Spectrum of continuous two-contours system

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Abstract. A deterministic continuous dynamical system is considered. This system contains two contours. The length of the ith contour equals c_i, $i = 1, 2$. There is a moving segment (cluster) on each contour. The length of the cluster, located on the ith contour, equals l_i, $i = 1, 2$. If a cluster moves without delays, then the velocity of the cluster is equal to 1. There is a common point (node) of the contours. Clusters cannot cross the node simultaneously, and therefore delays of clusters occur. A set of repeating system states is called a spectral cycle. Spectral cycles and values of average velocities of clusters have been found. The system belongs to a class of contour systems. This class of dynamical systems has been introduced and studied by A.P. Buslaev.

1 Introduction

In works of A.P. Buslaev basic approaches for describing flows on complex networks have been laid down. In accordance with these approaches a model is a dynamical system. The supporter of this system is a system of contours with a network structure. Particles (clusters) move on contours in accordance with some rules. Some limitations are imposed on the system. These limitations allow us to study the system analytically.

In [1] (Nagel, Schreckenberg, 1992), a transport model has been introduced. This model is a cellular automaton. In the model, particles move on an infinite or closed sequence of cells in accordance with given rules. In general case, Nagel–Schreckenberg model and its versions are too complicated for analytic research and were studied by simulation.

Analytic results for a simple version of Nagel–Schreckenberg model have been obtained in [2] (Belitzky, Ferrary, 2005) (a preprint of this paper has been published in 1999). These results have been obtained under assumption that, at any step, each particle moves onto a cell forward if the cell ahead is vacant and does not move if this cell is occupied. It is noted in [2] that the model is equivalent to the elementary cellular automaton 184 (CA 184) in classification of Wolfram, [3]. Results, similar to results of [2], have been obtained independently in [4] (Blank, 2000). In accordance with results of [3], [4], all particles move after some moment at every time for any initial state if the density of particles (the number of particles divided by the number of cells) is

not more than ½, then the average velocity of particles (the average number of transitions of a particle per time unit) equals $(1-\rho)/\rho$, where ρ is the density. In [5] (Gray, Griffeath, 2001), analytical results have been obtained for somewhat more general traffic model. In [6] (Kanai, 2008), analytical results have been obtained for a stochastic version of the traffic model. Some generalizations of results of [3]–[5] have been obtained in [7] (Blank, 2010).

A two-dimensional traffic model with a toroidal supporter (BML traffic model) has been introduced in [8] (Biham, Middelton, Levin, 1992). In this model, particle move in accordance with a rule, similar to the rule CA 184. Conditions of self-organization for BML model has been obtained in [9] (Austin, Benjamini, 2006).

The concept of a contour system has been introduced in [10] (Buslaev contour systems).

In [11] (Buslaev, Fomina, Tatashev, Yashina, 2018) the concept of spectrum of a contour system has been introduced for a deterministic dynamical system with a finite set of states. In such system, a sequence of states repeated periodically from some moment. This sequence of states is called a spectral cycle. The system, considered in [11], is a closed chain of contours. Particles move on each contour in accordance with the rule of the cellular automaton 240 (CA 240). There is one cluster on each contour. The spectrum of the system is a set of spectral cycles and corresponding values of clusters velocities.

In [12] (Buslaev, Tatashev, Yashina, 2016), [13] (Buslaev, Tatashev, 2017) and [14] (Buslaev, Tatashev, 2018), a discrete two-contours system was considered. In this system, particles move on contours in accordance with the rule of CA 184 or CA 240. In [12], [13], the following generalization was also considered. The supporter of the system contains N contours. There is one common point of the contours. In [12]–[14] theorems have been proved for different versions of movement rules. In [12]–[14], mainly, systems with contours of the same length were considered. For a system, containing contours of different lengths, in [13] conditions of self-organization (system resulting in a state of free movement from any initial state) have been obtained.

Buslaev contour systems were also studied in [13]–[24].

In this paper, a pair of contours is studied such that the lengths of the contours are different. There is a moving cluster on each contour. There exists a common point of contours (node). Delays occurs at the node. We have been found spectral cycles and obtained formulas for velocities of clusters.

