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Effects of Dynamic Variable - Value Ordering Heuristics on the Search Space of Sudoku Modeled as a Constraint Satisfaction Problem

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Abstract We carry out a detailed analysis of the effects of different dynamic variable and value ordering heuristics on the search space of Sudoku when the encoding method and the filtering algorithm are fixed. Our study starts by examining lexicographical variable and value ordering and evaluates different combinations of dynamic variable and value ordering heuristics. We eventually build up to a dynamic variable ordering heuristic that has two rounds of tie-breakers, where the second tie-breaker is a dynamic value ordering heuristic. We show that our method that uses this interlinked heuristic outperforms the previously studied ones with the same experimental setup. Overall, we conclude that constructing insightful dynamic variable ordering heuristics that also utilize a dynamic value ordering heuristic in their decision making process could improve the search effort for some NP-Complete problems.

Keywords: Backtracking-Search, Constraint Satisfaction Problems, Dynamic Variable Ordering Heuristics, NP-Completeness, Phase-Transition, Sudoku

1 Introduction

The manner in which the search space of a problem that has been modeled as a CSP (Constraint Satisfaction Problem) is explored depends on the various techniques that are built into the given CSP solver. These techniques might include filtering algorithms such as arc-consistency and path-consistency, random-restarts, back-jumping as well as dynamic variable ordering and dynamic value ordering heuristics.[0] All of these methods are important in their own right, but dynamic variable ordering and dynamic value ordering heuristics are especially important since they guide the backtracking search. In other words, combination of these heuristics could choose a variable and value pair that direct the search along a path

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where it is hard to recognize that no solution exists. Therefore, it is very important that these heuristics choose a variable and value pair that is more likely to either, direct the search along a path that is easy to determine that no solution exists, or direct the search along a path where a solution exists.

The study of dynamic variable ordering heuristics began with the dom heuristic of [3] that picks the next variable with the smallest domain size. A tie-breaker was introduced for the dom heuristic in [4] that breaks ties by picking the next variable with the highest initial degree, the number of unassigned neighbors. A more sophisticated version of this heuristic was already in use for graph coloring [5]. This heuristic breaks ties for the dom heuristic by picking the next variable with the current highest degree.[1] A variant of this heuristic was proposed in [6], where the next variable with the minimum value of current domain size over current degree is selected. This heuristic works well for instances where the variables have wide range of degree in combination with small variance in their domain sizes, since dom/deg gives equal importance to domain size and degree. On the other hand, there has not been much attention given to dynamic value ordering. The most prominent work on this front was the lvo heuristics of [4] with four different ranking functions that employ forward-checking.

In this paper, we investigate various dynamic variable and value ordering heuristics with respect to Sudoku modeled as a CSP. Sudoku was first studied as a CSP in [7] and has been been getting a lot of attention in this field since then [8, 9, 10, 11, 12, 13], because it is a highly constrained challenging combinatorial search problem. We generated numerous Sudoku puzzles of size 16 by 16 (order-4), 25 by 25 (order-5) and 36 by 36 (order-6) using the setup outlined in [9]. We encoded the Sudoku puzzles as a CSP using the modeling technique from [11, 12] and also employed their arc-consistency algorithm. We observed the mean depth, mean number of explored variables and mean number of instantiations within a fixed time as well as the percentage of solved puzzles and mean time. We did this for seven different heuristics, including dom [3], an interpretation of dom+deg [5] and sd-lrv-mfv [11]. We refrained from investigating dom/deg [6]heuristic since it was already shown to not perform well for multiple permutation problems such as Latin Squares in [14] and Sudoku in [11].

We showed that lexicographically choosing the next variable performs better than randomly choosing the next variable. We demonstrated that using a dynamic variable ordering heuristic performs better than either randomly or lexicographically selecting the next variable. We also showed that the width of the search tree decreased when the dynamic variable ordering heuristic was used, which agrees with the results in [3, 15] and [16]. We demonstrated that adding a dynamic value ordering heuristic on top of a dynamic variable ordering heuristic does not necessarily improve the search effort, but instead adding a tie-breaker for the dynamic variable ordering heuristic does in fact improve the search effort. We showed that combining a dynamic value ordering heuristic with a dynamic variable ordering heuristic, that already utilizes a tie-breaker, might actually hinder the performance. We demonstrated that adding a dynamic value ordering heuristic as a second round of tie-breaker for the dynamic variable ordering heuristic enhances the search effort. We also detected the easy-hard-easy “phase-transition” as was previously observed with Sudoku puzzles in [9, 11, 12] and [13]. This is a phenomenon with all NP-Complete problems.

As far as we know, there has been no dynamic variable ordering heuristic that utilizes a dynamic value ordering heuristic as a tie-breaker, of course other than the ones introduced in [11]. Our experiments showed that even if we use a well-known dynamic variable ordering heuristic that employs a tie-breaker, such as dom+deg, that there could still be more ties and employing a dynamic value ordering heuristic as a second round of tie-breaker has a positive affect in the outcome of the search effort. Overall, we demonstrated that coming up with insightful dynamic variable ordering heuristics that also use dynamic value ordering heuristic in their decision making process, can help to guide the search effort for some NP-Complete problems. In fact, the performance achieved at the critical point by our most sophisticated heuristic is far better than the performance of the methods utilized in [9] and [13] with the same experimental setup.

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[0] An excellent resource for most of these CSP techniques and many more is [1]. An excellent introduction written in Spanish is [2].

[1] This heuristic was initially called the Brelaz heuristic, then dom+futdeg heuristic and finally dom+deg heuristic. We should also note that in earlier papers the heuristic in [4] was called the dom+deg heuristic and the heuristic in [5] was called the dom+futdeg heuristic.

2 Methods

2.1 Encoding

We encode the Sudoku puzzles as a constraint satisfaction problem by using the natural combined model from [11, 12].

2.1.1 Natural Combined Model

The Sudoku puzzle is formulated on a 3D $\mathcal{S}(k*k)*(k*k)*(k*k)$ Boolean matrix, where $n=(k*k)$. We say that a Sudoku puzzle of order k is an n by n Sudoku puzzle.

