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FINITE GROUPS WITH THE SAME CONJUGACY CLASS SIZES AS A FINITE SIMPLE GROUP

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ABSTRACT. For a finite group H, let $cs(H)$ denote the set of non-trivial conjugacy class sizes of H and $OC(H)$ be the set of the order components of H. In this paper, we show that if S is a finite simple group with the disconnected prime graph and G is a finite group such that $cs(S) = cs(G)$, then $|S| = |G/Z(G)|$ and $OC(S) = OC(G/Z(G))$. In particular, we show that for some finite simple group S, $G \cong S \times Z(G)$.

1. Introduction

In this paper, all groups are finite and for a group G and $x \in G$, $C_G(x)$ and $cl_G(x)$ are the centralizer of x in G and the conjugacy class in G containing x, respectively and $cs(G)$ denotes the set of non-trivial conjugacy class sizes of G.

A. Camina and R. Camina in [10] found a nilpotent group G and a non-nilpotent group H such that $cs(G) = cs(H) = \{20, 10, 5, 4, 2\}$. This examples show that nilpotency can not be determined by cs.

In [33], Navarro by constructing some examples showed that solvability can not be recognized by cs.

J. Thompson (see [30, Problem 12.38]) conjectured that simplicity can be determined by cs in the class of finite centerless groups. In a series of papers [2], [4]-[7], [17] and [34], the veracity of

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Thompson's conjecture for some finite simple groups has been studied . In [9], A. Camina and R. Camina asked about the structure of a group with the same *cs* as a simple group. In the other word, it is interesting to know that whether simplicity can be determined by *cs* (up to an abelian direct factor).

In 2015, it has been shown that if G is a group with $cs(G) = cs(PSL_2(q))$, where q is a prime power, then $G \cong PSL_2(q) \times Z(G)$ [8]. In this paper, we continue this investigation for some other finite simple groups.

Throughout this paper, we use the following notation: For a natural number n, let $\pi(n)$ be the set of prime divisors of n and for a group H, let $\pi(H) = \pi(|H|)$. Also, $n.H$ denotes a central extension of H by a cyclic group of order n. For a prime r and natural numbers a and b, $|a|_r$ is the r-part of a, i.e., $|a|_r = r^t$ when $r^t \mid a$ and $r^{t+1} \nmid a$ and, $\gcd(a, b)$ and $\mathrm{lcm}(a, b)$ are the greatest common divisor of a and b and the lowest common multiple of a and b, respectively. For the set of primes π, x is named a π-element (π'-element) of a group H if $\pi(O(x)) \subseteq \pi$ ($\pi(O(x)) \subseteq \pi(H) - \pi$). For a group G, the prime graph $GK(G)$ of G is a simple graph whose vertices are the prime divisors of the order of G and two distinct prime numbers p and q are joined by an edge if G contains an element of order pq. Denote by $t(G)$ the number of connected components of the graph $GK(G)$ and denote by $\pi_i = \pi_i(G)$, $i = 1, \ldots, t(G)$, the i-th connected component of $GK(G)$. For a group G of an even order, let $2 \in \pi_1$. If $GK(G)$ is disconnected, then $|G|$ can be expressed as a product of co-prime positive integers $m_i(G)$, $i = 1, 2, \ldots, t(G)$, where $\pi(m_i(G)) = \pi_i(G)$, and if there is no ambiguity write m_i for showing $m_i(G)$. These m_i's are called the order components of G and the set of order components of G will be denoted by $OC(G)$. List of all finite simple groups with disconnected prime graph and their sets of order components have been obtained in [32] and [35].

2. **The main results**

Throughout this section, let S be a simple group with the disconnected prime graph and G be a group with $cs(S) = cs(G)$. Also, fix $\bar{G} = G/Z(G)$ and for $x \in G$, let $\bar{x}$ denote the image of x in $\bar{G}$.

Lemma 2.1. [18, Proposition 4] *Let H be a group. If there exists $p \in \pi(H)$ such that p does not divide any conjugacy class sizes of H, then the p-Sylow subgroup of H is an abelian direct factor of H.*

Lemma 2.2. *Let H be a group and $x, y \in H$.*

(i) *If $xy = yx$ and $\gcd(O(x), O(y)) = 1$, then $C_H(xy) = C_H(x) \cap C_H(y)$. In particular, $C_H(xy) \leq C_H(x)$ and $|cl_H(x)|$ divides $|cl_H(xy)|$.*

(ii) *If $K \leq H$, then $cs(KZ(H)) = cs(K)$.*

Proof. The proof of (i) is straightforward and we obtain (ii) from [3, Lemma 2.2] □

Definition 2.3. *A subgroup K of H is called isolated if for every $h \in H$, $K \cap K^h = 1$ or K and for all x in $K - \{1\}$, $C_H(x) \leq K$.*

