that

$$d(u_{m_k}, u_{n_k}) \geq \epsilon, \quad d(u_{m_k-1}, u_{n_k}) < \epsilon$$

and

- (B1.)  $\lim_{k \rightarrow \infty} d(u_{m_k-1}, u_{n_k+1}) = \epsilon$ ,
- (B2.)  $\lim_{k \rightarrow \infty} d(u_{m_k}, u_{n_k}) = \epsilon$ ,
- (B3.)  $\lim_{k \rightarrow \infty} d(u_{m_k-1}, u_{n_k}) = \epsilon$ .

In this paper, we first define a generalized  $C_\beta^\psi$ - rational contraction and then prove the existence and uniqueness of fixed points for self monotone map. We also consider a partially ordered set  $Y$  with comparable elements, and a complete metric  $d$  with set  $Y$  to deduce our main result. As application, we give an existence as well as uniqueness theorem for ordinary differential equation of second order and first degree with restricted boundary conditions.

## 2. FIXED POINT RESULT WITH PARTIAL ORDER

We define generalized  $C_\beta^\psi$ - rational contraction as follows:

**Definition 2.1.** A mapping  $f$  on a metric space  $(Y, d)$  is said to satisfy generalized  $C_\beta^\psi$ - rational contraction if, for all  $u, v \in Y$ ,

$$(1) \quad \frac{1}{2}d(u, fu) \leq d(u, v) \quad \text{implies} \quad \psi(d(fu, fv)) \leq \beta(M(u, v)),$$

where

$$(2) \quad M(u, v) = \max \left\{ d(u, v), \frac{d(u, fu)d(v, fv)}{[1 + d(u, v)]}, \frac{d(v, fv)[1 + d(u, fu)]}{[1 + d(u, v)]} \right\},$$

$\beta : [0, \infty) \rightarrow [0, \infty)$  is continuous function and  $\psi \in \Psi$ .

Main finding of this article is the following result.

**Theorem 2.1.** *Let  $(Y, d, \preceq)$  be a partially ordered complete metric space and let  $f : Y \rightarrow Y$  be a non-decreasing, monotone map satisfying generalized  $C_\beta^\psi$ -rational contraction. Also, suppose  $\beta : [0, \infty) \rightarrow [0, \infty)$  is continuous function and  $\psi \in \Psi$  satisfying*

$$(3) \quad 0 < \beta(\omega) < \psi(\omega), \quad \omega > 0.$$

Also assume that:

$$(4) \quad \text{For every } u, v \in Y, \text{ there exists } z \in Y, \text{ such that } u \preceq z \text{ and } v \preceq z.$$

If there exists  $u_0 \in Y$  such that  $u_0 \preceq fu_0$ , then  $f$  has a unique fixed point in  $Y$ .

*Proof.* Let  $u_0 \in Y$  satisfy  $u_0 \preceq fu_0$ . We define a sequence  $\{u_n\}$  as follows:

$$(5) \quad u_n = fu_{n-1}, \quad n \in \mathbb{N}.$$

If  $u_n = u_{n+1}$  for some  $n \in \mathbb{N}$ , then, clearly  $M(u_n, u_{n+1}) = 0$  and so,  $u_n$  is the fixed point of  $f$ . So, assume that  $u_n \neq u_{n+1}$  for all  $n \in \mathbb{N}$ . Let  $a_n = d(u_n, u_{n+1})$ . Then, clearly  $a_n > 0$ . Since  $u_0 \preceq fu_0 = u_1$  and  $f$  is non-decreasing, then

$$(6) \quad u_0 \preceq u_1 \preceq u_2 \cdots \preceq u_n \cdots$$

On taking  $u = u_n$  and  $v = fu_n = u_{n+1}$  in (1), we obtain that

$$\frac{1}{2}d(u_n, fu_n) = \frac{1}{2}d(u_n, u_{n+1}) \leq d(u_n, u_{n+1})$$

implies

$$(7) \quad \psi(d(fu_n, fu_{n+1})) = \psi(d(u_{n+1}, u_{n+2})) \leq \beta(M(u_n, u_{n+1})),$$

