CLASSIFYING FAMILIES OF CHARACTER DEGREE GRAPHS OF SOLVABLE GROUPS

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ABSTRACT. We investigate prime character degree graphs of solvable groups. In particular, we consider a family of graphs $\Gamma_{k,t}$ constructed by adjoining edges between two complete graphs in a one-to-one fashion. In this paper we determine completely which graphs $\Gamma_{k,t}$ occur as the prime character degree graph of a solvable group.

1. Introduction

We note that throughout this paper, G will be a finite solvable group. We denote $\mathrm{Irr}(G)$ for the set of irreducible characters of G, and $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. Write $\rho(G)$ to be the set of primes that divide degrees in $\mathrm{cd}(G)$. When working solely with graphs Γ (and not necessarily degree graphs of a finite solvable group G), we will also use the notation $\rho(\Gamma)$ to signify the vertex set. The degree graph of G, written as $\Delta(G)$, is the graph whose vertex set is $\rho(G)$. Two vertices p and q of $\rho(G)$ are adjacent in $\Delta(G)$ if there exists $a \in \mathrm{cd}(G)$ where pq divides a. We identify each vertex of a graph with a prime in $\rho(G)$. Throughout this paper, for simplicity, when labeling a vertex we associate that label also with a prime $p \in \rho(G)$. Character degree graphs have been studied in a variety of places; for example, see [2], [5], [6], [7], [8], and [9]. We expand upon the work done in [1].

Families of graphs have been considered and studied by way of direct products for some time. However, showing that a family of graphs cannot occur as the prime character degree graph of any solvable group was done recently in [1]. One of the main tools in that paper was to classify vertices as

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admissible. In particular, it was shown that the graph cannot occur when every vertex is admissible. Many arguments in this paper rely on this result.

In this paper, we construct a family of a graphs, denoted $\Gamma_{k,t}$ (for natural numbers $k \geq t \geq 1$), and inquire for which values of k and t does $\Gamma_{k,t}$ occur as the prime character degree graph of some solvable group. We construct $\Gamma_{k,t}$ by taking two complete graphs (uniquely determined by the number of vertices, of size k and t, respectively), and we place an edge between the two graphs injectively. That is, we attach edges uniquely from one complete graph to the other in a one-to-one fashion. The construction clearly gives a graph that satisfies Pálfy's condition and is at most diameter two. However, when determining if the graph can or cannot occur as the prime character degree graph of a solvable group, many of our arguments rely on facts about graphs that are diameter three. This has been studied more extensively in [5] and [9], and their techniques and results will play an important role.

For $k \geq t \geq 1$, we handle the cases separately for $t = 1$ (Proposition 3.1), $t = 2$ (Proposition 3.2), $t = 3$ (Proposition 3.3), and $t \geq 4$ (Proposition 3.4). We can sum up their parts in the result below:

Main Theorem. *The graph $\Gamma_{k,t}$ occurs as the prime character degree graph of a solvable group precisely when $t = 1$ or $k = t = 2$. Otherwise $\Gamma_{k,t}$ does not occur as the prime character degree graph of any solvable group.*

2. **Preliminaries**

Here we present some classic and more recent results that are immediately related to prime character degree graphs of solvable groups. Our aim is to keep this paper as self-contained as possible, and one can see [3] or [7] for more reading. Most of the results from this section, however, can be found in [1], [8], or [9].

Lemma 2.1 (Pálfy's condition). ([8]) *Let G be a solvable group and let π be a set of primes contained in $\Delta(G)$. If $|\pi| = 3$, then there exists an irreducible character of G with degree divisible by at least two primes from π. (In other words, any three vertices of the prime character degree graph of a solvable group span at least one edge.)*

Definition 2.2. ([1]) *A vertex p of a graph Γ is **admissible** if:*

(i) *the subgraph of Γ obtained by removing p and all edges incident to p does not occur as the prime character degree graph of any solvable group, and*

(ii) *none of the subgraphs of Γ obtained by removing one or more of the edges incident to p occur as the prime character degree graph of any solvable group.*

Lemma 2.3. ([1]) *If Γ is a graph in which every vertex is admissible, then Γ is not the prime character degree graph of any solvable group.*

Taking direct products is a method to generate graphs from smaller ones. We present the detailed construction below.

