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ON FINITE GROUPS HAVING A CERTAIN NUMBER OF CYCLIC SUBGROUPS

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Abstract. Let G be a finite group. In this paper, we study the structure of finite groups having $|G|-r$ cyclic subgroups for $3 \leq r \leq 5$.

1. Introduction

Let G be a finite group and $C(G)$ be the poset of cyclic subgroups of G. Some results show that the structure of $C(G)$ has an influence on the algebraic structure of G. In Main Theorem of [8], Tărnăuceanu proved that the finite group G has $|G|-1$ cyclic groups if and only if G is isomorphic to Z_3, Z_4, S_3, or D_8. In the end of that paper the author states the following problem:

Open Problem. Describe the finite group G satisfying $|C(G)| = |G| - r$ where $2 \leq r \leq |G|-1$.

In [9], Tărnăuceanu solved this open problem for $|C(G)| = |G| - 2$. In this paper, we describe the structure of finite groups with $|C(G)| = |G| - r$ in which $3 \leq r \leq 5$.

We summarize our notations. $cl(a)$ denotes the conjugacy class of a in G, $\pi(G)$ denotes the set of prime numbers dividing the order of G, $\phi(n)$ denotes the Euler function that counts the positive integers less than n that are relatively prime to n, $F(G)$ denotes the subgroup generated by all normal nilpotent subgroups of G, $O_p(G)$ denotes the unique maximal normal p-subgroup of G, $F_{p,q}$ denotes the Frobenius group of order pq and $o(x)$ denotes the order of x.

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2. Preliminaries

Lemma 2.1. *Let G be a finite group and $p \in \pi(G)$. Then the number of distinct subgroups of order p in G is $kp+1$ for some non-negative integer k.*

We denote with $c_n(G)$ the number of cyclic subgroups of order 2^n in a finite 2-group G. And, a group G of order p^m is said to be of maximal class if $m > 2$ and $cl(G) = m-1$.

Theorem 2.2. [1, Theorem 1.17] *Suppose that a 2-group G is neither cyclic nor of maximal class. Then $c_n(G)$ is even for $n > 1$ and $c_1(G) \equiv 3 (\mathrm{mod}\ 4)$.*

Remark 2.3 ([1])**.** *For each 2-group G, $c_2(G) = 1$ if and only if G is either cyclic or dihedral and $c_2(G) = 3$ if and only if G is either Q_8 or SD_{16}.*

Theorem 2.4. [1, Corollary 1.7 and Theorem 1.2] *Let G be a 2-group of maximal class. Then it is either D_{2^n}, Q_{2^n}, or $SD_{2^{n+1}}$ for $n \geq 3$.*

In this paper, $\Omega_n(G) = \langle x \in G | o(x) \leq p^n \rangle$ and $\mho_n(G) = \langle x^{p^n} | x \in G \rangle$. In the next theorem, 2-groups of order $> 2^3$ with $c_2(G) = 2$ are characterized.

Theorem 2.5. [1, Theorem 43.6] *and* [7, Theorem 5.1 and 5.2 and Proposition 1.4] *Suppose that a group G of order $2^m > 2^3$ has exactly two cyclic subgroups U and V of order 4; set $A = \langle U, V \rangle$. Then A is abelian of type $(4,2)$ and one of the following holds:*

(a) *$G \cong M_{2^m}$.*
(b) *G is abelian of type $(2^{m-1}, 2)$.*
(c) *$G = \langle a, b | a^{2^{m-2}} = b^8 = 1, a^b = a^{-1}, a^{2^{m-3}} = b^4 \rangle$, where $m \geq 5$.*
(d) *$G = D_{2^{m-1}} \times C_2$.*
(e) *$G = \langle b, t | b^{2^{m-2}} = t^2 = 1, b^t = b^{-1+2^{m-4}} u, u^2 = [u,t] = 1, b^u = b^{1+2^{m-3}}, m \geq 5 \rangle$.*

In the next theorems, 2-groups of order $> 2^4$ with $c_2(G) = 4$ are characterized.