The primal variable $x_{i,j}$ is represented by the slice $\{\mathcal{S}(i,j,v)|1 \leq v \leq n\}$. The initial domain of $x_{i,j}$ is denoted $D(x_{i,j})$ and is $\{v|\mathcal{S}(i,j,v)=True, 1 \leq v \leq n\}$.

The box dual variable $b_{v,p,q}$ where $\lceil i/k \rceil = p$, $\lceil j/k \rceil = q$ is represented by the slice $\{\mathcal{S}(i,j,v)|((p-1)*k+1) \leq i \leq p*k, ((q-1)*k+1) \leq j \leq q*k\}$. The initial domain of $b_{v,p,q}$ is denoted $D(b_{v,p,q})$ and, is $\{i,j|\mathcal{S}(i,j,v)=True, ((p-1)*k+1) \leq i \leq p*k, ((q-1)*k+1) \leq j \leq q*k\}$.

The column dual variable $c_{v,j}$ is represented by the slice $\{\mathcal{S}(i,j,v)|1 \leq i \leq n\}$. The initial domain of $c_{v,j}$ is denoted $D(c_{v,j})$ and is $\{i|\mathcal{S}(i,j,v)=True, 1 \leq i \leq n\}$.

The row dual variable $r_{v,i}$ is represented by the slice $\{\mathcal{S}(i,j,v)|1 \leq j \leq n\}$. The initial domain of $r_{v,j}$ is denoted $D(r_{v,j})$ and is $\{j|\mathcal{S}(i,j,v)=True, 1 \leq j \leq n\}$.

Then the $\neq$ constraints are enforced as follows: 1) on pairs of primal variables within each box, column and row. 2) on pairs of box dual variables of each value. 3) on pairs of column dual variables of each value. 4) on pairs of row dual variables of each value.

2.2 Filtering Algorithms

The following filtering algorithms were introduced in [11, 12] and work together with the natural combined model. Although the backtracking-search is being conducted on the primal variables, consistency checks on the other three view-points of the problem are performed whenever deemed necessary.

2.2.1 Redundantly Modeled Forward Checking Algorithm (RFC)

The RFC algorithm verifies whether or not assigning a value to a variable is consistent with respect to it's constraints. It also modifies the domains of the constrained variables accordingly. These checks and modifications are performed with respect to primal variables as well as the box, column and row dual variables.

Algorithm 1 Redundantly Modeled Forward Checking

```
Input: $(S, n, i, j, v)$[2]
Output: Is $\{S(i,j,v) ==$ True$\}$ consistent?
$S_{temp} \leftarrow S$
if (Remove-All-Except-v == True) then
    if (Remove-All-Other-v == True) then
        $S \leftarrow S_{temp}$
        return True
    end if
end if
return False

function Remove-All-Except-V
    for $k = 1...n$ do
        if ($S_{temp}(i,j,k) ==$ True $\wedge k \neq v$) then
            if (Exists-In-Box$(i,j,k) ==$ False) then
                return False
            end if
```

```
                if (Exists-In-Col($(i, j, k)$ == False) then
                    return False
                end if
                if (Exists-In-Row($(i, j, k)$ == False) then
                    return False
                end if
                $S_{temp}(i, j, k) \leftarrow$ False
            end if
        end for
        return True
    end function

    function Exists-In-Row($(p, q, r)$)
        for $k = 1...n$ do
            if ($S_{temp}(p, k, r) ==$ True $\wedge k \neq q$) then
                return True
            end if
        end for
        return False
    end function[3]

    function Remove-All-Other-V
        if (Remove-V-From-Box == False) then
            return False
        end if
        if (Remove-V-From-Col == False) then
            return False
        end if
        if (Remove-V-From-Row == False) then
            return False
        end if
        return True
    end function

    function Remove-V-From-Row
        for $k = 1...n$ do
            if ($S_{temp}(i, k, v) ==$ True $\wedge k \neq j$) then
                $t \leftarrow$ False
                for $l = 1...n$ do
                    if ($S_{temp}(i, k, l) ==$ True $\wedge l \neq v$) then
                        $t \leftarrow$ True
                    end if
                end for
                if t == False then
                    return False
                end if
                if (Exists-In-Box($(i, k, v)$ == False) then
                    return False
                end if
                if (Exists-In-Col($(i, k, v)$ == False) then
                    return False
                end if
                $S_{temp}(i, k, v) \leftarrow$ False
            end if
```

```
        end for
        return True
    end function[4]
```

2.2.2 Redundantly Modeled Arc-Consistency Algorithm One (RAC-1)

The RAC-1 algorithm takes the RFC algorithm one step further. Instead of just examining the constrained primal and dual variables of the instantiated variable, it checks all of the primal as well as all of the dual variables. It then verifies that they are consistent with the given assignment. Furthermore, it performs instantiations whenever the domain of some primal or dual variable becomes singleton, which forces it to repeat the whole process again. If the RAC-1 algorithm successfully executes, that is without finding an inconsistency, then the given puzzle is arc-consistent with respect to the primal and the box, column and row dual variables.

Algorithm 2 Redundantly Modeled Arc-Consistency One

```
Input: $(S, n)$ [5]
Output: Is $S$ arc-consistent?
$S_{temp} \leftarrow S$
$t \leftarrow 1$
while $t \neq 0$ do
    $t \leftarrow 0$
    if (Check-Primal-Variables == False) then
        return False
    end if
    if (Check-Box-Variables == False) then
        return False
    end if
    if (Check-Col-Variables == False) then
        return False
    end if
    if (Check-Row-Variables == False) then
        return False
    end if
end while
$S \leftarrow S_{temp}$
return True

function Check-Primal-Variables
    for $i = 1...n$ do
        for $j = 1...n$ do
            if $S_{temp}(i, j, 0) == 0$ then[6]
                $c \leftarrow 0$
                $v \leftarrow 0$
                for $k = 1...n$ do
                    if $S_{temp}(i, j, k) ==$ True then
                        $c \leftarrow c + 1$
                        $v \leftarrow k$
                    end if
                end for
                if c==0 then
```

---

[2] S, n, i, j, v and S_{temp} are global variables and accessible by all functions.

[3] Exists-In-Box(p, q, r) and Exists-In-Col(p, q, r) functions behave similarly.