where

$$\begin{aligned} M(u_n, u_{n+1}) &= \max \left\{ d(u_n, u_{n+1}), \frac{d(u_n, fu_n)d(u_{n+1}, fu_{n+1})}{[1 + d(u_n, u_{n+1})]}, \right. \\ &\quad \left. \frac{d(u_{n+1}, fu_{n+1})[1 + d(u_n, fu_n)]}{[1 + d(u_n, u_{n+1})]} \right\} \\ &= \max \left\{ d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2}), \right. \\ &\quad \left. \frac{d(u_n, u_{n+1})d(u_{n+1}, u_{n+2})}{[1 + d(u_n, u_{n+1})]} \right\}. \end{aligned}$$

Since  $\frac{d(u_n, u_{n+1})}{[1 + d(u_n, u_{n+1})]} < 1$  for all  $n \in \mathbb{N}$ , therefore

$$\frac{d(u_n, u_{n+1})d(u_{n+1}, u_{n+2})}{[1 + d(u_n, u_{n+1})]} < d(u_{n+1}, u_{n+2}),$$

and hence

$$M(u_n, u_{n+1}) \leq \max \{d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2})\}.$$

From (7), we have

$$(8) \quad \psi(d(u_{n+1}, u_{n+2})) \leq \beta(\max \{d(u_n, u_{n+1}), d(u_{n+1}, u_{n+2})\}).$$

If  $d(u_n, u_{n+1}) < d(u_{n+1}, u_{n+2})$ , then (8) gives a contradiction to condition (3) and hence

$$\psi(d(u_{n+1}, u_{n+2})) \leq \beta(d(u_n, u_{n+1})).$$

Since  $\psi$  and  $\beta$  are continuous functions, therefore

$$d(u_{n+1}, u_{n+2}) \leq d(u_n, u_{n+1}).$$

Similarly we get

$$d(u_n, u_{n+1}) \leq d(u_{n-1}, u_n).$$

Thus, we get a sequence  $\{d(u_n, u_{n+1})\}$  of functions, which is non-increasing and  $r \geq 0$  such that

$$(9) \quad \lim_{n \rightarrow \infty} d(u_n, u_{n+1}) = r.$$

However, by taking  $\lim_{n \rightarrow \infty}$  on both side of (8), we get  $\psi(r) \leq \beta(r)$ , which is a contradiction to (2). Thus we have  $r = 0$ , and hence

$$(10) \quad \lim_{n \rightarrow \infty} d(u_n, u_{n+1}) = r = 0.$$

Assume on contrary that sequence  $\{u_n\}$  is not Cauchy. Then for every  $\epsilon > 0$ , we can find subsequences of positive integers  $m_k$  and  $n_k$ , where  $n_k > m_k > k$ , for all  $k \in N$ , such that

$$(11) \quad d(u_{m_k}, u_{n_k}) > \epsilon \quad \text{and} \quad d(u_{m_k}, u_{n_k-1}) \leq \epsilon.$$

Also for this  $\epsilon > 0$ , the convergence of sequence  $\{d(u_n, u_{n+1})\}$  implies, there exists  $N_0 \in N$  such that  $d(u_n, u_{n+1}) < \epsilon$  for all  $n \geq N_0$ . Let  $N_1 = \max \{m_i, N_0\}$ . Then, for all  $m_k > n_k \geq N_1$ , we have

$$d(u_{n_k}, u_{n_k+1}) < \epsilon \leq d(u_{n_k}, u_{m_k}),$$

where  $m_k > n_k$  and hence

$$\frac{1}{2}d(u_{n_k}, u_{n_k+1}) \leq d(u_{n_k}, u_{m_k}).$$

Now from (1), on substituting  $u = u_{n_k}$  and  $v = u_{m_k}$ , we get

$$(12) \quad \psi(d(fu_{n_k}, fu_{m_k})) = \psi(d(u_{n_k+1}, u_{m_k+1})) \leq \beta(M(u_{n_k}, u_{m_k}))$$

