Essays on Credit Frictions and the Macroeconomy

Angus Foulis

A thesis submitted to the Department of Economics of the London School of Economics and Political Science for the degree of Doctor of Philosophy

July 2013

Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of the work carried out jointly by me and any other person is clearly identified in it).

The copyright of this thesis rests with the author. Quotation from it is permitted, provided that full acknowledgement is made. This thesis may not be reproduced without my prior written consent.

I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

I declare that my thesis consists of $44,886$ words

Statement of Conjoint Work

I confirm that the third chapter was jointly coauthored with Benjamin Nelson and Misa Tanaka of the Bank of England, and I contributed 70% of this work.

Abstract

The three chapters in this thesis consider the role macroprudential policy can play in economic booms and busts. The first two chapters concern the recent housing boom in the United States. Whilst it is popularly thought that a significant easing of credit standards caused the boom, the econometric attempts to establish this are largely inconclusive. The fall in real interest rates also fail to account for the magnitude of the boom, suggesting buyers' irrational exuberance. I approach this problem in a new way using tiered housing data that separately covers the price movements of cheap and expensive houses. During the US boom, the cheapest houses had the largest relative price gains in 51 of 52 metro areas studied. In the first chapter I use a simple model to show that this pattern could not have occurred without an easing of credit standards: without this, buyer exuberance or a fall in interest rates would produce the opposite pattern.

Chapter two examines alternative explanations for the tiered pattern, including changes in housing supply, speculation and differential income growth. I show that these variables are not responsible for the pattern, but that, in keeping the theory, there is a statistically and economically significant relationship between credit easing and the relative performance of low and high tier house prices. Taken together, the two chapters conclude that the housing boom would have significantly smaller if policy had prevented credit standards from easing.

The third chapter considers *credit traps*; a situation in which a severe financial crisis gives rise to a prolonged period of low lending to, and stagnation of, the real economy. We introduce a model in which credit traps are possible, then consider what macroprudential policy can do to help the economy escape from a trap, and to reduce the chances of falling into one.

Acknowledgements

I would like to thank my supervisor, Gianluca Benigno, for his continued encouragement and valuable feedback throughout my PhD. I am very grateful for his confidence in me from the beginning.

I am also grateful to have worked alongside two great co-authors at the Bank of England in Ben Nelson and Misa Tanaka, during a summer internship out of which the third chapter emerged. I found it very stimulating to be thinking about policy questions from within a policy institution and learned a great deal from both of them.

I am grateful to many others at LSE for useful conversations over the years, including Wouter den Haan, Johannes Boehm, Daniel Osorio, Michele Piffer and Jason Garred. I am particularly grateful to Gianni La Cava for sharing his wealth of knowledge about the US housing market with me, and also his data, without which Chapter 2 would not have been possible. I'm also incredibly grateful to my family for not only encouraging eleven years in higher education, but for taking a keen interest in the US housing market!

I would also like to thank the ESRC for providing financial support for my first four years at LSE, support without which the PhD would not have been possible. I also thank Fiserv, Inc. for providing the house price data used in Chapter 2.

Finally, the biggest thanks goes to Ana for all her support, love, and encouragement over the last four years. She taught me how to present, sat through many awful dry-runs, improved my writing style, but most importantly had unwavering confidence in me. This thesis is dedicated to Ana.

Contents

[1] This chapter is joint work with Benjamin Nelson and Misa Tanaka from the Bank of England.

List of Figures

List of Tables

Preface

Following the recent global financial crisis, there has been widespread recognition that governments and central banks should play a more active role in regulating financial markets. Many central banks are set to be granted macroprudential policy tools, though as their prior use has been highly limited, there is much uncertainty about how effective they would be. A crucial component of this is understanding what caused the crisis in the first place, and what, if anything, macroprudential policy could have done about it. The three chapters in this PhD thesis are all concerned with this issue, with the first two examining the causes of the boom and bust in the US housing market. The third considers theoretically what macroprudential policy can do to reduce the chances of a financial crisis occurring, and further what can be done to resurrect the economy after a severe crisis has occurred.

The global financial crisis began with the collapse of the US housing market, yet the causes of this dramatic boom and bust are still not well understood. In the words of two leading housing economists, writing in 2011:

> "The United States recently experienced house price growth of unprecedented scale...many researchers have tried to understand whether the most recent cycle was a bubble, or if rational theories can account for the variation in prices and quantities at the national level and across metropolitan areas (MSAs). Despite this work and the fact that we are now several years into the current housing crisis, researchers and policy makers still have conflicting views and limited knowledge about the causes of that extraordinary rise and decline in house prices."

Ferreira and Gyourko (2011)

Despite the widespread popular belief that easy credit caused the housing boom, researchers have not been able to show empirically that the extent of credit easing can account for the rise in prices. Another common explanation, the fall in real interest rates, also fails to account for the majority of the rise in prices. There could be a temptation to attribute the remainder of the boom to home buyers' irrational exuberance, however it is very hard to test this empirically as there is scant data on house price expectations during the boom. With the cause of the boom uncertain, there is thus great uncertainty about the efficacy of regulation in attenuating future housing cycles.

The first two chapters address this issue with a novel approach. A new dataset is introduced with separate house price indices for cheap ("low-tier") and expensive ("high-tier") houses for 52 US metro areas. I document a remarkable pattern: in 51 of the 52 cities, the cheapest houses had greater price growth during the boom. This pattern is used to the infer the underlying causes of the US housing boom. In Chapter 1, a theoretical model featuring two housing tiers is used to evaluate the three main proposed causes: a fall in interest rates, an easing of credit standards, and irrational exuberance on the part of homebuyers. This model is used to perform counterfactual analysis of what a fall in interest rates or irrational exuberance would look like if credit standards were not eased. In both cases, the model predicts that high tier houses prices would have greater growth, the opposite of what we see in the data. By contrast, with an easing of credit standards, low tier prices grow more, consistent with the recent US experience.

Chapter 2 tackles this issue empirically. First, the new dataset is used to introduce several new facts about the boom, the bust and the link between them. These facts can be accounted for in a parsimonious way using the theoretical prediction of Chapter 1 that an easing of non-price credit terms will have a relatively greater impact on low tier prices. Using this data I test this implication of the theory, finding statistically and economically significant relationships between two separate measures of credit easing, and relative changes in low and high tier prices, both during the boom and bust. Further, I augment the analysis of Chapter 1 by examining alternative explanations for the tiered pattern beyond the three considered, including changes in housing supply, speculators and differential income growth for low and high tier buyers. I show that these variables are not responsible for the pattern.

Taken together, the two chapters provide a strong case that there must have been a significant easing of non-price credit terms during the boom. Without this, we would not have observed the remarkable tiered pattern that we did. The implication for policy is that product regulation in the mortgage market could have significantly reduced the extent of the boom and bust in US housing. A back of the envelope calculation suggests that if non-price credit terms had been prevented from easing, the cheapest third of houses would have grown at least 55 percentage points less in nominal terms during the housing boom in the average city.

Chapter 3 moves away from the housing market and considers the link between the financial sector and the real economy. Motivated by post-crisis economic stagnation in the UK and a depressed banking sector, we consider the possibility of a *credit trap*: a steady state of the economy featuring permanently low output, bank lending, and financial sector net worth. We develop a simple overlapping generations model to perform counterfactual analysis regarding the appropriate policy actions if the economy is indeed stuck in a credit trap. We show that countercyclical leverage policy will be ineffective in a credit trap (in contrast to a 'normal' recession), and consider instead three unconventional credit policies, obtaining clear predictions about the relative efficacy of each. We also consider what policy can do to reduce the fall-out from a financial crisis, showing that a regulatory leverage ratio can increase the resilience of the economy, reducing the chances of it falling into a credit trap.

Chapter 1

The Role of Credit in the US Housing Boom

Whilst it is commonly believed that a major easing of credit standards caused the US housing boom, econometric attempts to show this have been largely inconclusive, as have attempts to explain the boom in terms of falling real interest rates. This has led some authors to speculate that the major cause was homebuyers' irrational exuberance. This paper introduces a novel way of distinguishing between these three proposed explanations of the boom by analysing the pattern of relative capital gains across different tiers of housing, sorted by value. In contrast to previous US housing booms, the cheaper houses within cities had significantly higher relative gains than more expensive houses. By using an Overlapping Generations model with a housing ladder, I show that this pattern could not have arisen through a fall in interest rates or buyer optimism without significant credit easing, establishing the necessity of credit easing. The results suggest that macroprudential tools that can prevent credit easing from occurring, such as a cap on maximum loan-to-value and loan-to-income ratios, could have reduced the nominal growth of the cheapest third of houses over the boom by at least 55 percent points. Chapter 2 tests this prediction empirically and examines alternative explanations for the tiered pattern beyond the three considered theoretically here.

1.1 Introduction

A major housing boom and bust occurred in the United States during the first decade of the twenty-first century (Figure 1.1). From the trough at the end of 1996 to the peak at the start of 2006, the Case-Shiller national house price index grew 86% in real terms. It then fell significantly to the end of 2011 with average prices only 10% higher than they were in 1996, putting them at the same level as those in 1987. The common explanation for this is a substantial relaxation of non-price credit terms such as the Loan-to-Value (LTV) and Loan-to-Income (LTI) ratios, with the growth in average LTV ratios shown in Figure 1.2. However, econometric attempts to establish the link between credit easing and the boom have been largely inconclusive. Glaeser et al (2010) and Coleman IV et al (2008) do not find a significant relationship between changes in LTV and house prices during the boom. A limitation of both papers is that they do not tackle the likely endogeneity between credit standards and house prices. This may arise for, on the one hand, an easing of mortgage credit can increase housing demand, thereby driving up prices. However, bubble conditions in the housing market could reduce the default concerns lenders have, thus increasing their desire to provide loans to risky borrowers.[1] Thus, rising prices could also lead to relaxed credit standards. Adelino et al (2012) and Favara and Imbs (2011) provide instruments to tackle this problem, based on changes in conforming loan limits and branching regulation, respectively. They find statistically significant relationships between changes in credit conditions and house prices during the boom, though the economic magnitudes are very small and explain less than 3 percentage points of house price growth over the period.

Attempts to explain the boom in terms of a fall in real interest rates (Glaeser et al 2010) have also failed to account for its magnitude. This has led to economists such as Glaeser et al (2010) to propose looking at the irrational exuberance of home-buyers.[2] This might seem appealing at first: survey evidence from Case and Shiller (2003) highlights the highly optimistic outlook buyers in the property market had concerning future capital gains, expecting an average annual gain of at least 11% for the next ten years in the four cities sampled.

In this chapter we consider theoretically the role of these three factors in the housing boom. The implications for policy are very different depending on the major cause, whether a fall in interest rates, an easing of non-price credit terms, or exogenous optimism. If the fall in interest rates was the main factor, then existing monetary policy tools could have been used to attenuate the boom. If an easing of credit standards was the primary culprit, through looser LTV and LTI ratios, interest rate policy may be largely ineffective as well as undesirable, given its impact on the wider economy. New macroprudential policies such as LTV or LTI caps, as well as different capital requirements for loans to the housing market, could prove to be useful and efficient by targeting just the housing market. We are left with irrational exuberance: if this were indeed responsible for the boom, there is no obvious policy prescription. Given the various possibilities, each with different implications, it is crucial to determine the primary cause so that future damaging booms may be attenuated using appropriate policy measures.

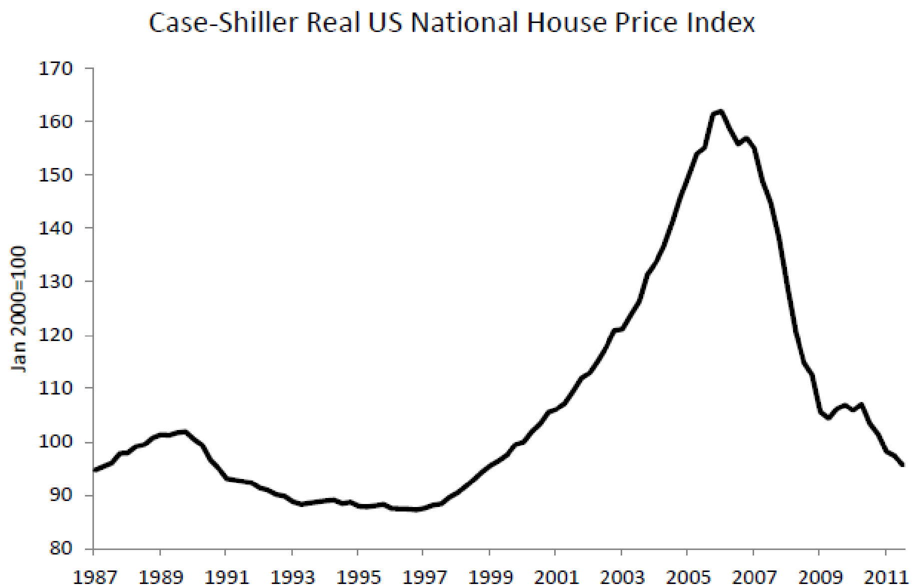
An econometric approach cannot be used to help determine the contribution of buyer optimism to the housing boom, as there is insufficient time series data on future price expectations.[3] It is neither possible to eyeball this from the aggregate data, as a boom caused by optimism will look very similar to a boom caused by a fall in interest rates or an easing of credit standards. A residual

[1] See Brueckner et al (2012). They formalise this mechanism showing theoretically that a positive shift in lenders' house price expectations reduces default concerns, thereby spuring lending. They also find tentative evidence for this mechansim during the boom in the US, showing that borrower riskiness rises when a proxy for price expectations improves.

[2] This is to be distinguished from any irrational exuberance on the part of mortgage lenders, which has been studied indirectly by Mian and Sufi (2009). In this paper we focus on the exuberance of buyers.

[3] Case et al (2012) have recently produced time-series for future price expectations from 2003 until 2012, however it only covers four US cities.

Figure 1.1: Case-Shiller National Data

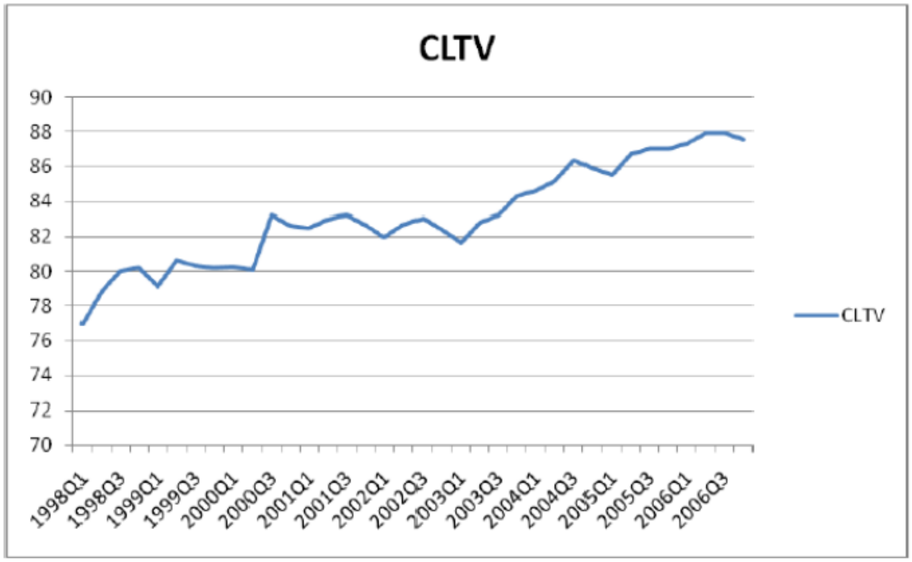


component of the boom, unexplained by credit and interest rate studies, cannot be attributed to buyer optimism due to serious endogeneity concerns regarding credit supply and house prices. In this paper I shall not offer any novel instruments to resolve this econometric issue or unveil any new data regarding price expectations. Rather, I shall argue for the necessity of the credit channel through analysis of tiered housing data.

In addition to their famous repeat-sales aggregate house price index, Case and Shiller have produced an index that breaks the housing data within each city down into three equal tiers, sorted by value.[4] A weighted average of this across 17 publicly available Case-Shiller cities is shown in Figure 1.3. A striking pattern emerges regarding relative capital gains: prior to the boom, all price tiers grow at the same rate; during the boom, the cheapest houses see the highest relative gains, followed by the middle tier, with the most expensive houses experiencing the smallest increase. The pattern is not a feature of aggregation. I augment the public data with additional purchased data from Fiserv giving tiered Case-Shiller house price indices during the boom for a total of 52 US cities, covering 26 states, with graphs given in the appendix to chapter 2. Remarkably, low tier prices grew relatively more than the high tier prices during the boom in 51 of the 52 cities. The pattern is not a quirk of the way the data is constructed; indeed, using Case-Shiller data, Mayer (1993) documented that in the 1970s and 80s the opposite pattern occurred with high tier house prices growing relatively more in Atlanta, Chicago, Dallas and Oakland. Poterba (1991) has similar findings. Finally, using different methods, Smith and Tesarek (1991) show that in the 1970s boom in Houston, high tier houses had the greatest appreciation. Indeed such was the prevalence

[4] A detailed description of this data is given in Chapter 2.

Figure 1.2: Credit Easing: Average for Case-Shiller 19 MSAs



Note: The CLTV measures the percentage of the value of the house that can be met with a loan. The remaining percentage is required as a down-payment. This graph comes from Coleman IV et al (2008).

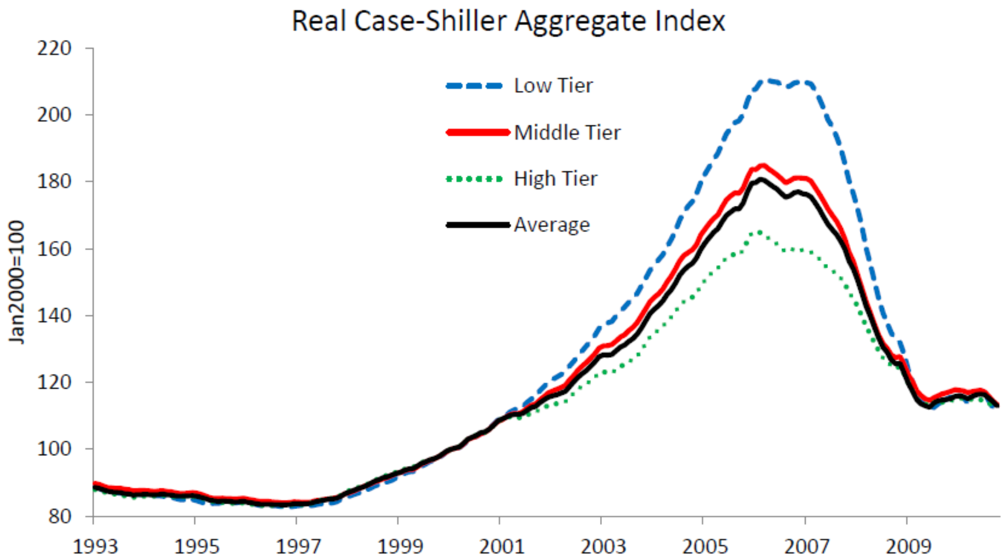
of this pattern, Mayer (1993) suggests a theory based on an extension of Stein (1995) showing why it's inevitable that high tier house prices will grow more than low tier prices during a boom. The recent experience was not inevitable.

Tiered housing data tells us something new about the housing market because the buyers of houses in different tiers are different in a crucial respect. Specifically, buyers of low tier houses are likely to be credit constrained when buying, whilst the buyers of high tier houses are not. Mayer and Engelhardt (1996) show that first time buyers, who buy cheaper houses, make significantly smaller percentage down-payments than repeat buyers, who purchase more expensive houses.[5] The reason for this is that first-time buyers tend to be younger, typically have low wealth, and must save on average for 2.5 years to accumulate their deposit. By contrast, repeat buyers are older and have built up housing equity in their present house through paying down their mortgage, and are therefore able to make larger relative down-payments. Engelhardt (1996) demonstrates the burden of saving for the deposit on a first house, using panel data to show that people actually reduce their expenditure on food whilst saving for this first purchase.

This heterogeneity between buyers of cheap and expensive houses affects not only how they respond to credit easing but also their response to other shocks. The relative price growth of the

[5] In Table 1.2 in the appendix we provide average low and high tier LTV data for 26 of our 52 cities during different stages of the recent boom, with data coming from various editions of the American Housing Survey. In 25 of the 26 cities, the average low tier LTV is higher than the average high tier LTV. This is survey data with a small sample size, so comparisons should only be made between low and high tier buyers *within* a city, not between cities, or over time.

Figure 1.3: Tiered Case-Shiller Data



tiers during the housing boom can thus be used to infer the underlying shocks. We argue that the price of low tier housing could not have grown relatively more than the high tier in the absence of an easing of non-price credit terms.

Home-buyers constrained by a maximum LTV ratio make the minimum percentage down-payment that they're required to and would like to make an even smaller down-payment for the purpose of smoothing consumption across their life rather than reducing their expenditure on food for a few years prior to buying. Being credit-constrained reduces the responsiveness of the price they pay to changing factors in the housing market, such as expectations of future price growth. Ultimately, any increase in the price that constrained house buyers pay *must come from a reduction in current consumption*, as it requires a greater absolute deposit. As people are limited by their ability to reduce their expenditure on food, there is a limit to how much they can respond in the absence of credit easing, regardless of how optimistic they are. This contrasts with unconstrained home buyers who make a greater deposit than the minimum required. If they want to pay more for a house, they can do so *without affecting current consumption*. They thus have the resources to respond more elastically to optimism over future price growth or a fall in the interest rate.

For both constrained and unconstrained house buyers, a reduction in the interest rate or an increase in future resale prices reduces the lifetime cost of owning a house, inducing a greater price paid. However, the increase in price paid will be *greater* for the unconstrained buyer as they have the resources to increase the price paid without affecting current consumption, thereby pushing up demand. An easing of the down-payment requirement will have no direct effect on unconstrained buyers; relaxing this, on the other hand, will increase the price that constrained buyers can pay for a house.