[4] Remove-V-From-Box and Remove-V-From-Col functions behave similarly.

```
                    return False
                else if c==1 then
                    if RFC($S_{temp}, n, i, j, v$) == True then
                        $t \leftarrow 1$
                    else
                        return False
                    end if
                end if
            end if
        end for
    end for
    return True
end function[7]
```

2.3 Search Heuristics

The following heuristics are utilized with respect to primal variables.

2.3.1 Heuristic-1 (RAN&LEX)

Randomly select the unassigned variable; and lexicographically select the values in its domain.

2.3.2 Heuristic-2 (LEX&LEX)

Lexicographically select the unassigned variable; and lexicographically select the values in its domain.

2.3.3 Heuristic-3 (SD&LEX)

Lexicographically select the unassigned variable with the smallest domain size; and lexicographically select the values in its domain.

2.3.4 Heuristic-4 (SD&MFV)

Lexicographically select the unassigned variable with the smallest domain size; and order the values in it's domain that occur from most frequently to least frequently in the domains of all of the assigned variables.

2.3.5 Heuristic-5 (SD-LRV&LEX)

Among the variables with the smallest domain size, lexicographically select the variable that has the least number of fixed variables among those with which it shares a constraint; and lexicographically select the values in its domain.

2.3.6 Heuristic-6 (SD-LRV&MFV)

Among the variables with the smallest domain size, lexicographically select the variable that has the least number of fixed variables among those with which it shares a constraint; order the values in its domain that occur from most frequently to least frequently in the domains of all of the assigned variables.

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[5] S, n and S_{temp} are global variables and accessible by all functions.

[7] Check-Box-Variables, Check-Col-Variables and Check-Row-Variables functions behave similarly.

[6] We assume that the integer puzzle is located at $v = 0$ of $S(i, j, v)$ for simplicity.

2.3.7 Heuristic-7 (SD-LRV-MFV&MFV)

Among the variables with the smallest domain size that also have the least number of fixed variables among those with which it shares a constraint, lexicographically select the one that contains a value in its domain that occurs the most frequent times in the domains of all of the assigned variables; order the values in its domain that occur from most frequently to least frequently in the domains of all of the assigned variables.

Heuristic-1 serves the purpose of understanding how the backtracking search performs when the next variable to be assigned is chosen uniformly at random. Heuristic-3 is clearly the dom heuristic of [3]. Heuristic-4 serves the purpose of understanding how a dynamic variable ordering heuristic works with no tie-breaker, but with a value ordering heuristic. Heuristic-5 can actually be viewed as an interpretation of the Brelaz (dom+deg) heuristic of [5]. Heuristic-6 serves the purpose of examining how a dynamic variable ordering heuristic that has a tie-breaker works with a value ordering heuristic.

We should also note that we did experiment with the opposite tie-breaker, as was done in [11], the most number of constrained variables (MRV) and the opposite dynamic value ordering heuristic, order the values in it's domain that occur from least frequently to most frequently in the domains of all of the assigned variables (LFV). However, none of the three possible combinations of these tie-breakers (LRV and MRV) and dynamic value ordering heuristics (MFV and LFV) worked as well as Heuristic-7.

3 Experimental Setup

We implemented the modeling method, the filtering algorithm and the search heuristics in C++. Then we generated the test instances by following the methodology that was outlined in [9] for Sudoku puzzles. We first randomly filled 5%-25% percent of the empty puzzle board without invalidating the puzzle and then randomly chose one of our algorithms to complete the puzzle. We generated 100 fully solved Sudoku puzzles of order 4, 5 and 6, which correspond to 16 by 16, 25 by 25 and 36 by 36 size Sudoku puzzles, respectively.

Then we removed a cell from a given fully solved puzzle with probability p, where $p = 0$ implies a fully solved puzzle and $p = 1$ implies an empty one. We did this for all of the 100 fully solved puzzles of order 4 and 5 for each probability p, from $p = 0.05$ through $p = 0.95$ with 0.05 increments. We used the Mersenne Twister Pseudo-Random generator mt19937 in C++ for generating our random numbers, that were uniformly distributed between 0 and 1. A total of 1900 puzzles were generated for Sudoku puzzles of order 4 and 5. We only generated 100 Sudoku puzzles of order 6 at the hard region so that we can verify our observations carry over to larger puzzles.

All of the puzzles we generated are guaranteed to have a solution, but they can have more than one solution. It would take a tremendous amount of time to verify that each puzzle has a unique solution, because we would have to check the possibility of a solution at every single path. Furthermore, the puzzles become so sparse at higher probabilities that it is impossible to guarantee that they have a unique solution.

We set the time limit to 30 seconds for Sudoku puzzles of order 4 and to 360 seconds for Sudokdu puzzles of order 5 since these were the time limits used in [9] and [13]. We set the time limit to 720 seconds for Sudoku puzzles of order 6.

We kept track of six different parameters at each probability p: 1) Percentage of puzzles solved 2) Mean time 3) Mean depth of the search tree 5) Mean number of explored variables 6) Mean number of instantiations. If no solution was found within the allocated time then the given time limit and the calculated values for each parameter up to that time limit were used in the calculations. We used a computer that possesses an Intel Dual Core (Four Threads) I3-3227U CPU at 1.90GHZ x64 based processor with 4GB of RAM to run our simulations.

4 Empirical Analysis

4.1 Order 4 Puzzles

The performance of our seven heuristics when solving order 4 puzzles at different probabilities are presented in tables 1 2 and 3 as well as in the corresponding graphs. The results of the methods in [9] and [13] are also included as graphs that show the percentage of solved puzzles at each probability versus mean time in seconds.

We observe that Heuristic-1 performed the worst with respect to being able to solve all of the instances at each probability within the allocated time. Heuristic-1 is followed by Heuristic-2 and Heuristic-3 where they also had an overall worse performance than the methods in [9] and [13]. However, when they were able to solve the given instances, they did so much faster than the methods in [9] and [13]. This can be seen in the corresponding graphs that show the mean time in seconds. On the other hand, Heuristic-5 and Heuristic-7 were able to solve all of the instances at each probability within the allocated time. And they did so in a very fast manner as can be seen by the corresponding graphs that show the mean time in seconds.