Definition 2.4. *A graph Γ is a **direct product** if it can be constructed in the following way:*

We start with graphs A and B. We have that $\rho(\Gamma) = \rho(A) \cup \rho(B)$, where $\rho(A)$ and $\rho(B)$ are disjoint sets. There is an edge between vertices p and q in Γ if any of the following are satisfied:

  (i) $p, q \in \rho(A)$ and there is an edge between p and q in A,

  (ii) $p, q \in \rho(B)$ and there is an edge between p and q in B, or

(iii) $p \in \rho(A)$ and $q \in \rho(B)$.

Lemma 2.5. *Complete graphs occur as $\Delta(G)$ for some solvable group G.*

The following two results are due to Lewis. They will aid in constructions throughout this paper.

Lemma 2.6. *([7]) For every positive integer N, there is a solvable group G so that $\Delta(G)$ has two complete, connected components: one having an isolated vertex, and the other having N vertices.*

Lemma 2.7. *([5]) Let G be a solvable group and let $p \in \rho(G)$. If P is a normal Sylow p-subgroup of G, then $\rho(G/P') = \rho(G) \setminus \{p\}$.*

The above Lemma 2.7 implies the subgraph $\Delta(G/P')$ is a subgraph of $\Delta(G)$ without the vertex p, all edges incident to p, and possibly without edges between vertices adjacent to p.

The following was presented in [9]. Suppose Γ is a graph of diameter three. We can partition $\rho(\Gamma)$ into four nonempty disjoint sets: $\rho(\Gamma) = \rho_1 \cup \rho_2 \cup \rho_3 \cup \rho_4$. One can do this in the following way: find vertices p and q where the distance between them is three. Let ρ_4 be the set of all vertices that are distance three from the vertex p, which will include the vertex q. Let ρ_3 be the set of all vertices that are distance two from the vertex p. Let ρ_2 be the set of all vertices that are adjacent to the vertex p and some vertex in ρ_3. Finally, let ρ_1 consist of p and all vertices adjacent to p that are not adjacent to anything in ρ_3. This notation is not unique, and depends on the vertices p and q. Using this notation, we have the following result.

Proposition 2.8. *([9]) Let G be a solvable group where $\Delta(G)$ has diameter three. One then has the following:*

  (i) $|\rho_3| \geq 3$,

  (ii) $|\rho_1 \cup \rho_2| \leq |\rho_3 \cup \rho_4|$,

(iii) if $|\rho_1 \cup \rho_2| = n$, then $|\rho_3 \cup \rho_4| \geq 2^n$, and

(iv) G has a normal Sylow p-subgroup for exactly one prime $p \in \rho_3$.

Next we state the main theorem from [1]. It gives rise to two families of graphs that satisfy Pálfy's condition. We will concern ourselves with one of them, which is our motivation of this paper.

Theorem 2.9. *([1]) Let Γ be a graph satisfying Pálfy's condition with $k \geq 5$ vertices. Assume that there exists two vertices p_1 and p_2 in Γ such that*

  (i) both p_1 and p_2 are of degree two,

  (ii) p_1 is adjacent to p_2, and

(iii) p_1 and p_2 share no common neighbor.

Then Γ is not the prime character degree graph of any solvable group.

3. **Main Results**

Similar to a direct product, here we construct the graph $\Gamma_{k,t}$. This will yield a family of graphs that we investigate and classify in our Main Theorem. We say that a graph Γ is in the family $\Gamma_{k,t}$ if it can be constructed in the following way:

We start with complete graphs A and B. Let A have k vertices (arbitrarily labeled $a_1, a_2, \ldots, a_k$), and let B have t vertices (arbitrarily labeled $b_1, b_2, \ldots, b_t$). Without loss of generality, we fix $k \geq t$.