Theorem 2.6. [6, Theorem 2.1 and Proposition 1.3 and 1.4] *and* [7, Theorem 2.1] *Let G be a 2-group of order $> 2^4$ with $c_2(G) = 4$ and $|\Omega_2(G)| = 2^4$. Then one of the following holds:*

(a) *$G \cong D_8 * C$ (the central product) where C is cyclic of order ≥ 4.*
(b) *$G \cong Q_8 S$ where S is cyclic of order ≥ 16 and Q_8 is normal in G.*
(c) *$G = \langle E, a, b \rangle$, in which $E \cong E_8$, $o(b) = 8$, $o(a) = 2^n$ and $a^{2^{n-2}} = v$ is of order 4. Moreover $A = \Omega_2(G) = \langle E, v \rangle \cong C_4 \times C_2 \times C_2$.*
(d) *$G = \langle E, a \rangle$, in which $E \cong E_8$, $o(a) = 2^n$ and $a^{2^{n-2}} = v$ is of order 4. Moreover $A = \Omega_2(G) = \langle E, v \rangle \cong C_4 \times C_2 \times C_2$.*

In case(d) of Theorem 2.6, if $n = 2$, then $a = v$ and so $G \cong C_4 \times C_2 \times C_2$ contradicting $|G| > 2^4$. Hence, G has some element of order at least 8.

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Theorem 2.7. [6, Theorem 2.2] *Let G be a 2-group of order $> 2^4$ with $c_2(G) = 4$ and $|\Omega_2(G)| > 2^4$. If G has a quaternion subgroup Q, then Q is normal in G, $C = C_G(Q)$ is cyclic of order 2^n, $n \geq 2$, $G = (Q * C)\langle t\rangle$, where t is an involution such that $Q\langle t\rangle \cong SD_{2^4}$ and $\langle t\rangle C \cong D_{2^{n+1}}$. We have $|Z(G)| = 2$, $|G| = 2^{n+3}$, and $\Omega_2(G) = G$.*

In Theorem 2.7, since $Q\langle t\rangle \cong SD_{2^4}$ is a subgroup of G, then G has some element of order 8.

Theorem 2.8. [6, Theorem 2.4, 2.5, and 2.6] *Let G be a 2-group of order $> 2^4$ with $c_2(G) = 4$ and $|\Omega_2(G)| > 2^4$. Suppose that G has no subgroup isomorphic to Q_8. Then*

(a) *$G = \Omega_2(G)$ and $\Omega_2(G) = B\langle t\rangle$ where B is abelian of type $(2^m, 2, 2)$, $m \geq 2$, and t is an involution acting invertingly on B.*
(b) *$|G : \Omega_2(G)| \geq 4$ and $G = \langle a, b, t | a^8 = b^8 = t^2 = 1; a^2 = v; a^4 = z; b^2 = ev; a^b = a^{-1}u; e^2 = u^2 = [e, v] = [u, v] = [e, u] = [a, u] = [t, e] = [t, u] = 1, e^a = ez; e^b = ez; u^b = uz; v^t = v^{-1}; a^t = eva^{-1}; b^t = euvb^{-1}\rangle$.*
(c) *$|G : \Omega_2(G)| = 2$ and $G = \langle b, e, t | b^8 = e^2 = t^2 = 1; (tb)^2 = a; a^{2^n} = 1, n \geq 3, a^{2^{n-2}} = v; a^{2^{n-1}} = z, b^2 = uv, u^2 = [b, e] = [a, e] = [a, u] = [u, e] = [t, e] = [t, u] = 1, u^b = uz; a^b = a^{-1}; a^t = a^{-1}\rangle$.*

In the previous theorems, we observe that each 2-group of order $> 2^4$ with $c_2(G) = 4$ has some element of order at least 8, except for some cases of Theorem 2.6(a) and Theorem 2.8(a).

Lemma 2.9. *Let G be a finite group with the Sylow p-subgroup P and $\pi(G) = \{p, q\}$. Assume that $a \in G$ and p divides $o(a)$ if and only if a is of prime power order. Then G is a Frobenius group with kernel P.*

Proof. Since P is normal in G, then $G = PQ$ where Q is a Sylow q-subgroup of G. On the other hand, if $x \in C_Q(P)$, p and q divide $o(x)$ which is a contradiction. Therefore, By Problem 7.1 of [5], G is a Frobenius group with kernel P. □