An additional pertinent feature of the housing market is the housing ladder. People typically move from lower to higher quality houses as age, income, and wealth increase.[6] This channel acts as a mechanism for transmitting capital gains from the low to the high tier: as people move up the housing ladder, any increase in the price received for the sold low quality house can be used for a greater deposit for the purchase of a higher quality house. In practice, low tier buyers are credit constrained and at the bottom of the housing ladder, whilst high tier buyers are unconstrained *and* higher up the housing ladder.

An easing of non-price credit terms directly affects only the low tier buyers, and whilst the high tier buyers have an indirect effect through transmitted capital gains, the relative price increase will be greater for the low tier. By contrast, with buyer optimism or a fall in the interest rate, the relative price growth of the high tier will be greater. As high tier buyers are unconstrained, there is a greater direct effect on the price they pay from the change in expectations or the interest rate. In addition, there is the indirect positive effect on the price from transmitted capital gains. Thus, the pattern observed in the US of the low tier growing relatively more than the high tier *could not have occurred without an easing of non-price credit terms.* If credit standards had not been reduced, and the boom was caused by a fall in interest rates or buyer optimism alone, we would have observed exactly the *opposite* pattern.[7]

I shall not attempt to disentangle the likely endogenous relationship between credit easing, house price expectations, and interest rates to point to an ultimate cause of the housing boom. We do not need to do this to be able to draw useful policy implications. The results we develop show that the whole cycle could not have taken place as it did if credit standards were not relaxed. The growth of the high tier places a lower bound on how much low tier prices could have been attenuated, if these were the three factors driving the boom.[8] This bound implies an average 55 percentage point reduction in the nominal growth rate of the low tier to the peak of the housing boom, across the 52 cities studied (the reduction in low tier price growth required for the high tier to grow more in the average city).

In this paper we present a housing model comprised of two types of houses to buy and a housing ladder that formalises the above arguments. The remainder of Section 1.1 discusses the modelling approach, whilst Section 1.2 elaborates on a basic model with only one type of house to buy and develops the intuition regarding the differing responses of constrained and unconstrained house buyers. Section 1.3 extends this model to include two types of houses available for purchase and a housing ladder, and it presents the main results of the paper. Section 1.4 shows robustness to alternative explanations, whilst Section 1.5 establishes the negative welfare cost of boom and bust cycles, and the benefit that policy can bring. Section 1.6 concludes.

[6] In the last decade an average of 66% of recent home-buyers say they have moved to a higher quality house (American Housing Survey).

[7] The approach taken here is similar in spirit to Landvoigt et al (2012) in seeking to distinguish between different candidate explanations for the US housing boom by looking at the differential effect each has on relative price gains within a metro area.

[8] It is a lower bound as the actual growth in the low tier contributed to the growth of the high tier.

1.1.1 Modelling approach

The standard existing theoretical literature on the housing market, such as Iacoviello (2005), Kiyotaki et al (2007), and the user cost models in the style of Poterba (1984), are completely silent on the relative capital gains across different housing tiers, as these models only include one type of house. However, if these models included tiered housing, the relative capital gains would be *identical*. This is because they treat housing as an infinitely divisible[9] asset, with individuals choosing how much housing to buy, rather than whether or not to buy a specific house of fixed size.[10] Consequently, in equilibrium, all agents adjust the amount of housing and non-housing consumption during each period in such a way that they all have the same intratemporal MRS between the two: all agents are thus marginal buyers. Therefore, each unit of housing has the same price in terms of consumption, so a house of size (alternatively quality) S costs exactly half that of a house of size $2S$. Let the price of a unit of housing be p. Thus a house of size S costs $P(S) = pS$. Suppose due to a shock the price changes from p to p'. Then the house of size S now costs $p'S$ and the relative capital gain is given by

$$\frac{p'S - pS}{pS} = \frac{p' - p}{p}$$

which is independent of S so all houses have exactly the same relative capital gains. Such a modelling approach is clearly of no use for our purposes.

To tackle the US experience, we build a housing model with an indivisible housing stock. One approach is that of Landvoigt et al (2012) who use an assignment model that maps a continuum of house buyers into a continuum of indivisible houses.[11] Equilibrium house prices adjust to assign movers to the distribution of houses. Movers differ along three dimensions: wealth, income and age; whilst it is assumed that all the features that matter for the quality of a house, such as the neighbourhood and structure, can be combined into a unique quality index. Estimating the quality index from microdata for San Diego, they numerically solve for the change in the distribution of house prices over the boom from 2000-2005, comparing their results to the data.

For tractability, we instead make the choice over housing discrete rather than continuous. Previous work in this area has been done by Ortalo-Magne and Rady (1999) in an OLG set-up with agents living for 4 periods. To keep their analysis tractable, they assume that low tier buyers buy a house as soon as they can afford the required down-payment. The low tier buyers do not take interest rates and future house prices into account when making their purchase, thereby rendering their model unsuitable for our purposes. By making simplifications along other dimensions (agents live 2 rather than 4 periods), we are able to present a model in which interest rates and future house prices influence the decision of both low and high tier buyers.

[9] A key requirement for this is that there exist a costless linear technology for turning different types of house into each other, allowing people to get exactly the amount of housing they want. This allows that, for example, a house of size S and a house of size $2S$ can be combined to produce, say, two houses of size $1.5S$. When housing is treated as indivisible, this rearrangement is not possible.

[10] This point is made in detail in Landvoigt et al (2012).

[11] It is possible to have a continuous housing choice with indivisible housing. The key is that the distribution of the housing stock is fixed and cannot be altered.

1.2 Model With One Type of House to Buy

1.2.1 Set-Up

We first develop a model in which agents can choose between renting and buying one type of house. This simple set-up shows the different responses of constrained and unconstrained home buyers to not only changes in credit conditions, but also to buyer optimism over future prices and the interest rate. In the next section, we extend the model to include two types of houses to buy, thereby demonstrating that the tiered pattern observed in the data cannot have occurred without credit easing.

As discussed in the introduction, in order for differential responses across housing tiers to be possible, the choice over housing must be indivisible. We model this with a discrete choice over housing: agents choose whether to buy or not, not how much housing services to buy. To keep the model tractable and bring out the intuition, we use an overlapping generations model with homogeneous agents living for two periods, being born without assets, and leaving no bequests. We model this in a partial equilibrium setting to simplify the model and keep the focus on the housing market. Agents born at time t have exogenous income flow $y_{0,t}$, $y_{1,t}$ and must allocate this between housing and non-housing consumption. Agents can rent a house in each period at exogenous rental price R_t. This gives them u_R units of utility per period. Alternatively, agents can buy one house when young and sell it when old, with the agent renting for the last period of life.[12] Living in a house gives u_L units of utility per period, with $u_L \geq u_R$ reflecting a weak preference for owning a house over renting.

Agents can save between the first and second period of life at a risk-free rate r_t. However, the only borrowing available to young agents is borrowing secured against their house, with the same risk-free rate r_t. Consistent with the evidence regarding the importance of the down-payment constraint on first time buyers, a minimum down payment γP_t is required with $\gamma \in (0, 1)$ and P_t the price of the house when bought.[13] Equivalently, the maximum LTV allowed in the model is $(1 - \gamma)$. Whilst γ is exogenous in the model, agents can endogenously choose any LTV $\leq (1 - \gamma)$. In a standard housing model with a continuous choice over how much housing to buy, agents will typically buy as much housing as they can and the constraint will always bind. However, with the discrete choice set-up here, the LTV constraint *need not* bind in equilibrium. This allows us to examine the contrasting price response when the constraint does and does not bind.

The utility function for non-housing consumption is an increasing concave function:

$$u'(C) > 0, u''(C) < 0 \tag{1.1}$$

There is no uncertainty in the model, so for both renters and buyers, the lifetime utility from

[12] We do this so that a high resale price boosts the non-housing consumption of the agent and hence their utility. Alternatively, we could specify that the agent's non-housing consumption occurs at the end of each period.

[13] In Section 4 the analysis is redone with a maximum LTI ratio instead, producing analogous results.

non-housing consumption is given by

$$\max_{b_{0,t}\geq 0} u(x^i_{0,t} - b_{0,t}) + \beta u\left(x^i_{1,t} + (1+r_t)b_{0,t}\right)$$

where $x_{j,t}$ are the resources available for non-housing consumption after housing expenses have been paid for the period.

Definition 1 *An agent is constrained under housing choice i iff*

$$u'(C^i_{0,t}) > \beta(1+r_t)u'\left(C^i_{1,t}\right)$$

An agent is unconstrained under housing choice i iff

$$u'(C^i_{0,t}) = \beta(1+r_t)u'\left(C^i_{1,t}\right)$$

Constrained agents have lower non-housing consumption when young than they would like, and they wish to borrow from future income to smooth this consumption but cannot ($b^*_{0,t} = 0$). Unconstrained agents wish to save ($b^*_{0,t} \geq 0$) so the credit constraint does not affect them and they are able to spread consumption as desired, with their Euler equation holding with equality.

When an agent rents in both periods of their life, the resources available for non-housing consumption in each period are simply given by income minus the rental price: $x^R_{j,t} = y_{j,t} - R_t$. When an agent buys a house when young, selling it when old

$$\begin{aligned} x^L_{0,t} &= y_{0,t} - \gamma P_t \\ x^L_{1,t} &= y_{1,t} - R_{t+1} - (1+r_t)(1-\gamma)P_t + P_{t+1} \end{aligned}$$

A young household consumes what's left out of income after they've made their down-payment. When old, they sell their house, paying off the remainder including interest, and pay the rental cost. If the agent is constrained, they make the minimum down-payment γP_t. If they are unconstrained, they save some first period income, which pays the same rate of interest as their mortgage. Thus, unconstrained agents effectively make a down-payment larger than γP_t.

To pin down equilibrium prices in the model, we assume that there are more people interested in buying houses than houses available to be bought, with a perfectly elastic rental market unlimited in size.[14] Specifically, we assume there is a constant mass N of people in each generation and a fixed mass M of houses available to buy with $N > M$. In equilibrium we must therefore have agents indifferent between owning a house and renting:

$$u(C^L_{0,t}) + \beta u\left(C^L_{1,t}\right) + u_L + \beta u_R = u(C^R_{0,t}) + \beta u\left(C^R_{1,t}\right) + u_R + \beta u_R \tag{1.2}$$

[14] A responsive housing supply would not affect price responses in this model as all agents are homogeneous, with the demand for housing a step function in its price. However, in a fuller model with heterogeneous agents and a downwards sloping demand curve for housing, changes in the housing supply will affect prices. The role of supply in the housing boom is tackled empirically in Chapter 2.

In the appendix we give conditions under which there is a unique P_t that solves this for a given P_{t+1}. If there is a positive premium for owning rather than renting, $u_L > u_R$, then utility from non-housing consumption will be greater for renters.

1.2.2 Analytic Solutions in Special Cases

In general there is no analytic solution to (1.2) though in a special case the equilibrium relationship is identical to the user cost model.

Proposition 2 [15] *Suppose there is no utility premium from owning a house, $u_L = u_R$ and agents are unconstrained both when buying and renting. Then*

$$P_t = R_t + \frac{P_{t+1}}{1+r_t} \tag{1.3}$$

In this special case, the price of housing is like any other asset: the value of the asset today is the sum of dividends (the rental payment avoided) plus the discounted future resale price. The intuition for the result is straightforward: when there is no utility premium from owning, lifetime utility from non-housing consumption must be equal for renters and buyers. As they are unconstrained in both cases, all that matters is the present value of resources available for lifetime consumption. The present value of the cost of renting and buying are thus equated, which is precisely the user cost model.

The above proposition holds for any utility function satisfying (1.1). In the special case of log utility we can generalise the user cost model to the case of a positive housing utility premium.

Example 3 *Suppose $u(C) = \log(C)$ and agents are unconstrained both when buying and renting. Then*

$$P_t = R_t + \frac{P_{t+1}}{1+r_t} + \left(y_{0,t} - R_t + \frac{y_{1,t} - R_{t+1}}{1+r_t}\right)\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right) \tag{1.4}$$

In this case, the price paid for a house is greater than in the user cost model when there is a positive utility premium from owning a house: $u_L > u_R$. Intuitively, this greater price is required to lower the non-housing consumption of the buyer to offset the greater utility living in a house brings.

1.2.3 Dynamic Equilibrium

The solution to (1.2) is an equilibrium price of housing today as a function of the price in the following period: $P_t(P_{t+1})$. It can be shown that the current price is sufficiently insensitive to future prices that the model has a unique steady state, which it jumps to instantly following any unanticipated shock.

[15] Proofs of all propositions are given in the appendix.

Proposition 4 *With equilibrium in the housing market given by (1.2), in all cases, assuming* $r_t > 0$,

$$0 < \frac{dP_t}{dP_{t+1}} \leq \frac{1}{1+r_t} < 1$$

Further, under conditions given in the appendix, the economy has a unique steady state P^* *which it jumps to, and no bubbles are possible.*

The user cost model (1.3) has a unique steady state under the assumption of no bubbles. Specifically a transversality condition has to be assumed

$$\lim_{s\to\infty} \frac{P_{t+s}}{\prod_{i=0}^{s-1}(1+r_{t+i})} = 0$$

and so ultimately only the rental dividend from owning a house matters. Bubbles can only arise in the standard user cost model because agents can borrow an unlimited amount against future housing, so a high expected resale price can be translated into a high price today. By contrast, bubbles cannot arise in this model due to the borrowing constraint. Because home-buyers must make a minimum down-payment, $C^L_{0,T} \leq y_{0,T} - \gamma P_T$ and, with non-negative non-housing consumption, we must have $P_T \leq \frac{y_{0,T}}{\gamma}$. Thus prices are bounded in all periods and cannot become arbitrarily large.[16]

1.2.4 Constrained vs Unconstrained Buyers

We now explore the contrasting responses of house prices to a given shock when home-buyers are constrained and unconstrained. The tiered data on the US housing boom we have shows the *relative* gains across different types of housing. We are thus interested in relative price responses to common shocks for constrained and unconstrained buyers.

The key difference between constrained and unconstrained buyers is captured in the first order condition: the Euler equation holds with equality for unconstrained agents and inequality for constrained agents

$$\begin{aligned} u'(C^{L,c}_{0,t}) &> \beta(1+r_t)u'\left(C^{L,c}_{1,t}\right) \\ u'(C^{L,u}_{0,t}) &= \beta(1+r_t)u'\left(C^{L,u}_{1,t}\right) \end{aligned}$$

The rental market is identical in both cases so the differential response to common shocks highlights the impact made by the presence of a credit constraint. It is important to emphasise that the results presented below are very general, holding for any utility function that satisfies (1.1).

[16] This argument is formalised in the proof in the appendix.

Shock to γ

We begin with a shock to the down-payment requirement γ, which provides the starkest contrast between these two cases. An increase in γ means a smaller maximum LTV, representing a tightening of credit conditions.

Proposition 5 *Suppose the home-buyer is constrained in equilibrium with price P_t^c. Then*

$$\frac{dP_t^c}{d\gamma} < 0$$

Suppose the home-buyer is unconstrained in equilibrium with price P_t^u. Then

$$\frac{dP_t^u}{d\gamma} = 0$$

The proposition says that a loosening of credit standards (fall in γ) increases the current price of housing when the buying agent is constrained, but that it has no impact on the price when the buying agent is unconstrained. Clearly then $\frac{dP_t^c}{d\gamma}\frac{1}{P_t^c} < \frac{dP_t^u}{d\gamma}\frac{1}{P_t^u} = 0$ so the relative price responses have a clear ordering.

The difference between these cases is about the ability to smooth non-housing consumption. Suppose the minimum required down-payment is 10%. An unconstrained agent makes a larger payment than this, say 20%, and spreads non-housing consumption in the desired manner across time. A reduction in the minimum down-payment to 5% has no impact on their utility as a non-binding constraint has been relaxed. There is thus no change in the equilibrium price that leaves them indifferent to renting.

Contrast this with a constrained agent. They make the minimum down-payment they can and would make a smaller payment if they could. They cannot smooth non-housing consumption across time in the desired manner (we recall Engelhardt's study showing that first time buyers reduce their consumption of food in order to put together the down-payment). Holding house prices constant, a relaxation of the constraint increases utility as it allows them to transfer lifetime resources from consumption when old to consumption when young. To ensure equilibrium and leave them indifferent to renting, the price paid must increase.

Shock to P_{t+1}

The difficulty in knowing how much of the boom in US house prices can be attributed to buyer irrational exuberance is exacerbated by scant data on price expectations, and in the aggregate it is difficult to differentiate a boom caused by expectations rather than fundamental economic variables. Here we show that a given degree of irrational exuberance will differentially affect constrained and unconstrained buyers. Whilst in our model rational agents perceive that (1.2) is repeated every period in the future, to examine irrational exuberance we depart from this and allow the current generation to have arbitrary expectations of the resale price for their house next period.

Proposition 6 *Suppose the home-buyer is constrained in equilibrium with price P_t^c. Then*

$$0 < \frac{dP_t^c}{dP_{t+1}^c} < \frac{1}{1+r_t}$$

Suppose the home-buyer is unconstrained in equilibrium with price P_t^u. Then

$$\frac{dP_t^u}{dP_{t+1}^u} = \frac{1}{1+r_t}$$

For both constrained and unconstrained buyers, a higher expected resale price results in a higher price paid today, though the absolute size of the response is always smaller for the constrained buyer. In both cases, a higher resale price directly raises consumption when old, increasing lifetime utility, resulting in an increase in the equilibrium price.

The difference in the size of the price responses is determined by whether the agent is able to smooth non-housing consumption. An unconstrained agent can increase the price they pay for a house today without affecting first period consumption by making a smaller deposit. As they are unconstrained, what matters for lifetime non-housing utility is the present value of lifetime resources available for consumption. In this, the resale price is discounted at the market rate $1+r_t$ and so if P_{t+1}increases by 1 unit, P_t must increase by $\frac{1}{1+r_t}$ units to leave their lifetime consumption unaltered.

In contrast to the unconstrained case, the constrained agent is already making the minimum down-payment possible. If they pay more for a house today, the absolute size of their deposit must increase, which comes out of current non-housing consumption. As they are constrained, consumption when young is lower than they would like and the further reduction exacerbates this. The burden of a higher purchase price falls disproportionately on consumption when young, greatly hurting their lifetime utility. This results in only a small increase in P_t being required to offset the benefit of an increase in P_{t+1}, keeping them indifferent to renting.

It can be further shown that

$$\frac{dP_t^c}{dP_{t+1}^c} = \frac{\frac{1}{1+r_t}}{\gamma\left[\frac{u'(C_{0,t}^L)}{\beta(1+r_t)u'(C_{¿,t}^L)} - 1\right] + 1}$$

For the constrained buyer $\frac{u'(C_{0,t}^L)}{\beta(1+r_t)u'(C_{¿,t}^L)} > 1$ and further, the more constrained they are, the greater $\frac{u'(C_{0,t}^L)}{\beta(1+r_t)u'(C_{¿,t}^L)}$ is above 1 and hence the smaller $\frac{dP_t^c}{dP_{t+1}^c}$ is. Thus, the more constrained a home-buyer is, the less responsive is the price they'll pay to optimism about the future resale price. This is because the more constrained the buyer is, the more an increase in the deposit paid when young hurts their lifetime utility. A highly constrained buyer will have an absolute price response significantly lower than that of the unconstrained buyer. We now turn to *relative* price responses.

Corollary 7 *Suppose the expected relative capital gains are equal for constrained and unconstrained* *buyers:*[17]

$$\frac{dP^c_{t+1}}{P^c_t} = \frac{dP^u_{t+1}}{P^u_t}$$

Then the price response of unconstrained buyers is relatively greater:

$$0 < \frac{dP^c_t}{P^c_t} < \frac{dP^u_t}{P^u_t}$$

The proposition states that the absolute increase P_t for a given change in P_{t+1} is greater for unconstrained agents. The corollary says that if the expected capital gains for constrained and unconstrained buyers are proportionate, then the relative increase in price is greater for unconstrained buyers. Thus if both expected prices to increase by 10%, then the relative price increase would be greater for the unconstrained buyer. To see the reason for this intuitively, we make use of the following decomposition for discrete price changes:

$$\frac{\Delta P_t}{P_t} = \frac{\Delta P_t}{\Delta P_{t+1}} \frac{\Delta P_{t+1}}{P_t}$$

The relative increase in P_t following an increase in P_{t+1} is given by the product of the absolute increase in P_t in response to a given change in P_{t+1} and the increase in P_{t+1} relative to P_t. From the proposition, the absolute increase in P_t for a given change in P_{t+1} is greater in the unconstrained case, thus with proportionate expected price increases, the relative gain is greater in the unconstrained case.

Shock to r_t

We now consider how having a constrained buyer affects the responsiveness of the equilibrium price to a fall in the interest rate. We contrast the relative price responses of the model above, in which the buyer is constrained by a maximum LTV ratio $1 - \gamma$, with that in which the buyer does not face a borrowing constraint, and so is unconstrained.

Proposition 8 *Let P^c_t be the equilibrium house price with a constrained buyer and P^u_t be the equilibrium house price when the buyer does not face a down-payment constraint, and so is unconstrained. Then*

$$0 < \left(\frac{-dP^c_t}{dr_t}\right) \frac{1}{P^c_t} < \left(\frac{-dP^u_t}{dr_t}\right) \frac{1}{P^u_t}$$

In both cases, a decrease in the interest rate increases the price paid today, with unconstrained agents having a greater relative response.[18] When the agent facing the borrowing constraint is just unconstrained, with desired borrowing exactly 0, the prices in both cases are identical and

[17] Unlike for changes in γ and r we need such a restriction here as the constrained and unconstrained groups are not being hit by a change in a common variable. As the expectations could differ between the groups, for the result we need them to increase in a comparable manner.

[18] The result does not depend on any assumption about how the interest rate the renter can save at moves with the mortgage rate the buyer faces. There is no spread in the model, but if there were and the rates were independent, the result still goes through.

so too are the responses to a change in the interest rate. The proof establishes that the relative price response decreases as the agent becomes more constrained (γ increases), thus showing that the relative price response will always be lower for the constrained agent.