Letting $\Gamma_{k,t} := \Gamma$, we have that $\rho(\Gamma_{k,t}) = \rho(A) \cup \rho(B)$, where $\rho(A)$ and $\rho(B)$ are disjoint sets. There is an edge between vertices p and q in $\Gamma_{k,t}$ if any of the following are satisfied:

(i) $p, q \in \rho(A)$,

(ii) $p, q \in \rho(B)$, or

(iii) $p = a_i \in \rho(A)$ and $q = b_i \in \rho(B)$ for some $1 \leq i \leq t$.

Notice how this construction generates a graph that satisfies Pálfy's condition and is at most diameter two. We now proceed towards proving the propositions needed for the Main Theorem.

Proposition 3.1. *Let $k \geq 1$. The graph $\Gamma_{k,1}$ occurs as $\Delta(G)$ for some solvable group G.*

Proof. Observe that $\Gamma_{1,1}$ occurs by Lemma 2.5. For $k \geq 2$, we consider the following: let A be a graph consisting of two complete, connected components: one having an isolated singleton, and the other having $k - 1$ vertices. Let B be a singleton. Notice that A occurs as $\Delta(G_A)$ for some solvable group G_A by Lemma 2.6, and B occurs as $\Delta(G_B)$ for some solvable group G_B by Lemma 2.5. Therefore, the direct product of A and B occurs as $\Delta(G)$ for some solvable group G. Finally, observe that $\Gamma_{k,1}$ is precisely the direct product of A and B. $\qquad\square$
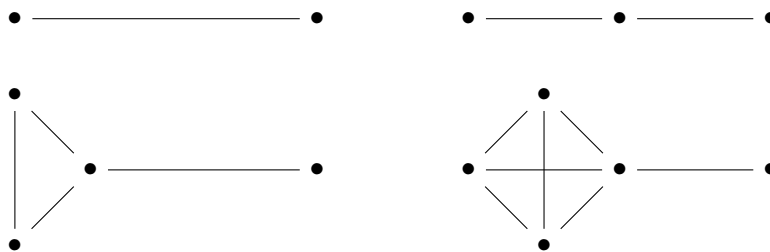

FIGURE 1. Graphs in Proposition 3.1

It was shown in [2] that $\Gamma_{2,2}$, represented in Figure 2, occurs as $\Delta(G)$ for some solvable group G. It is in fact a direct product. The rest of the cases for $t = 2$ can be extrapolated from Theorem 2.9.

FIGURE 2. The graph $\Gamma_{2,2}$

Proposition 3.2. ([1]) *Let $k \geq 3$. The graph $\Gamma_{k,2}$ is not the prime character degree graph of any solvable group.*

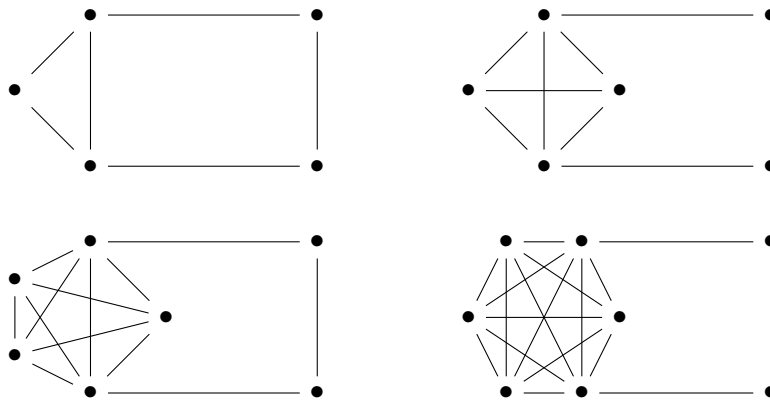*Proof.* Observe that Theorem 2.9 yields two families, one of which is $\Gamma_{k,2}$. □