3. Main Theorem

Theorem 3.1. *Let G be a finite group. Then*

(1) *$|C(G)| = |G| - 3$ if and only if $G \cong Z_5$, Q_8, or D_{10}.*
(2) *$|C(G)| = |G| - 4$ if and only if $G \cong Z_8$, $Z_3 \times Z_3$, $Z_6 \times Z_2$, $Z_4 \times Z_2 \times Z_2$, D_{16}, $(Z_4 \times Z_2) \rtimes Z_2$, $(Z_3 \times Z_3) \rtimes Z_2$, $D_8 \times Z_2 \times Z_2$, or $D_8 * Z_4$(the central product D_8 and Z_4).*
(3) *$|C(G)| = |G| - 5$ if and only if $G \cong Z_7$, $Z_3 \times Z_6$, Dic_{12}, D_{14}, or $F_{5,4}$(where Dic_{12} is the dicyclic group of order 12).*

Proof. We know that

$$|G| = \sum_{i=1}^{k} n_i \phi(d_i) \text{ and } |C(G)| = \sum_{i=1}^{k} n_i \tag{3.1}$$

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in which d_i's are the positive divisors of $|G|$ and n_i's are the number of cyclic subgroups of order d_i in G for $1 \leq i \leq k$. If $|C(G)| = |G| - r$ then we can obtain that

$$\sum_{i=1}^{k} n_i(\phi(d_i) - 1) = r. \tag{3.2}$$

Suppose that $n = (n_1, \ldots, n_k)$, $d = (d_1, \ldots, d_k)$, P_q is a Sylow q-subgroup of G, and A is a cyclic subgroup of order 4. Applying equation 3.2 and Lemma 2.1, we distinguish several cases for r, n, and d.

(1) $r = 3$.

`Case (1):` $n = (1, m)$ and $d = (5, 2)$.

If $m = 0$, then $G \cong Z_5$. Otherwise, by Lemma 2.9, G is a Frobenius group with kernel P_5 of order 5. By Theorem 13.3(1),(3) of [3], we deduce that $G \cong D_{10}$.

`Case (2):` $n = (3, m)$ and $d = (4, 2)$.

G is a 2-group and $c_2(G) = 3$. By Remark 2.3 we obtain that $G \cong Q_8$ or SD_{16} and since SD_{16} has some elements of order 8, then $G \cong Q_8$.

`Case (3):` $n = (2, 1, m)$ and $d = (4, 3, 2)$.

By Lemma 2.9 and Theorem 13.3(1) of [3] $G \cong S_3$ which contradicts the hypothesis.

`Case (4):` $n = (1, 1, 1, m)$ and $d = (6, 4, 3, 2)$.

Observe that A and $P_3 \cong Z_3$ are normal in G and so $AP_3 \cong Z_4 \times Z_3 \cong Z_{12}$ is a subgroup of G which is impossible by the hypothesis.

`Case (5):` $n = (2, 1, m)$ and $d = (6, 3, 2)$.

Assume that $G = P_2P_3$ in which P_3 is normal of order 3 and P_2 is an elementary abelian 2-group. Since each element of order 6 is a product of an element of order 3 and an element of order 2, then these elements belong to $C_G(P_3)$ and $C_G(P_3) \cong Z_3 \times Z_2 \times \cdots \times Z_2 \subseteq P_3P_2$. Furthermore, by Theorem 2.2 $c_1(C_G(P_3)) = 1$ or $4k + 3$, therefore the number of cyclic subgroups of order 6 is 1 or $4k + 3$ and so G can not have exactly 2 subgroups of order 6.

(2) $r = 4$.

`Case (1):` $n = (1, 1, m)$ and $d = (5, 3, 2)$.

Observe that $P_3 \cong Z_3$ and $P_5 \cong Z_5$ are normal in G and so $P_3P_5 \cong Z_{15}$ is a subgroup of G, which is impossible.

`Case (2):` $n = (1, 1, m)$ and $d = (8, 4, 2)$.

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Since $c_2(G) = 1$, then by Remark 2.3, $G \cong Z_8$ or D_{16}.

`Case (3):` $n = (1, 1, m)$ and $d = (5, 4, 2)$.

Observe that A and $P_5 \cong Z_5$ are normal in G and so $AP_5 \cong Z_{20}$ is a subgroup of G, which is impossible.

`Case (4):` $n = (4, m)$ and $d = (3, 2)$.