where,

$$M(u_{n_k}, u_{m_k}) = \max \left\{ d(u_{n_k}, u_{m_k}), \frac{d(u_{n_k}, fu_{n_k})d(u_{m_k}, fu_{m_k})}{[1+d(u_{n_k}, u_{m_k})]}, \frac{d(u_{m_k}, fu_{m_k})[1+d(u_{n_k}, fu_{n_k})]}{[1+d(u_{n_k}, u_{m_k})]} \right\}$$

$$(13) \quad = \max \left\{ d(u_{n_k}, u_{m_k}), \frac{d(u_{n_k}, u_{n_{k+1}})d(u_{m_k}, u_{m_{k+1}})}{[1+d(u_{n_k}, u_{m_k})]}, \frac{d(u_{m_k}, u_{m_{k+1}})[1+d(u_{n_k}, u_{n_{k+1}})]}{[1+d(u_{n_k}, u_{m_k})]} \right\}.$$

On using Lemma 1.2 and letting  $k \rightarrow \infty$  in (12) and (13), we obtain  $\psi(\epsilon) \leq \beta(\epsilon)$ , that's a contradiction to (3) and hence by Lemma 1.1, we get  $\epsilon = 0$ . This contradicts the assumption that  $\epsilon > 0$ . Therefore our assumption is wrong. Hence  $\{u_n\}$  is Cauchy. Since  $Y$  is complete, so  $\{u_n\}$  converges with all its subsequences to some limiting value, say  $z \in Y$ .

Now assume for every  $n \in N$

$$d(u_n, z) < \frac{1}{2}d(u_n, u_{n+1})$$

and

$$d(u_{n+1}, z) < \frac{1}{2}d(u_{n+1}, u_{n+2}).$$

Then we have

$$\begin{aligned} d(u_n, u_{n+1}) &\leq d(u_n, z) + d(u_{n+1}, z) \\ &< \frac{1}{2} [d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2})] \\ &\leq d(u_n, u_{n+1}), \end{aligned}$$

this is a contradiction. Hence we must have  $d(u_n, z) \geq \frac{1}{2}d(u_n, u_{n+1})$  or  $d(u_{n+1}, z) \geq \frac{1}{2}d(u_{n+1}, u_{n+2})$ , for all  $n \in N$ . Thus for a sub-sequence  $\{n_k\}$  of  $N$ , we obtain

$$\frac{1}{2}d(u_{n_k}, fu_{n_k}) = \frac{1}{2}d(u_{n_k}, u_{n_{k+1}}) \leq d(u_{n_k}, z), \quad k \in N,$$

which implies

$$(14) \quad \psi(d(fu_{n_k}, fz)) = \beta(M(u_{n_k}, z)),$$

where

$$(15) \quad M(u_{n_k}, z) = \max \left\{ d(u_{n_k}, z), \frac{d(u_{n_k}, fz)d(u_{n_k}, fu_{n_k})}{[1+d(u_{n_k}, z)]}, \frac{d(z, fz)[1+d(u_{n_k}, fu_{n_k})]}{[1+d(u_{n_k}, z)]} \right\}.$$

Both, on letting  $k \rightarrow \infty$ , and using (15) in (14), we get

$$\psi(d(z, fz)) \leq \beta(d(z, fz)).$$

Lemma 1.1 implies that  $d(z, fz) = 0$ . That is,  $fz = z$ .

To establish uniqueness, we suppose on contradictory that for all  $u, v \in Y$ ,  $u = fu$  and  $v = fv$  provided  $u \neq v$ . Now we discuss following two case for both elements.

*Case 1.* Without loss of generality, suppose that  $u \preceq v$  are comparable. Then

$$0 = \frac{1}{2}d(u, fu) \leq d(u, v),$$

implies that

$$(16) \quad \psi(d(fu, fv)) = \psi(d(u, v)) \leq \beta(M(u, v)) = \beta(d(u, v)),$$

Thus from (2) and Lemma 1.1, we get  $d(u, v) = 0$ , *i.e.*,  $u = v$ .

*Case 2.* Assume that  $u$  and  $v$  are not comparable then from (4), there exists some  $z \in Y$  comparable to  $u$  and  $v$  such that  $fz = z$  is comparable  $u = fu$  and  $v = fv$ .