FIGURE 3. Graphs in Proposition 3.2

Proposition 3.3. *Let $k \geq 3$. The graph $\Gamma_{k,3}$ is not the prime character degree graph of any solvable group.*

Proof. We proceed by induction on k. As our base case, we will show that $\Gamma_{3,3}$ does not occur as the prime character degree graph of any solvable group. We label the vertices a_i and b_i for $1 \leq i \leq 3$, where the a's and b's each form a complete graph, with edges between a_i and b_i. For the sake of contradiction, suppose that the solvable group G is a counterexample with $|G|$ minimal such that $\Delta(G) = \Gamma_{3,3}$.

We will show that a_1 is admissible. Removing the edge between a_1 and a_i (with $1 \leq i \leq 3$ and $i \neq 1$) violates Pálfy's condition with a_1, a_i, and b_j, where $1 \leq j \leq 3$, $j \neq 1$, and $j \neq i$. Removing the edge between a_1 and b_1 reduces to a graph of diameter three. This graph cannot occur since $|\rho_3| = 2$, which contradicts Proposition 2.8(i). Removing the vertex a_1 with all incident edges yields the graph $\Gamma_{3,2}$, which does not occur by Proposition 3.2. Thus, a_1 is admissible. By symmetry, we note that every remaining vertex is also admissible. Hence, $\Gamma_{3,3}$ cannot occur by Lemma 2.3.

Next we consider $k \geq 4$. As the inductive hypothesis, we suppose that $\Gamma_{k-1,3}$ does not occur as the prime character degree graph of any solvable group. We now proceed to the inductive step. In order to see that $\Gamma_{k,3}$ also does not occur as the prime character degree graph of any solvable group, one again shows that every vertex is admissible. We start by labeling the vertices a_i and b_j for $1 \leq i \leq k$ and $1 \leq j \leq 3$, where the a's and b's each form a complete graph, with edges between a_j and b_j. For the sake of contradiction, suppose that the solvable group G is a counterexample with $|G|$ minimal such that $\Delta(G) = \Gamma_{k,3}$.

First we consider the vertex a_1. Removing a_1 and all incident edges yields a graph of diameter three such that $|\rho_3| = 2$, which contradicts Proposition 2.8(i). Removing the edge between a_1 and a_i, where $2 \leq i \leq k$, violates Pálfy's condition with a_1, a_i, and b_j, where $1 \leq j \leq 3$, $j \neq 1$, and $j \neq i$. Removing the edge between a_1 and b_1 again gives a graph of diameter three, which once again contradicts Proposition 2.8(i). Thus, a_1 is admissible, and by symmetry so too are a_2 and a_3.

Next we consider the vertex a_k. Removing a_k and all incident edges yields the graph $\Gamma_{k-1,3}$, which we are assuming does not occur as the prime character degree graph of any solvable group by our inductive hypothesis. Removing the edge between a_k and a_i for $1 \leq i \leq k-1$ again violates Pálfy's condition with a_k, a_i, and b_j, where $1 \leq j \leq 3$ and $j \neq i$. Thus, a_k is admissible. By symmetry, we also get that a_i is admissible for $4 \leq i \leq k-1$.

Finally we consider the vertex b_1. Removing b_1 and all incident edges gives us the graph $\Gamma_{k,2}$, which does not occur as the prime character degree graph of any solvable group by way of Proposition 3.2. Removing the edge between b_1 and b_j for $2 \leq j \leq 3$ violates Pálfy's condition with b_1, b_j, and a_k. Removing the edge between b_1 and a_1 was already considered above and shown to result in a graph which cannot occur. Thus, b_1 is admissible, and by symmetry we now have that b_2 and b_3 are admissible as well.