For the unconstrained buyer $1+r_t$ is the rate at which they discount second period consumption. A decrease in r_t then raises utility, increasing the price that leaves them indifferent to renting. This is similar to the user cost model (1.3) in which a lower interest rate results in future capital gains being discounted at a lower rate, increasing the price paid today.

For the constrained buyer, a decrease in r_t results in them having to pay a lower rate of interest on their mortgage, which directly raises their second period consumption. However, as they are constrained, their primary concern is low first period consumption, so the change in second period consumption does not greatly impact their lifetime utility. Consequently, whilst the price that leaves them indifferent to renting increases, the relative increase is not as great as for the unconstrained buyer.

1.2.5 Summary

The results developed in this section are key building blocks when coming to the full model with two types of houses available for purchase. We have shown under very general conditions, with minimal assumptions on the utility function and no assumptions on the rental market, that relative price responses are markedly different for constrained and unconstrained house buyers. Specifically, unconstrained buyers have greater relative price reactions to changes in interest rates and expected future prices, as well as a smaller (zero) reaction to changes in the down-payment requirement. In the next section, these results are combined with a housing ladder to generate predictions about the relative price movements of low and high tier houses in response to common shocks.

1.3 Model With Two Types of Housing to Buy

In this section, we extend the model to include two types of house available for purchase to enable comparison with the US experience. The houses available are H for high quality, and L for low quality. In addition to these, there is still the option of renting. We assume that living in each environment for 1 period delivers respective utility levels u_H, u_L and u_R with

$$u_H > u_L \geq u_R$$

so there is a utility premium for living in the high quality housing. In fact, this is what distinguishes a high quality house from a low quality house.

We shall assume parameter values that result in unconstrained high tier buyers and constrained low tier buyers, consistent with the US evidence. Directly applied, the results from the previous section imply that the price of high tier houses will grow relatively more in response to a fall in interest rates and high future price expectations, whilst the low tier will grow relatively more in response to a fall in down-payment requirements. This may seem enough to complete our argument, though the analysis is further complicated by the presence of the housing ladder. Recall from the American Housing Survey that 66% of recent movers moved to a higher quality house. A price increase in low tier houses can then be transmitted to high tier houses through the realised capital gains of those moving from the low to a higher tier. It is important to incorporate these effects in our model, so we include two types of housing to buy as well as the option of moving between houses.[19]

To keep the model simple and the analysis clear, we continue to assume that agents live for two periods. As agents rent in the last period of their life, they can only buy one house. Thus, for movement up the housing ladder to be possible, some agents must be born owning a house. These agents do not pay for the house they inherit. However, for anticipated capital gains to affect the price paid for housing today, we require agents to expect to sell their house when old. To enable these to be mutually consistent we introduce agent heterogeneity and allow that some agents face a constant probability of death before old age. Formally, we suppose there are two groups of agents (each a continuum):

- **Group A**: These agents are born without housing and have the set-up of agents in Section 1.2, choosing between renting their whole life or buying a house when young and selling it when old.
- **Group B**: These agents are identical to group A agents *except* they are born owning a low tier house.

We assume that group A agents face a constant probability $(1-q)$ of dying before old age, with this probability independent of their housing choice (group B agents reach old age with certainty). As there is a continuum of group A agents, a constant fraction of each of their cohorts will die before old age. Given this, we can costlessly transfer the low tier houses of the home-owners who die to the next cohort of group B. However, as the group A agents don't know *who* will die, anticipated capital gains play a role in their decision when considering how much to pay for a low tier house.

More formally, with group A agents dying between young and old age with probability $(1-q)$, lifetime utility becomes

$$\begin{aligned} V &= u(C_{0,t}) + u(H_{0,t}) + q \cdot \beta(u(C_{1,t}) + u(H_{1,t})) + (1-q) \cdot 0 \\ &= u(C_{0,t}) + u(H_{0,t}) + (q\beta)\,(u(C_{1,t}) + u(H_{1,t})) \end{aligned}$$

The agents in group A are thus identical to the agents in the model of Section 1.2 (so all the results go through) *except* that they discount the future more, placing weight $q\beta$ as opposed to β on future utility when young.[20]

[19] The Landvoigt et al (2012) paper also features links between the markets for different houses. They find that strong buyer demand for low quality houses can "spillover" onto the demand for higher quality houses, affecting their price too.

[20] The differential discount rates for the low and high tier buyers are not responsible for any of the results in this section: they all hold for any $q \in (0,1)$.

To keep the analysis simple we limit the housing options available to both groups. Group A-who are born without a house-can either rent their whole life, or buy L when young and sell it when old. Group B agents-who are born owning a low tier house-can either live in L when young and sell it when old, or move up the housing ladder, selling their low tier house to help fund the buying of a high tier house, which they sell when old. For completeness, we list the resources available for non-housing consumption in each period for each of groups A and B under each available housing choice.[21]

Group A Resources Under Each Housing Choice

$$
\begin{aligned}
x_{j,t}^{A,R} &= y_{j,t} - R_j \\
x_{0,t}^{A,L} &= y_{0,t} - \gamma P_t^L \\
x_{1,t}^{A,L} &= y_{1,t} - R_{t+1} - (1+r_t)(1-\gamma)P_t^L + P_{t+1}^L
\end{aligned}
$$

As in Section 1.2, when group A agents rent their whole life, their available resources for non-housing consumption in each period are given by income minus the rental cost. When group A agents buy a low tier house when young and sell it when old-housing option $x_{j,t}^{A,L}$-they have analogous resources to the model of Section 1.2.

Group B Resources Under Each Housing Choice

$$
\begin{aligned}
x_{0,t}^{B,L} &= y_{0,t} \\
x_{1,t}^{B,L} &= y_{1,t} - R_{t+1} + P_{t+1}^L \\
x_{0,t}^{B,H} &= y_{0,t} + P_t^L - \gamma P_t^H \\
x_{1,t}^{B,H} &= y_{1,t} - R_{t+1} - (1+r_t)(1-\gamma)P_t^H + P_{t+1}^H
\end{aligned}
$$

Under their first housing option ($x_{j,t}^{B,L}$) group B agents continue to live in the low tier house they are born with when young, selling it when old. In this case their resources when young is simply their income, as they have no housing costs, and their resources when old are given by income plus the resale value of their house, minus the cost of the rental accommodation they live in when old. Alternatively, they can sell this low tier house when young and move up the housing ladder to a high tier house (option $x_{j,t}^{B,H}$). In this case they can put their income and the funds from selling their house$-P_t^L-$towards the downpayment on the high tier house, γP_t^H. When old, they sell this high tier house at price P_{t+1}^H, pay off the rest of their mortgage and rent for the remainder of their life.

Under conditions given in the appendix, the equilibrium is pinned down by group A being indifferent between buying L and renting, and group B being indifferent between staying in L

[21] This is prior to any saving that may be done by the agents to smooth non-housing consumption between the two periods of their lives.

and trading up to H. As no agent goes from owning a high tier house to a low tier house, the equilibrium is recursive: P_t^L is determined independently of P_t^H from the indifference of group A agents in exactly the manner of the simple model of Section 1.2. Taking this price as an input, P_t^H is then determined by the indifference of group B agents.

In the rest of the analysis, we use log utility, which simplifies the model and gives a clean intuitive expression of how the housing ladder affects the price of high tier housing.

Proposition 9 *Suppose $u(C_t) = \log(C_t)$ and high tier buyers are unconstrained.*[22] *Then*

$$P_t^H = \frac{P_{t+1}^H}{1+r_t} + P_t^L - \frac{P_{t+1}^L}{1+r_t} \frac{1}{\exp\left(\frac{u_H - u_L}{1+\beta}\right)} + \left(y_{0,t} + \frac{y_{1,t} - R_{t+1}}{1+r_t}\right)\left(1 - \frac{1}{\exp\left(\frac{u_H - u_L}{1+\beta}\right)}\right) \tag{1.5}$$

The proposition is a clear extension of (1.4) when there was only one type of house to buy. The price paid, P_t^H, depends on its resale price discounted at the gross interest rate $(1 + r_t)$. Further, part of the price is independent of the resale price and represents the utility gained by living in a high rather than low tier house. The difference here is the terms in P_t^L, P_{t+1}^L which are due to the housing ladder. An increase in P_t^L directly increases the income of the high tier buyer, allowing more consumption at unchanged P_t^H. Thus P_t^H must rise to keep the agent indifferent between buying and staying in the low tier house, maintaining equilibrium. Counteracting this is the foregone sale price P_{t+1}^L that the agent would have realised had they stayed in the low tier house, before selling it in the following period. This resale price is discounted not only by the gross rate of interest, but also by a utility term reflecting the lower housing utility from staying in the low tier house for an extra period.

1.3.1 High vs Low Tier Prices

We now show that the pattern observed in the US data could not have occurred without credit easing. We do this by comparing the relative growth in the low and high tiers in response to credit easing, buyer optimism about future prices, and a fall in interest rates. As a result of the housing ladder, any change can have direct and indirect effect on the high tier price. For a change in variable x we have the following price response:

$$\frac{dP_t^H}{dx} = \frac{\partial P_t^H}{\partial x} + \frac{\partial P_t^H}{\partial P_t^L}\left(\frac{dP_t^L}{dx}\right)$$

The term $\frac{\partial P_t^H}{\partial P_t^L}$ gives the strength of the capital gains transmission up the housing ladder, with $\frac{dP_t^L}{dx}$ the full effect of the variable on P_t^L.

[22] Precisely, group B agent are unconstrained both when buying and when they stay living in L.

Credit Easing

We first consider the impact on tiered housing of a change in the down-payment requirement γ by itself.

Proposition 10 *Suppose low tier buyers are constrained when buying and high tier buyers are unconstrained when buying and would be unconstrained if they didn't buy. Then the relative price change is greater for low tier housing:*

$$0 < \left(\frac{-dP_t^H}{d\gamma}\right)\frac{1}{P_t^H} < \left(\frac{-dP_t^L}{d\gamma}\right)\frac{1}{P_t^L}$$

The proposition shows that a decrease in γ, representing an easing of credit conditions, results in higher prices for the high and the low tier, with a greater relative price increase for low tier housing. As discussed in the previous section, with the low tier buyer constrained, an easing of credit conditions at unchanged prices allows them to better smooth consumption, thereby improving lifetime utility. In equilibrium, to keep them indifferent to renting, the price of low tier housing must increase. The easing of credit conditions has no *direct* effect on high tier housing as they are already able to smooth consumption as desired. However, there is an *indirect* effect due to the transmission of capital gains via the housing ladder. This group simultaneously sells a low tier house as they buy the high tier house. A greater price for the low tier house results in higher consumption and utility at unchanged prices, so in equilibrium the high tier price must increase to leave group B agents indifferent between buying and staying in a low tier house.

To understand the intuition for the relative price responses, we look at the relationship between the low and high tier prices in equilibrium:

$$P^H = \left(1 + \frac{1}{r}\left(1 - \frac{1}{\exp\left(\frac{u_H - u_L}{1+\beta}\right)}\right)\right)P^L + \frac{(1+r_t)}{r_t}\left(y_{0,t} + \frac{y_{1,t} - R_{t+1}}{1+r_t}\right)\left(1 - \frac{1}{\exp\left(\frac{u_H - u_L}{1+\beta}\right)}\right)$$

Given $u_H > u_L$, $1+\frac{1}{r}\left(1 - \frac{1}{\exp\left(\frac{u_H-u_L}{1+\beta}\right)}\right) > 1$ so an increase in P^L results in an *absolute* increase in P^H greater than the increase in P^L. However, because housing is a consumption good, P^H has a fixed component independent of P^L reflecting the preference for living in a high tier house. This results in the elasticity of P^H wrt P^L, $\frac{\partial P^H}{\partial P^L}\frac{P^L}{P^H}$ being less than 1. A given rise in P^L then results in a proportionately smaller rise in P^H. Thus the *relative* increase in P^H following an decrease in γ is smaller than the relative increase in P^L.

Consequently, the pattern observed in the US data of low tier house prices growing more than high tier house prices is consistent with an easing of down-payment requirements.[23]

[23] Landvoigt et al (2012) find in their assignment model for San Diego that lower downpayment requirements lead to higher captial gains during the boom for the least expensive houses. The intuition is similar to that developed here with the buyers of less expensive houses more likely to be credit constrained. Consequently, a relaxation of these constrants will have a greater impact on the price of these houses.

Buyer Optimism

We now consider the relative pattern between the tiers when there is optimism about future prices for both low and high tier buyers and no change in down-payment requirements or interest rates. For the purpose of a fair experiment, we suppose proportionate expected price increases for the low and high tier.

Proposition 11 *Suppose low tier buyers are constrained when buying and high tier buyers are unconstrained when buying and would be unconstrained if they didn't buy. Further, suppose the relative expected price increases are the same for low and high tier buyers:*

$$\frac{dP^H_{t+1}}{P^H_t} = \frac{dP^L_{t+1}}{P^L_t}$$

Then the relative price increase is greater for high tier houses:

$$0 < \frac{dP^L_t}{P^L_t} < \frac{dP^H_t}{P^H_t}$$

The proposition states that both the low and the high tiers experience price growth following an increase in future expected prices, *but the relative growth is greater for the high tier.* The increase in future prices increases non-housing consumption when old for both low and high tier buyers, so both prices must increase to maintain equilibrium in their respective markets. However, as the low tier buyer is constrained, their price response is muted compared to direct response of the high tier buyer. In addition to this, the housing ladder results in an indirect increase in P^H_t from the capital gains of high tier buyers. The overall result is a greater relative price increase for the high tier houses.

It follows that the pattern observed in the US *could not have been generated by buyer optimism alone.* If this were the only change in the market, we would have observed greater relative price growth in the high tier, which is contrary to what we see in the data.

Fall in the Interest Rate

We now consider the impact of a change in the interest rate by itself.

Proposition 12 *Suppose low tier buyers are constrained when buying and high tier buyers are unconstrained when buying and would be unconstrained if they didn't buy. Then the relative price change is greater for high tier housing:*

$$0 < \left(\frac{-dP^L_t}{dr_t}\right)\frac{1}{P^L_t} < \left(\frac{-dP^H_t}{dr_t}\right)\frac{1}{P^H_t}$$

The proposition shows that following a fall in interest rates with other variables held constant, low and high tier house prices will increase, with relative growth greater for high tier houses. The fall in interest rates gives a boost to the low tier buyer by reducing their mortgage payments when old, resulting in P_t^L increasing in equilibrium. However, this increase is muted because the utility gain is small as it does not affect the consumption of the constrained buyer when young. The fall in r_t reduces the discount rate of the unconstrained high tier buyer, increasing the value of future capital gains as in the user cost model. From Proposition 8, for comparable constrained and unconstrained buyers, this results in a larger relative price response for the unconstrained buyer. In the tiered model we add to this the transmission of capital gains from P_t^L to P_t^H resulting in a greater relative price increase for the high tier.

The US housing boom *could not have been caused by a fall in interest rates alone* without a change in any other variables. Whilst a fall in interest rates by itself can explain a rise in house prices, it cannot in and of itself explain why the low tier has grown relatively more than the high tier during the US boom.

1.3.2 Summary & Policy Implications

During the US housing boom, low tier house prices grew significantly more in relative terms than high tier prices across 51 of 52 cities. If the housing boom were caused by a fall in interest rates alone, we would have witnessed the opposite, with growth in the high surpassing that of the low tier. If this were due to home-buyers' irrational exuberance, the high tier would also have grown relatively more than the low tier. In short, unless there was an easing of non-price credit terms, and just a fall in the interest rate or buyer exuberance, the low tier *would not have grown relatively more than the high tier*. Given that this did occur, we can conclude that neither of these two explanations could have caused the boom without a significant easing of non-price credit terms.

It may be objected that, in practice, the variables are all endogenous and that a lowering of interest rates drove a search for yield and an easing of non-price credit terms, in turn fuelling price rises and buyer optimism. This is not denied. The fall in global interest rates may well have been the driver behind the whole process, with the easing maximum LTV ratios a symptom of this. Regardless, if the whole chain could not have happened without a reduction of non-price credit terms, it shows that the intervention of policy could have attenuated the housing boom by preventing this reduction. As shown in the model, if there had not been an easing of non-price credit terms, the low tier would have grown relatively *less* than the high tier, with the high tier growing even less due to smaller passed on capital gains. The relative growth in the low tier over the high tier thus places a lower bound on the contribution of the easing of non-price credit terms in the housing boom, and what macroprudential policy such as an LTV cap could have achieved in attenuating the boom. This counter-factual calculation on the 52 cities we have data for results in the low tier growing by 55 percentage points less in nominal terms on average during the boom.

In summary, whilst we cannot disentangle the complex endogenous relationships between house price expectations, interest rates, and non-price credit terms, we can assert that if these were the three factors driving the US housing boom, an LTV cap would have significantly attenuated it.

1.4 Robustness

We now consider the robustness of the prior analysis to alternative explanations.

1.4.1 Greater Buyer Optimism for Low Tier Buyers

In the previous section, it was shown that in the absence of credit easing, if low and high tier buyers had proportionate expectations about price growth across the tiers, the high tier would grow relatively more than the low tier, contrary to what was observed in the data. However, this result could be reversed if low tier buyers were sufficiently more optimistic than high tier buyers. We might expect this to be the case because of differing levels of housing market experience.[24] Specifically, low tier buyers are likely to be younger, and inexperienced in the housing market. With less experience, they may think prices only ever go up resulting in wild expectations of future house price increases. By contrast, high tier buyers will likely be older repeat buyers, thereby having more experience with the housing market and having lived through previous housing busts. This institutional memory could temper the capital gains they expect. In order to deal with this challenge, we first calibrate the model to provide a bound on how much more optimistic low tier buyers would need to be than high tier buyers to generate the pattern observed in the data. We then turn to the available indirect evidence on this.

Model Bound

In the prior analysis of the model we implicitly assumed low and high tier buyers had common expectations of P^L_{t+1}(for the high tier buyers, P^L_{t+1} matters as it represents the price they could eventually sell their low tier house for had they not bought a high tier house). To address this challenge, we now relax this assumption and allow expectations specific to each group, $P^{L,A}_{t+1}, P^{L,B}_{t+1}$ which will not be equal in general. For a fair test we assume that the high tier buyers of group B expect proportionate price growth in the low and high tiers:

$$\frac{dP^{H,B}_{t+1}}{P^H_t} = \frac{dP^{L,B}_{t+1}}{P^L_t}$$

Recall, for the low tier buyer (those in group A), the price responsiveness is given by

$$(1+r_t)\frac{dP^L_t}{dP^{L,A}_{t+1}} = \frac{1}{1+\gamma\left[\frac{u'(C^{L,A}_{0,t})}{\beta(1+r_t)u'(C^{L,A}_{1,t})} - 1\right]} < 1$$

This price responsiveness is dampened as the buyer becomes more constrained. Crucially, as $P^{L,A}_{t+1}$ increases, the *buyer becomes more constrained.* This is due to a direct effect that increases consumption when old, and an indirect effect coming through the equilibrium increase in P_t, that

[24] I'd like to thank John Van Reenen for suggesting this alternative explanation.

further reduces consumption when young. Thus as $P_{t+1}^{L,A}$ increases the responsiveness of P_t^L *decreases*. Formally, for a constrained buyer[25]

$$\frac{d^2 P_t^L}{d(P_{t+1}^{L,A})^2} < 0.$$

We build upon this insight and consider the impact of large *discrete* changes in $P_{t+1}^{L,A}$ over the length of the housing boom. As we shall show below, with a large discrete increase in $P_{t+1}^{L,A}$, the total price response $\frac{(1+r_t)\Delta P_t^L}{\Delta P_{t+1}^{L,A}}$ can be significantly less than 1. The following theorem establishes this this quantity is a key bound.

Theorem 13 *Suppose the low tier buyer is constrained when buying and the high tier buyer is unconstrained when buying and staying in L.*

Suppose

$$\frac{\Delta P_{t+1}^{H,B}}{P_t^H} \geq \frac{\Delta P_{t+1}^{L,A}}{P_t^L}\left(\frac{(1+r_t)\Delta P_t^L}{\Delta P_{t+1}^{L,A}}\right)$$

Then

$$\frac{\Delta P_t^H}{P_t^H} > \frac{\Delta P_t^L}{P_t^L}$$

The theorem provides a lower bound on the high tier expectations required relative to low tier expectations in order for the high tier to grow relatively more than the low tier. For example, if $\left(\frac{(1+r_t)\Delta P_t^L}{\Delta P_{t+1}^{L,A}}\right) = 0.6$, then so long as high tier expectations were at least 60% of low tier expectations, high tier prices would grow relatively more than low tier prices during the housing boom.

To quantify $\frac{(1+r_t)\Delta P_t^L}{\Delta P_{t+1}^{L,A}}$ we calibrate[26] the model under the conservative assumption that the low tier buyer is initially just unconstrained (i.e. their desired equilibrium borrowing is exactly 0).[27] We also assume that initially, prior to the increase in expectations, $P_t^L = P_{t+1}^L$. In Figure 1.4 we graph the results for the calibrated model, both for a constrained buyer and an unconstrained buyer, with ΔP_{t+1}^L on the x-axis and ΔP_t^L on the y-axis, with both taken relative to the initial price.

For the unconstrained buyer, $\Delta P_t^{L,U} = \frac{\Delta P_{t+1}^{L,U}}{1+r_t}$, thus, taking the ratio of the two graphs for a given ΔP_{t+1}^L gives:

$$\frac{\Delta P_t^{L,C}}{\Delta P_t^{L,U}} = \frac{\Delta P_t^{L,C}}{\left(\frac{\Delta P_{t+1}^L}{1+r_t}\right)} = (1+r_t)\frac{\Delta P_t^{L,C}}{\Delta P_{t+1}^{L,C}}$$

The ratio of the constrained over the unconstrained graphs for a given ΔP_{t+1}^L thus measures $\frac{(1+r_t)\Delta P_t^{L,C}}{\Delta P_{t+1}^{L,C}}$. With the buyer initially unconstrained in the equilibrium we have picked, the two lines initially grow at the same rate. Then, as the expected future increase in prices becomes larger, the responsiveness of the constrained buyer decreases as they become more constrained.