Lemma 2.4. *For every $i \geq 2$,*

(i) *$\frac{|S|}{m_i} \in cs(S)$ and $\frac{|S|}{m_i}$ is maximal and minimal in $cs(S)$ by divisibility.*
(ii) *For every $\alpha \in cs(S)$, either $\alpha = \frac{|S|}{m_i}$ or $m_i \mid \alpha$.*

Proof. Since S is a simple group with the odd order component m_i, [35] shows that S contains a subgroup, namely H_i, of the order m_i which is abelian and isolated. So for every $x \in H_i - \{1\}$, $C_S(x) = H_i$ and hence, $|cl_S(x)| = \frac{|S|}{m_i} \in cs(S)$. If $y \in S - \{1\}$ such that $|cl_S(y)| \mid \frac{|S|}{m_i}$, then $m_i \mid |C_S(y)|$. Since m_i is an odd order component of S, we get that $|C_S(y)| = m_i$ and hence, $|cl_S(y)| = \frac{|S|}{m_i}$, so $\frac{|S|}{m_i}$ is minimal in $cs(S)$ by divisibility. Also, since H_i is an abelian π_i-Hall subgroup of S, we can see at once that $\frac{|S|}{m_i}$ is maximal in $cs(S)$ by divisibility and hence, (i) follows.

For proving (ii), let $\alpha \neq \frac{|S|}{m_i}$. Since $\alpha \in cs(S)$, there exists an element $y \in S - \{1\}$ such that $|cl_S(y)| = \alpha$. On the contradiction, suppose that $m_i \nmid \alpha$. Thus $\gcd(|C_S(y)|, m_i) \neq 1$. So there exists a prime q such that $q \mid \gcd(|C_S(y)|, m_i)$ and hence, $q \in \pi_i$ and $C_S(y)$ contains a non-trivial element x of order q. Therefore, $O(xy) = \mathrm{lcm}(O(y), q)$. Since π_i is a connected component of $GK(S)$, we get that y is a π_i-element of S and hence, as mentioned in (i), y can be considered as an element of H_i which is an abelian π_i-Hall subgroup of S. Therefore, $m_i \mid |C_S(y)|$, so $|cl_S(y)| \mid \frac{|S|}{m_i}$ and hence, by minimality of $\frac{|S|}{m_i}$ in $cs(S)$, we get that $|cl_S(y)| = \frac{|S|}{m_i}$, which is a contradiction. This shows that $m_i \mid \alpha$, as wanted in (ii). □

Lemma 2.5. *$|S| \mid |G/Z(G)|$.*

Proof. By Lemma 2.4(i), for every $i \geq 2$, $\frac{|S|}{m_i} \in cs(S)$. Also, since π_1 is a connected component of $GK(S)$, for every non-trivial π_1-element $x_1 \in S$, $|C_S(x_1)| \mid m_1$ and hence $\Pi_{i=2}^{t(S)} m_i \mid |cl_S(x_1)|$. Therefore, $\mathrm{lcm}\{\alpha : \alpha \in cs(S)\} = |S|$. On the other hand, for every $y \in G$, $Z(G) \leq C_G(y)$ and hence, $|cl_G(y)| \mid [G : Z(G)] = |G/Z(G)|$. Therefore, $|S| = \mathrm{lcm}\{\alpha : \alpha \in cs(S) = cs(G)\} \mid |G/Z(G)|$, as wanted. □

Chen in [17] proved that if G is centerless, then $|S| = |G|$ and $OC(S) = OC(G)$. In the following, we are going to prove the similar facts for an arbitrary group G with $cs(S) = cs(G)$.

Lemma 2.6. *Let $i \geq 2$.*

(i) *For every π_i-element $x \in G - Z(G)$, $|cl_G(x)| = \frac{|S|}{m_i}$.*
(ii) *For every π_i-element $\bar{x} \in \bar{G}$, $|cl_{\bar{G}}(\bar{x})| = \frac{|S|}{m_i}$.*

Proof. If there exists a non-central π_i'-element $y \in G$ such that $|cl_G(y)| = \frac{|S|}{m_i}$, then

$$\gcd(m_i, |cl_G(y)|) = 1$$

and Lemma 2.5 shows that $m_i \mid |C_G(y)/Z(G)|$ and hence, for every $q \in \pi_i$, $C_G(y)$ contains a q-Sylow subgroup of G, so we can assume that for a given non-central q-element z, $z \in C_G(y)$ (up to conjugation). Thus Lemma 2.2(i) guarantees that $|cl_G(y)| = \frac{|S|}{m_i}, |cl_G(z)| \mid |cl_G(yz)|$ and hence, by maximality and minimality of $\frac{|S|}{m_i}$ in $cs(S) = cs(G)$, $|cl_G(z)| = |cl_G(y)| = |cl_G(yz)| = \frac{|S|}{m_i}$. On the other hand, every non-central π_i-element x of G can be written as a product of some π_i-elements of the prime power orders which their orders are co-prime and at least one of them is non-central, so by Lemma 2.2(i), $\frac{|S|}{m_i} \mid |cl_G(x)|$. Thus maximality of $\frac{|S|}{m_i}$ in $cs(G)$ by divisibility forces $|cl_G(x)| = \frac{|S|}{m_i}$, as desired.