If $P_3 \cong Z_3$, then by Main Theorem of [2] G is a Frobenius group with kernel P_3 of order 3 and Theorem 13.3(1) of [3] $G \cong S_3$ which is impossible. Otherwise, $Z_3 \times Z_3 \subseteq P_3$ and $Z_3 \times Z_3$ has 4 subgroups of order 3, then by Main Theorem of [2], either $G \cong Z_3 \times Z_3$ or $G \cong (Z_3 \times Z_3) \rtimes Z_2$.

`Case (5):` $n = (3, 1, m)$ and $d = (4, 3, 2)$.

By Lemma 2.9 G is a Frobenius group with kernel P_3 of order 3. On the other hand, by Theorem 13.3(1) of [3] $P \cong Z_2$ which is a contradiction.

`Case (6):` $n = (3, 1, m)$ and $d = (6, 3, 2)$.

Since $P_3 \cong Z_3 \cong \langle x \rangle$ is normal in G and P_2 is an elementary abelian 2-group, then $G' \subseteq P_3$ and $|cl(x)| = |G|/|C_G(x)| \leq |G'| \leq 3$, and either $x \in Z(G)$ or $|cl(x)| = 2$. On the other hand, since G has 3 cyclic subgroups of order 6, we can obtain that $C_G(x) \cong Z_2 \times Z_2 \times Z_3$. Thus $G \cong Z_6 \times Z_2$ or $(Z_6 \times Z_2) \rtimes Z_2$, but using GAP [4], we know that $(Z_6 \times Z_2) \rtimes Z_2$ has some element of order 4.

`Case (7):` $n = (1, 2, 1, m)$ and $d = (6, 4, 3, 2)$.

Since G has 2 cyclic subgroups of order 4, then $c_2(P_2) \leq 2$ and so by Theorem 2.5, $P_2 \cong Z_4 \times Z_2$, $D_8 \times Z_2$, Z_4, or D_8 and $Q \cong Z_3$. Using GAP [4], we obtain that such group does not exist.

`Case (8):` $n = (2, 1, 1, m)$ and $d = (6, 4, 3, 2)$.

Observe that A and $P_3 \cong Z_3$ are normal in G and so $AP_3 \cong Z_{12}$ which is impossible by the hypothesis.

`Case (9):` $n = (4, m)$ and $d = (4, 2)$.

Observe that G is a 2-group with $c_2(G) = 4$ and since G does not some element of order 8, then by Theorems 2.6-2.8 and using GAP[4], we obtain that $G \cong Z_4 \times Z_2 \times Z_2$, $(Z_4 \times Z_2) \rtimes Z_2$, $D_8 \times Z_2 \times Z_2$, or $D_8 * Z_4$(Central product).

(3) $r = 5$.

`Case (1):` $n = (1, m)$ and $d = (7, 2)$

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If $m = 0$, then $G \cong Z_7$. Otherwise, by Lemma 2.9, G is a Frobenius group with kernel P_7 of order 7. By Theorem 13.3(1) of [3] G is isomorphic to D_{14}.

`Case (2):` $n = (1, 1, 1, m)$ and $d = (5, 4, 3, 2)$

We observe that P_3 and P_5 are normal in G and $P_3P_5 \cong Z_{15}$ is a subgroup of G which is impossible.

`Case (3):` $n = (1, 2, m)$ and $d = (5, 4, 2)$

By Lemma 2.9 G is a Frobenius group with kernel P_5 of order 5. By Theorem 13.3(1) of [3], we deduce that $G \cong F_{5,4}$.

`Case (4):` $n = (1, 1, 1, m)$ and $d = (6, 5, 3, 2)$.

Observe that P_3, P_5 are normal in G. Thus, $P_3P_5 \cong Z_{15}$ which is a contradiction.

`Case (5):` $n = (1, 1, 1, m)$ and $d = (8, 4, 3, 2)$.

Since A and P_3 are normal in G, then $AP_3 \cong Z_4 \times Z_3 \cong Z_{12}$ is a subgroup of G which is impossible.

`Case (6):` $n = (1, 2, m)$ and $d = (8, 4, 2)$.

Since G is a 2-group and $c_3(G) = 1$, then by Theorem 2.2, G is of maximal class and so by Theorem 2.4, G is isomorphic to either D_{2^n}, Q_{2^n}, or $SD_{2^{n+1}}$ for $n \geq 3$. However, we have $c_2(G) = 2$ and so this case is impossible by Theorem 2.5.