[25] For an unconstrained buyer $\frac{d^2 P_t}{dP_{t+1}^2} = 0$

[26] Details on the calibration are given in the appendix.

[27] This is conservative, because if they were initially constrained, this would dampen the price response further.

Figure 1.4: Model Response to Changing Expectations

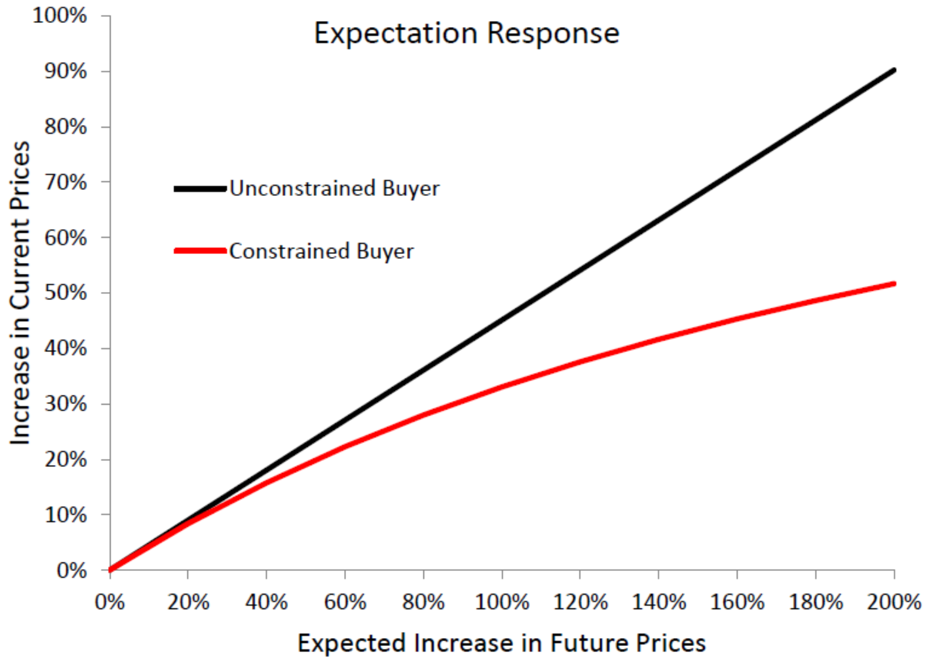


Table 1.1: Response of Constrained Buyer

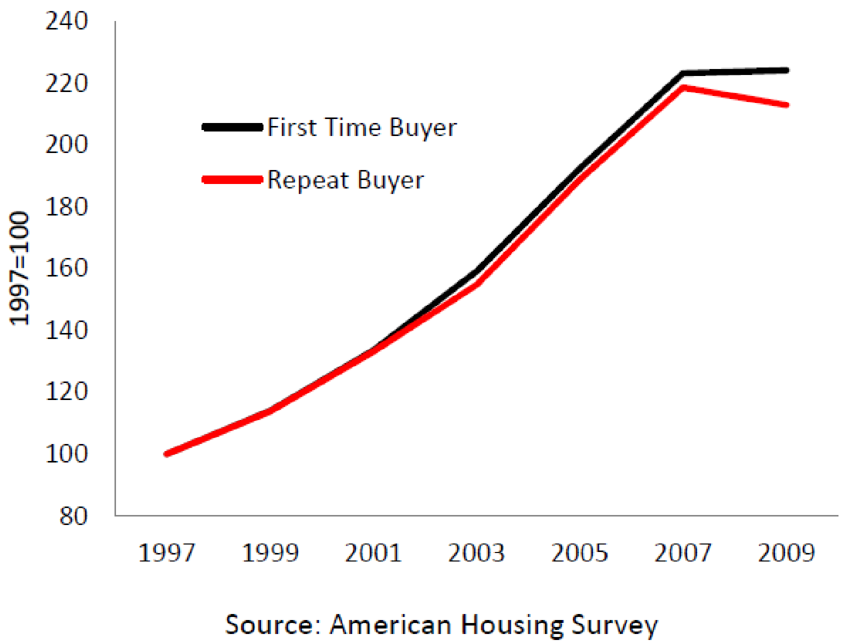
$\frac{\Delta P^L_{t+1}}{P^L_t}$	$\frac{(1+r_t)\Delta P^L_t}{\Delta P^L_{t+1}}$
100%	0.73
150%	0.64
200%	0.57

The greater the expected increase in future prices, the smaller $(1 + r_t)\frac{\Delta P^{L,C}_t}{\Delta P^{L,C}_{t+1}}$ becomes. Summary data from the chart is presented in Table 1.1.

From the table, if the low tier price was expected to increase by 200% over the course of the boom (a tripling), high tier prices would have greater growth if they were expected to grow at least 114% (57% of 200%). This expected low tier price growth is comparable to the average expected price growth of those surveyed in Case and Shiller (2003).[28] This demonstrates that for the observed pattern to be caused by buyer optimism alone, low tier buyers would need to be significantly more optimistic than the high tier buyers. We now turn to the available evidence on buyer expectations during the boom.

[28] Their average expected increase was over 11.7% for 10 years-growth which would amount to a 202% increase in prices over the period.

Figure 1.5: Perceived Nominal House Prices



Indirect Evidence

A major issue in assessing the role of optimism during the US housing boom is the lack of sufficiently thorough time series data for buyer expectations.[29] Further, there is no time series data that distinguishes between the capital gains expected by low and high tier buyers. Instead, to assess this, we use an indirect approach using data from the American Housing Survey (AHS). The AHS is a longitudinal study taking place every second year, surveying around 50,000 housing units nationally in each edition. The study asks homeowners what they *think* their house is worth in each year. By controlling for the same households throughout the sample period, we can track how the *perceived* market value of the same housing units changes over time. We proxy low tier buyers with first time buyers and high tier buyers with repeat buyers.[30] Using the AHS data we can compare how the perceived market value of their house varies between these two groups. Given that buyers are likely to rely upon aggregate house price indices, either for their city, or nationally, differences in optimism between the two groups would likely show up in differences in how they perceive the value of their house has changed over time. Specifically, if low tier buyers were carried away by market inexperience that resulted in highly optimistic price growth expectations, they'd likely also overestimate how much the value of their house had increased during the boom to date. The results for the two groups are shown below in Figure 1.5.

[29] As noted in the introduction, the survey in Case et al (2012) doesn't start until 2003 and only covers four cities.
[30] A simple approach just looking at the cheapest and most expensive houses within the AHS sample will conflate high tier buyers in cheaper places like Atlanta, with low tier buyers in more expensive places like San Francisco.

We see that throughout the boom, the two series are very close together, with the perceived increase in value at the peak of the boom just 2% higher for first time buyers. This evidence is of course indirect and deals with perceptions, not future expectations, but it does suggest that inexperienced home owners were not significantly more optimistic than experienced ones.[31]

Summary

In summary, the calibrated model shows that for buyer optimism to have caused the tiered pattern in the data without an easing of credit standards, low tier buyers would have needed to be significantly more optimistic about future price growth than high tier buyers. The available indirect evidence on price expectations does not suggest that such a difference exists between the two groups. We thus conclude that the boom was not caused by optimism alone.

1.4.2 LTI

In our model, agents are credit constrained by a maximum LTV ratio. Here we show that our results are unaffected if agents are instead constrained by a maximum LTI ratio. HMDA data shows during the housing boom that house buyers in all our cities with lower income, who buy cheaper houses, had higher LTIs than buyers with greater income who buy the more expensive houses.[32] Consequently, a relaxation of LTI limits only directly affects the low tier buyers. There is direct evidence[33] in the HMDA data of an increase in LTI ratios over time, though this likely understates the true magnitude of the increase due to the increased use of stated income loans. The share of no/low-documentation mortgage purchases in the US went from 18% in 2001 to 49% in 2006 (Credit Suisse 2007). These loans have rightly been labelled "liar loans": a 2006 study found that out of a sample of stated income loans, over 60% had overstated their income by 50% or more (Credit Suisse 2007).

Our model can be modified in a simple way to incorporate a LTI constraint rather than a LTV constraint. Recall, both when buying a house and renting, the utility from non-housing consumption is given by

$$\max_{b_{0,t}\geq 0} u(x^i_{0,t} - b_{0,t}) + \beta u\left(x^i_{1,t} + (1 + r_t)b_{0,t}\right)$$

where $x_{j,t}$ are the resources available for non-housing consumption after housing expenses have been paid for the period.

With the rental market unchanged, we modify the housing expenses when an agent buys a house

[31] Interestingly, the first time buyers did not believe that the value of their house fell during 2007-2009, contrasting with the repeat buyers. This may reflect their inexperience of housing market busts.

[32] Data on 26 of these is given in Table 1.3 in the appendix, for both 1997 and 2006.

[33] Evidence for 26 of these cities is given in Table 1.3 in the appendix. In the full sample of 51 cities we have data on, the average LTI for the low tier buyers increased from 2.43 in 1997 to 3.51 in 2006.

when young:

$$\begin{aligned} x_{0,t}^L &= y_{0,t} + \delta y_{0,t} - P_t \\ x_{1,t}^L &= y_{1,t} - R_{t+1} - (1+r_t)\delta y_{0,t} + P_{t+1} \end{aligned}$$

where δ represents the exogenous maximum LTI ratio permitted in the market.[34] The buying agent pays the price of the house P_t up front, and the maximum total resources available to them are given by $y_{0,t} + \delta y_{0,t}$, their income when young, plus the maximum loan that can be secured against it.[35] The agent need not take out the maximum loan they can however, with the effective size of loan they take out given by $\delta y_{0,t} - b_{0,t}$ with $b_{0,t} \geq 0$. When they are old, they sell the house receiving P_{t+1}, pay rental cost R_{t+1}, receive income $y_{1,t}$ and pay back the interest on the loan taken out $(1+r_t)(\delta y_{0,t} - b_{0,t})$. An agent is constrained when buying if they take out the maximum loan they can against their income.

An increase in δ represents a loosening of the credit constraint here, allowing an agent to take out a bigger loan and transfer resources from consumption when old to consumption when young to better smooth consumption. Agents constrained by an LTV or an LTI constraint similarly suffer in having lower non-housing consumption than they would like when young. We thus obtain an analogous series of results when agents are LTI constrained.

Proposition 14 *When $u(C) = \log(C)$, and the low tier buyer is constrained and the high tier buyer is unconstrained when buying and would be unconstrained if they didn't buy, we have:*

(i) Credit result:

$$0 < \left(\frac{dP_t^H}{d\delta}\right)\frac{1}{P_t^H} < \left(\frac{dP_t^L}{d\delta}\right)\frac{1}{P_t^L}$$

(ii) Interest rate result:

$$0 < \left(\frac{-dP_t^L}{dr_t}\right)\frac{1}{P_t^L} < \left(\frac{-dP_t^H}{dr_t}\right)\frac{1}{P_t^H}$$

(iii) Expectations result: if expectations for growth in both tiers are proportionate

$$\frac{dP_{t+1}^H}{P_t^H} = \frac{dP_{t+1}^L}{P_t^L}$$

then

$$0 < \frac{dP_t^L}{P_t^L} < \frac{dP_t^H}{P_t^H}$$

A relaxation of the LTI constraint results in low tier prices growing relatively more than high tier prices[36], whilst buyer optimism or a fall in interest rates results in the high tier growing relatively more than the low tier.

[34] We suppose that $\delta y_{0,t} \leq P_t$ so the maximum permitted LTV is 100%.

[35] Income when young is the relevant income in practice for setting the LTI against: people generally cannot borrow based on expectations of a higher salary many years into the future.

[36] Note that an increase in the maximum LTI δ represents an *easing* of credit standards.

Thus, if in fact low tier buyers were constrained by an LTI rather than an LTV constraint, our analysis is unchanged: the observed pattern in the US with the low tier growing relatively more than the high tier could not have occurred without an easing of the credit constraint.

1.4.3 Other Potential Explanations

We have used the tiered pattern to discriminate between three explanations of the housing boom. There are other possible factors that could generate the pattern beyond the three considered, such as greater income growth for low tier buyers, a surge in speculators buying low tier houses, or a surge in house building for high tier houses. In the next chapter we tackle these and alternative explanations empirically, showing they were not responsible for the tiered pattern, thus reinforcing the conclusions of this chapter.

1.5 Welfare

We have argued that the US housing boom would have been significantly attenuated if non-price credit terms had not been eased. Here we look at the welfare implications of housing booms and busts, showing first that they have a welfare cost which is increasing in the size of the boom and bust cycle. We then show that policy which limits the easing of the down payment requirement is welfare improving.

1.5.1 Welfare Cost of Boom and Bust

It may not be obvious that preventing the reduction of non-price credit terms will be welfare improving. Indeed, *all else equal,* easing the constraint on constrained house buyers is welfare improving, as it allows them to better smooth consumption throughout their lifetime. However, all else is not equal, and in equilibrium, from (1.2), prices will adjust to leave buyers indifferent to renting, leaving their lifetime utility unchanged.[37] However, whilst the lifetime utility of those buying will not be affected by a shock to the housing market, the utility of the old will be affected as the price they sell for will be different to what they expected. During a boom and bust cycle, there will be winners and losers in the housing market. Those that buy before and sell during the boom for a higher price than expected are the winners, whilst those that buy during the boom and sell during the bust suffer with a lower selling price than anticipated.

[37] It does however improve their utility when young. From the proof of Lemma (24) we have that

$$-1 < \frac{dP_t^L}{d\gamma}\frac{\gamma}{P_t^L} < 0$$

Hence,

$$-\frac{dC_{0,t}^L}{d\gamma} = P_t^L(\gamma) + \gamma\frac{dP_t^L}{d\gamma} > 0$$

Showing that consumption when young increases as γ decreases.

For simplicity, we suppose there is only one type of house to buy in addition to a rental market. To examine the welfare question in a simple way, we assume the boom is caused by an exogenous generic positive shock to house prices. We model the bust as a negative shock, with the price reverting to the pre-boom level. We show that the net utility effect of a symmetric boom and bust cycle, starting and finishing with the same price, is *negative*. That is, whilst some win and some lose from a boom and bust, the losers lose more than the winners win. It is notable that we get this result without considering many other negative effects of housing busts, such as possible resultant banking crises. We summarise this in a proposition, which holds regardless of whether the buyer is constrained or unconstrained.

Proposition 15 *Suppose the buyer's utility from consumption is increasing and concave:* $u'(.) > 0$, $u''(.) < 0$. *Let* $V(P_{t+1}, \widetilde{P}_{t+1})$ *be the lifetime utility of a buyer who expects to sell at price* P_{t+1} *but sells at price* $\widetilde{P}_{t+1}$. *Consider a symmetric boom and bust, with the selling price rising from* P_{t+1} *to* $P_{t+1} + x$, $(x > 0)$ *then in the bust falling from* $P_{t+1} + x$ *back to* P_{t+1}, *with the change in price unanticipated on both occasions.*

The impact on lifetime utility for the buyer who sold during the boom is

$$V(P_{t+1}, P_{t+1} + x) - V(P_{t+1}, P_{t+1}) > 0$$

Whilst the impact on the buyer who sold during the bust is

$$V(P_{t+1} + x, P_{t+1}) - V(P_{t+1} + x, P_{t+1} + x) < 0$$

We show that the net welfare cost of the boom and bust is negative:

$$[V(P_{t+1}, P_{t+1} + x) - V(P_{t+1}, P_{t+1})] + [V(P_{t+1} + x, P_{t+1}) - V(P_{t+1} + x, P_{t+1} + x)] < 0$$

Further,

$$\frac{d}{dx}\left([V(P_{t+1}, P_{t+1} + x) - V(P_{t+1}, P_{t+1})] + [V(P_{t+1} + x, P_{t+1}) - V(P_{t+1} + x, P_{t+1} + x)]\right) < 0$$

so, the net welfare cost of the boom and bust is increasing in the amplitude of the cycle.

The intuition for the result is based on diminishing marginal utility. The increase in consumption for the buyer who sells during the boom is equal to the decrease in consumption for the buyer who sells in the bust. However, because of diminishing marginal utility, the increase in utility for the boom seller is smaller than the decrease in utility for the bust seller. Because we are considering a symmetric boom and bust, the size of this fall in aggregate utility is increasing in the amplitude of the boom cycle.

1.5.2 Impact of Policy

Thus far in the paper we have taken the minimum down payment requirement, γ, as an exogenous parameter to perform the counterfactual analysis. However, bubble conditions with high prices could reduce lenders' concerns about borrowers defaulting, easing the credit standards they lend at (Brueckner et al 2012). We now extend the model to consider the utility benefit of macroprudential policy in a context where credit standards fall in response to higher prices. Specifically, we look at the utility benefit of macroprudential policy that keeps γ fixed at its initial pre-boom level by comparing it to the net utility loss when γ varies in addition to P_{t+1}. For the latter, we assume that when prices are higher, the required minimum percentage down payment decreases:

$$\gamma'(P_{t+1}) < 0$$

For the change in γ to play a role, we focus on the case of constrained buyers. Given this, the easing of γ further raises P_t. We first calculate the expected net utility cost of the boom when γ is endogenous. The endogeneity of γ does not change the utility benefit for the pre-boom buyer, as the boom was not expected. Rather, the difference between the cases arises in the utility cost suffered by those who buy in the boom and sell in the bust. As they bought during the boom, γ is lower than before the boom ($\gamma(P_{t+1}+x) < \gamma(P_{t+1})$), further increasing the price they initially paid for their house. We can show that this results in a higher utility cost of the boom when γ is endogenous.

Proposition 16 *Let $V(P_{t+1}, \widetilde{P}_{t+1}, \gamma(P_{t+1}))$ be the lifetime utility of a buyer who expects to sell at price P_{t+1} but sells at price $\widetilde{P}_{t+1}$, where γ is endogenous, with $\gamma'(P_{t+1}) < 0$. Consider a symmetric boom and bust, with the selling price rising from P_{t+1} to $P_{t+1}+x$, $(x > 0)$ then in the bust falling from $P_{t+1}+x$ back to P_{t+1}, with the change a shock in both occasions.*

Then

$$\begin{aligned}
& [V(P_{t+1}, P_{t+1}+x, \gamma(P_{t+1})) - V(P_{t+1}, P_{t+1}, \gamma(P_{t+1}))] \\
& + [V(P_{t+1}+x, P_{t+1}, \gamma(P_{t+1}+x)) - V(P_{t+1}+x, P_{t+1}+x, \gamma(P_{t+1}+x))] \\
< \; & [V(P_{t+1}, P_{t+1}+x, \overline{\gamma}) - V(P_{t+1}, P_{t+1}, \overline{\gamma})] \\
& + [V(P_{t+1}+x, P_{t+1}, \overline{\gamma}) - V(P_{t+1}+x, P_{t+1}+x, \overline{\gamma})] \\
< \; & 0
\end{aligned}$$

where $\overline{\gamma} := \gamma(P_{t+1})$ represents the central bank fixing a minimum percentage down payment requirement at the pre-boom level.

Thus, the net utility loss from the boom and bust is lower when the central bank sets a binding minimum down payment requirement, than when they don't and this is allowed to fall during the boom.

The intuition for the result is straightforward. As discussed, in both cases the utility gain from the boom is the same. The loss in utility from the bust arises because consumption is lower than expected due to the drop in resale price. Due to decreasing marginal utility, the utility loss worsens as the level of consumption decreases. When γ is endogenous, consumption is lower for two reasons. First, the lower gamma increased the price paid for the house during the boom. Second, a lower percentage of the price was paid during the first period, resulting in a higher remainder to be repaid in the second period. Thus, with endogenous γ the utility loss from a housing bust is greater than if policy limits the decline in γ.

1.5.3 Summary

We have shown that a symmetric housing boom and bust results in a net welfare loss, with the losers who sell in the bust outweighing the winners who sell during the boom. Further, the size of the welfare loss is increasing in the amplitude of the housing cycle. When (as seems likely in practice) credit standards endogenously fall during the boom, an LTV cap that prevents the reduction in credit standards reduces the welfare cost of the boom and bust.

1.6 Conclusion

The bust of the housing market in America triggered enormous financial and ultimately fiscal consequences around the world that continue to be felt today. Policy-makers are desperate to avoid a repeat of this in the years to come, and central banks around the world are being given new macroprudential policy tools to try and attenuate the next bubble. Despite widespread recognition that there was substantial easing of credit standards in America during the housing boom, its causal role is still not well understood, in part because of the sheer complexity of the financial operations that took place at the time. This paper contributes to the debate by highlighting the information that can be inferred from the relative growth of different sections of the housing market during the boom. We show that when buyers of cheaper houses are constrained by maximum LTI or LTV ratios, a fall in interest rates or increased buyer optimism alone would result in greater relative price growth for expensive houses. The fact that we witnessed the opposite pattern in the US housing boom tells us that neither of these two explanations could have caused the boom without a significant easing of non-price credit terms. A simple calculation demonstrates that if non-price credit terms had not been relaxed during the boom, and it was caused by either falling interest rates or buyer exuberance, the nominal growth of low tier house prices would have been at least 55 percentage points less. This suggests a highly significant benefit from the future use of macroprudential tools.

Table 1.2: Low and High Tier Average Loan to Value Ratio

Metro	Low T.	High T.	Year
Phoenix, AZ	90.04	75.99	2001
Los Angeles, CA	100.10	57.83	2001/02
Santa Ana, CA	85.51	71.40	2001
San Diego, CA	79.70	74.10	2001
San Jose, CA	84.87	73.86	1997
Oakland, CA	82.70	71.60	1997
Riverside, CA	95.59	86.49	2001
San Francisco, CA	69.13	76.51	1997
Sacramento, CA	89.76	73.79	2003
Denver, CO	99.02	79.82	2003
Washington, DC	88.66	74.54	1997
Miami,FL	85.36	75.43	2001
Tampa,FL	82.00	76.45	1997
Atlanta, GA	95.00	78.92	2003
Chicago, IL	83.63	65.94	2001/02
Boston, MA	81.86	69.41	1997
Detroit, MI	92.36	69.72	2001/02
Minneapolis, MN	92.14	80.73	1997
Rochester, NY	84.33	79.68	1997
New York, NY	78.05	54.76	2001/02
Cincinnati, OH	94.65	75.28	1997
Columbus, OH	88.43	81.38	2001
Portland, OR	86.47	74.55	2001
Philadelphia, PA	86.52	78.66	2001/02
Providence, RI	85.88	73.91	1997
Seattle, WA	91.40	73.03	2003
Milwaukee, WI	89.88	79.18	2001

Source: AHS, Fiserv inc. Tiered breakpoints are used to group the low and high tier buyers in each city.