Now assume that for every non-central π_i'-element $y \in G$, $|cl_G(y)| \neq \frac{|S|}{m_i}$. Thus since $\frac{|S|}{m_i} \in cs(S) = cs(G)$, there exists a non-central π_i-element $z \in G$ such that $|cl_G(z)| = \frac{|S|}{m_i}$. Lemma 2.2(i) allows us to assume that z is an element of order p^α for some $p \in \pi_i$. If there exists $q \in \pi_i - \{p\}$, then since $\gcd(m_i, |cl_G(z)|) = 1$, we deduce that $C_G(z)$ contains every q-element of G (up to conjugation). Also, $|S| \mid |G/Z(G)|$ and hence, G contains some non-central q-elements. Let w be a non-central q-element of $C_G(z)$. Then by Lemma 2.2(i), $|cl_G(z)| = \frac{|S|}{m_i}, |cl_G(w)|$ divide $|cl_G(zw)|$, so maximality and minimality of $\frac{|S|}{m_i}$ in $cs(S) = cs(G)$ forces $|cl_G(w)| = |cl_G(zw)| = |cl_G(z)| = \frac{|S|}{m_i}$. Now let u be a non-central p-element of G. Since $|cl_G(w)| = \frac{|S|}{m_i}$ and $p \in \pi_i$, we have $C_G(w)$ contains a p-Sylow subgroup of G, so we can assume that $u \in C_G(w)$. Thus by Lemma 2.2(i), $|cl_G(w)| = \frac{|S|}{m_i}, |cl_G(u)|$ divide $|cl_G(uw)|$, so maximality and minimality of $\frac{|S|}{m_i}$ in $cs(S) = cs(G)$ forces $|cl_G(u)| = |cl_G(uw)| = |cl_G(u)| = \frac{|S|}{m_i}$. The same reasoning as above shows that for every non-central π_i-element x of a prime power order, $|cl_G(x)| = \frac{|S|}{m_i}$. Also since every non-central π_i-element x of G can be written as a product of some π_i-elements of the prime power orders which their orders are co-prime and at least one of them is non-central, by Lemma 2.2(i), $\frac{|S|}{m_i} \mid |cl_G(x)|$. Thus maximality of $\frac{|S|}{m_i}$ in $cs(G)$ by divisibility forces $|cl_G(x)| = \frac{|S|}{m_i}$, as desired.

Now let $\pi_i = \{p\}$. Since $\gcd(p, |cl_G(z)|) = 1$, we deduce that $C_G(z)$ contains a p-Sylow subgroup P of G and hence, $z \in Z(P) - Z(G)$. If there exists a non-central p-element u of G such that $|cl_G(u)| \neq \frac{|S|}{m_i}$, then Lemma 2.4(ii) shows that $m_i \mid |cl_G(u)|$ and hence,

$$|G/Z(G)|_p = |C_G(u)/Z(G)|_p |cl_G(u)|_p > |S|_p, \tag{2.1}$$

because $u \in C_G(u) - Z(G)$. On the other hand, our assumption implies that if y is a non-central π_i'-element of G, then $|cl_G(y)| \neq \frac{|S|}{m_i}$. Thus Lemma 2.4(ii) shows that $m_i \mid |cl_G(y)|$. Also by (2.1), $p \mid |C_G(y)/Z(G)|$, so $C_G(y)$ contains a non-central p-element v. Thus Lemma 2.2(i) shows that $|cl_G(y)|, |cl_G(v)| \mid |cl_G(yv)|$, so $m_i \mid |cl_G(yv)|$ and hence, $|cl_G(v)| \neq \frac{|S|}{m_i}$. Consequently, Lemma 2.4(ii) forces $m_i \mid |cl_G(v)|$. This shows that $|cl_G(v)|_p = |cl_G(vy)|_p = |cl_G(y)|_p = |m_i|_p = |S|_p$ and hence, $|C_G(v)|_p = |C_G(vy)|_p = |C_G(y)|_p \geq |G/Z(G)|_p/|S|_p \neq 1$, by (2.1). On the other hand, Lemma 2.2(i) yields that $C_G(vy) = C_G(v) \cap C_G(y) \leq C_G(v), C_G(y)$ and without loss of generality, we can assume that $C_P(vy) \in \mathrm{Syl}_p(C_G(vy))$. Thus the facts that "$C_G(vy) \leq C_G(v), C_G(y)$ and $|C_G(vy)|_p = |C_G(v)|_p = |C_G(y)|_p$" guarantee that $C_P(v) = C_P(yv) = C_P(y)$. But v is a central

p-element in $C_G(vy)$, so $v \in C_P(vy) \leq P$. Therefore, $Z(P) \leq C_P(v)$ and hence, $z \in Z(P) \leq C_P(y)$. Note that $\gcd(O(y), p) = 1$. Thus Lemma 2.2(i) gives $|cl_G(y)|, |cl_G(z)| \mid |cl_G(yz)|$, so the above statements show that $m_i, \frac{|S|}{m_i} \mid |cl_G(yz)|$. Therefore, $|S| \mid |cl_G(yz)|$ and consequently, $|cl_G(yz)| \notin cs(S)$, which is a contradiction. This contradiction shows that for every non-central π_i-element $u \in G$, $|cl_G(u)| = \frac{|S|}{m_i}$, as wanted in (i).