When a heuristic at some probability has the same or almost the same average depth, average number of explored nodes and average number of instantiations, it means that the search backtracked few times or it did not backtrack at all. This is the case at all of the probabilities for Heuristic-5 and Heuristic-7 and most of the probabilities for Heuristic-4 and Heuristic-6. The reason for this occurrence cannot really be attributed to the choice of a dynamic variable and value ordering heuristics, but it is due to the modeling choice as well as the constraint propagation algorithm that is being employed. In fact, the reason that the first three heuristics actually performed this well is due to this. In other words, the constraint propagation algorithm along with the encoding method overpowered the variable ordering aspect of our solver when solving order 4 puzzles.

Another observation is that the average number of instantiations that are roughly double the average number of explored nodes starting from Heuristic-3 and on. This is due to using the smallest domain heuristic in variable selection that allows the width of the search tree to be narrower and consequently results in a deeper search-tree. This observation concurs with the results in [3, 15] and [16] that the width of the search-tree decreases when the smallest domain heuristic is utilized. This difference will be more striking with order 5 puzzles.

4.2 Order 5 Puzzles

The performance of our seven heuristics when solving order 5 puzzles at different probabilities are presented in tables 4 5 and 6 as well as in the corresponding graphs. The results of the methods in [9] and [13] are also included as graphs that show the percentage of solved puzzles at each probability versus mean time in seconds.

Heuristic-1 cannot solve any instances at probabilities 0.55, 0.6, 0.65 and 0.8 within the allocated time. Heuristic-2 performs some what better than Heuristic-1, but it still performs worse than the methods in [9] and [13]. On the other hand, the difference in performance between Heuristic-2 and Heuristic-3 is quite striking at each probability as can be observed from the corresponding graphs and tables. Heuristic-3 allowed the search effort to go further down in the search tree, which as a consequence resulted in more nodes being explored in the allocated time if no solution was found; or a solution was found with less number of explored nodes. This was also true with respect to the instantiations at each probability. As a result, the search was not stuck at a certain depth and more puzzles were solved at each probability within the given time limit. As we also observed with order-4 puzzles, the width of the search tree was minimized when Heuristic-3 was employed. This can be recognized with respect to the ratio between the average number of instantiations over the average number of explored nodes at each probability, which was approximately three with Heuristic-2 and two with Heuristic-3. This observation, once again, concurs with the results in [3, 15] and [16] that the width of the search-tree decreases when the smallest domain heuristic is utilized.

We also witness the easy-hard-easy “phase-transition” that is a phenomenon with all NP-Complete problems. The hard region or the critical point of Sudoku is around $p = 0.55$. It is around this region where the most difficult puzzles reside.

We observe from the corresponding graphs that Heuristic-5 significantly improved the search effort at the critical point of $p = 0.55$ compared to Heuristic-4 which performed similar to Heuristic-3. In fact, the difference of performance between Heuristic-3 and Heuristic-4 at the hard region was not statistically significant. This experiment distinguishes the importance of using a tie-breaker for the smallest domain heuristic over just coupling the smallest domain heuristic with a value ordering heuristic.

We also notice from the corresponding graphs that although Heuristic-6 performed a little better at the critical point of $p = 0.55$ compared to Heuristic-5, it actually performed worse at the subsequent probabilities of the heavy-tail region. This experiment demonstrates that adding value ordering heuristic without really gathering insight from the variable ordering heuristic does not necessarily yield better performance and it may actually hinder it.

As can be seen from the corresponding graphs, Heuristic-7 achieved a significant improvement over both Heuristic-5 and Heuristic-6. The performance at the critical point of $p = 0.55$ is the most profound compared to the previous six heuristics. Heuristic-7, on average, was able to go further down in the search tree, traversed less number of nodes and made less number of instantiations. This demonstrates that a second round of tie-breaker for the smallest domain heuristic could be very useful. Indeed, the value ordering heuristic could be used for this purpose and it executes well for this intent. In other words, considering the values in the domains of the variables when breaking ties is relevant.

We also observe from the corresponding graphs that the methods in [9] and [13] did as well as our Heuristic-4 at the critical point of $p = 0.55$ whereas our Heuristic-7 clearly out performed them at this significant point.

4.3 Order 6 puzzles

The performance of our seven heuristics when solving order 6 puzzles at the hard region are presented in table 7. We detect similar patterns with order 6 puzzles that we also observed with order 4 and 5 puzzles with respect to average depth, average nodes and average instantiations. However, this time the differences among the heuristics are more profound, because there are more unassigned variables with domain sizes that are greater than two and as a result, Heuristic-1 and Heuristic-2 perform even worse. Although Heuristic-3 and Heuristic-4 cannot solve any instances within the allocated time, they still perform better than Heuristic-1 and Heuristic-2 since they were able to go further down the search-tree and thus explored more nodes on average. Heuristic-7 solves the most instances and it is followed by Heuristic-6 and Heuristic-5. The performance of Heuristic-7 when solving order 6 puzzles at the hard region is further evidence that considering the values in the domains of the variables when selecting the next variable to instantiate does in fact make a difference in guiding the search effort.

	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
Heuristic-1	5	9	16	27	39	53	70	83	98	118
Heuristic-2	4	9	17	27	44	61	77	95	113	137
Heuristic-3	5	11	22	41	63	83	103	123	138	156
Heuristic-4	6	12	22	44	69	87	109	131	147	164
Heuristic-5	5	11	20	39	60	84	104	126	140	156
Heuristic-6	6	13	22	44	70	89	112	132	148	165
Heuristic-7	6	13	24	45	70	90	113	134	149	167

Table 1: Average depth of the search tree at each probability for order 4 puzzles

	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
Heuristic-1	28	160	2,273	4,269	5,609	7,020	11,222	8,493	5,897	5,328
Heuristic-2	6	87	442	2,339	6,825	3,267	5,471	9,774	4,065	1,482
Heuristic-3	6	27	94	3,338	3,295	3,897	3,403	124	138	156
Heuristic-4	9	50	158	97	1,997	175	129	145	148	164
Heuristic-5	5	18	25	52	62	98	105	126	140	156
Heuristic-6	9	16	30	64	3,436	96	112	137	4,431	165
Heuristic-7	9	16	49	50	76	102	114	135	149	167

Table 2: Average number of explored nodes at each probability for order 4 puzzles