2 System description

We consider a discrete dynamical system containing two *contours 1 and 2*, figure 1. The length of the contour i is equal to c_i, $i = 1, 2$. There is a moving segment *cluster* (cluster i) on the contour i ($i = 1, 2$). The contours have a common point *(node)*. At any time, each cluster moves in the direction of movement. The cluster i *passes* the distance c_i per c_i time units (the velocity equals 1) if there is no delay, $i = 1, 2$. The length of the cluster i is equal to $0 < l_i < c_i$, $i = 1, 2$. The coordinate system $[0, c_i]$ is introduced on the contour i, $i = 1, 2$. The direction of the coordinate axis is reverse to the direction of movement by modulo c_i, $i = 1, 2$. The coordinate of the node is equal to 0 for both contours. The state of the system is the vector

$$(\alpha_1(t),\ \alpha_2(t)),$$

where $\alpha_i(t)$ is the coordinate of the leading point of the cluster i, $i = 1, 2$. We say that the cluster i *covers the node* at time t if $c_i - l_i < \alpha_i(t) < 1$, $i = 1, 2$. We say that the cluster i *is at the node* at time t if $\alpha_i(t) = 0$, $i = 1, 2$. *Admissible* states of the system are only the states such that no more than cluster covers the node. If at time t, the cluster i, is at the node, and the other cluster covers the node, then a delay of the cluster i occurs, $i = 1, 2$. If both clusters are at the node, i.e., the system is in the state (0, 0), then a *conflict* occurs. In the case of a conflict, only one cluster moves in accordance with a *conflict resolution rule*. We suppose that the conflict resolution rule is the following. If a conflict occurs at the initial time $t = 0$, or a conflict occurs at time $t = t_0$ and there are no delays at the time interval [0, t), then, at time $t = t_0$, the cluster 1 moves. Assume that a conflict occurs at the time $t = t_0$, i.e., $\alpha_1(t_0) = \alpha_2(t_0) = 0$. Suppose the latest delay in the time interval [0, t_0) occurs at time $t = t_1$, and, at time t_1, the cluster i_0, $i_0 = 1, 2$, does not

move. Then, at time t_0, the same cluster i_0 does not move and the other cluster moves.

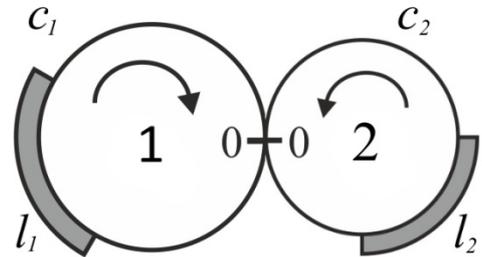The *initial* state of the system is given. This state should be admissible.

Fig. 1. A two-contours system.

3 Concepts of spectral cycle, velocity and free movement

A *spectral cycle* is a set of states, repeated periodically. Suppose T is the *period* of the cycle; H_i is the distance that the cluster i passes on a spectral cycle for the period), $i = 1, 2$. The number

$$v_i = \frac{H_i}{T}$$

is called *the average velocity of the cluster i on the spectral cycle, $i = 1, 2$*.

The system is in state of free *movement* from some moment t_0 if both clusters move at any moment $t \geq t_0$. If the system results in a state of free movement, then the velocity of clusters is equal to 1.

4 Behavior of the system

We have proved the following theorems, which give conditions of free movement.

Theorem 1. *If $\frac{c_2}{c_1}$ is an irrational number, then the system does not result in a state of free movement from any initial state.*

Theorem 2. *Suppose $\frac{c_2}{c_1}$ is a rational number, and d is the greatest common divisor of c_1 and c_2, i.e., d is the greatest number such that there exist natural numbers k_1, k_2 satisfying the condition $k_1 d = c_1$, $k_2 d = c_2$. Then the following is true.*

If the condition holds

$$l_1 + l_2 \leq d, \qquad (1)$$

then the system results in the state of free movement over a finite time from any initial state (self-organization).

If (1) does not result in the state of free movement over a finite time from no initial state.

Assume that the condition (1) does not hold.

Let us introduce auxiliary parameters g_1, g_2 b_1, b_2 and describe a way to calculate these parameters.

We shall see that, if inequality (1) does not hold, there is one spectral cycle or there are two spectral cycles depending on values of a_1, a_2, b_1, b_2. The average velocities of clusters depend on these parameters.

Suppose A is the set of system states such that one cluster does not move in that state; A_i is the set of system states such that the cluster i does not move, $i = 1, 2$. The set A_1 contains states $(0, \alpha_2)$, $c_2 - l_2 < \alpha_2 < 1$. The set A_2 contains states $(\alpha_1, 0)$, $c_1 – l_1 < \alpha_1 < 1$. We have $A = A_1 \cup A_2 \cup (0, 0)$. Since the condition of self-organization does not hold, then the system results in the states set A from any initial state over a finite time, and comes out of the set A through the state $(0, c_2 - l_2)$ or $(c_1 - l_1, 0)$.