Now let $\bar{x}$ be a π_i-element of $\bar{G}$. So there exists a non-central π_i-element $y \in G$ such that $\bar{x} = \bar{y}$. Therefore, (i) shows that $|cl_G(y)| = \frac{|S|}{m_i}$. Fix $C_{\bar{G}}(\bar{y}) = C/Z(G)$. Note that $C_G(y)$ is a normal subgroup in C, because for every $g \in C$, $(\bar{g})^{-1}\bar{y}\bar{g} = \bar{y}$, so there exists $z \in Z(G)$ such that $g^{-1}yg = yz$ and hence, for every $h \in C_G(y)$, $(g^{-1}hg)^{-1}y(g^{-1}hg) = y$ which means that $g^{-1}hg \in C_G(y)$. If there exists a π_i'-element $\bar{g} \in C_{\bar{G}}(\bar{y})$, then we can assume that g is a π_i'-element of G and $g^{-1}yg = yz$, for some $z \in Z(G)$. Since $O(y) = O(yz) = \mathrm{lcm}(O(y), O(z))$, we get that z is a π_i-element of $Z(G)$. Also, $ygy^{-1} = gz$ and hence, $O(g) = O(gz) = \mathrm{lcm}(O(g), O(z))$, so $O(z) \mid O(g)$. This forces $z = 1$ and hence, $g \in C_G(y)$. If there exists $g \in G$ such that $\gcd(O(g), m_i), \gcd(O(g), \frac{|S|}{m_i}) \neq 1$ and $\bar{g} \in C_{\bar{G}}(\bar{y})$, then we have $g = g_1g_2 = g_2g_1$, where g_1 is a π_i-element and g_2 is a π_i'-element of G. Therefore, $\bar{y} \in C_{\bar{G}}(\bar{g_1}) \cap C_{\bar{G}}(\bar{g_2})$. So the above statements show that $g_2 \in C_G(y)$. Also, if $g_1 \notin C_G(y)$, then we have $g_1 \in C - C_G(y)$ and hence, $p \mid |C/C_G(y)|$, for some $p \in \pi_i$. Since $C \leq G$, we get that $|C/C_G(y)| \mid |cl_G(y)| = \frac{|S|}{m_i}$, so $p \mid \frac{|S|}{m_i}$, which is a contradiction. This shows that $g_1 \in C_G(y)$. The same reasoning shows that every π_i-element of C lies in $C_G(y)$ and hence, $C = C_G(y)$, so $|cl_{\bar{G}}(\bar{y})| = |cl_G(y)| = \frac{|S|}{m_i}$, as wanted in (ii). □

Remark 2.7. *Let $q \in \pi(S)$ and Q be a q-Sylow subgroup of S. Then since S is simple and $Z(Q) \neq 1$, we deduce that there exists a non-trivial element $\alpha \in cs(S) = cs(G)$ such that $q \nmid \alpha$, so if $\alpha = |cl_G(y)|$ for some $y \in G$, then y is a non-central element of G and $C_G(y)$ contains a q-Sylow subgroup of G.*

Theorem 2.8. (i) $|G/Z(G)| = |S|$.
(ii) $OC(G/Z(G)) = OC(S)$.