Clearly,

$$0 = d(u, u) = \frac{1}{2}d(u, fu) < d(u, w)$$

implies that

$$(17) \quad \psi(d(fu, fw)) \leq \beta(M(u, w)),$$

where

$$\begin{aligned} M(u, w) &= \max \left\{ d(u, w), \frac{d(u, fu)d(w, fw)}{1 + d(u, w)}, \frac{d(w, fw)[1 + d(u, fu)]}{1 + d(u, w)} \right\} \\ &= \max \{d(u, w), 0, 0\} = d(u, w). \end{aligned}$$

Hence, from (17),

$$\psi(d(fu, fw)) \leq \beta(d(u, w)).$$

Consequently, we have

$$\psi(d(u, w)) \leq \beta(d(u, w)).$$

On using Lemma 1.1, we have  $d(u, w) = 0$ .

Similarly, we can obtain  $d(v, w) = 0$ . This implies that  $u = v$ .

This completes the proof of Theorem 2.1. □

**Theorem 2.2.** *Let  $(Y, d, \preceq)$  be a partially ordered complete metric space and let  $f : Y \rightarrow Y$  be a non-decreasing, monotone map such that for all  $u, v \in Y$ ,*

$$(18) \quad \frac{1}{2}d(u, fu) \leq d(u, v) \quad \text{implies} \quad \psi(d(fu, fv)) \leq \beta(N(u, v)),$$

and

$$(19) \quad N(u, v) = a_1d(u, v) + a_2 \frac{d(u, fu)d(v, fv)}{[1 + d(u, v)]} + a_3 \frac{d(v, fv)[1 + d(u, fu)]}{[1 + d(u, v)]},$$

where  $\psi \in \Psi$ ,  $a_i \geq 0$ ,  $\sum a_i < 1$ , for all  $i = 1, 2, 3$  and  $\beta : [0, \infty) \rightarrow [0, \infty)$  is continuous function such that

$$(20) \quad 0 < \beta(\omega) < \psi(\omega), \quad \omega > 0.$$

Also assume that, for every  $u, v \in Y$ , there exists  $z \in Y$ , such that  $u \preceq z$  and  $v \preceq z$ . If there exists  $u_0 \in Y$  such that  $u_0 \preceq fu_0$ , then  $f$  has a unique fixed point in  $Y$ .

*Proof.* Given that  $f : Y \rightarrow Y$  be monotone, nondecreasing map such that for all  $u, v \in Y$ ,

$$\frac{1}{2}d(u, fu) \leq d(u, v) \quad \text{implies} \quad \psi(d(fu, fv)) \leq \beta(N(u, v)),$$

and

$$\begin{aligned} N(u, v) &= a_1d(u, v) + a_2 \frac{d(u, fu)d(v, fv)}{[1 + d(u, v)]} + a_3 \frac{d(v, fv)[1 + d(u, fu)]}{[1 + d(u, v)]} \\ &= \sum a_i \cdot \max \left\{ d(u, v), \frac{d(u, fu)d(v, fv)}{[1 + d(u, v)]}, \frac{d(v, fv)[1 + d(u, fu)]}{[1 + d(u, v)]} \right\}. \end{aligned}$$

Since all  $a_i \geq 0$  and  $\sum a_i < 1$ , for all  $i = 1, 2, 3$ , then

$$\begin{aligned} N(u, v) &\leq \max \left\{ d(u, v), \frac{d(u, fu)d(v, fv)}{[1 + d(u, v)]}, \frac{d(v, fv)[1 + d(u, fu)]}{[1 + d(u, v)]} \right\} \\ &= M(u, v). \end{aligned}$$

Rest of the proof follows directly from main result (Theorem 2.1).  $\square$

If we take  $a_2 = a_3 = 0, a_1 = 1$  in Theorem 2.2, we obtain following result of Yan et al. [5] satisfying weaker type of  $C_\beta^\psi$ - condition.