Table 1.3: Low and High Tier Average Loan to Income Ratio

Metro	1997 Low T.	1997 High T.	2006 Low T.	2006 High T.
Phoenix, AZ	2.45	1.27	3.75	1.67
Los Angeles, CA	2.78	1.61	3.96	2.14
Santa Ana, CA	2.74	1.54	3.92	2.05
San Diego, CA	2.88	1.60	3.92	2.04
San Jose, CA	2.71	1.82	4.16	2.49
Oakland, CA	2.77	1.69	4.06	2.43
Riverside, CA	2.62	1.27	3.81	1.98
San Francisco, CA	2.89	1.67	4.00	2.14
Sacramento, CA	2.79	1.38	3.97	1.95
Denver, CO	2.72	1.50	3.55	1.78
Washington, DC	2.68	1.38	3.99	2.31
Miami,FL	2.13	0.82	3.17	1.53
Tampa,FL	2.05	1.07	3.14	1.33
Atlanta, GA	2.37	1.44	3.31	1.70
Chicago, IL	2.44	1.43	3.29	1.80
Boston, MA	2.50	1.43	3.91	1.85
Detroit, MI	2.15	1.53	2.99	1.77
Minneapolis, MN	2.34	1.39	3.69	1.81
Rochester, NY	1.90	0.98	2.29	1.24
New York, NY	2.23	1.19	3.45	1.82
Cincinnati, OH	2.19	1.35	2.85	1.57
Columbus, OH	2.31	1.35	2.92	1.54
Portland, OR	2.73	1.45	3.78	1.68
Philadelphia, PA	2.13	1.43	2.98	1.67
Providence, RI	2.40	1.00	3.95	1.89
Seattle, WA	2.78	1.62	3.89	2.05
Milwaukee, WI	2.05	1.43	3.09	1.62

Source: HMDA. See appendix to Chapter 2 for details.

1.A Proofs With One Type of House To Buy

1.A.1 User Cost Model as Special Case

We restate the proposition. Suppose there is no utility premium from owning a house, $u_L = u_R$, and agents are unconstrained both when buying and renting. Then

$$P_t = R_t + \frac{P_{t+1}}{1+r_t}$$

Proof of Proposition 2. With the agent unconstrained when buying and renting:

$$\begin{aligned} \beta(1+r_t)u'(C^L_{1,t}) &= u'(C^L_{0,t}) \\ \beta(1+r_t)u'(C^R_{1,t}) &= u'(C^R_{0,t}) \end{aligned} \tag{1.6}$$

From (1.2) with $u_L = u_R$ we have that

$$u(C^L_{0,t}) + \beta u(C^L_{1,t}) = u(C^R_{0,t}) + \beta u(C^R_{1,t}) \tag{1.7}$$

So the lifetime utility from non-housing consumption is the same when buying and renting in equilibrium.

Now in general, for an unconstrained consumer $C_{0,t} = x_{0,t} - b^*_{0,t}$ and $C_{1,t} = x_{1,t} + (1+r_t)b^*_{0,t}$ so it follows that

$$C_{0,t} + \frac{C_{1,t}}{1+r_t} = x_{0,t} + \frac{x_{1,t}}{1+r_t}$$

Thus

$$\begin{aligned} C^L_{0,t} + \frac{C^L_{1,t}}{1+r_t} &= y_{0,t} - \gamma P_t + \frac{y_{1,t} - R_{t+1} + P_{t+1}}{1+r_t} - (1-\gamma)P_t \\ &= y_{0,t} + \frac{y_{1,t} - R_{t+1} + P_{t+1}}{1+r_t} - P_t \end{aligned}$$

Similarly

$$C^R_{0,t} + \frac{C^R_{1,t}}{1+r_t} = y_{0,t} - R_t + \frac{y_{1,t} - R_{t+1}}{1+r_t}$$

We now show that $C^R_{1,t} = C^L_{1,t}$.

Suppose not. Suppose $C^R_{1,t} > C^L_{1,t}$. Then it follows that $u(C^R_{1,t}) > u(C^L_{1,t})$ as $u' > 0$. Then from (1.7) we have that $u(C^R_{0,t}) < u(C^L_{0,t})$ and so $C^R_{0,t} < C^L_{0,t}$.

Now, given $u'' < 0$, as $C^R_{1,t} > C^L_{1,t}$ we have that $u'(C^R_{1,t}) < u'(C^L_{1,t})$

Thus from (1.6) it follows that

$$\begin{aligned} u'(C^L_{0,t}) &= \beta(1+r_t)u'(C^L_{1,t}) \\ &> \beta(1+r_t)u'(C^R_{1,t}) \\ &= u'(C^R_{0,t}) \end{aligned}$$

Hence, it follows that $C^L_{0,t} < C^R_{0,t}$. But $C^R_{0,t} < C^L_{0,t}$, a contradiction. Hence we can't have $C^R_{1,t} > C^L_{1,t}$. By symmetry of argument, we can't have $C^R_{1,t} < C^L_{1,t}$, hence we must have $C^R_{1,t} = C^L_{1,t}$. From (1.7) it follows that $C^L_{0,t} = C^R_{0,t}$. Thus

$$\begin{aligned}
C^L_{0,t} + \frac{C^L_{1,t}}{1+r_t} &= C^R_{0,t} + \frac{C^R_{1,t}}{1+r_t} \text{ so} \\
y_{0,t} + \frac{y_{1,t} - R_{t+1} + P_{t+1}}{1+r_t} - P_t &= y_{0,t} - R_t + \frac{y_{1,t} - R_{t+1}}{1+r_t} \text{ so} \\
P_t &= R_t + \frac{P_{t+1}}{1+r_t}
\end{aligned}$$

This completes the proof of the proposition. ■

1.A.2 Generalisation of User Cost Model with Log Utility

Here we prove Example 3

Recall, it states that when $u(C) = \log(C)$ and agents are unconstrained both when buying and renting, then

$$P_t = R_t + \frac{P_{t+1}}{1+r_t} + \left(y_{0,t} - R_t + \frac{y_{1,t} - R_{t+1}}{1+r_t}\right)\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)$$

Proof. With the agent unconstrained when buying and renting:

$$\begin{aligned}
\beta(1+r_t)u'(C^L_{1,t}) &= u'(C^L_{0,t}) \\
\beta(1+r_t)u'(C^R_{1,t}) &= u'(C^R_{0,t})
\end{aligned}$$

With log utility, we have

$$C^i_{0,t}\beta(1+r_t) = C^i_{1,t}$$

As they are unconstrained $C_{0,t} + \frac{C_{1,t}}{1+r_t} = x_{0,t} + \frac{x_{1,t}}{1+r_t}$ (as discussed in the prior proof).
Then we have

$$C_{0,t}\left[1 + \frac{\beta(1+r_t)}{(1+r_t)}\right] = x_{0,t} + \frac{x_{1,t}}{1+r_t}$$

Hence, for both buyers and renters, we have

$$\begin{aligned}
C_{0,t} &= \frac{1}{1+\beta}\left[x_{0,t} + \frac{x_{1,t}}{1+r_t}\right] \\
C_{1,t} &= \frac{\beta(1+r_t)}{1+\beta}\left[x_{0,t} + \frac{x_{1,t}}{1+r_t}\right]
\end{aligned}$$

Given the discounted lifetime resources available for non-housing consumption in each case, by

(1.2) in equilibrium we have

$$\begin{aligned}
& \log\left(\frac{1}{(1+\beta)}\left(y_{0,t}-P_t+\frac{y_{1,t}+P_{t+1}-R_{t+1}}{(1+r_t)}\right)\right) \\
& +\beta\log\left(\frac{\beta(1+r_t)}{(1+\beta)}\left(y_{0,t}-P_t+\frac{y_{1,t}+P_{t+1}-R_{t+1}}{(1+r_t)}\right)\right)+u_L+\beta u_R \\
= & \log\left(\frac{1}{1+\beta}\left(y_{0,t}-R_t+\frac{y_{1,t}-R_{t+1}}{(1+r_t)}\right)\right) \\
& +\beta\log\left(\frac{\beta(1+r_t)}{1+\beta}\left(y_{0,t}-R_t+\frac{y_{1,t}-R_{t+1}}{(1+r_t)}\right)\right)+(1+\beta)u_R
\end{aligned}$$

Thus

$$\begin{aligned}
& (1+\beta)\log\left(y_{0,t}-P_t+\frac{y_{1,t}+P_{t+1}-R_{t+1}}{(1+r_t)}\right)+u_L-u_R \\
= & (1+\beta)\log\left(y_{0,t}-R_t+\frac{y_{1,t}-R_{t+1}}{(1+r_t)}\right) \\
& \log\left(y_{0,t}-P_t+\frac{y_{1,t}+P_{t+1}-R_{t+1}}{(1+r_t)}\right)+\frac{u_L-u_R}{(1+\beta)} \\
= & \log\left(y_{0,t}-R_t+\frac{y_{1,t}-R_{t+1}}{(1+r_t)}\right)
\end{aligned}$$

Taking exponentials of both sides

$$\left(y_{0,t}-P_t+\frac{y_{1,t}+P_{t+1}-R_{t+1}}{(1+r_t)}\right)=\left(y_{0,t}-R_t+\frac{y_{1,t}-R_{t+1}}{(1+r_t)}\right)\left(\frac{1}{\exp\left(\frac{u_L-u_R}{(1+\beta)}\right)}\right)$$

Rearranging gives

$$\begin{aligned}
P_t &= \frac{P_{t+1}}{(1+r_t)}+\left(y_{0,t}+\frac{y_{1,t}-R_{t+1}}{(1+r_t)}\right)-\left(y_{0,t}-R_t+\frac{y_{1,t}-R_{t+1}}{(1+r_t)}\right)\left(\frac{1}{\exp\left(\frac{u_L-u_R}{(1+\beta)}\right)}\right) \\
&= R_t+\frac{P_{t+1}}{(1+r_t)}+\left(y_{0,t}-R_t+\frac{y_{1,t}-R_{t+1}}{(1+r_t)}\right)\left(1-\frac{1}{\exp\left(\frac{u_L-u_R}{(1+\beta)}\right)}\right)
\end{aligned}$$

This completes the proof. ■

1.A.3 Equilibrium and Dynamic Equilibrium

Here we establish conditions under which the solution to (1.2) is a function, $P_t(P_{t+1})$, and prove Proposition 4. We prove both through a series of lemmas.

Lemma 17 *Suppose P_{t+1} is such that*[38]

$$u(0)+\beta u\left(y_{1,t}-R_{t+1}-(1+r_t)\frac{y_{0,t}}{\gamma}+P_{t+1}\right)+u_L-u_R<u(C^R_{0,t})+\beta u\left(C^R_{1,t}\right)$$

then there is a function $P_t(P_{t+1})$ that solves the housing market equilibrium condition (1.2).

Proof. For a given $P_{t+1}\geq 0$, let

$$\begin{aligned} g(P_t,P_{t+1}) \quad : \quad &= u(C^L_{0,t})+\beta u(C^L_{1,t})+u_L-u_R \\ &-\left(u(C^R_{0,t})+\beta u\left(C^R_{1,t}\right)\right) \end{aligned}$$

Then we have equilibrium in the housing market when $g(P_t,P_{t+1})=0$. To show existence, we thus show that there is a $P_t\geq 0$ that solves $g(P_t,P_{t+1})=0$.

Suppose $P_t=0$. We show that $g(0,P_{t+1})>0$ so, if the price of a house is 0 everybody prefers to buy a house to renting.

Under these conditions, when buying

$$\begin{aligned} \widehat{x}^L_{0,t} &= y_{0,t} \\ \widehat{x}^L_{1,t} &= y_{1,t}-R_{t+1}+P_{t+1} \\ \widehat{x}^R_{0,t} &= y_{0,t}-R_t \\ \widehat{x}^R_{1,t} &= y_{1,t}-R_{t+1} \end{aligned}$$

Claim

Let

$$V(\widehat{x}_{0,t},\widehat{x}_{1,t}):=\max_{b_{0,t}} u\left(\widehat{x}_{0,t}-b_{0,t}\right)+\beta u\left(\widehat{x}_1+b_{0,t}(1+r_t)\right)$$

Then

$$\frac{dV}{d\widehat{x}_{0,t}}>0 \text{ and } \frac{dV}{d\widehat{x}_{1,t}}>0$$

Proof of Claim

$$\begin{aligned} \frac{dV}{d\widehat{x}_{0,t}} &= u'\left(\widehat{x}_{0,t}-b_{0,t}\right)\left[1-\frac{db_{0,t}}{d\widehat{x}_{0,t}}\right]+\beta u'\left(\widehat{x}_1+b_{0,t}(1+r_t)\right)\left[(1+r_t)\frac{db_{0,t}}{d\widehat{x}_{0,t}}\right] \\ &= u'\left(\widehat{x}_{0,t}-b_{0,t}\right)+\frac{db_{0,t}}{d\widehat{x}_{0,t}}\left[(1+r_t)\beta u'(C_{1,t})-u'(C_{0,t})\right] \\ &= u'\left(\widehat{x}_{0,t}-b_{0,t}\right)>0 \end{aligned}$$

To see the last step if the agent is unconstrained when buying $(1+r_t)\beta u'(C_{1,t})-u'(C_{0,t})=0$. Otherwise, if they're constrained when buying $b_{0,t}\equiv 0$ and $\frac{db_{0,t}}{d\widehat{x}_{0,t}}=0$.

[38] This is guaranteed if

$$\lim_{C\to 0} u(C)=-\infty$$

As with log utility.

The proof for $\frac{dV}{d\widehat{x}_{1,t}}$ is similar. This completes the proof of the claim.

Thus, by the claim, as $\widehat{x}^L_{0,t} > \widehat{x}^R_{0,t}$ and $\widehat{x}^L_{1,t} \geq \widehat{x}^R_{1,t}$ we have $V(\widehat{x}^L_{0,t}, \widehat{x}^L_{1,t}) > V(\widehat{x}^R_{0,t}, \widehat{x}^R_{1,t})$. Thus $g(0, P_{t+1}) > 0$ as $u_L \geq u_R$.

We now show that for sufficiently high P_t, $g(P_t, P_{t+1}) < 0$.

Suppose $P_t = \frac{y_{0,t}}{\gamma}$, then the buying agent's entire income when young is spent in the housing deposit. Their consumption when young is thus 0. If their lifetime utility is then $-\infty$ (as with $u(C) = \log(C)$) then $g(\frac{y_{0,t}}{\gamma}, P_{t+1}) < 0$

Otherwise, if lifetime utility is still well defined, by the assumption for the lemma we have $g(\frac{y_{0,t}}{\gamma}, P_{t+1}) < 0$.

Now, as $\frac{\partial g(P_t, P_{t+1})}{\partial P_t}$ is well defined, $g(P_t, P_{t+1})$ is continuous in P_t. Hence, by the Intermediate Value Theorem, there exists $P^*_t \in (0, \frac{y_{0,t}}{\gamma})$ with $g(P^*_t, P_{t+1}) = 0$. Hence, an equilibrium exists.

We now show uniqueness.

For this it is enough to show that $\frac{\partial g(P_t, P_{t+1})}{\partial P_t} < 0$. The change in P_t only affects $V(\widehat{x}^L_{0,t}, \widehat{x}^L_{1,t})$ (i.e. the change in price doesn't affect the utility of the renting agent).

Now

$$\begin{aligned}
\widehat{x}^L_{0,t} &= y_{0,t} - \gamma P_t \\
\widehat{x}^L_{1,t} &= y_{1,t} - R_{t+1} - (1-\gamma)(1+r_t)P_t + P_{t+1}
\end{aligned}$$

Thus $\frac{\partial \widehat{x}^L_{0,t}}{\partial P_t} = -\gamma < 0$ and $\frac{\partial \widehat{x}^L_{0,t}}{\partial P_t} = -(1-\gamma)(1+r_t) < 0$

Hence, by the claim, holding the rest constant, as P_t increases, $\widehat{x}^L_{0,t}$ and $\widehat{x}^L_{1,t}$ decrease, decreasing $V(\widehat{x}_{0,t}, \widehat{x}_{1,t})$, in turn decreasing $g(P_t, P_{t+1})$. This shows that $\frac{\partial g(P_t, P_{t+1})}{\partial P_t} < 0$ establishing uniqueness which completes the proof of the lemma. ■

Lemma 18 *In the dynamic equilibrium of the housing market, regardless of whether the house-buying agent is constrained or unconstrained, we have, assuming* $r_t > 0$,

$$0 < \frac{dP_t}{dP_{t+1}} \leq \frac{1}{1+r_t} < 1$$

Proof. This is proved below in the proof of Proposition 6 ■

Lemma 19 *Suppose that*[39]

$$u(0) + \beta u\left(y_{1,t} - R_{t+1} + (1 - (1-\gamma)(1+r_t))\frac{y_{0,t}}{\gamma}\right) + u_L - u_R < u(C^R_{0,t}) + \beta u\left(C^R_{1,t}\right) \tag{1.8}$$

and $r_t > 0$, *then the model has a unique steady state.*

Proof. Let

$$g(P_t) := u(C^L_{0,t}) + \beta u(C^L_{1,t}) + u_L - u_R - \left(u(C^R_{0,t}) + \beta u\left(C^R_{1,t}\right)\right)$$

[39] This is guaranteed if

$$\lim_{C \to 0} u(C) = -\infty$$

As with log utility.

Where we have held $P_t \equiv P_{t+1}$. Given this, we have

$$\begin{aligned}
\widehat{x}^L_{0,t} &= y_{0,t} - \gamma P_t \\
\widehat{x}^L_{1,t} &= y_{1,t} - R_{t+1} + (1 - (1-\gamma)(1+r_t))\, P_t \\
\widehat{x}^R_{0,t} &= y_{0,t} - R_t \\
\widehat{x}^R_{1,t} &= y_{1,t} - R_{t+1}
\end{aligned}$$

Then we have a steady-state equilibrium in the housing market when $g(P_t) = 0$. To show existence, we thus show that there is a $P_t \geq 0$ that solves $g(P_t) = 0$.

Suppose $P_t = 0$. We show that $g(0) > 0$ so, if the price of a house today and tomorrow is 0 everybody prefers to buy a house to renting.

Under these conditions, when buying

$$\begin{aligned}
\widehat{x}^L_{0,t} &= y_{0,t} \\
\widehat{x}^L_{1,t} &= y_{1,t} - R_{t+1}
\end{aligned}$$

Then $\widehat{x}^L_{0,t} > \widehat{x}^R_{0,t}$, and $\widehat{x}^L_{1,t} = \widehat{x}^R_{1,t}$. Hence by the claim in the previous lemma, it follows that $V(\widehat{x}^L_{0,t}, \widehat{x}^L_{1,t}) > V(\widehat{x}^R_{0,t}, \widehat{x}^R_{1,t})$. Thus $g(0, P_{t+1}) > 0$ as $u_L \geq u_R$.

We now show that for sufficiently high P_t, $g(P_t) < 0$.

Suppose $P_t = \frac{y_{0,t}}{\gamma}$, then the buying agent's entire income when young is spent in the housing deposit. Their consumption when young is thus 0. If their lifetime utility is then $-\infty$ (as with $u(C) = \log(C)$) then $g(\frac{y_{0,t}}{\gamma}) < 0$

Otherwise, if lifetime utility is still well defined, by the assumption for the lemma we have $g(\frac{y_{0,t}}{\gamma}) < 0$.

Now, as $\frac{dg(P_t)}{dP_t}$ is well defined, $g(P_t)$ is continuous. Hence, by the Intermediate Value Theorem, there exists $P_t^* \in (0, \frac{y_{0,t}}{\gamma})$ with $g(P_t^*) = 0$. Hence, an equilibrium exists.

We now show uniqueness.

For this it is enough to show that $g'(P_t) < 0$.

Now

$$\begin{aligned}
g(P_t) &= u\left(y_{0,t} - \gamma P_t - b_{0,t}\right) + \beta u\left(y_{1,t} - R_{t+1} + P_t\left(1 - (1-\gamma)(1+r_t\right) + b_{0,t}(1+r_t)\right) \\
&\quad + u_L - u_R - \left(u(C^R_{0,t}) + \beta u\left(C^R_{1,t}\right)\right)
\end{aligned}$$

Thus

$$
\begin{aligned}
g'(P_t) &= u'(C_{0,t})\left(-\gamma - \frac{db_{0,t}}{dP_t}\right) + \beta u'(C_{1,t})\left((1-(1-\gamma)(1+r_t) + \frac{db_{0,t}}{dP_t}(1+r_t)\right) \\
&= \frac{db_{0,t}}{dP_t}\left[(1+r_t)\beta u'(C_{1,t}) - u'(C_{0,t})\right] \\
&\quad -\gamma u'(C_{0,t}) + \beta u'(C_{1,t})\left((1-(1-\gamma)(1+r_t)\right) \\
&= -\gamma u'(C_{0,t}) + \beta u'(C_{1,t})\left((1-(1-\gamma)(1+r_t)\right) \\
&= \gamma\left[(1+r_t)\beta u'(C_{1,t}) - u'(C_{0,t})\right] + \beta u'(C_{1,t})\left((1-(1+r_t)\right) \\
&= \gamma\left[(1+r_t)\beta u'(C_{1,t}) - u'(C_{0,t})\right] - r_t\beta u'(C_{1,t}) < 0
\end{aligned}
$$

The last line follows as the first term is non-positive and $r_t > 0$. Further, we have used (as with the proof of the claim) that $\frac{db_{0,t}}{dP_t}\left[(1+r_t)\beta u'(C_{1,t}) - u'(C_{0,t})\right] = 0$.