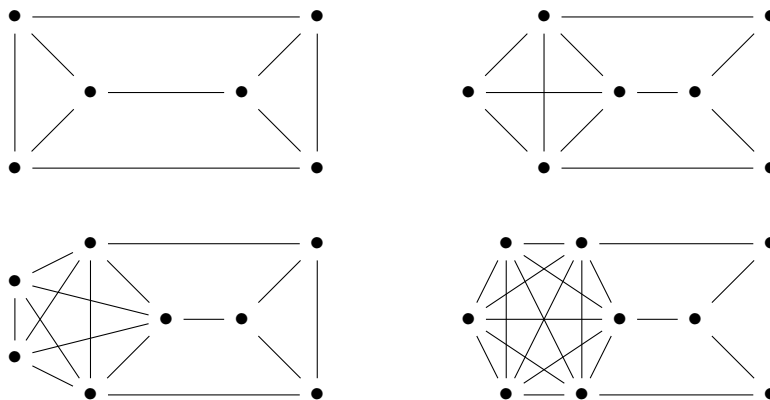Since all vertices are admissible, we then invoke Lemma 2.3 to get our desired result. $\square$

FIGURE 4. Graphs in Proposition 3.3

Proposition 3.4. *Let $k \geq t \geq 4$. The graph $\Gamma_{k,t}$ is not the prime character degree graph of any solvable group.*

Proof. We proceed by induction on both k and t. For our base case, we will show that $\Gamma_{k,4}$ does not occur as the prime character degree graph of any solvable group. To do this, one must first consider $\Gamma_{4,4}$ and then induct on k. To see that $\Gamma_{4,4}$ does not occur, we start by labeling the vertices a_i and b_i for $1 \leq i \leq 4$, where the a's and b's each form a complete graph, with edges between a_i and b_i. For the sake of contradiction, suppose that the solvable group G is a counterexample with $|G|$ minimal such that $\Delta(G) = \Gamma_{4,4}$.

We will show that a_1 is admissible. Removing the edge between a_1 and a_i (with $1 \leq i \leq 4$ and $i \neq 1$) violates Pálfy's condition with a_1, a_i, and b_j, where $1 \leq j \leq 4$, $j \neq 1$, and $j \neq i$. Removing the edge between between a_1 and b_1 yields a graph of diameter three. Notice that $|\rho_1 \cup \rho_2| = 4$ and $|\rho_3 \cup \rho_4| = 4$, which contradicts Proposition 2.8(iii). Removing the vertex a_1 and all incident edges yields the graph $\Gamma_{4,3}$, which does not occur by Proposition 3.3. Thus a_1 is admissible. By symmetry, notice that every remaining vertex is also admissible. Hence $\Gamma_{4,4}$ cannot occur by Lemma 2.3.

Next we consider $k \geq 5$. As the inductive hypothesis, we assume that $\Gamma_{k-1,4}$ does not occur as the prime character degree graph of any solvable group. To see that $\Gamma_{k,4}$ does not occur as the prime character degree graph of any solvable group, we start by labeling the vertices a_i and b_j for $1 \leq i \leq k$ and $1 \leq j \leq 4$, where the a's and b's each form a complete graph, with edges between a_j and b_j. For the sake of contradiction, suppose that the solvable group G is a counterexample with $|G|$ minimal such that $\Delta(G) = \Gamma_{k,4}$.