**Corollary 2.1.** *Let  $(Y, d, \preceq)$  be a partially ordered complete metric space and let  $f : Y \rightarrow Y$  be a non-decreasing map such that for all  $u, v \in Y$ ,*

$$\frac{1}{2}d(u, fu) \leq d(u, v) \quad \text{implies} \quad \psi(d(fu, fv)) \leq \beta(d(u, v)),$$

where  $\psi \in \Psi$  and  $\beta : [0, \infty) \rightarrow [0, \infty)$  is a continuous function such that

$$0 < \beta(\omega) < \psi(\omega), \quad \omega > 0.$$

Also assume that for every  $u, v \in Y$ , there exists  $z \in Y$ , such that  $u \preceq z$  and  $v \preceq z$ . If there exists  $u_0 \in Y$  such that  $u_0 \preceq fu_0$ , then  $f$  has a unique fixed point in  $Y$ .

If we take  $\psi(\omega) = \omega$  and  $\beta(\omega) = \omega$  in Theorem 2.2, we get the following new result.

**Corollary 2.2.** *Let  $(Y, d, \preceq)$  be a partially ordered complete metric space and let  $f : Y \rightarrow Y$  be a non-decreasing map such that for all  $u, v \in Y$ ,*

$$\frac{1}{2}d(u, fu) \leq d(u, v) \quad \text{implies} \quad d(fu, fv) \leq N(u, v),$$

and

$$N(u, v) = a_1d(u, v) + a_2 \frac{d(u, fu)d(v, fv)}{[1 + d(u, v)]} + a_3 \frac{d(v, fv)[1 + d(u, fu)]}{[1 + d(u, v)]},$$

where  $a_i \geq 0$ ,  $\sum a_i < 1$ , for all  $i = 1, 2, 3$ . Also assume that for every  $u, v \in Y$ , there exists  $z \in Y$ , such that  $u \preceq z$  and  $v \preceq z$ . If there exists  $u_0 \in Y$  such that  $u_0 \preceq fu_0$ , then  $f$  has a unique fixed point in  $Y$ .

**Remark 2.1.** If we take  $a_2 = 0$  in Corollary 2.2, we obtain the result of Dass and Gupta [24] in fame work of partially ordered metric spaces satisfying  $C$ -condition.

**Remark 2.2.** If we take  $a_1 = 0$  and  $a_2 = 0$  in Corollary 2.2, we get new result in the sense of partially ordered metric spaces satisfying  $C$ - condition.

### 3. APPLICATION: EXISTENCE OF SOLUTION OF SECOND ORDER BOUNDARY VALUE PROBLEM

We consider following second order differential equation with boundary condition

$$(21) \quad -\frac{d^2u}{d\omega^2} = f(\omega, u(\omega)), \quad \omega \in L = [0, 1], \quad u \in [0, \infty), \\ u(0) = u'(1) = 0.$$

If  $u \in C^2(L)$  is zero of (21), then  $u \in C(L)$  is also a zero of following integral equation

$$u(\omega) = \int_0^T G(\omega, \theta) f(\theta, u(\theta)) d\theta \quad \text{for all } \omega \in L,$$

where  $G(\omega, \theta)$  is the Green function given by

$$G(\omega, \theta) = \begin{cases} \omega, & \text{if } 0 \leq \omega \leq \theta \leq 1, \\ \theta, & \text{if } 0 \leq \theta \leq \omega \leq 1. \end{cases}$$

**Theorem 3.1.** Consider a second order differential equation (21) with a map  $f : L \times R \rightarrow R$ . Assume that  $f$  is weakly increasing with respect to second variable and continuous. If there exist  $\lambda \in (0, 2]$  such that

$$f(\omega, u) - f(\omega, v) \leq \lambda \sqrt{\log[(u - v)^2 + 1]}, \quad u \geq v,$$

then there exist a unique non negative solution for the problem (21).

*Proof.* If we let  $S = \{u \in C(L), L = [0, 1] : u(\omega) \geq 0\}$  be a cone, and  $(S, d)$  be a metric space with metric defined as  $d(u, v) = \sup \{|u(\omega) - v(\omega)| : \omega \in L\}$ ; for all  $u, v \in E$ , then clearly  $(S, d)$  is complete.

Define  $H : C(L) \rightarrow C(L)$  by

$$(Hu)(\omega) = \int_0^1 G(\omega, \theta) f(\theta, u(\theta)) d\theta.$$

If  $u \in C(L)$  is a fixed point of  $H$ , then  $u \in C^1(L)$  is a zero of (21).