	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
Heuristic-1	90	384	8,024	15,574	20,969	25,509	40,781	29,485	21,714	19,225
Heuristic-2	9	206	1,061	7,663	20,936	8,934	15,501	27,426	11,076	3,267
Heuristic-3	9	46	170	6,640	6,532	7,741	6,715	125	139	157
Heuristic-4	13	91	297	154	3,928	266	150	161	149	165
Heuristic-5	7	28	34	67	65	114	108	127	142	157
Heuristic-6	12	20	41	86	6844	104	113	143	8771	165
Heuristic-7	14	20	76	57	83	117	116	136	149	167

Table 3: Average number of instantiations at each probability for order 4 puzzles

	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
Heuristic-1	22	49	69	101	138	170	210	248	288	326
Heuristic-2	20	48	70	102	141	169	216	260	280	320
Heuristic-3	26	69	102	154	203	250	300	346	387	427
Heuristic-4	25	70	107	162	218	263	321	361	404	446
Heuristic-5	25	69	108	153	206	253	301	348	391	429
Heuristic-6	27	72	118	163	219	263	313	362	398	442
Heuristic-7	26	74	116	162	216	266	319	362	404	443

Table 4: Average depth of the search tree at each probability for order 5 puzzles

	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
Heuristic-1	47,044	133,714	135,957	142,250	140,868	142,004	156,206	134,533	113,051	73,159
Heuristic-2	40,706	135,905	134,490	106,990	90,329	101,750	97,151	107,546	204,505	138,951
Heuristic-3	38,130	158,853	150,619	70,106	18,159	41,850	8,509	7,726	579	457
Heuristic-4	35,990	166,221	133,956	45,065	44,181	51,999	39,672	22,142	1,436	8,958
Heuristic-5	19,949	113,120	81,071	60,149	37,510	51,910	8,603	389	14,176	473
Heuristic-6	26,080	112,179	107,804	77,751	43,837	23,859	31,291	7,038	20,272	6,979
Heuristic-7	27,539	101,227	73,391	37,723	19,038	14,087	12,826	16,465	514	662

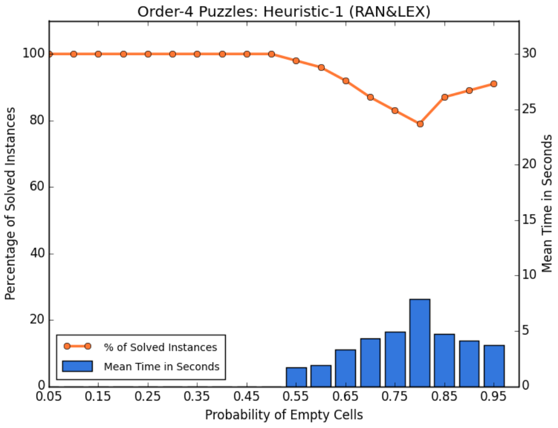
Table 5: Average number of explored nodes at each probability for order 5 puzzles

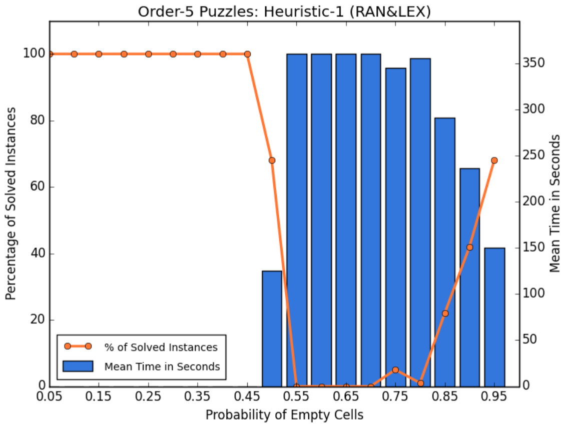
	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
Heuristic-1	176,119	515,880	535,476	566,626	561,713	561,895	611,545	520,519	433,597	278,044
Heuristic-2	112,075	391,026	395,833	300,883	258,723	290,502	286,054	306,336	597,076	466,662
Heuristic-3	76,250	317,693	301,200	140,112	36209	85714	16741	15164	778	494
Heuristic-4	71,970	332,428	267,866	90,186	88,936	105,301	79,123	44,225	2,664	17,513
Heuristic-5	39,886	226,222	162,081	120,190	74,894	103,666	16,927	438	28,022	536
Heuristic-6	52,148	224,313	215,561	155,410	87,599	47,529	63,880	13,741	42,252	13,553
Heuristic-7	55,610	202,314	146,721	75,322	37,920	24,852	26,190	32,610	626	883

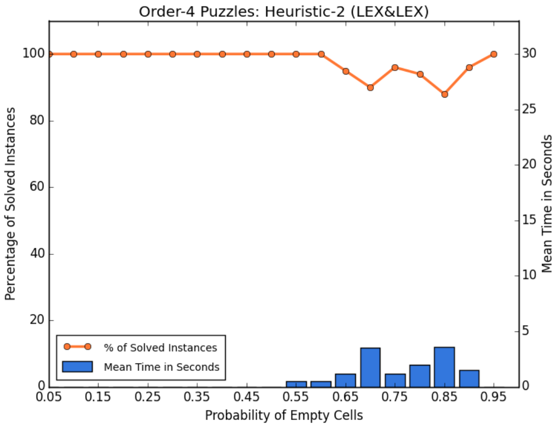
Table 6: Average number of instantiations at each probability for order 5 puzzles

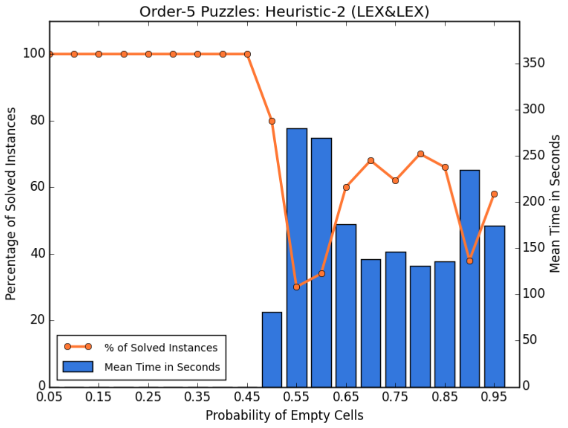
	Solved	Avg. Time	Avg. Depth	Avg. Nodes	Avg. Inst.
Heuristic-1	0%	720 secs	302	123,294	580,268
Heuristic-2	0%	720 secs	207	144,089	472,677
Heuristic-3	0%	720 secs	433	205,125	410,570
Heuristic-4	0%	720 secs	464	217,037	446,432
Heuristic-5	2%	691 secs	459	190,939	390,949
Heuristic-6	4%	677 secs	487	181,660	363,736
Heuristic-7	11%	623 secs	481	165,015	333,751