Assume that, at time t_0, the system is in the state $(0, c_2 - l_2)$. Let b be a number such that $0 < b < l_2$. Suppose there exist integer non-negative numbers k, s satisfying the equality

$$a_1 k – a_2 s = c_2 – l_2 + b. \qquad (2)$$

Due to that (1) does not hold, there exists k and s satisfying the condition.

Let $k_0(j)$ be the minimum integer non-negative value of k such that there exists s satisfying the condition (2). Then at time $t_0 + c_1 k_0(j)$ the system results in the state $(0, c_2 - b) \in A_1$ if the system has not resulted in another state belonging to A.

Denote by $g(b)$ the value $c_1 k_0(b)$, $0 < b < l_2$. If for a fixed value 0 there do not exist integer non-negative solutions of (1), then we assume that $g_1(b) = \infty$. Let b be an integer number such that $-l_1 < b < 0$. Assume that there exist integer non-negative numbers k, s satisfying (2). Let $k_0(j)$ be a minimum integer non-negative value of k such that there exists s satisfying the condition. Then at time $t_0 = c_1 k_0(b)+|b|$ the system results in the state $(c_1 - |b|, 0) \in A_2$ if the system has not resulted in another state belonging to A. Denote by $g_1(b)$ the value $c_1 k_0(b) + |b|$, $-l_1 < b < 0$. If for a given value b there do not exist integer non-negative solutions of (2), we assume that $g_1(b) = \infty$. Suppose there are exist integer non-negative numbers k, s such that $c_1 k - c_2 s = c_2 - l_2$. Let $k_0(0)$ be the minimum integer non-negative number k such that there exist s satisfying the condition. Then at time $t_0 + k_0(b)$ the system results in the state $(0, 0)$ if the system has not resulted in another state belonging to A. Denote by $g_1(0)$ the value $c_1 k_0(0)$. If for $b = 0$ there do not exist integer non-negative solutions of (2), then we assume that $g_1(0) = \infty$.

Suppose $g_1 = g_2(b_1) = \min g_1(b)$, $-l_1 < b < l_2$; then, at time $t_0 + g_1$, the system results either in the state $(0, c_2 - b_1)$ if $0 < b_1 < l_2$, or in the state $(c_1 - |b_1|, 0)$ if $-l_1 < b_1 \leq 0$, or in the state $(0, 0)$ if $b_1 = 0$.

Assume that at time t_0, the system is in the state $(c_1 - l_1, 0)$. Let b be an integer number such that $0 < b < l_1$. Assume that there are exist integer non-negative numbers k, s such that

$$c_2 s - c_1 s = c_1 - l_1 + b. \qquad (3)$$

Suppose $s_0(j)$ is the minimum integer non-negative number s such there exists integer non-negative number k satisfying the condition (3). Then, at time $t_0 + s_0(j)$, the system results in the state $(c_1 - b, 0) \in A_2$ if the system has not resulted in another state belonging to the set A. Denote by $g_2(b)$ the value $c_2 s_0(j)$, $0 < b < l_1$. If, for given b, there are no integer non-negative numbers satisfying (3), then we suppose $g_1(b) = \infty$. Let b be an integer number such that $-l_2 < b < 0$. Assume that there are integer non-negative

numbers k, s satisfying (3). Let $s_0(j)$ be the minimum integer s such that there exists k satisfying the condition. Then at time $t_0 + c_2 s_0(b) + |b|$ the system results in the state $(0, c_2 - |b|) \in A_1$ if the system has not resulted in another state belonging the set A. Denote by $g_2(b)$ the value $c_2 s_0(j) + |j|$, $-l_2 < j < 0$. If, for given b, there are no integer non-negative numbers k, s satisfying (4), then we assume that $g_1(0) = \infty$. Suppose that there exist integers non-negative numbers such that $c_2 s - c_1 k = c_1 - l_1$, and $s_0(0)$ is the minimum value of s such that there exists k satisfying the condition. Then the system, at time $t_0 = c_2 s_0(0)$, results in the state $(0, 0)$ if the system has not resulted in another state belonging to the set A. If, for $b = 0$, there exist no integer non-negative numbers satisfying (3), then we assume that $g_1(0) = \infty$.

Denote by $g_2(0)$ the value $c_2 b_0(0)$. Suppose $g_2 = g_2(b) = \min(b)$, $-l_2 < j < l_1$. Then, at time $t_0 + g_2$, the system results either in the state $(c_1 - b_2, 0)$ if $0 < b_2 < l_1$, or in the state $(0, c_2 - b_2)$ if $-l_2 < b_2 < 0$, or in the state $(0, 0)$ if $b_2 = 0$.