Clearly, with assumption on  $f$  and elements  $u, v \in E$ , we obtain

$$(Hu)(\omega) = \int_0^1 G(\omega, \theta) f(\theta, u(\theta)) d\theta \geq \int_0^1 G(\omega, \theta) f(\theta, v(\theta)) d\theta = (Hv)(\omega).$$



Since  $G(\omega, \theta) > 0$ , for  $\omega \in L$ . This proves that  $H$  is also weakly increasing mapping.

Also, for all  $u, v \in E$  with  $u \geq v$  implies that

$$(22) \quad \sup \{|u(\omega) - v(\omega)|, \omega \in L\} \geq \sup \{|Hu(\omega) - u(\omega)|, \omega \in L\},$$

and so, in term of metric

$$(23) \quad d(u, v) \geq d(Hu, u) \geq \frac{1}{2}d(Hu, u).$$

This implies

$$\begin{aligned} d(Hu, Hv) &= \sup_{\omega \in L} |(Hu)(\omega) - (Hv)(\omega)| = \sup_{\omega \in L} ((Hu)(\omega) - (Hv)(\omega)) \\ &= \sup_{\omega \in L} \int_0^1 G(\omega, \theta) [f(\theta, u(\theta)) - f(\theta, v(\theta))] d\theta \\ &= \sup_{\omega \in L} \int_0^1 G(\omega, \theta) \lambda \sqrt{\log[(u(\theta) - v(\theta))^2 + 1]} d\theta \\ &= \sup_{\omega \in L} \int_0^1 G(\omega, \theta) \lambda \sqrt{\log[d(u, v)^2 + 1]} d\theta \\ (24) \quad &= \lambda \sqrt{\log[d(u, v)^2 + 1]} \sup_{\omega \in L} \int_0^1 G(\omega, \theta) d\theta. \end{aligned}$$

It is easy to calculate that

$$\int_0^1 G(\omega, \theta) d\theta = \frac{-\omega^2}{2} + \omega,$$

and so

$$(25) \quad \sup_{\omega \in L} \int_0^1 G(\omega, \theta) d\theta = \frac{1}{2}.$$

On using (25) in (24), we get

$$(26) \quad d(Hu, Hv) \leq \frac{\lambda}{2} \sqrt{\log[d(u, v)^2 + 1]}.$$

Since,  $\lambda \in (0, 2]$ , we obtain

$$d(Hu, Hv) \leq \sqrt{\log[d(u, v)^2 + 1]},$$

and that

$$(27) \quad d(Hu, Hv)^2 \leq \log[d(u, v)^2 + 1].$$

Assuming  $\psi(\omega) = \omega^2$  and  $\beta(\omega) = \log[\omega^2 + 1]$ . Then clearly,  $\psi \in \Psi$ , and for all  $u > 0$ ,  $\psi(u) > \beta(u)$ .

Relation (27) implies that

$$\psi(d(Hu, Hv)) \leq \beta(d(u, v))$$

$$\begin{aligned} &\leq \beta \left( \max \left\{ d(u, v), \frac{d(u, fu)d(v, fv)}{[1 + d(u, v)]}, \frac{d(v, fv)[1 + d(u, fu)]}{[1 + d(u, v)]} \right\} \right) \\ &= \beta(M(u, v)). \end{aligned}$$

Also,

$$H(0) = \int_0^1 G(\omega, \theta) f(\theta, 0) d\theta \geq 0.$$

Thus one by one all assumptions of Theorem 2.1 are satisfied and therefore, the function  $H$  has a unique non negative solution.  $\square$

#### 4. CONCLUSION

In this manuscript, we have first defined a generalized  $C_{\beta}^{\psi}$ -rational contraction and then derived our main result Theorem 2.1. Some consequence results (Corollary 2.1, 2.2) and Remarks 2.1, 2.2 flaunted that our result is a proper generalization and extension of some previous existing results. As an application of our main result, we have presented an example to find the existence and uniqueness of solutions of second order boundary value problem.

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