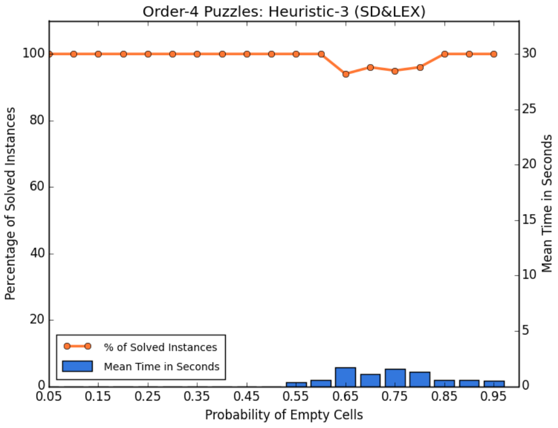
Table 7: 100 order 6 puzzles at the hard region with a time out of 720 seconds

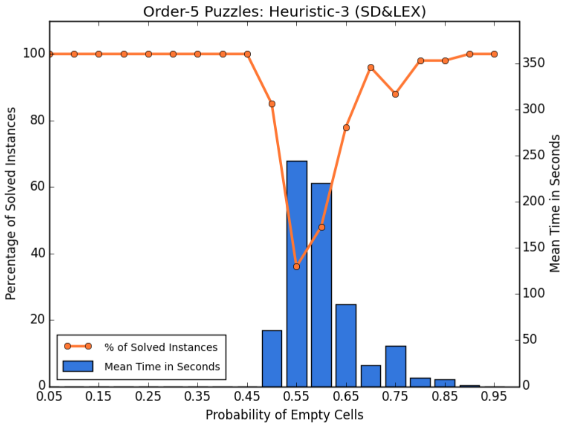

Order-4 Puzzles: Heuristic-1 (RAN&LEX)
Percentage of Solved Instances
0
20
40
60
80
100
Mean Time in Seconds
0
5
10
15
20
25
30
Probability of Empty Cells
0.05
0.15
0.25
0.35
0.45
0.55
0.65
0.75
0.85
0.95
% of Solved Instances
Mean Time in Seconds

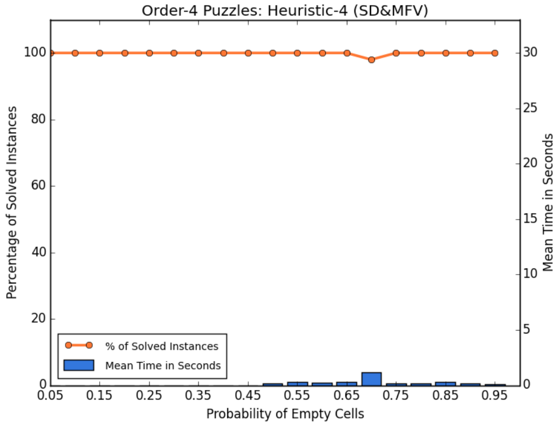

Order-5 Puzzles: Heuristic-1 (RAN&LEX)
Percentage of Solved Instances
0
20
40
60
80
100
Mean Time in Seconds
0
50
100
150
200
250
300
350
Probability of Empty Cells
0.05
0.15
0.25
0.35
0.45
0.55
0.65
0.75
0.85
0.95
% of Solved Instances
Mean Time in Seconds


Order-4 Puzzles: Heuristic-2 (LEX&LEX)
Percentage of Solved Instances
0
20
40
60
80
100
Mean Time in Seconds
0
5
10
15
20
25
30
Probability of Empty Cells
0.05
0.15
0.25
0.35
0.45
0.55
0.65
0.75
0.85
0.95
% of Solved Instances
Mean Time in Seconds


Order-5 Puzzles: Heuristic-2 (LEX&LEX)
Percentage of Solved Instances
0
20
40
60
80
100
Mean Time in Seconds
0
50
100
150
200
250
300
350
Probability of Empty Cells
0.05
0.15
0.25
0.35
0.45
0.55
0.65
0.75
0.85
0.95
% of Solved Instances
Mean Time in Seconds


Order-4 Puzzles: Heuristic-3 (SD&LEX)
Percentage of Solved Instances
0
20
40
60
80
100
Mean Time in Seconds
0
5
10
15
20
25
30
Probability of Empty Cells
0.05
0.15
0.25
0.35
0.45
0.55
0.65
0.75
0.85
0.95
% of Solved Instances
Mean Time in Seconds


Order-5 Puzzles: Heuristic-3 (SD&LEX)
Percentage of Solved Instances
0
20
40
60
80
100
Mean Time in Seconds
0
50
100
150
200
250
300
350
Probability of Empty Cells
0.05
0.15
0.25
0.35
0.45
0.55
0.65
0.75
0.85
0.95
% of Solved Instances
Mean Time in Seconds