First we show that a_1 is admissible. Removing the edge between a_1 and a_i (where $1 \leq i \leq k$ and $i \neq 1$) violates Pálfy's condition with a_1, a_i, and b_j, where $1 \leq j \leq 4$, $j \neq 1$, and $j \neq i$. Removing the edge between a_1 and b_1 yields a graph of diameter three (which we denote by $\Delta(H)$ for some solvable group H). Notice that Proposition 2.8(ii) forces $\rho_3 = \{a_2, a_3, a_4\}$, and by Proposition 2.8(iv), we know that H has a normal Sylow p-subgroup for exactly one prime $p \in \rho_3$. Without loss of generality, let $p = a_2$, and call the normal subgroup P. By Lemma 2.7, we know that $\rho(H/P') = \rho(H) \setminus \{p\}$. Observe that $\Delta(H/P')$ is a connected subgraph obtained from $\Delta(H)$ by removing the vertex a_2 and all incident edges, and possibly edges between vertices adjacent to a_2. However, in $\Delta(H/P')$, we have that $|\rho_3| = 2$, which contradicts Proposition 2.8(i). Thus the edge between a_1 and b_1 cannot be lost. Removing the vertex a_1 and all incident edges yields a graph of diameter three. By following an argument similar to the above, we can again get that $|\rho_3| = 2$, which cannot happen. Thus a_1 is admissible. By symmetry, a_2, a_3, and a_4 are admissible as well.

Following similar arguments, one can show that b_j is admissible for $1 \leq j \leq 4$ (relying on $\Gamma_{k,3}$ from Proposition 3.3), as well as a_i for $5 \leq i \leq k$ (relying on $\Gamma_{k-1,4}$ from the above inductive hypothesis). Thus, all vertices are admissible, and we get our result from Lemma 2.3.

Finally, for $k \geq t \geq 5$, we verify that $\Gamma_{k,t}$ does not occur. The inductive hypothesis requires that $\Gamma_{k,t-1}$ does not occur as the prime character degree graph of any solvable group. We then need to

show $\Gamma_{k,t}$ does not occur, which requires induction. First we will verify that $\Gamma_{t,t}$ does not occur; we will follow a similar argument as above, mimicking the proof for $\Gamma_{4,4}$. Label the vertices a_i and b_i for $1 \leq i \leq t$, where the a's and b's each form a complete graph, with edges between a_i and b_i. For the sake of contradiction, suppose that the solvable group G is a counterexample with $|G|$ minimal such that $\Delta(G) = \Gamma_{t,t}$.

We start by considering the vertex a_1. Removing the edge between a_1 and a_i (with $1 \leq i \leq t$ and $i \neq 1$) violates Pálfy's condition with a_1, a_i, and b_j, where $1 \leq j \leq t$, $j \neq 1$, and $j \neq i$. Removing the edge between between a_1 and b_1 yields a graph of diameter three. Notice that $|\rho_1 \cup \rho_2| = t$ and $|\rho_3 \cup \rho_4| = t$, which contradicts Proposition 2.8(iii). Removing the vertex a_1 and all incident edges yields the graph $\Gamma_{t,t-1}$, which does not occur by the inductive hypothesis. Thus a_1 is admissible. By symmetry, notice that every remaining vertex is also admissible. Hence $\Gamma_{t,t}$ cannot occur by Lemma 2.3.

Then we suppose that for $k > t \geq 5$, we have that $\Gamma_{k-1,t}$ does not occur as the prime character degree graph of any solvable group. Showing $\Gamma_{k,t}$ does not occur employs the same argument as for $\Gamma_{k,4}$, using Proposition 2.8(iv) and Lemma 2.7 multiple times before getting a contradiction with Proposition 2.8(i). To start, we label the vertices a_i and b_j for $1 \leq i \leq k$ and $1 \leq j \leq t$, where the a's and b's each form a complete graph, with edges between a_j and b_j. For the sake of contradiction, suppose that the solvable group G is a counterexample with $|G|$ minimal such that $\Delta(G) = \Gamma_{k,t}$.