Proof. (i) By Lemma 2.5, $|S| \mid |G/Z(G)|$. Now let there exist $q \in \pi(G/Z(G)) - \pi(S)$. Then since $q \nmid |S|$, we get that q does not divide any conjugacy class sizes of G and hence, Lemma 2.1 forces the q-Sylow subgroup of G to be a subgroup of $Z(G)$, so $q \notin \pi(G/Z(G))$, which is a contradiction. This yields that $\pi(G/Z(G)) = \pi(S)$. If $q \in \pi(G/Z(G)) = \pi(S)$ such that $|G/Z(G)|_q \neq |S|_q$, then we have $|G/Z(G)|_q > |S|_q$, by Lemma 2.5. If $q \notin \pi_2$, then since for a non-central π_2-element $y \in G$, $q \mid |C_G(y)/Z(G)|$, we can assume that $C_G(y)$ contains a non-central q-element z. But $|cl_G(y)| = \frac{|S|}{m_2}$, by Lemma 2.6(i) and hence, Lemmas 2.2(i) and 2.4(i) show that $|cl_G(z)| = |cl_G(yz)| = \frac{|S|}{m_2}$, so $C_G(y) = C_G(z)$. Now let Q be a q-Sylow subgroup of G containing z. Then Remark 2.7 forces $C_G(Q)$ to contain a non-central element w. Without loss of generality, we can assume that w is of a prime power order. Since $C_G(Q) \leq C_G(z) = C_G(y)$, we get that $w \in C_G(y)$. On the other hand, $q \nmid |cl_G(w)|$ and hence, $|cl_G(w)| \neq \frac{|S|}{m_2}$, so it can be concluded from Lemmas 2.4(ii) and 2.6(i) that $m_2 \mid |cl_G(w)|$ and w is a π_2'-element of $C_G(y)$. Thus Lemma 2.2(i) shows that $|cl_G(w)|, |cl_G(y)| = \frac{|S|}{m_2} \mid |cl_G(wy)|$ and hence, $|S| \mid |cl_G(wy)|$, so $|cl_G(wy)| \notin cs(S) = cs(G)$, which is a contradiction. This guarantees that if $q \notin \pi_2$, then $|G/Z(G)|_q = |S|_q$. Now assume that $q \in \pi_2$. By considering the elements of $cs(S) = cs(G)$, we can find a non-central element $v \in G$ of a prime power order such that $|cl_G(v)| = \frac{|S|}{k_1}$, where k_1 is a divisor of m_1. By Lemma 2.6(i), we can assume that v is a π_1-element of G and by our assumption, $q \mid |C_G(v)/Z(G)|$ and hence, we can assume that $C_G(v)$ contains a non-central q-element x. Since $q \in \pi_2$, we conclude from Lemma 2.6(i) that $|cl_G(x)| = \frac{|S|}{m_2}$ and by Lemma 2.2(i), $|cl_G(x)|, |cl_G(v)| \mid |cl_G(vx)|$, so $|S| \mid |cl_G(vx)|$ and hence, $|cl_G(vx)| \notin cs(S) = cs(G)$, which is a contradiction. This shows that $|G/Z(G)|_q = |S|_q$, as wanted in (i). Now we are going to prove (ii). Let $1 \leq i, j \leq t(S)$ such that $i \neq 1, j$. If there exist $p \in \pi_i$ and $q \in \pi_j$ such that p and q are adjacent in $GK(\bar{G})$, then $\bar{G}$ contains a p-element $\bar{x}$ such that $q \mid |C_{\bar{G}}(\bar{x})|$. Also by Lemma 2.6(ii), $|cl_{\bar{G}}(\bar{x})| = \frac{|S|}{m_i}$, so $|cl_{\bar{G}}(\bar{x})|_q = |S|_q = |\bar{G}|_q$, and hence, $|C_{\bar{G}}(\bar{x})|_q = 1$, which is a contradiction. Thus p and q are not adjacent in $GK(\bar{G})$. On the other hand, (i) and Lemma 2.6(ii) show that the order of the centralizer of every non-trivial π_i-element of $\bar{G}$ is m_i and hence, π_i is a connected component of $\bar{G}$. Now let $p, q \in \pi_1$ such that p and q are adjacent in $GK(S)$. So $cs(S) = cs(G)$ contains an element α such that $|\alpha|_p < |S|_p$ and $|\alpha|_q < |S|_q$. Let y be an element of G such that $|cl_G(y)| = \alpha$. Then since $|G/Z(G)| = |S|$, we get that $p, q \mid |C_G(y)/Z(G)|$ and since $C_G(y)/Z(G) \leq C_{\bar{G}}(\bar{y}) \leq C_{\bar{G}}(\bar{y^m})$, for every natural number m, we can assume that $O(\bar{y})$ is a prime power and $p, q \mid |C_{\bar{G}}(\bar{y})|$. So if p or $q \mid O(\bar{y})$, then p and q are adjacent in $GK(\bar{G})$ and if $O(\bar{y})$ is a power of a prime r, where $r \notin \{p, q\}$, then we have p, r and q, r are adjacent in $GK(\bar{G})$. This shows that there exists a path between p and q and hence, for every path in $GK(S)$ between elements of π_1, there exists a path in $GK(\bar{G})$ between elements of π_1, so since π_js, for $j \geq 2$, are connected components of $GK(\bar{G})$, we get that π_1 is a connected component of $GK(\bar{G})$, too. Also $|\bar{G}| = |S|$ and hence, $OC(\bar{G}) = OC(S)$, as desired in (ii). □

Definition 2.9. [17] *For a group H, the number of isomorphism classes of groups with the same set $OC(H)$ of order components is denoted by $h(OC(H))$. If $h(OC(H)) = k$, then H is called k-recognizable by the set of its order components and if $k = 1$, then H is simply called OC-characterizable or OC-recognizable.*

In many papers, it has been shown that many finite simple groups with disconnected prime graphs are OC-characterizable, for example see [11]-[29].