This shows that $g'(P_t) < 0$ establishing uniqueness which completes the proof of the lemma. ■

Lemma 20 *Let P^* be the unique steady state price. Then, for given P_{t+1}, the function $P_t(P_{t+1})$ satisfies*

$$|P_t(P_{t+1}) - P^*| \leq \frac{1}{(1+r)}|P_{t+1} - P^*|$$

Proof. By above results, $\exists P^* : P_t(P^*) = P^*$ (this is just the steady state solution).

Let

$$f(z) := P_t(z) - P^*$$

Thus $f(P^*) = 0$. Further $f'(z) = P_t^{'}(z) \leq \frac{1}{(1+r)} \equiv k, say$

Claim: If $P_{t+1} > P^*$, then $P_t(P_{t+1}) > P^*$, and if $P_{t+1} < P^*$, then $P_t(P_{t+1}) < P^*$.

This follows as $P_t(P^*) = P$ and from the above result that $P_t'(P_{t+1}) > 0$. This completes the proof of the claim.

We now proceed through the various cases.

(i) Suppose $P_{t+1} > P^*$. From the derivative result it follows that

$$\int_{P^*}^{P_{t+1}} f'(z)dz \leq \int_{P^*}^{P_{t+1}} kdz$$

And so

$$f(P_{t+1}) - f(P^*) \leq k(P_{t+1} - P^*)$$

But $f(P^*) = 0$, so

$$P_t(P_{t+1}) - P^* \leq k(P_{t+1} - P^*)$$

As both sides of the inequality are positive (LHS following from the above claim)

$$|P_t(P_{t+1}) - P^*| \leq k\,|(P_{t+1} - P^*)|$$

(ii) Suppose $P_{t+1} = P^*$ Then it's trivially true that

$$|P_t(P_{t+1}) - P^*| \leq k\,|(P_{t+1} - P^*)|$$

(iii) Suppose finally that $P_{t+1} < P^*$ then

$$\int_{P_{t+1}}^{P^*} f'(z)dz \leq \int_{P_{t+1}}^{P^*} kdz$$

Thus

$$f(P^*) - f(P_{t+1}) \leq k(P^* - P_{t+1})$$

So

$$k(P_{t+1} - P^*) \leq f(P_{t+1}) = P_t(P_{t+1}) - P^*$$

But, from above claim, given that $P_{t+1} < P^*$, we have $P_t\,(P_{t+1}) < P^*$, so RHS<0. But $k(P_{t+1} - P^*) = -k\,|P_{t+1} - P^*|$ so

$$-k\,|P_{t+1} - P^*| \leq P_t(P_{t+1}) - P^*$$

And then as RHS<0,

$$-k\,|P_{t+1} - P^*| \leq P_t(P_{t+1}) - P^* \leq k\,|P_{t+1} - P^*|$$

And so

$$|P_t(P_{t+1}) - P^*| \leq k\,|P_{t+1} - P^*|$$

This completes the proof. ■

Corollary 21 *Let P_T be finite, and P^* be the steady state equilibrium of the system. Then, given $r_t > 0$ we have*

$$\lim_{s\to\infty} P_{T-s} = P^*$$

Proof. Claim

$$\forall s \geq 0\,|P_{T-s} - P^*| \leq k^s\,|P_T - P^*|$$

Proof of Claim

We prove this by induction.

Base Case $s = 0$. This is trivially true.

Inductive Step $s = n$. Suppose $|P_{T-n} - P^*| \leq k^n\,|P_T - P^*|$

By the corollary $|P_{T-n-1} - P^*| \leq k\,|P_{T-n} - P^*|$

Thus $|P_{T-n-1} - P^*| \leq k\,(k^n\,|P_{T-1} - P^*|) = k^{n+1}\,|P_T - P^*|$,completing the inductive step. Thus, by induction, the claim is established.

We now conclude the proof.

As $k \in (0,1)$ when $r_t > 0$, $\lim_{s\to\infty} k^s \left|P_T - P^*\right| = 0$, thus by the sandwich theorem $\lim_{s\to\infty} \left|P_{T-s} - P^*\right| = 0$ and so $\lim_{s\to\infty} P_{T-s} = P^*$ completing the proof. ∎

Lemma 22 *The model jumps instantly to its steady state price and no bubbles are possible.*

Proof. As discussed in the text, no bubbles are possible, because on all dates T we must have $P_T \leq \frac{y_{0,T}}{\gamma}$ for consumption when young to be non-negative. Thus, arbitrarily far into the future, the price of housing must be finite. Thus, by the corollary, (with parameters fixed at their current level), we must have $P_t = P^*$. That is, with no future shocks anticipated, the price of housing must be at its steady state level. This completes the proof of the lemma. ∎

1.A.4 Constrained vs Unconstrained Response to Change in Max LTV

We restate the proposition. Suppose the home-buyer is constrained in equilibrium with price P_t^c. Then

$$\frac{dP_t^c}{d\gamma} < 0$$

Suppose the home-buyer is unconstrained in equilibrium with price P_t^u. Then

$$\frac{dP_t^u}{d\gamma} = 0$$

Proof of Proposition 5. Let

$$g(P_t^*(\gamma),\gamma) := u(C_{0,t}^L) + \beta u(C_{1,t}^L) + u_L + \beta u_R - \left(u(C_{0,t}^R) + \beta u(C_{1,t}^R) + (1+\beta)u_R\right)$$

Then, in equilibrium we have

$$g(P_t^*(\gamma),\gamma) \equiv 0$$

Thus

$$\frac{\partial g(P_t^*(\gamma),\gamma)}{\partial P_t^*}\frac{dP_t^*}{d\gamma} + \frac{\partial g(P_t^*(\gamma),\gamma)}{\partial \gamma} = 0$$

Thus

$$\frac{dP_t^*}{d\gamma} = \frac{-\frac{\partial g(P_t^*(\gamma),\gamma)}{\partial \gamma}}{\frac{\partial g(P_t^*(\gamma),\gamma)}{\partial P_t^*}}$$

(i) we first suppose the agent is constrained when buying.

As the agent is constrained $b_{0,t}^* = 0$ and

$$\begin{aligned} C_{0,t}^L &= y_{0.t} - \gamma P_t^* \\ C_{1,t}^L &= y_{1.t} - R_{t+1} + \left(1 - (1+r_t)(1-\gamma)\right) P_t^* \end{aligned}$$

Thus

$$\frac{\partial g(P_t^*(\gamma),\gamma)}{\partial P_t^*} = -u'(C_{0,t}^L)\gamma + \beta u'(C_{1,t}^L)\left(1 - (1+r_t)(1-\gamma)\right)$$

And
$$\frac{\partial g(P_t^*(\gamma),\gamma)}{\partial \gamma} = -u'(C_{0,t}^L)P_t^* + \beta u'(C_{1,t}^L)(1+r_t)P_t^*$$

Thus
$$\frac{dP_t^*}{d\gamma} = \frac{-\left(-u'(C_{0,t}^L)P_t^* + \beta u'(C_{1,t}^L)(1+r_t)P_t^*\right)}{-u'(C_{0,t}^L)\gamma + \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \tag{1.9}$$

Further

$$\begin{aligned} -u'(C_{0,t}^L)P_t^* + \beta u'(C_{1,t}^L)(1+r_t)P_t^* &< 0 \text{ iff} \\ \beta u'(C_{1,t}^L)(1+r_t)P_t^* &< u'(C_{0,t}^L)P_t^* \text{ iff} \\ \beta u'(C_{1,t}^L)(1+r_t) &< u'(C_{0,t}^L) \end{aligned}$$

But this is the condition for the agent being constrained so $\frac{\partial g(P_t^*(\gamma),\gamma)}{\partial \gamma} < 0$.
Similarly

$$\begin{aligned} -u'(C_{0,t}^L)\gamma + \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right) &< 0 \text{ iff} \\ \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right) &< u'(C_{0,t}^L)\gamma \end{aligned}$$

Now

$$1-(1-\gamma)(1+r_t) < \gamma(1+r_t) \text{ iff} \tag{1.10}$$
$$1 < \gamma(1+r_t) + (1-\gamma)(1+r_t) \text{ iff} \tag{1.11}$$
$$1 < (1+r_t) \tag{1.12}$$

Which is true.
Putting this together,

$$\begin{aligned} & \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right) \\ < \; & \beta u'(C_{1,t}^L)\gamma(1+r_t) \\ < \; & u'(C_{0,t}^L)\gamma \end{aligned}$$

Thus $\frac{\partial g(P_t^*(\gamma),\gamma)}{\partial P_t^*} < 0$.
It thus follows from (1.9) that
$$\frac{dP_t^*}{d\gamma} < 0$$

(ii) Unconstrained buyer

$$\frac{dP_t^*}{d\gamma} = \frac{-\left(u'(C_{0,t}^L)\frac{\partial C_{0,t}^L}{\partial\gamma} + \beta u'(C_{1,t}^L)\frac{\partial C_{1,t}^L}{\partial\gamma}\right)}{u'(C_{0,t}^L)\frac{\partial C_{0,t}^L}{\partial P_t^*} + \beta u'(C_{1,t}^L)\frac{\partial C_{1,t}^L}{\partial P_t^*}}$$

$$= \frac{-\left(\beta u'(C_{1,t}^L)(1+r_t)\frac{\partial C_{0,t}^L}{\partial\gamma} + \beta u'(C_{1,t}^L)\frac{\partial C_{1,t}^L}{\partial\gamma}\right)}{\beta u'(C_{1,t}^L)(1+r_t)\frac{\partial C_{0,t}^L}{\partial P_t^*} + \beta u'(C_{1,t}^L)\frac{\partial C_{1,t}^H}{\partial P_t^*}} \tag{1.13}$$

$$= \frac{-\left((1+r_t)\frac{\partial C_{0,t}^L}{\partial\gamma} + \frac{\partial C_{1,t}^L}{\partial\gamma}\right)}{(1+r_t)\frac{\partial C_{0,t}^L}{\partial P_t^*} + \frac{\partial C_{1,t}^L}{\partial P_t^*}} \tag{1.14}$$

Where we have used the fact that as the agent is unconstrained when buying,

$$\beta u'(C_{1,t}^L)(1+r_t) = u'(C_{0,t}^L)$$

Further,

$$C_{0,t}^L + \frac{C_{1,t}^L}{1+r_t} \equiv y_{0,t} + \frac{y_{1,t} - R_{t+1}}{1+r_t} - \frac{r_t P_t^*}{1+r_t}$$

Thus

$$\frac{\partial C_{0,t}^L}{\partial\gamma} + \frac{\left(\frac{\partial C_{1,t}^L}{\partial\gamma}\right)}{1+r_t} = 0 \text{ and}$$

$$\frac{\partial C_{0,t}^L}{\partial P_t^*} + \frac{\left(\frac{\partial C_{1,t}^L}{\partial P_t^*}\right)}{1+r_t} = \frac{-r_t}{1+r_t}$$

Thus

$$\frac{dP_t^*}{d\gamma} = \frac{-0}{\left(\frac{-r_t}{1+r_t}\right)} = 0$$

This completes the unconstrained case and the proof of the proposition. ■

1.A.5 Constrained vs Unconstrained Price Response to Optimism With Max LTV Constraint

We prove an extension of the formula, also demonstrating the price response formula for the constrained buyer. Suppose the home-buyer is constrained in equilibrium with price P_t^c. Then

$$0 < \frac{dP_t^c}{dP_{t+1}^c} = \frac{\frac{1}{1+r_t}}{\gamma\left[\frac{u'(C_{0,t}^L)}{\beta(1+r_t)u'(C_{1,t}^L)} - 1\right] + 1} < \frac{1}{1+r_t}$$

Suppose the home-buyer is unconstrained in equilibrium with price P_t^u. Then

$$\frac{dP_t^u}{dP_{t+1}^u} = \frac{1}{1+r_t}$$

Proof of Proposition 6. Let

$$g(P_t(P_{t+1}), P_{t+1}) := u(C_{0,t}^L) + \beta u(C_{1,t}^L) + u_L + \beta u_R - \big(u(C_{0,t}^R) + \beta u(C_{1,t}^R) + (1+\beta)u_R\big)$$

Then, in equilibrium we have

$$g(P_t(P_{t+1}), P_{t+1}) \equiv 0$$

Thus

$$\frac{\partial g(P_t(P_{t+1}), P_{t+1})}{\partial P_t}\frac{dP_t}{dP_{t+1}} + \frac{\partial g(P_t(P_{t+1}), P_{t+1})}{\partial P_{t+1}} = 0$$

And

$$\frac{dP_t}{dP_{t+1}} = \frac{-\frac{\partial g(P_t(P_{t+1}), P_{t+1})}{\partial P_{t+1}}}{\frac{\partial g(P_t(P_{t+1}), P_{t+1})}{\partial P_t}}$$

We consider the two cases separately.

(i) Agent constrained when buying

As the agent is constrained $b_{0,t}^* = 0$ and

$$\begin{aligned}
C_{0,t}^L &= y_{0,t} - \gamma P_t \\
C_{1,t}^L &= y_{1,t} - R_{t+1} - (1+r_t)(1-\gamma)P_t + P_{t+1}
\end{aligned}$$

Thus

$$\frac{\partial g(P_t(P_{t+1}), P_{t+1})}{\partial P_t} = -u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)(1+r)(1-\gamma)$$

And

$$\frac{\partial g(P_t(P_{t+1}), P_{t+1})}{\partial P_{t+1}} = \beta u'(C_{1,t}^L)$$

Thus

$$\begin{aligned}
\frac{dP_t}{dP_{t+1}} &= \frac{-\frac{\partial g(P_t(P_{t+1}), P_{t+1})}{\partial P_{t+1}}}{\frac{\partial g(P_t(P_{t+1}), P_{t+1})}{\partial P_t}} \\
&= \frac{-\beta u'(C_{1,t}^L)}{-u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)(1+r_t)(1-\gamma)} \\
&= \frac{\beta u'(C_{1,t}^L)}{u'(C_{0,t}^L)\gamma + \beta u'(C_{1,t}^L)(1+r_t)(1-\gamma)} > 0
\end{aligned}$$

Dividing top and bottom by $\beta u'(C^L_{1,t})(1+r_t)$,

$$\frac{dP_t}{dP_{t+1}} = \frac{\frac{1}{(1+r_t)}}{\frac{u'(C^L_{0,t})\gamma}{\beta(1+r_t)u'(C^L_{1,t})} + (1-\gamma)} = \frac{\frac{1}{(1+r_t)}}{\gamma\left[\frac{u'(C^L_{0,t})}{\beta(1+r_t)u'(C^L_{1,t})} - 1\right] + 1}$$

As the agent is constrained, $\beta(1+r_t)u'(C^L_{1,t}) < u'(C^L_{0,t})$, and so the term in [.] is positive, resulting in the denominator being greater than 1. It follows that

$$\frac{dP_t}{dP_{t+1}} < \frac{1}{(1+r_t)}$$

And this completes the constrained case.

(ii) Unconstrained buyer.

We have

$$\frac{dP_t}{dP_{t+1}} = \frac{-\left(u'(C^L_{0,t})\frac{\partial C^L_{0,t}}{\partial P_{t+1}} + \beta u'(C^L_{1,t})\frac{\partial C^L_{1,t}}{\partial P_{t+1}}\right)}{u'(C^L_{0,t})\frac{\partial C^L_{0,t}}{\partial P_t} + \beta u'(C^L_{1,t})\frac{\partial C^L_{1,t}}{\partial P_t}}$$

As the agent is unconstrained when buying,

$$\beta(1+r_t)u'(C^L_{1,t}) = u'(C^L_{0,t})$$

And so

$$\begin{aligned}
\frac{dP_t}{dP_{t+1}} &= \frac{-\left(\beta(1+r_t)u'(C^L_{1,t})\frac{\partial C^L_{0,t}}{\partial P_{t+1}} + \beta u'(C^L_{1,t})\frac{\partial C^L_{1,t}}{\partial P_{t+1}}\right)}{\beta(1+r_t)u'(C^L_{1,t})\frac{\partial C^L_{0,t}}{\partial P_t} + \beta u'(C^L_{1,t})\frac{\partial C^L_{1,t}}{\partial P_t}} \\
&= \frac{-u'(C^L_{1,t})\left(\beta(1+r_t)\frac{\partial C^L_{0,t}}{\partial P_{t+1}} + \beta\frac{\partial C^L_{1,t}}{\partial P_{t+1}}\right)}{u'(C^L_{1,t})\left(\beta(1+r_t)\frac{\partial C^L_{0,t}}{\partial P_t} + \beta\frac{\partial C^L_{1,t}}{\partial P_t}\right)} \\
&= \frac{-\left((1+r_t)\frac{\partial C^L_{0,t}}{\partial P_{t+1}} + \frac{\partial C^L_{1,t}}{\partial P_{t+1}}\right)}{(1+r_t)\frac{\partial C^L_{0,t}}{\partial P_t} + \frac{\partial C^L_{1,t}}{\partial P_t}}
\end{aligned}$$

Now as the agent is unconstrained when buying, we have the following identity:

$$C^L_{0,t} + \frac{C^L_{1,t}}{1+r_t} \equiv y_{0,t} + \frac{y_{1,t} - R_{t+1} + P_{t+1}}{1+r_t} - P_t$$

Thus, differentiating with respect to P_t :

$$\frac{\partial C^L_{0,t}}{\partial P_t} + \frac{\left(\frac{\partial C^L_{1,t}}{\partial P_t}\right)}{1+r_t} = -1$$

Similarly

$$\frac{\partial C^L_{0,t}}{\partial P_{t+1}} + \frac{\left(\frac{\partial C^L_{1,t}}{\partial P_{t+1}}\right)}{1+r_t} = \frac{1}{1+r_t}$$

Thus

$$\begin{aligned}
\frac{dP_t}{dP_{t+1}} &= \frac{-\left((1+r_t)\frac{\partial C^L_{0,t}}{\partial P_{t+1}} + \frac{\partial C^L_{1,t}}{\partial P_{t+1}}\right)}{(1+r_t)\frac{\partial C^L_{0,t}}{\partial P_t} + \frac{\partial C^L_{1,t}}{\partial P_t}} \\
&= \frac{-1}{-(1+r_t)} \\
&= \frac{1}{(1+r_t)}
\end{aligned}$$

This completes the proof for the constrained case and the proof of the proposition. ■

Proof of Corollary 7

Here we show a stronger result: that if expected relative capital gains are at least as great for unconstrained buyers

$$\frac{dP^c_{t+1}}{P^c_t} \leq \frac{dP^u_{t+1}}{P^u_t}$$

Then the price response of unconstrained buyers is relatively greater:

$$0 < \frac{dP^c_t}{P^c_t} < \frac{dP^u_t}{P^u_t}$$

Proof. To implement this mathematically, we suppose that P^c_{t+1} is a function of P^u_{t+1}. The expectations assumption can then be implemented as

$$\frac{dP^c_{t+1}}{dP^u_{t+1}} \leq \frac{P^c_t}{P^u_t}$$

Then, using Proposition 6

$$\frac{dP^c_t}{dP^u_{t+1}} = \frac{dP^c_t}{dP^c_{t+1}}\frac{dP^c_{t+1}}{dP^u_{t+1}} \leq \left(\frac{dP^c_t}{dP^c_{t+1}}\right)\frac{P^c_t}{P^u_t} < \left(\frac{dP^u_t}{dP^u_{t+1}}\right)\frac{P^c_t}{P^u_t}$$

Thus

$$\frac{dP^c_t}{dP^u_{t+1}}\frac{1}{P^c_t} < \frac{dP^u_t}{dP^u_{t+1}}\frac{1}{P^u_t}$$

This completes the proof of the proposition. ■

1.A.6 Constrained vs Unconstrained Response to Changes in Interest Rate When Constrained by Max LTV

Here we prove Proposition 8

We restate the proposition. Let $P_t^{c,*}$ be the price when the agent buying a house is constrained and $P_t^{u,*}$ be the price when the agent buying the house is unconstrained. Then

$$0 > \left(\frac{dP_t^{c,*}}{dr_t}\right)\frac{1}{P_t^{c,*}} > \left(\frac{dP_t^{u,*}}{dr_t}\right)\frac{1}{P_t^{u,*}}$$

The proof is highly involved so we proceed via a number of lemmas. We first define the following functions.

For the constrained agent, let

$$g^c(P_t^{c,*}(r_t), r_t) := u(C_{1,t}^{L,c}) + \beta u(C_{1,t}^{L,c}) + u_L + \beta u_R - \big(u(C_{0,t}^R) + \beta u(C_{1,t}^R) + (1+\beta)u_R\big)$$

And for the unconstrained agent let

$$g^u(P_t^{u,*}(r_t), r_t) := u(C_{0,t}^{L,u}) + \beta u(C_{1,t}^{L,u}) + u_L + \beta u_R - \big(u(C_{0,t}^R) + \beta u(C_{1,t}^R) + (1+\beta)u_R\big)$$

In g^c when the agent is buying $b_{0,t} \equiv 0$ so the agent cannot borrow or save. In g^u, $b_{0,t}$ can take any value when the agent buys, thus they are unconstrained.

Then we have

$$\begin{aligned}
\left(\frac{dP_t^{c,*}}{dr_t}\right)\frac{1}{P_t^{c,*}} &= \frac{-\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial r_t}}{\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial P_t^{c,*}}P_t^{c,*}} \\
\left(\frac{dP_t^{u,*}}{dr_t}\right)\frac{1}{P_t^{u,*}} &= \frac{-\frac{\partial g^u(P_t^{u,*}(r_t),r_t)}{\partial r_t}}{\frac{\partial g^u(P_t^{u,*}(r_t),r_t)}{\partial P_t^{u,*}}P_t^{u,*}}
\end{aligned}$$

Lemma 23 *When $b_{0,t}^* = 0$ in the 'constrained' model then*

$$\left(\frac{dP_t^{c,*}}{dr_t}\right)\frac{1}{P_t^{c,*}} = \left(\frac{dP_t^{u,*}}{dr_t}\right)\frac{1}{P_t^{u,*}}$$

Proof. When $b_{0,t}^* = 0$ the agent is just unconstrained. Thus the constrained and unconstrained buyers would be solving an identical problem (as they have the same parameters) so $P_t^{c,*} \equiv P_t^{u,*}$. Thus

$$\frac{dP_t^{c,*}}{dr_t} = \frac{dP_t^{u,*}}{dr_t}$$

This completes the proof of the lemma. ■

Lemma 24 *In the constrained model with desired $b_0 \leq 0$ (i.e. the agent wishes to borrow unsecured* *but cannot):*

$$\frac{d\left[u'(C^L_{0,t}) - \beta u'(C^L_{1,t})(1+r_t)\right]}{d\gamma} > 0$$

This says that as γ is increased, the agent becomes more constrained. In particular, it follows that if the agent is constrained and γ is increased, then the agent will still be constrained.