We show that a_1 is admissible. Removing the edge between a_1 and a_i (where $1 \leq i \leq k$ and $i \neq 1$) violates Pálfy's condition with a_1, a_i, and b_j, where $1 \leq j \leq t$, $j \neq 1$, and $j \neq i$. Removing the edge between a_1 and b_1 yields a graph of diameter three (which we denote by $\Delta(H)$ for some solvable group H). Notice that Proposition 2.8(ii) forces $\rho_3 = \{a_2, a_3, \ldots, a_t\}$, and by Proposition 2.8(iv), we know that H has a normal Sylow p_1-subgroup for exactly one prime $p_1 \in \rho_3$. Without loss of generality, let $p_1 = a_2$, and call the normal subgroup P_1. By Lemma 2.7, we know that $\rho(H/P_1') = \rho(H) \setminus \{p_1\}$. Observe that $\Delta(H/P_1')$ is a connected subgraph obtained from $\Delta(H)$ by removing the vertex a_2 and all incident edges, and possibly edges between vertices adjacent to a_2. However, notice that $\Delta(H/P_1')$ is also of diameter three. For this graph, Proposition 2.8(ii) forces $\rho_3 = \{a_3, \ldots, a_t\}$, and by Proposition 2.8(iv) we know that H/P_1' has a normal Sylow p_2-subgroup for exactly one prime $p_2 \in \rho_3$. Without loss of generality, let $p_2 = a_3$, and call the normal subgroup P_2. Observe one can show that $P_1 P_2 = P_1 \times P_2$ is normal in H. Notice this implies that P_2 is also a normal Sylow p-subgroup of H, where $p \in \rho_3$. This contradicts Proposition 2.8(iv), since we now have two such subgroups: P_1 and P_2. Finally, removing the vertex a_1 and all incident edges yields a graph of diameter three. By following an argument similar to the above, we can again get that $|\rho_3| = 2$, which cannot happen. Thus a_1 is admissible. By symmetry, a_j is admissible for all $1 \leq j \leq t$.

One can show that b_j is admissible, which relies on the graph $\Gamma_{k,t-1}$ from the inductive hypothesis. Furthermore, showing a_i is admissible ($t+1 \leq i \leq k$) relies on the graph $\Gamma_{k-1,t}$, which we are supposing does not occur. Since all vertices are admissible, our result follows by way of Lemma 2.3.   □
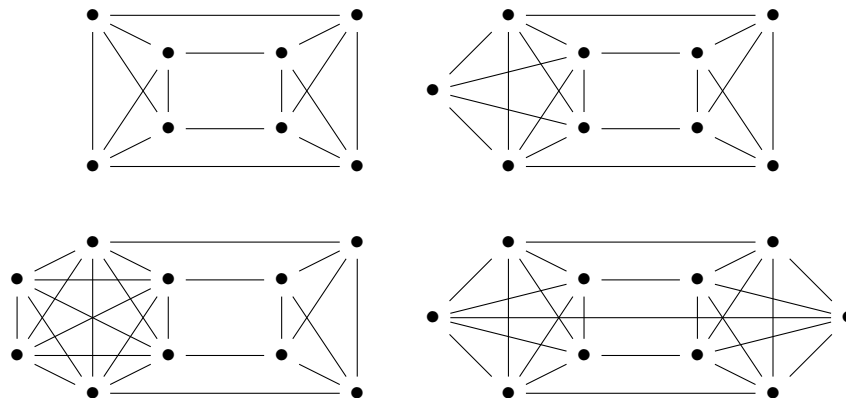
FIGURE 5. Graphs in Proposition 3.4

As consequence of Propositions 3.1, 3.2, 3.3, and 3.4, our Main Theorem is proved. We state it again below:

Theorem 3.5. *The graph $\Gamma_{k,t}$ occurs as the prime character degree graph of a solvable group precisely when $t = 1$ or $k = t = 2$. Otherwise $\Gamma_{k,t}$ does not occur as the prime character degree graph of any solvable group.*

Corollary 3.6. *Let $k \geq t \geq 2$. No connected proper subgraph of $\Gamma_{k,t}$ with the same vertex set is the prime character degree graph of any solvable group.*

The case for the disconnected subgraph can also be considered. The group structure of two connected components is fully outlined in [4].

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