Corollary 2.10. *If S is OC-characterizable, then $G/Z(G) \cong S$.*

Proof. Since by Theorem 2.8(ii), $OC(\bar{G}) = OC(S)$, the result follows from the OC-recognizability of S. □

Definition 2.11. [31] *A central extension of a group H is a group K such that $K/Z(K) \cong H$. A central extension of H which is perfect is called a covering group of H. Also, if K is a covering group of H such that $K \ncong H$, then K is named a proper covering group of H. It was shown by Schur that there is a unique covering group of the maximal order, called the full covering group of H. The center of the full covering group of H is denoted by $M(H)$ and it is called the Schur multiplier of H.*

Theorem 2.12. *If S is OC-characterizable and there is no proper covering group of S with the same cs as $cs(S)$, then $G \cong S \times Z(G)$.*

Proof. On the contrary, suppose that G is the smallest group such that $cs(S) = cs(G)$ and $G \ncong S \times Z(G)$. Then since by Corollary 2.10, $G/Z(G) \cong S$ and $G'Z(G)/Z(G)$ is a normal subgroup of $G/Z(G)$, we get that $G'Z(G)/Z(G) = 1$ or $G/Z(G)$. Also G is non-solvable and hence, the former case can not occur. Thus $G'Z(G)/Z(G) = G/Z(G)$ and hence, $G'Z(G) = G$, so Lemma 2.2(ii) shows that $cs(S) = cs(G) = cs(G'Z(G)) = cs(G')$. Thus our assumption forces $G' = G$, so G is perfect and hence, G is a proper covering group of S with the same cs as $cs(S)$. This is a contradiction with our assumption. Therefore $G \cong S \times Z(G)$, as desired. (Note that some part of this proof is similar to that of in [8].) □

Theorem 2.13. *If S is OC-characterizable and $M(S) = 1$, then $G \cong S \times Z(G)$.*

Proof. It follows immediately from Theorem 2.12. □

Theorem 2.14. *If S is one of the groups in Table 1 (up to isomorphism), then $G \cong S \times Z(G)$.*

Proof. Since S is OC-characterizable, by the references stated in the third column of Table 1, we deduce from Theorem 2.13 that if $M(S) = 1$, then $G \cong S \times Z(G)$, as desired. In the following, we are going to study the remaining cases, with the help of [1]. For this aim let H be a covering group of S such that $S \ncong H$. Then considering $M(S)$ shows that:

- If $S = M_{12}$, then $H = 2.M_{12}$ and hence, $792 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.
- If $S = J_2$, then $H = 2.J_2$ and hence, $5040 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.
- If $S = HS$, then $H = 2.HS$ and hence, $30800 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.
- If $S = RU$, then $H = 2.RU$ and hence, $57002400 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.
- If $S = J_3$, then $H = 3.J_3$ and hence, $620160 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.
- If $S = McL$, then $H = 3.McL$ and hence, $99792000 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.
- If $S = Suz$, then $H = 2.Suz, 3.Suz$ or $6.Suz$ and hence, 4670265600 or $415134720 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.
- If $S = M_{22}$, then $H = 2.M_{22}, 3.M_{22}, 4.M_{22}, 6.M_{22}$ or $12.M_{22}$ and hence, $27720 \in cs(S) - cs(H)$, $12320 \in cs(S) - cs(H)$ or $2310 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.

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- If $S = Suz(8)$, then $H = 2.Suz(8)$ or $H = 4.Suz(8)$ and hence, $455 \in cs(S) - cs(H)$, so $cs(H) \neq cs(S)$.
- If $S = G_2(3)$, then $H = 3.G_2(3)$ and hence, $2184 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.
- If $S = G_2(4)$, then $H = 2.G_2(4)$ and hence, $131040 \in cs(H) - cs(S)$, so $cs(H) \neq cs(S)$.
- If $S = PSL_n(q)$, where $n > 2$ is prime, $(n, q) \neq (3, 2), (3, 4)$ and $\gcd(n, q-1) = n$, then $H = n.PSL_n(q) \cong SL_n(q)$. Let $GF(q)$ be a field with q elements and $(GF(q))^* = GF(q) - \{0\}$. Then $(GF(q))^*$ is a cyclic group of order $q - 1$. Since $n \mid q - 1$, we can assume that $(GF(q))^*$ contains an element ξ of the order n. Set $x = \mathrm{diag}(1, \xi, \xi^2, \dots, \xi^{n-1}) \in SL_n(q)$ and $Z = Z(SL_n(q))$, where $\mathrm{diag}(1, \xi, \xi^2, \dots, \xi^{n-1})$ means a diagonal matrix with numbers $1, \xi, \xi^2, \dots, \xi^{n-1}$ on a diagonal. We can check at once that $C_{PSL_n(q)}(xZ) = \langle \tau_1 Z \rangle \rtimes \langle \tau_2 Z \rangle$, where for $a_1, \dots, a_{n-1} \in (GF(q))^*$, $\tau_1 = \mathrm{diag}(a_1, \dots, a_{n-1}, (a_1 \cdots a_{n-1})^{-1})$ and
$$\tau_2 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix},$$
and hence, $|cl_{PSL_n(q)}(xZ)| = \frac{|SL_n(q)|}{n(q-1)^{n-1}} \in cs(PSL_n(q))$. We claim that $\frac{|SL_n(q)|}{n(q-1)^{n-1}} \notin cs(SL_n(q))$. If not, then there exists $y \in SL_n(q)$ such that $|C_{SL_n(q)}(y)| = n(q-1)^{n-1}$. Since $y \in C_{SL_n(q)}(y)$ and $\gcd(|C_{SL_n(q)}(y)|, q) = 1$, we get that y is a semi-simple element of $SL_n(q)$, so there exists a maximal torus T of $SL_n(q)$ containing y. Thus $T \leq C_{SL_n(q)}(y)$ and hence, $|T|$ divides $n(q-1)^{n-1}$. This forces $|T| = (q-1)^{n-1}$ and hence, we can assume that $y = \mathrm{diag}(y_1, \dots, y_{n-1}, y_n = (y_1 \cdots y_{n-1})^{-1})$ for some $y_1, \dots, y_{n-1} \in (GF(q))^*$. Since $|C_{SL_n(q)}(y)| = n(q-1)^{n-1}$, we can check at once that y_is are distinct and hence, $C_{SL_n(q)}(y) = T$, so $|C_{SL_n(q)}(y)| = (q-1)^{n-1}$, which is a contradiction. This shows that $cs(S) \neq cs(H)$.
- If $S = PSU_n(q)$, where $n > 2$ is prime, $(n, q) \neq (3, 2), (5, 2)$ and $\gcd(n, q+1) = n$, then $H = n.PSU_n(q) \cong SU_n(q)$ and hence, by replacing $GF(q)$ with $GF(q^2)$ in the case $S = PSL_n(q)$, we can see that $cs(H) \neq cs(S)$.