Proof.

$$\begin{aligned}
&\frac{d\left[u'(C^L_{0,t}) - \beta u'(C^L_{1,t})(1+r_t)\right]}{d\gamma} \\
= \quad & u''(C^L_{0,t})\frac{dC^L_{0,t}}{d\gamma} - \beta(1+r_t)u''(C^L_{1,t})\frac{dC^L_{1,t}}{d\gamma}
\end{aligned}$$

As $b_{0^*} \leq 0$,

$$\begin{aligned}
C^L_{0,t} &= y_{0\,t} - \gamma P^{c,*}_t \\
C^L_{1,t} &= y_{1\,t} - R_{t+1} + P^{c,*}_t\left(1-(1+r_t)(1-\gamma)\right)
\end{aligned}$$

So

$$\begin{aligned}
\frac{dC^L_{0,t}}{d\gamma} &= -\left(P^{c,*}_t + \gamma\frac{dP^{c,*}_t}{d\gamma}\right) \\
\frac{dC^L_{1,t}}{d\gamma} &= \frac{dP^{c,*}_t}{d\gamma}\left(1-(1+r_t)(1-\gamma)\right) + P^{c,*}_t(1+r_t)
\end{aligned}$$

And so

$$\begin{aligned}
&\frac{d\left[u'(C^L_{0,t}) - \beta u'(C^L_{1,t})(1+r_t)\right]}{d\gamma} \qquad (1.15)\\
= \quad & -u''(C^L_{0,t})\left(P^{c,*}_t + \gamma\frac{dP^{c,*}_t}{d\gamma}\right) - \beta(1+r_t)u''(C^L_{1,t})\left(\frac{dP^{c,*}_t}{d\gamma}\left(1-(1+r_t)(1-\gamma)\right) + P^{c,*}_t(1+r_t)\right)
\end{aligned}$$

We show that

$$P^{c,*}_t + \gamma\frac{dP^{c,*}_t}{d\gamma} > 0$$

To do this, from (1.9) we have that

$$\frac{dP^{c,*}_t}{d\gamma} = \frac{-P^{c,*}_t\left(u'(C^L_{0,t}) - \beta(1+r_t)u'(C^L_{1,t})\right)}{u'(C^L_{0,t})\gamma - \beta u'(C^L_{1,t})\left(1-(1+r_t)(1-\gamma)\right)}$$

with the denominator positive and term in brackets on the numerator non-negative.

Thus

$$\begin{aligned}
& P_t^{c,*} + \gamma \frac{dP_t^{c,*}}{d\gamma} \qquad (1.16) \\
= \quad & \frac{P_t^{c,*}\left[u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)\right] - \gamma P_t^{c,*}\left(u'(C_{0,t}^L) - \beta(1+r_t)u'(C_{1,t}^L)\right)}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
= \quad & \frac{P_t^{c,*}\left[-\beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right) + \gamma\beta(1+r_t)u'(C_{1,t}^L)\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
= \quad & \frac{P_t^{c,*}\beta u'(C_{1,t}^L)\left[(1+r_t)\gamma - (1-(1+r_t)(1-\gamma))\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
= \quad & \frac{P_t^{c,*}\beta u'(C_{1,t}^L)\left[(1+r_t)(\gamma+1-\gamma) - 1\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
= \quad & \frac{P_t^{c,*}\beta u'(C_{1,t}^L)r_t}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} > 0
\end{aligned}$$

Incidentally, this establishes that

$$0 < -\frac{dP^{c,*}}{d\gamma}\frac{\gamma}{P^{c,*}} < 1 \qquad (1.17)$$

Turning to the other term,

$$\begin{aligned}
& \frac{dP^{c,*}}{d\gamma}\left(1-(1+r_t)(1-\gamma)\right) + P^{c,*}(1+r) \qquad (1.18) \\
= \quad & \frac{\begin{array}{c}-\left(1-(1+r_t)(1-\gamma)\right)P_t^{c,*}\left[u'(C_{0,t}^L) - \beta(1+r_t)u'(C_{1,t}^L)\right] \\ +P^{c,*}(1+r_t)\left[u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)\right]\end{array}}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
= \quad & \frac{-\left(1-(1+r_t)(1-\gamma)\right)P_t^{c,*}u'(C_{0,t}^L) + P^{c,*}(1+r_t)u'(C_{0,t}^L)\gamma}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
= \quad & \frac{P_t^{c,*}u'(C_{0,t}^L)\left[(1+r_t)\gamma - (1-(1+r_t)(1-\gamma))\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \qquad (1.19) \\
= \quad & \frac{P_t^{c,*}u'(C_{0,t}^L)\left[(1+r_t)(\gamma+1-\gamma) - 1\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \qquad (1.20) \\
= \quad & \frac{P_t^{c,*}u'(C_{0,t}^L)r_t}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \qquad (1.21)
\end{aligned}$$

Thus $\frac{dP_t^{c,*}}{d\gamma}\left(1-(1+r_t)(1-\gamma)\right) + P_t^{c,*}(1+r_t) > 0$
and so from (1.15) using $u''(.) < 0$,

$$\frac{d\left[u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)(1+r_t)\right]}{d\gamma} > 0$$

This completes the proof of the lemma. ■

Lemma 25

$$\begin{aligned}
\frac{\partial g^c(P_t^{c,*}(r_t), r_t)}{\partial r_t} &= -\beta P_t^{c,*}(1-\gamma)u'(C_{1,t}^L) - T(r_t) \\
\frac{d\left[\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial r_t}\right]}{d\gamma} &= \frac{\beta P_t^{c,*}\left[-(1-\gamma)P_t^{c,*}u''(C_{1,t}^L)u'(C_{0,t}^L)r_t + u'(C_{1,t}^L)\left[u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)\right]\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)}
\end{aligned}$$

Where

$$T(r_t) := u'(C_{0,t}^R)\frac{\partial C_{0,t}^R}{\partial r_t} + \beta u'(C_{1,t}^R)\frac{\partial C_{1,t}^R}{\partial r_t}$$

Proof.

$$\frac{\partial g^u(P^{u,*}(r_t), r_t)}{\partial r_t} = u'(C_{0,t}^L)\frac{\partial C_{0,t}^L}{\partial r_t} + \beta u'(C_{1,t}^L)\frac{\partial C_{1,t}^L}{\partial r_t} - T(r_t)$$

As the agent is constrained when buying

$$\begin{aligned}
C_{0,t}^L &= y_{0\,t} - \gamma P_t^{c,*} \\
C_{1,t}^L &= y_{1\,t} - R_{t+1} + P^{c,*}\left(1-(1+r_t)(1-\gamma)\right)
\end{aligned}$$

And so

$$\begin{aligned}
\frac{\partial C_{0,t}^L}{\partial r_t} &= 0 \\
\frac{\partial C_{1,t}^L}{\partial r_t} &= -P_t^{c,*}(1-\gamma)
\end{aligned}$$

Thus

$$\frac{\partial g^c(P_t^{c,*}(r_t), r_t)}{\partial r_t} = -\beta P_t^{c,*}(1-\gamma)u'(C_{1,t}^L) - T(r_t)$$

This completes the first part of the lemma.

Turning to the second part:

$$\begin{aligned}
&\frac{d\left[\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial r_t}\right]}{d\gamma} \\
&= -\beta P_t^{c,*}(1-\gamma)u''(C_{1,t}^L)\frac{dC_{1,t}^L}{d\gamma} + \beta u'(C_{1,t}^L)\left[P_t^{c,*} - \frac{dP^{c,*}}{d\gamma}(1-\gamma)\right] \\
&= -\beta P_t^{c,*}(1-\gamma)u''(C_{1,t}^L)\left[\frac{dP_t^{c,*}}{d\gamma}\left(1-(1+r_t)(1-\gamma)\right) + P_t^{c,*}(1+r_t)\right] \\
&\quad + \beta u'(C_{1,t}^L)\left[P_t^{c,*} - \frac{dP_t^{c,*}}{d\gamma}(1-\gamma)\right]
\end{aligned}$$

We compute the terms in the square brackets.

Recall

$$\frac{dP_t^{c,*}}{d\gamma} = \frac{-P_t^{c,*}\left(u'(C_{0,t}^L) - \beta(1+r_t)u'(C_{1,t}^L)\right)}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)}$$

Hence

$$
\begin{aligned}
& P_t^{c,*} - \frac{dP^{c,*}}{d\gamma}(1-\gamma) \\
= \quad & \frac{\left[u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)\right] P_t^{c,*} + (1-\gamma)P_t^{c,*}\left(u'(C_{0,t}^L) - \beta(1+r_t)u'(C_{1,t}^L)\right)}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
= \quad & \frac{P_t^{c,*}u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right) P_t^{c,*} - (1-\gamma)P_t^{c,*}\beta(1+r_t)u'(C_{1,t}^L)}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
& \frac{P_t^{c,*}u'(C_{0,t}^L) - P_t^{c,*}\beta u'(C_{1,t}^L)\left[(1+r_t)(1-\gamma) + (1-(1+r_t)(1-\gamma))\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
= \quad & \frac{P_t^{c,*}u'(C_{0,t}^L) - P_t^{c,*}\beta u'(C_{1,t}^L)}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
= \quad & \frac{P_t^{c,*}\left[u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} > 0
\end{aligned}
$$

Now from (1.18)

$$
\frac{dP^{c,*}}{d\gamma}\left(1-(1+r_t)(1-\gamma)\right) + P^{c,*}(1+r_t) = \frac{P_t^{c,*}u'(C_{0,t}^L)r_t}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)}
$$

Thus

$$
\begin{aligned}
& \frac{d\left[\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial r_t}\right]}{d\gamma} \\
= \quad & \frac{-\beta P_t^{c,*}(1-\gamma)u''(C_{1,t}^L)P_t^{c,*}u'(C_{0,t}^L)r_t}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
& + \frac{\beta u'(C_{1,t}^L)P_t^{c,*}\left[u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
& \frac{\beta P_t^{c,*}\left[-(1-\gamma)P_t^{c,*}u''(C_{1,t}^L)u'(C_{0,t}^L)r_t + u'(C_{1,t}^L)\left[u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)\right]\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
> \quad & 0
\end{aligned}
$$

This completes the proof of the lemma. ∎

Lemma 26 $T(r_t) \geq 0$ *Further, if the agent is unconstrained when renting then*

$$
T(r_t) = \beta u'(C_{1,t}^R)(y_{0,t} - R_t - C_{0,t}^R)
$$

Proof.

$$
T(r_t) := u'(C_{0,t}^R)\frac{\partial C_{0,t}^R}{\partial r_t} + \beta u'(C_{1,t}^R)\frac{\partial C_{1,t}^R}{\partial r_t}
$$

(i) Suppose the rate an agent can borrow at does not move with the mortgage interest rate. In this case, a change in the mortgage rate has no impact on the utility of the agent who rents, thus

$T(r_t) = 0$.

(ii) Suppose the rates move together and the agent is constrained when renting.

Then

$$\begin{aligned} C^R_{0,t} &= y_{0,t} - R_t \\ C^R_{1,t} &= y_{1,t} - R_{t+1} \end{aligned}$$

And so

$$\begin{aligned} \frac{\partial C^R_{0,t}}{\partial r_t} &= 0 \\ \frac{\partial C^R_{1,t}}{\partial r_t} &= 0 \end{aligned}$$

Thus $T(r_t) = 0$.

(iii) Suppose the rates move together and the agent is unconstrained when renting. In this case a change in r_t will have an impact on the utility of an agent who rents. As the agent is unconstrained, $u'(C^R_{0,t}) = (1+r_t)\beta u'(C^R_{1,t})$ and $b^*_{0,t} \geq 0$

Thus

$$\begin{aligned} T(r_t) &= u'(C^R_{0,t})\frac{\partial C^R_{0,t}}{\partial r_t} + \beta u'(C^R_{1,t})\frac{\partial C^R_{1,t}}{\partial r_t} \\ &= (1+r_t)\beta u'(C^R_{1,t})\frac{\partial C^R_{0,t}}{\partial r_t} + \beta u'(C^R_{1,t})\frac{\partial C^R_{1,t}}{\partial r_t} \\ &= \beta u'(C^R_{1,t})\left[(1+r_t)\frac{\partial C^R_{0,t}}{\partial r_t} + \frac{\partial C^R_{1,t}}{\partial r_t}\right] \end{aligned}$$

Now, as the agent is unconstrained, it follows that

$$C^R_{0,t} + \frac{C^R_{1,t}}{1+r_t} = y_{0,t} - R_t + \frac{y_{1,t} - R_{t+1}}{1+r_t}$$

And so

$$(1+r_t)C^R_{0,t} + C^R_{1,t} \equiv (1+r_t)\left(y_{0,t} - R_t\right) + y_{1,t} - R_{t+1}$$

Differentiate both sides wrt r:

$$C^R_{0,t} + (1+r_t)\frac{\partial C^R_{0,t}}{\partial r_t} + \frac{\partial C^R_{1,t}}{\partial r_t} = (y_{0,t} - R_t)$$

Thus

$$(1+r_t)\frac{\partial C^R_{0,t}}{\partial r_t} + \frac{\partial C^R_{1,t}}{\partial r_t} = (y_{0,t} - R_t) - C^R_{0,t}$$

Thus

$$T(r_t) = \beta u'(C^R_{1,t})\left[(y_{0\,t} - R_t) - C^R_{0,t}\right]$$

Now $(y_{0,t}-R_t)-C^R_{0,t}=b^*_{0,t}$ and as the agent is unconstrained they are a saver, so $(y_{0,t}-R_t)-C^R_{0,t}\geq 0$. Thus $T(r_t)\geq 0$

This completes the proof of the lemma. ■

Lemma 27

$$
\begin{aligned}
-\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial P_t^{c,*}}P_t^{c,*} &= u'(C^L_{0,t})\gamma P_t^{c,*}(r_t)-\beta u'(C^L_{1,t})\left(1-(1+r_t)(1-\gamma)\right)P_t^{c,*}(r_t) \\
\frac{d\left[-\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial P_t^{c,*}}P_t^{c,*}\right]}{d\gamma} &= \frac{P_t^{c,*}(\gamma)P_t^{c,*}(\gamma)\beta r_t}{u'(C^L_{0,t})\gamma-\beta u'(C^L_{1,t})\left(1-(1+r_t)(1-\gamma)\right)} \\
&\quad\cdot\left[-u''(C^L_{0,t})u'(C^L_{1,t})\gamma-u''(C^L_{1,t})u'(C^L_{0,t})\left(1-(1+r_t)(1-\gamma)\right)\right]
\end{aligned}
$$

Proof. When the agent is constrained

$$
\begin{aligned}
C^L_{0,t} &= y_{0\,t}-\gamma P_t^{c,*} \\
C^L_{1,t} &= y_{1\,t}-R_{t+1}+P_t^{c,*}\left(1-(1+r_t)(1-\gamma)\right)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{\partial C^L_{0,t}}{\partial P_t^{c,*}} &= -\gamma \\
\frac{\partial C^L_{1,t}}{\partial P_t^{c,*}} &= \left(1-(1+r_t)(1-\gamma)\right)
\end{aligned}
$$

Now

$$
\begin{aligned}
-\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial P_t^{c,*}}P_t^{c,*} &= -u'(C^L_{0,t})\frac{\partial C^L_{0,t}}{\partial P_t^{c,*}}P_t^{c,*}(r_t)-\beta u'(C^L_{1,t})\frac{\partial C^L_{1,t}}{\partial P_t^{c,*}}P_t^{c,*}(r_t) \\
&= u'(C^L_{0,t})\gamma P_t^{c,*}(r_t)-\beta u'(C^L_{1,t})\left(1-(1+r_t)(1-\gamma)\right)P_t^{c,*}(r_t)
\end{aligned}
$$

We now turn to the second term.

To ease notation, let

$$
\begin{aligned}
H(\gamma) &: = u'(y_{0,t}-h(\gamma))h(\gamma) \\
h(\gamma) &: = \gamma P^{c,*}(\gamma)
\end{aligned}
$$

And

$$
\begin{aligned}
F(\gamma) &: = \beta u'(y_{1,t}-R_{t+1}+f(\gamma))f(\gamma) \\
f(\gamma) &: = \left(1-(1+r_t)(1-\gamma)\right)P_t^{c,*}(\gamma)
\end{aligned}
$$

Then

$$-\frac{\partial g^c(P_t^{c,*}(r_t), r_t)}{\partial P_t^{c,*}} P_t^{c,*} = H(\gamma) - F(\gamma)$$

And

$$\frac{d\left[-\frac{\partial g^c(P_t^{c,*}(r_t), r_t)}{\partial P_t^{c,*}} P_t^{c,*}\right]}{d\gamma} = H'(\gamma) - F'(\gamma)$$

We calculate these in turn.

$$\begin{aligned} H'(\gamma) &= -u''(y_{0,t} - h(\gamma))h'(\gamma)h(\gamma) + u'(y_{0,t} - h(\gamma))h'(\gamma) \\ &= h'(\gamma)\left[-u''(y_{0,t} - h(\gamma))h(\gamma) + u'(y_{0,t} - h(\gamma))\right] \end{aligned}$$

Using (1.16)

$$\begin{aligned} h'(\gamma) &= P_t^{c,*} + \gamma \frac{dP_t^{c,*}}{d\gamma} \\ &= \frac{P_t^{c,*} \beta u'(C_{1,t}^L) r_t}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1 - (1 + r_t)(1 - \gamma)\right)} > 0 \end{aligned}$$

We compute an analogous expression for $F'(\gamma)$:

$$\begin{aligned} F'(\gamma) &= \beta u''(y_{1,t} - R_{t+1} + f(\gamma))f'(\gamma)f(\gamma) + \beta u'(y_{1,t} - R_{t+1} + f(\gamma))f'(\gamma) \\ &= \beta f'(\gamma)\left[u''(y_{1,t} - R_{t+1} + f(\gamma))f(\gamma) + u'(y_{1,t} - R_{t+1} + f(\gamma))\right] \end{aligned}$$

And

$$f'(\gamma) = (1 + r_t)P_t^{c,*}(\gamma) + \left(1 - (1 + r_t)(1 - \gamma)\right)\frac{P_t^{c,*}(\gamma)}{d\gamma}$$

Now from (1.18)

$$\frac{dP^{c,*}}{d\gamma}\left(1 - (1 + r_t)(1 - \gamma)\right) + P^{c,*}(1 + r_t) = \frac{P_t^{c,*} u'(C_{0,t}^L) r_t}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1 - (1 + r_t)(1 - \gamma)\right)}$$

Thus

$$f'(\gamma) = \frac{P_t^{c,*} u'(C_{0,t}^L) r_t}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1 - (1 + r_t)(1 - \gamma)\right)} > 0$$

Thus, combining results

$$\begin{aligned}
& H'(\gamma) - F'(\gamma) \\
= \quad & h'(\gamma)\left[-u''(y_{0,t} - h(\gamma))h(\gamma) + u'(y_{0,t} - h(\gamma))\right] \\
& -\beta f'(\gamma)\left[u''(y_{1,t} - R_{t+1} + f(\gamma))f(\gamma) + u'(y_{1,t} - R_{t+1} + f(\gamma))\right] \\
= \quad & \frac{P_t^{c,*}\beta u'(C_{1,t}^L)r_t\left[-u''(C_{0,t}^L)\gamma P_t^{c,*}(\gamma) + u'(C_{0,t}^L)\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
& -\frac{\beta P_t^{c,*}u'(C_{0,t}^L)r_t\left[u''(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)P_t^{c,*}(\gamma) + u'(C_{1,t}^L)\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
= \quad & \frac{P_t^{c,*}(\gamma)\beta r_t\left[\begin{array}{c} -u''(C_{0,t}^L)u'(C_{1,t}^L)\gamma P_t^{c,*}(\gamma) + u'(C_{1,t}^L)u'(C_{0,t}^L) \\ -u''(C_{1,t}^L)u'(C_{0,t}^L)\left(1-(1+r_t)(1-\gamma)\right)P_t^{c,*}(\gamma) - u'(C_{0,t}^L)u'(C_{1,t}^L) \end{array}\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
= \quad & \frac{P_t^{c,*}(\gamma)\beta r_t\left[-u''(C_{0,t}^L)u'(C_{1,t}^L)\gamma P_t^{c,*}(\gamma) - u''(C_{1,t}^L)u'(C_{0,t}^L)\left(1-(1+r_t)(1-\gamma)\right)P_t^{c,*}(\gamma)\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
= \quad & \frac{P_t^{c,*}(\gamma)P_t^{c,*}(\gamma)\beta r_t\left[-u''(C_{0,t}^L)u'(C_{1,t}^L)\gamma - u''(C_{1,t}^L)u'(C_{0,t}^L)\left(1-(1+r_t)(1-\gamma)\right)\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)}
\end{aligned}$$

This completes the proof of the lemma. ■

Lemma 28

$$\frac{d\left[\frac{-\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial r_t}}{\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial P_t^{c,*}}P_t^{c,*}}\right]}{d\gamma} > 0$$

Proof. Let

$$\begin{aligned}
W(\gamma) \quad &: \quad = \frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial r_t} \\
V(\gamma) \quad &: \quad = -\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial P_t^{c,*}}P_t^{c,*}
\end{aligned}$$

Then

$$\begin{aligned}
\frac{d\left[\frac{-\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial r_t}}{\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial P_t^{c,*}}P_t^{c,*}}\right]}{d\gamma} \quad &= \quad \left(\frac{W(\gamma)}{V(\gamma)}\right)' \\
&= \quad \frac{W'(\gamma)V(\gamma) - W(\gamma)V'(\gamma)}{V(\gamma)^2}
\end{aligned}$$

Thus

$$\frac{d\left[\frac{-\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial r_t}}{\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial P_t^{c,*}}P_t^{c,*}}\right]}{d\gamma} > 0 \text{ iff } W'(\gamma)V(\gamma) > W(\gamma)V'(\gamma)$$

We show this.