`Case (7):` $n = (5, m)$ and $d = (4, 2)$.

Since G is a 2-group and $c_2(G) = 5$, then by Theorem 2.2, G is of maximal class and so by Theorem 2.4, G is isomorphic to either D_{2^n}, Q_{2^n}, or $SD_{2^{n+1}}$ for $n \geq 3$. Since D_{2^n}, Q_{2^n}, and SD_{2^n} have some element of order 8 for $n \geq 4$, then G is isomorphic to D_8 or Q_8 which is impossible by Remark 2.3.

`Case (8):` $n = (4, 1, m)$ and $d = (6, 3, 2)$.

We can see that $G = P_2P_3$ in which P_3 is normal of order 3 and P_2 is an elementary abelian 2-group. Additionally, $C_G(P_2) \cong Z_3 \times Z_2 \times \cdots \times Z_2$. Since each element of order 6 is a product of an element of order 3 and an element of order 2, then such elements belong to $C_G(P_2)$. Thus, since by Theorem 2.2 $c_1(C_Q(P_2)) = 1$ or $4k + 3$, then G does not have exactly 4 subgroups of order 6.

`Case (9):` $n = (3, 1, 1, m)$ and $d = (6, 4, 3, 2)$.

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Since A and P_3 are normal in G, then $AP_3 \cong Z_4 \times Z_3 \cong Z_{12}$ is a subgroup of G which is a contradiction.

`Case (10):` $n = (2, 2, 1, m)$ and $d = (6, 4, 3, 2)$.

We can see that $G = P_2P_3$ in which P_3 is normal of order 3 and P_2 is a 2-group with $c_2(P) = 2$. If x is of order 4, then $x \notin C_Q(P_2)$, $C_Q(P_2)$ is an elementary abelian 2-group, and $C_G(P_2) \cong Z_3 \times Z_2 \times \cdots \times Z_2$. Since each element of order 6 belongs to $C_G(P_2)$ and by Theorem 2.2 $c_1(C_Q(P_2)) = 1$ or $4k+3$, then G does not exactly 2 subgroups of order 6.

`Case (11):` $n = (1, 4, m)$ and $d = (4, 3, 2)$.

By Lemma 2.9 AP_3 is a Frobenius group with kernel A of order 4. By Theorem 13.3(1) of [3] we obtain that $|P_3| = 3$ and so AP_3 is a Frobenius group of order 12 contradicting the fact that A_4 is the only Frobenius group of order 12 and it does not have any element of order 4.

`Case (12):` $n = (4, 1, m)$ and $d = (4, 3, 2)$.

By Lemma 2.9 G is a Frobenius group with kernel P_3 of order 3 and by Theorem 13.3(1) of [3] G is of order 6 which contradicts the hypothesis.

`Case (13):` $n = (1, 4, m)$ and $d = (6, 3, 2)$.

Since G has 4 subgroups of order 3, then $P_3 \cong Z_3$ or $Z_3 \times Z_3$. If $P_3 \cong Z_3$, since G has 1 subgroup of order 6, then $C_G(P_3) \cong Z_6$ is a subset of $F(G)$. Therefore $F(G) = O_2(G) \times O_3(G) = O_2(G) \times P_3$ and so P_3 is normal in G which is a contradiction.

Now, $P_3 \cong Z_3 \times Z_3$ has 4 subgroups of order 3, then P_3 is normal in G. Assume that B is the cyclic subgroup of order 6 in G. Since P_2 is an elementary abelian 2-group, then $G' \subseteq P_3$ and so $|P_2| = |cl(y)| \leq |G'| \leq 9$ for $y \in P_3 \backslash B$. Using GAP [4], we get that $G \cong Z_3 \times Z_6$.

`Case (14):` $n = (1, 3, 1, m)$ and $d = (6, 4, 3, 2)$.

Observe that $G = P_2P_3$ where $c_2(P_2) \leq 3$ and $P_3 \cong Z_3$ is normal in G. By Remark 2.3 and Theorem 2.5 and 2.4, P_2 is isomorphic to Q_8, D_8, Z_4, $Z_2 \times D_8$, or $Z_2 \times Z_4$. Using GAP[4], we conclude that $G \cong Dic_{12}$, that is the dicyclic group of order 12.

□

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