Order-4 Puzzles: Heuristic-4 (SD&MFV)
Percentage of Solved Instances
Mean Time in Seconds
Probability of Empty Cells
% of Solved Instances
Mean Time in Seconds
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20
40
60
80
100
0
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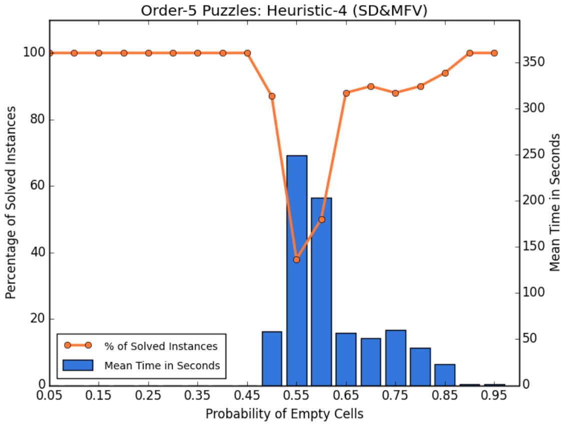
Order-5 Puzzles: Heuristic-4 (SD&MFV)
Percentage of Solved Instances
Mean Time in Seconds
Probability of Empty Cells
% of Solved Instances
Mean Time in Seconds
0
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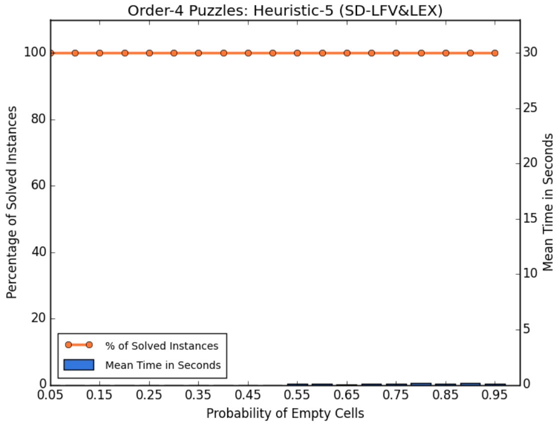
Order-4 Puzzles: Heuristic-5 (SD-LFV&LEX)
Percentage of Solved Instances
Mean Time in Seconds
Probability of Empty Cells
% of Solved Instances
Mean Time in Seconds
0
20
40
60
80
100
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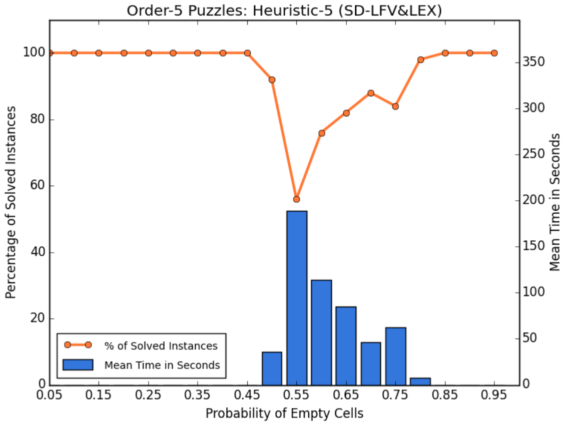
Order-5 Puzzles: Heuristic-5 (SD-LFV&LEX)
Percentage of Solved Instances
Mean Time in Seconds
Probability of Empty Cells
% of Solved Instances
Mean Time in Seconds
0
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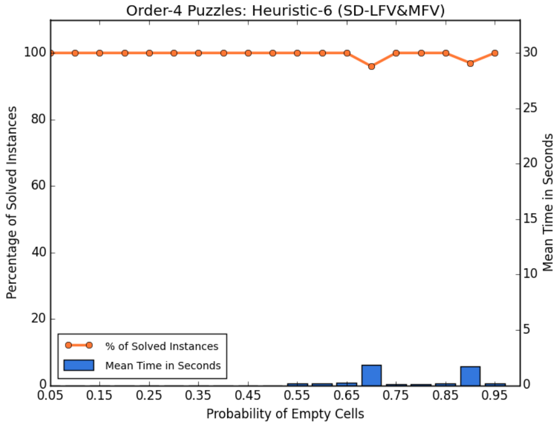
Order-4 Puzzles: Heuristic-6 (SD-LFV&MFV)
Percentage of Solved Instances
Mean Time in Seconds
Probability of Empty Cells
% of Solved Instances
Mean Time in Seconds
0
20
40
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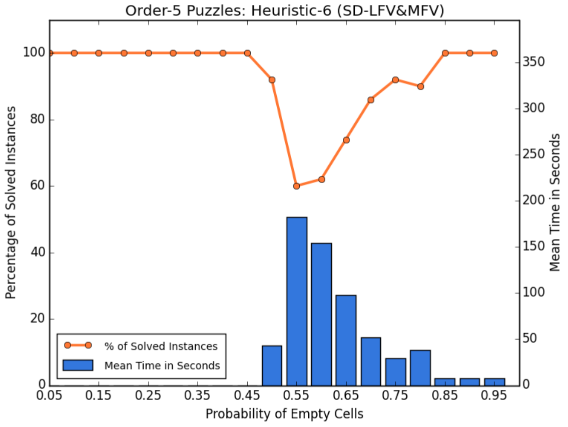
Order-5 Puzzles: Heuristic-6 (SD-LFV&MFV)
Percentage of Solved Instances
Mean Time in Seconds
Probability of Empty Cells
% of Solved Instances
Mean Time in Seconds
0
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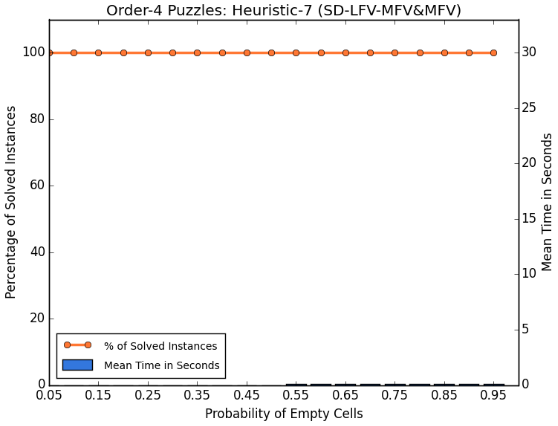
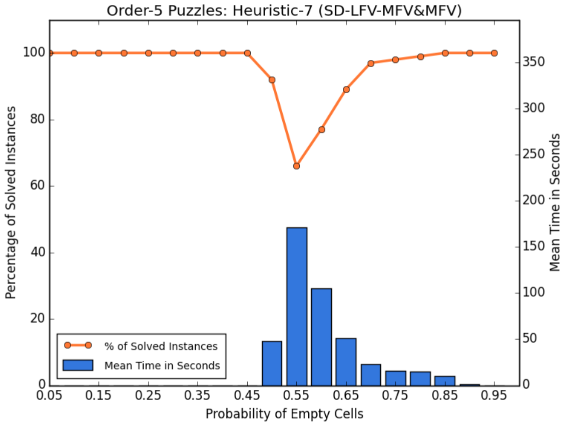