The above consideration shows that if S is one of the groups mentioned in Table 1 with $M(S) \neq 1$, then there is no proper covering group of S with the same cs as $cs(S)$ and hence, Theorem 2.12 forces $G \cong S \times Z(G)$, as desired. Note that if $S \cong PSL_3(2) \cong PSL_2(7)$, then it has been shown in [8] that $G \cong S \times Z(G)$. So theorem follows. □

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Table 1.

Group S	Conditions	References for OC-recognition	$\|M(S)\|$
M_{11}, M_{23}, M_{24} J_1, J_4, He, HN, Ly, Th Co_2, Co_3, Fi_{23}, M		[14]	1
M_{12}, J_2, HS, RU		[14]	2
J_3, McL		[14]	3
Suz		[14]	6
M_{22}		[14]	12
${}^2G_2(3^{2n+1}), {}^2F_4(2^{2n+1})$ $Sz(2^{2n+1})$	$n \geq 1$ $n \geq 2$	[13]	1 1
$Sz(8)$		[13]	4
$G_2(q)$	$q \neq 2, 3, 4$	[12],[15]	1
$G_2(3)$		[12],[15]	3
$G_2(4)$		[12],[15]	2
$E_8(q)$		[16]	1
$E_6(q)$	$\gcd(3, q-1) = 1$	[29]	1
$F_4(q)$	$q > 2$	[21],[23]	1
$PSL_n(q)$	$n > 2$ is prime and $(n, q) \neq (3, 2), (3, 4)$	[26]	$\gcd(n, q-1)$
$PSL_3(2) \cong PSL_2(7)$		[8]	2
$PSU_n(q)$	$n > 2$ is prime and $(n, q) \neq (3, 2)$, $(5, 2)$	[24]	$\gcd(n, q+1)$
$C_n(2^m)$	either $n = 2$ and $m > 2$ or $n = 2^u \geq 4$	[22] [25]	1
$D_{n+1}(2)$	$n > 3$ is prime	[19]	1
${}^2D_n(2^m)$	either $m = 1$ and $n = 2^u + 1 \geq 5$ or $n = 2^u \geq 4$	[20] [27]	1
${}^2E_6(q)$	$\gcd(3, q+1) = 1$ and $q \neq 2$	[28]	1
${}^3D_4(q^3)$		[11]	1

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References

[1] R. Abbott and et al., Atlas of finite group representations-version 3, brauer.maths.qmul.ac.uk/Atlas/v3/.

[2] N. Ahanjideh, Thompson's conjecture for finite simple groups of Lie type B_n and C_n, *J. Group Theory*, **19** (2016) 713–733.

[3] N. Ahanjideh, Groups with the given set of the lengths of conjugacy classes, *Turkish J. Math.*, **39** (2015) 507–514.

[4] N. Ahanjideh, Thompson's conjecture for some finite simple groups with connected prime graph, *Algebra Logic*, **51** (2013) 451–478.

[5] N. Ahanjideh, On the Thompson's conjecture on conjugacy classes sizes, *Int. J. Algebr Comput.*, **23** (2013) 37–68.

[6] N. Ahanjideh, On Thompson's conjecture for some finite simple groups, *J. Algebra*, **344** (2011) 205–228.