Theorem 3. *Suppose inequalities $b_1 \geq 0$, $b_2 < 0$ hold; then there exist a unique spectral cycle, and this cycle contains the state $(0, c_2 - l_2)$. The period of the cycle is equal to $g_1 + l_2 - b_1$. Average velocities of clusters are equal to*

$$v_1 = \frac{g_1}{g_1 + l_1 - b_1}, v_2 = 1$$

Proof. Since (2) does not hold, the system does not result in the state of free movement. Hence the system results in a state, belonging the set A_1 , over a finite time, and, after this, in the state $(0, c_2 - l_2)$, or a state, belonging the set A_2 , and, after this, in the state $(c_1 - l_1, 0)$. From each of these state, the system result in a state, belonging the set $A_1 \cup (0, 0)$, and, returning in the set A, will be only in the set $A_1 \cup (0, 0)$. On the spectral cycle, the system is in states, not belonging to the set A_1 (both clusters move in these states), during g_1 time units and the system is in states, belonging to the set A_1 (only the cluster 2 moves in these states), during $l_1 - b_1$ time units. From this, Theorem 3 follows.

Theorem 4. *Suppose inequalities $b_1 < 0$, $b_2 \geq 0$ hold; then there exists a unique spectral cycle, and this cycle contains the state $(c_1 - l_1, 0)$. The period of this spectral cycle equals $g_2 + l_1 - b_2$. Velocities of clusters are equal to*

$$v_1 = 1, v_2 = \frac{g_2}{g_2 + l_1 - b_2},$$

Theorem 5. *Suppose inequalities $b_1 < 0$, $b_2 < 0$ hold. Then there a unique spectral cycle. The spectral cycle contains the states $(c_1 - l_1, 0)$ and $(0, c_2 - l_2)$. The period of the spectral cycle equals $g_1 + g_2 + l_1 + l_2 - |b_1| - |b_2|$. Velocities of clusters are equal to*

$$v_1 = 1 - \frac{c_2 - |b_2|}{g_1 + g_2 + l_1 + l_2 - |b_1| - |b_2|},$$

$$v_2 = 1 - \frac{c_1 - |b_1|}{g_1 + g_2 + l_1 + l_2 - |b_1| - |b_2|}.$$

Theorems 4, 5 are proved similarly to Theorem 3.

Theorem 6. *Suppose inequalities $b_1 \geq 0$, $b_2 \geq 0$ hold. Then there are two spectral cycles. One of these cycles contains the state $(0, c_2 - l_2)$. The period of this cycle equals $g_1 + l_2 - b_1$. On this cycle, velocities of clusters are equal to*

$$v_1 = \frac{g_1}{g_1 + l_2 - b_1}, v_2 = 1.$$

The other cycle contains the state $(c_1 - l_1, 0)$. The period of this cycle equals $g_2 + l_1 - b_2$. On this cycle, velocities of clusters are equal to

$$v_1 = 1, v_2 = \frac{g_2}{g_2 + l_1 - b_2}.$$

Proof. Depending on the initial state, the system results in a state of the set A_1 or in the set A_2. If the system is in the state of the set A_1, then, returning to the set A, the system will only in states of $A_1 \cup (0, 0)$. If the system is in the state of the set A_2, then, returning to the set A, the system will only in states of $A_2 \cup (0, 0)$. The rest of the proof of Theorem 6 is similar to the proof of Theorem 3.

Example 1. Suppose $c_1 = 4$, $l_1 = 2$, $c_2 = 6$, $l_2 = 2$.

In this case we have

$$b_1 = 0, g_1 = 4, b_2 = 0, g_2 = 6.$$

There exist two spectral cycles. On one of these cycles, the clusters move with velocities

$$v_1 = \frac{2}{3}, v_2 = 1.$$

The period of this cycle equals $T = 6$.

On the other spectral cycle, the clusters move with velocities

$$v_1 = 1, v_2 = \frac{3}{4}..$$

The period of this cycle equals $T = 8$.

5 Conclusion

We study the spectrum of two-contours system with contours of different lengths. The length of the contour i is equal to c_i, $i = 1,2$. There is a moving cluster on each contour. The length of the cluster, moving on the contour i, is equal to l_i, $i = 1,2$. There is a common point of the contours. This point is called the node. More than one cluster cannot cross the node simultaneously. A cluster stops if it comes to the node at time such that at this time the other cluster crosses the node. If clusters come to the node simultaneously, then only one the cluster moves in accordance with a conflict resolution rule. A set of states such that these states are repeated periodically is called a spectral cycle. We say that the system has the property of self-organization if the system results in the state of free movement over a finite time. In this paper, we have obtained conditions of self-organization. We have proved that, if the condition of self-organization does not hold, then, depending on c_1, c_2, l_1, l_2, there are one or two spectral cycles. Formulas for average velocities of clusters have been obtained.

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