Order-4 Puzzles: Heuristic-7 (SD-LFV-MFV&MFV)
Percentage of Solved Instances
0
20
40
60
80
100
Mean Time in Seconds
0
5
10
15
20
25
30
Probability of Empty Cells
0.05
0.15
0.25
0.35
0.45
0.55
0.65
0.75
0.85
0.95
% of Solved Instances
Mean Time in Seconds


Order-5 Puzzles: Heuristic-7 (SD-LFV-MFV&MFV)
Percentage of Solved Instances
0
20
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60
80
100
Mean Time in Seconds
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350
Probability of Empty Cells
0.05
0.15
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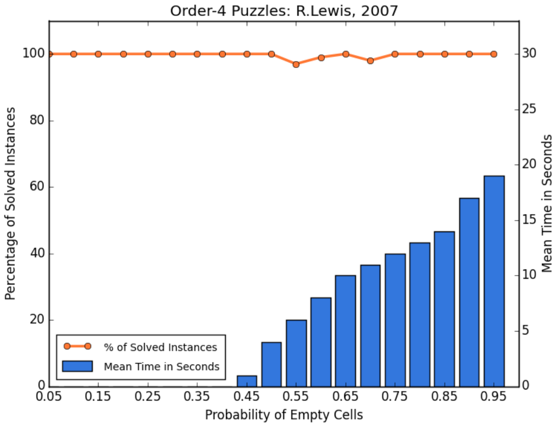
0.55
0.65
0.75
0.85
0.95
% of Solved Instances
Mean Time in Seconds


Order-4 Puzzles: R.Lewis, 2007
Percentage of Solved Instances
0
20
40
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Mean Time in Seconds
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Probability of Empty Cells
0.05
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0.25
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0.75
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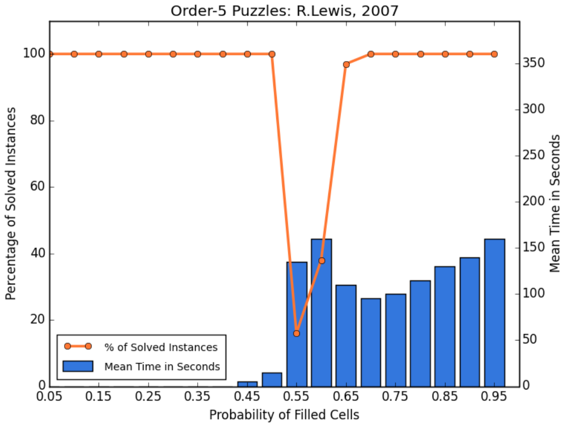
0.95
% of Solved Instances
Mean Time in Seconds


Order-5 Puzzles: R.Lewis, 2007
Percentage of Solved Instances
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20
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Mean Time in Seconds
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350
Probability of Filled Cells
0.05
0.15
0.25
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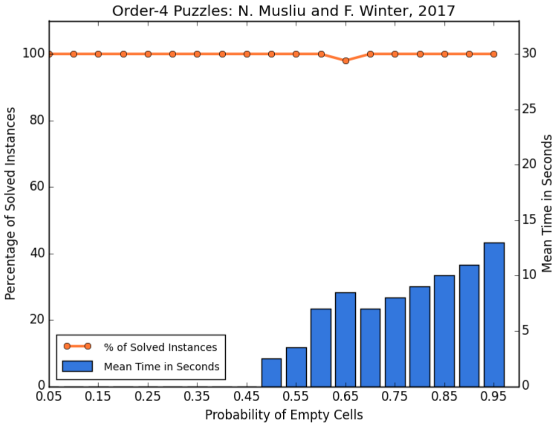
0.45
0.55
0.65
0.75
0.85
0.95
% of Solved Instances
Mean Time in Seconds


Order-4 Puzzles: N. Musliu and F. Winter, 2017
Percentage of Solved Instances
0
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40
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Mean Time in Seconds
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Probability of Empty Cells
0.05
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0.75
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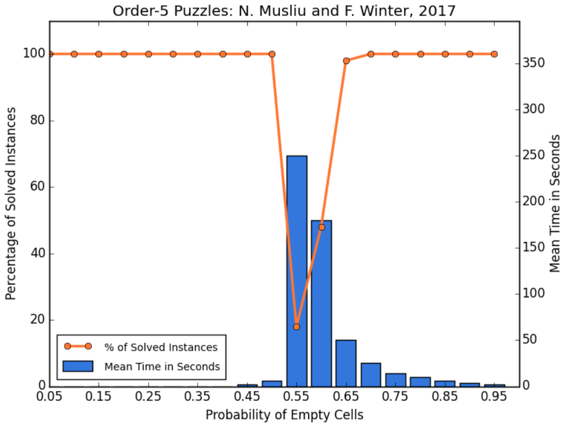
0.95
% of Solved Instances
Mean Time in Seconds


Order-5 Puzzles: N. Musliu and F. Winter, 2017
Percentage of Solved Instances
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Mean Time in Seconds
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Probability of Empty Cells
0.05
0.15
0.25
0.35
0.45
0.55
0.65
0.75
0.85
0.95
% of Solved Instances
Mean Time in Seconds

5 Conclusion

We conducted a detailed analysis of how dynamic variable and value ordering heuristics affect the search effort for Sudoku when the encoding method and the filtering algorithm are fixed. One of the most striking insights we gained from this experiment is the importance of incorporating a dynamic value ordering heuristic into the decision making process of a dynamic variable ordering heuristic. We observed that as the search space got smaller, there were still many more ties even after the first tie-breaker. If at this point the dynamic variable ordering heuristic makes a decision without considering the values in those variables domain, then it might very well go down a path where it is hard to recognize that no solution exists. However, by also considering the values in that variables domain, it can gain more insight on which path to guide the search. For instance, if there are some values that can only be fixed to few variables then it is better to guide the search with variables that have those values in their domains. So that we can either reach a solution faster or backtrack faster if it fails. Of course, this is especially true when the number of values in the domains of all the variables come from a discrete set of values, which is the case for Sudoku. However, we believe that the insights gained from this study can be carried over to other NP-Complete problems that are modeled as constraint satisfaction problems. This is because there are still many more ties left after applying a dynamic variable ordering heuristics and traditionally they are broken lexicographically without consulting any heuristic. We hope to further study the effects of dynamic variable and value ordering heuristics with more sophisticated constraint propagation algorithms.

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