[7] N. Ahanjideh and M. Ahanjideh, On the validity of Thompson's conjecture for finite simple groups, *Commun. Algebra*, **41** (2013) 4116–4145.

[8] Z. Akhlaghi, M. Khatami, T. Le, J. Moori and H. P. Tong-Viet, A dual version of Huppert's conjecture on conjugacy class sizes, *J. Group Theory* , **18** (2015) 115–131.

[9] A. R. Camina and R. D. Camina, The influence of conjugacy class sizes on the structure of finite groups: a survey, *Asian-Eur. J. Math.*, **4** (2011) 559–588.

[10] A. R. Camina and R. D. Camina, Recognising nilpotent groups, *J. Algebra*, **300** (2006) 16–24.

[11] G. Chen, Characterization of ${}^3D_4(q)$, *Southeast Asian Bull. Math.*, **25** (2001) 389–401.

[12] G. Chen, Characterization of Lie type group $G_2(q)$ by its order components, *J. Southwest China Normal Univ. (Natural Science)*, **26** (2001) 503–509.

[13] G. Chen, A new characterization of Suzuki-Ree groups, *Sci. China Ser. A*, **40** (1997) 807–812.

[14] G. Chen, A new characterization of sporadic simple groups, *Algebra Colloq.*, **3** (1996) 49–58.

[15] G. Chen, A new characterization of $G_2(q)$, *J. Southwest China Normal Univ.*, **21** (1996) 47–51.

[16] G. Chen, A new characterization of $E_8(q)$, *J. Southwest China Normal Univ.*, **21** no. 3 (1996) 215–217.

[17] G. Chen, On Thompson's conjecture, *J. Algebra*, **185** (1996) 184–193.

[18] D. Chillag and M. Herzog, On the length of the conjugacy classes of finite groups, *J. Algebra*, **131** (1990) 110–125.

[19] M. R. Darafsheh, Characterization of the groups $D_{p+1}(2)$ and $D_{p+1}(3)$ using order components, *J. Korean Math. Soc.*, **47** (2010) 311–329.

[20] M. R. Darafsheh and A. Mahmiani, A characterization of the group ${}^2D_n(2)$ where $n = 2^m + 1 \geq 5$, *J. Appl. Math. Comput.*, **31** (2009) 447–457.

[21] A. Iranmanesh and B. Khosravi, A characterization of $F_4(q)$ where q is an odd prime power, *Lecture Notes London Math. Soc.*, **304** (2003) 277–283.

[22] A. Iranmanesh and B. Khosravi, A characterization of $C_2(q)$, where $q > 5$, *Commentationes Mathematicae Universitatis Carolinae*, **43** (2002) 9–21.

[23] A. Iranmanesh and B. Khosravi, A characterization of $F_4(q)$ where $q = 2^n, (n > 1)$, *Far East J. Math. Sci.*, **2** (2000) 853–859.

http://dx.doi.org/10.22108/ijgt.2017.21236

[24] B. Khosravi, B. Khosravi and B. Khosravi, A new characterization of $PSU(p,q)$, *Acta Math. Hungar.*, **107** (2005) 235–252.

[25] A. Khosravi and B. Khosravi, r-Recognizability of $B_n(q)$ and $C_n(q)$ where $n = 2^m \geq 4$, *J. Pure Appl. Algebra*, **199** (2005) 149–165.

[26] A. Khosravi and B. Khosravi, A new characterization of $PSL(p,q)$, *Commun. Algebra*, **32** (2004) 2325–2339.

[27] A. Khosravi and B. Khosravi, A characterization of ${}^2D_n(q)$, where $n = 2^m \geq 4$, *Int. J. Math. Game Theory Algebra*, **13** (2003) 253–265.

[28] B. Khosravi and B. Khosravi, A characterization of ${}^2E_6(q)$, *Kumamoto J. Math.*, **16** (2003) 1–11.

[29] B. Khosravi and B. Khosravi, A characterization of $E_6(q)$, *Algebras, Groups and Geom.*, **19** (2002) 225–243.

[30] E. I. Khukhro and V.D. Mazurov, *Unsolved problems in group theory: The Kourovka Notebook*, Sobolev Institute of Mathematics, Novosibirsk, 17th edition, 2010.

[31] P. Kleidman and M. Liebeck, *The subgroup structure of finite classical groups*, Cambridge University Press, 1990.

[32] A. S. Kondratiev, Prime graph components of finite simple groups, *Math. USSR-Sb.*, **67** (1990) 235–247.

[33] G. Navarro, The set of conjugacy class sizes of a finite group does not determine its solvability, *J. Algebra*, **411** (2014) 47–49.

[34] A. V. Vasilev, On Thompson's conjecture, *Sib. Elektron. Mat. Izv.*, **6** (2009) 457–464.

[35] J. S. Williams, Prime graph components of finite groups, *J. Algebra*, **69** (1981) 487–513.

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