From Lemmas 25,27 we have

$$\begin{aligned}
W(\gamma) &= -\beta P_t^{c,*}(1-\gamma)u'(C_{1,t}^{L,c}) - T(r_t) \\
W'(\gamma) &= \frac{\beta P_t^{c,*}\left[-(1-\gamma)P_t^{c,*}u''(C_{1,t}^L)u'(C_{0,t}^L)r_t + u'(C_{1,t}^L)\left[u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)\right]\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
V(\gamma) &= u'(C_{0,t}^L)\gamma P_t^{c,*} - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)P_t^{c,*} \\
V'(\gamma) &= \frac{P_t^{c,*}P_t^{c,*}\beta r_t\left[-u''(C_{0,t}^L)u'(C_{1,t}^L)\gamma - u''(C_{1,t}^L)u'(C_{0,t}^L)\left(1-(1+r_t)(1-\gamma)\right)\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)}
\end{aligned}$$

Thus $W'(\gamma)V(\gamma) > W(\gamma)V'(\gamma)$ iff

$$\begin{aligned}
& \frac{\beta P_t^{c,*}\left[-(1-\gamma)P_t^{c,*}u''(C_{1,t}^L)u'(C_{0,t}^L)r_t + u'(C_{1,t}^L)\left[u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)\right]\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)} \\
& \left[u'(C_{0,t}^L)\gamma P_t^{c,*}(r_t) - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)P_t^{c,*}(r_t)\right] \\
> & \left[-\beta P_t^{c,*}(1-\gamma)u'(C_{1,t}^{L,c}) - T(r_t)\right] \\
& \frac{P_t^{c,*}(\gamma)P_t^{c,*}(\gamma)\beta r_t\left[-u''(C_{0,t}^L)u'(C_{1,t}^L)\gamma - u''(C_{1,t}^L)u'(C_{0,t}^L)\left(1-(1+r_t)(1-\gamma)\right)\right]}{u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)}
\end{aligned}$$

Iff

$$\begin{aligned}
& \left[-(1-\gamma)P_t^{c,*}u''(C_{1,t}^L)u'(C_{0,t}^L)r_t + u'(C_{1,t}^L)\left[u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)\right]\right] \\
& \left[u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)\right] \\
> & \left[-\beta P_t^{c,*}(1-\gamma)u'(C_{1,t}^L) - T(r_t)\right] \\
& r_t\left[-u''(C_{0,t}^L)u'(C_{1,t}^L)\gamma - u''(C_{1,t}^L)u'(C_{0,t}^L)\left(1-(1+r_t)(1-\gamma)\right)\right]
\end{aligned}$$

We move all the terms in $u''(C_{1,t}^L)$ to the LHS of the inequality. Collecting these:

$$\begin{aligned}
& -(1-\gamma)P_t^{c,*}u''(C_{1,t}^L)u'(C_{0,t}^L)r_t\left[u'(C_{0,t}^L)\gamma - \beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right)\right] \\
& +u''(C_{1,t}^L)u'(C_{0,t}^L)\left(1-(1+r_t)(1-\gamma)\right)r_t\left[-\beta P_t^{c,*}(1-\gamma)u'(C_{1,t}^L) - T(r_t)\right] \\
= & -(1-\gamma)P_t^{c,*}u''(C_{1,t}^L)u'(C_{0,t}^L)r_t u'(C_{0,t}^L)\gamma \\
& +(1-\gamma)P_t^{c,*}u''(C_{1,t}^L)u'(C_{0,t}^L)r_t\beta u'(C_{1,t}^L)\left(1-(1+r_t)(1-\gamma)\right) \\
& -u''(C_{1,t}^L)u'(C_{0,t}^L)\left(1-(1+r_t)(1-\gamma)\right)r_t\beta P_t^{c,*}(1-\gamma)u'(C_{1,t}^L) \\
& -T(r_t)u''(C_{1,t}^L)u'(C_{0,t}^L)\left(1-(1+r_t)(1-\gamma)\right)r_t \\
= & -(1-\gamma)P_t^{c,*}u''(C_{1,t}^L)u'(C_{0,t}^L)r_t u'(C_{0,t}^L)\gamma - T(r_t)u''(C_{1,t}^L)u'(C_{0,t}^L)\left(1-(1+r_t)(1-\gamma)\right)r_t \\
= & -u''(C_{1,t}^L)u'(C_{0,t}^L)r_t\left[u'(C_{0,t}^L)\gamma P_t^{c,*}(1-\gamma) + T(r_t)\left(1-(1+r_t)(1-\gamma)\right)\right]
\end{aligned}$$

Thus $W'(\gamma)V(\gamma) > W(\gamma)V'(\gamma)$ iff

$$\begin{aligned}
& u'(C^L_{1,t})\left[u'(C^L_{0,t}) - \beta u'(C^L_{1,t})\right]\left[u'(C^L_{0,t})\gamma - \beta u'(C^L_{1,t})\left(1-(1+r_t)(1-\gamma)\right)\right] \\
& -u''(C^L_{1,t})u'(C^L_{0,t})r_t\left[u'(C^L_{0,t})\gamma P^{c,*}_t(1-\gamma) + T(r_t)\left(1-(1+r_t)(1-\gamma)\right)\right] \\
> & \left[-\beta P^{c,*}_t(1-\gamma)u'(C^L_{1,t}) - T(r_t)\right]\left(-r_t u''(C^L_{0,t})u'(C^L_{1,t})\gamma\right)
\end{aligned}$$

Now $u' > 0$, $u'' < 0$ and $T(r_t) \geq 0$ and as the agent is constrained when buying

$$\begin{aligned}
u'(C^L_{0,t}) - \beta u'(C^L_{1,t}) &> 0 \\
u'(C^L_{0,t})\gamma - \beta u'(C^L_{1,t})\left(1-(1+r_t)(1-\gamma)\right) &> 0
\end{aligned}$$

Hence

$$\begin{aligned}
u'(C^L_{1,t})\left[u'(C^L_{0,t}) - \beta u'(C^L_{1,t})\right]\left[u'(C^L_{0,t})\gamma - \beta u'(C^L_{1,t})\left(1-(1+r_t)(1-\gamma)\right)\right] &> 0 \\
\left[-\beta P^{c,*}_t(1-\gamma)u'(C^L_{1,t}) - T(r_t)\right]\left(-r_t u''(C^L_{0,t})u'(C^L_{1,t})\gamma\right) &< 0
\end{aligned}$$

Thus, a sufficient condition for $W'(\gamma)V(\gamma) > W(\gamma)V'(\gamma)$ holding is that

$$\begin{aligned}
-u''(C^L_{1,t})u'(C^L_{0,t})r_t\left[u'(C^L_{0,t})\gamma P^{c,*}_t(1-\gamma) + T(r_t)\left(1-(1+r_t)(1-\gamma)\right)\right] &> 0 \text{ iff} \\
u'(C^L_{0,t})\gamma P^{c,*}_t(1-\gamma) + T(r_t)\left(1-(1+r_t)(1-\gamma)\right) &> 0 \text{ iff} \\
u'(C^L_{0,t})\gamma P^{c,*}_t(1-\gamma) + T(r_t) &> T(r_t)(1+r_t)(1-\gamma)
\end{aligned}$$

We now show this.

If the agent is constrained when renting, or the interest rate they can borrow at does not move with the mortgage rate, then $T(r_t) = 0$ and the condition holds.

Otherwise,

$$T(r_t) = \beta u'(C^R_{1,t})\left[(y_{0,t} - R_t) - C^R_{0,t}\right] \geq 0$$

Given $T(r_t) \geq 0$ it is sufficient to show that

$$u'(C^L_{0,t})\gamma P^{c,*}_t(1-\gamma) > \beta(1+r_t)u'(C^R_{1,t})\left[(y_{0,t} - R_t) - C^R_{0,t}\right](1-\gamma) \tag{1.22}$$

As the agent is unconstrained when renting,

$$u'(C^R_{0,t}) = \beta(1+r_t)u'(C^R_{1,t})$$

Hence it is sufficient to show that

$$u'(C^L_{0,t})\gamma P^{c,*}_t > u'(C^R_{0,t})\left[(y_{0,t} - R_t) - C^R_{0,t}\right]$$

We first show that $u'(C^L_{0,t}) > u'(C^R_{0,t})$

To see this, suppose it didn't hold. Then $u'(C^R_{0,t}) \geq u'(C^L_{0,t})$ so $C^R_{0,t} \leq C^L_{0,t}$ Further as the agent is constrained when buying,

$$\beta(1+r_t)u'(C^R_{1,t}) = u'(C^R_{0,t}) \geq u'(C^L_{0,t}) > \beta(1+r_t)u'(C^L_{1,t})$$

So $u'(C^R_{1,t}) > u'(C^L_{1,t})$ giving $C^R_{1,t} < C^L_{1,t}$.

Thus

$$u(C^R_{0,t}) + \beta u(C^R_{1,t}) < u(C^L_{0,t}) + \beta u(C^L_{1,t}) \leq u(C^L_{0,t}) + \beta u(C^L_{1,t}) + u_L - u_R$$

This is a contradiction, as then the market is not in equilibrium as everyone prefers to buy a house than to rent.

Thus we must have

$$\begin{aligned} u'(C^L_{0,t}) &> u'(C^R_{0,t}) \\ C^L_{0,t} &< C^R_{0,t} \end{aligned}$$

Now

$$\begin{aligned} \gamma P^{c,*}_t &> (y_{0,t} - R_t) - C^R_{0,t} \text{ iff} \\ C^R_{0,t} + R_t &> y_{0,t} - \gamma P^{c,*}_t \end{aligned}$$

But this holds as

$$C^L_{0,t} = y_{0,t} - \gamma P^{c,*}_t$$

Hence

$$u'(C^L_{0,t})\gamma P^{c,*}_t > u'(C^R_{0,t})\gamma P^{c,*}_t > u'(C^R_{0,t})\left[(y_{0,t} - R_t) - C^R_{0,t}\right]$$

We've thus shown (1.22), completing the proof of the lemma. ■

Lemma 29

$$0 > \left(\frac{dP^{c,*}_t}{dr_t}\right)\frac{1}{P^{c,*}_t}$$

Proof.

$$\left(\frac{dP^{c,*}_t}{dr_t}\right)\frac{1}{P^{c,*}_t} = \frac{-\frac{\partial g^c(P^{c,*}_t(r_t), r_t)}{\partial r_t} r_t}{\frac{\partial g^c(P^{c,*}_t(r_t), r_t)}{\partial P^{c,*}_t} P^{c,*}_t}$$

From Lemmas 27,25

$$\begin{aligned} \frac{\partial g^c(P^{c,*}_t(r_t), r_t)}{\partial r_t} &= -\beta P^{c,*}_t(1-\gamma)u'(C^{L,c}_{1,t}) - T(r_t) < 0 \\ -\frac{\partial g^c(P^{c,*}_t(r_t), r_t)}{\partial P^{c,*}_t} P^{c,*}_t &= u'(C^L_{0,t})\gamma P^{c,*}_t(r_t) - \beta u'(C^L_{1,t})\left(1-(1+r_t)(1-\gamma)\right)P^{c,*}_t(r_t) > 0 \end{aligned}$$

To see the second inequality, note that as the agent is constrained when buying

$$u'(C^L_{0,t}) > \beta(1+r_t)u'(C^L_{1,t})$$

Thus

$$\begin{aligned} u'(C^L_{0,t})\gamma P^{c,*}_t(r_t) &> \beta(1+r_t)u'(C^L_{1,t})\gamma P^{c,*}(r_t) \\ &> \beta u'(C^L_{1,t})\left(1-(1+r_t)(1-\gamma)\right)P^{c,*}_t(r_t) \end{aligned}$$

This last line follows because

$$\begin{aligned} (1+r_t)\gamma &> (1-(1+r_t)(1-\gamma)) \text{ iff} \\ (1+r_t)(\gamma+1-\gamma) &> 1 \text{ iff} \\ r_t &> 0 \end{aligned}$$

which we assume.

Thus $0 > \left(\frac{dP^{c,*}_t}{dr_t}\right)\frac{1}{P^{c,*}_t}$. This completes the proof of the lemma. ∎

Proof of Proposition 12. From Lemma 23 when the agent is just unconstrained

$$\left(\frac{dP^{c,*}_t}{dr_t}\right)\frac{1}{P^{c,*}_t} = \left(\frac{dP^{u,*}_t}{dr_t}\right)\frac{1}{P^{u,*}_t}$$

Let this correspond to $\overline{\gamma}$ i.e. $b_{0,t}(\overline{\gamma}) = 0$ (i.e. for desired borrowing for the buying agent). From Lemma 24 as γ is increased from this point, the agent becomes constrained when buying and is constrained for all higher γ. From Lemma 28 $\left(\frac{dP^{c,*}_t}{dr_t}\right)\frac{1}{P^{c,*}_t}$ increases as γ increases. Further, $\left(\frac{dP^{u,*}_t}{dr_t}\right)\frac{1}{P^{u,*}_t}$ is clearly unaffected by changes in γ.

Thus for a constrained agent, $\gamma > \overline{\gamma}$ and

$$\left[\left(\frac{dP^{c,*}_t}{dr_t}\right)\frac{1}{P^{c,*}_t}\right]_\gamma > \left[\left(\frac{dP^{c,*}_t}{dr_t}\right)\frac{1}{P^{c,*}_t}\right]_{\gamma=\overline{\gamma}} = \left(\frac{dP^{u,*}_t}{dr_t}\right)\frac{1}{P^{u,*}_t}$$

Finally, from Lemma 29

$$0 > \left(\frac{dP^{c,*}_t}{dr_t}\right)\frac{1}{P^{c,*}_t}$$

This completes the proof of the proposition. ∎

1.B Proofs With Two Types of Houses to Buy

1.B.1 Conditions for Market Equilibrium in Tiered Model

Here we give conditions on the housing market that ensure that equilibrium in the tiered housing model is given by group A agents indifferent between renting and a low tier house, and group B agents indifferent between staying in a low tier house and moving to a high tier house.

Proposition 30 *Let the mass of X be denoted by $|X|$. Suppose*

$$\begin{aligned} |A| &> |L| \\ |L| &> |B| \\ |B| &> |H| > 0 \\ 1-q &= \frac{|B|}{|L| - (|B| - |H|)} \\ |B| &< \frac{|L| + |H|}{2} \end{aligned}$$

Where $1-q$ is the probability of death for group A agents between young and old age. And e.g. $|A|$ is the mass of new young agents born each period in group A. Then equilibrium in the tiered model is given by indifference for group A between renting and buying and for group B between staying in the low tier house and trading up to the high tier house.

Proof. As $|B| > |H|$ and only group B agents can buy high tier houses we must have indifference between staying in L and trading up to H for this group. Otherwise, if they all preferred to buy H, there would be excess demand for H, whilst if they all preferred to stay in L, there would be excess supply of H, neither an equilibrium.

As $|A| > |L|$, in equilibrium we cannot have all group A agents preferring to move to a L house than renting, as then there would be excess demand for L. If on the other hand, they all preferred to rent, the total demand for L would be $|B| - |H|$, given by the group B agents who don't trade up. But $|L| > |B|$ so $|L| > (|B| - |H|)$ so the market for L does not clear and we do not have an equilibrium. Thus, in equilibrium, group A agents must be indifferent between buying L and renting.

We now show that markets can clear with these indifference conditions for both groups A and B holding. Consider the following allocation: in each generation $|H|$ of group B buy a H house, and $|B| - |H|$ stay in the low tier house. Further, $|L| - (|B| - |H|)$ group A agents buy the low tier house, with the remainder renting. Then total demand for $H = |H|$ so the market for H clears. Total demand for $L = (|B| - |H|) + |L| - (|B| - |H|) = |L|$ so the market for L clears. We must also verify that the intergenerational allocation of houses is correct. Fraction $1-q$ of the young group A agents die before old age, and this is independent of whether they bought L or not. Thus, fraction $1-q$ of the low tier houses they hold are transferred to the next cohort of group B. Given our assumptions, the total transferred to group B is given by

$$\begin{aligned} & [|L| - (|B| - |H|)](1-q) \\ = \; & [|L| - (|B| - |H|)] \frac{|B|}{|L| - (|B| - |H|)} \\ = \; & |B| \end{aligned}$$

Thus the intergenerational allocation of houses is correct.

Finally, we must verify that $q \in (0,1)$ so is indeed a probability. As $|L| > |B| > 0$ we clearly have $1-q>0$. Further $1-q<1$ iff $|B| < |L| - (|B| - |H|)$. This holds as $|B| < \frac{|L|+|H|}{2}$. Thus q is a well defined probability. Thus, as markets clear and agents' choices are optimal, we have an equilibrium. This completes the proof. ■

1.B.2 Proof of Proposition 9: High Tier Price Formula With Log Utility

This proposition states that with $u(C_t) = \log(C_t)$ and group B buyers unconstrained both when trading up to a high tier house and staying in their low tier house:

$$P_t^H = \frac{P_{t+1}^H}{1+r_t} + P_t^L - \frac{P_{t+1}^L}{1+r_t} \frac{1}{\exp\left(\frac{u_H - u_L}{1+\beta}\right)} + \left(y_{0,t} + \frac{y_{1,t} - R_{t+1}}{1+r_t}\right)\left(1 - \frac{1}{\exp\left(\frac{u_H - u_L}{1+\beta}\right)}\right)$$

Proof. For the group B agents that don't move, the pdv of lifetime resources they have for non-housing consumption is given by

$$y_{0,t} + \frac{y_{1,t} - R_{t+1} + P_{t+1}^L}{(1+r_t)}$$

This reflects the fact that they sell their house when old, and rent for the last period of life.

For the group B agent that trade up to the high tier house, the pdv of lifetime resources they have for non-housing consumption is given by

$$y_{0,t} + P_t^L - P_t^H + \frac{y_{1,t} + P_{t+1}^H - R_{t+1}}{(1+r_t)}$$

This reflects that they sell a low tier house and buy a high tier house when young, then sell this high tier house when old, renting for the last period of their life.

As with the proof of Example 3, with group B agents unconstrained under both housing scenarios they face, and log utility, in equilibrium we have the following relationship between the pdv's of lifetime utility under the two scenarios:

$$\begin{aligned} & (1+\beta)\log\left(y_{0,t} + P_t^L - P_t^H + \frac{y_{1,t} + P_{t+1}^H - R_{t+1}}{(1+r_t)}\right) + u_H - u_L \\ = \quad & (1+\beta)\log\left(y_{0,t} + \frac{y_{1,t} - R_{t+1} + P_{t+1}^L}{(1+r_t)}\right) \end{aligned}$$

Thus, rearranging and applying the exponential function to both sides

$$\left(y_{0,t} + P_t^L - P_t^H + \frac{y_{1,t} + P_{t+1}^H - R_{t+1}}{(1+r_t)}\right) = \left(y_{0,t} + \frac{y_{1,t} - R_{t+1} + P_{t+1}^L}{(1+r_t)}\right) \frac{1}{\exp\left(\frac{u_H - u_L}{1+\beta}\right)}$$

Rearranging for P_t^H we have

$$P_t^H = \frac{P_{t+1}^H}{1+r_t} + P_t^L - \frac{P_{t+1}^L}{(1+r_t)} \frac{1}{\exp\left(\frac{u_H - u_L}{1+\beta}\right)} + \left(y_{0,t} + \frac{y_{1,t} - R_{t+1}}{(1+r_t)}\right)\left(1 - \frac{1}{\exp\left(\frac{u_H - u_L}{1+\beta}\right)}\right)$$

This completes the proof of the proposition. ■

1.B.3 Low vs High Tier Response to Change in LTV Constraint: Proof of Proposition 10

We restate the proposition. Suppose low tier buyers are constrained when buying and high tier buyers are unconstrained when buying and would be unconstrained if they didn't buy. Further, suppose $u(C) = \log(C)$. Then the relative price change is greater for low tier housing:

$$0 > \left(\frac{dP_t^H}{d\gamma}\right)\frac{1}{P_t^H} > \left(\frac{dP_t^L}{d\gamma}\right)\frac{1}{P_t^L}$$

Proof. Under the given conditions, by Proposition 9

$$P_t^H = \frac{P_{t+1}^H}{1+r_t} + P_t^L - \frac{P_{t+1}^L}{(1+r_t)} \frac{1}{\exp\left(\frac{u_H - u_L}{1+\beta}\right)} + \left(y_{0,t} + \frac{y_{1,t} - R_{t+1}}{(1+r_t)}\right)\left(1 - \frac{1}{\exp\left(\frac{u_H - u_L}{1+\beta}\right)}\right)$$

From our prior analysis we concluded that absent any shocks, P_t^L, P_{t+1}^L will be at their steady state value P^L, thus the pricing equation becomes

$$P_t^H = \frac{P_{t+1}^H}{(1+r_t)} + \left(\frac{(1+r_t)\exp\left(\frac{u_H-u_L}{(1+\beta)}\right) - 1}{(1+r_t)\exp\left(\frac{u_H-u_L}{(1+\beta)}\right)}\right)P^L + \left(y_{0,t} + \frac{y_{1,t} - R_{t+1}}{(1+r_t)}\right)\left(1 - \frac{1}{\exp\left(\frac{u_H-u_L}{(1+\beta)}\right)}\right)$$

Following additional prior analysis, if this system is repeated infinitely into the future (or at least expected to be) it will jump to its steady state value P^H. This will satisfy

$$P^H = \left(\frac{(1+r_t)\exp\left(\frac{u_H-u_L}{(1+\beta)}\right) - 1}{r_t \exp\left(\frac{u_H-u_L}{(1+\beta)}\right)}\right)P^L + \left(\frac{1+r_t}{r_t}\right)\left(y_{0,t} + \frac{y_{1,t} - R_{t+1}}{(1+r_t)}\right)\left(1 - \frac{1}{\exp\left(\frac{u_H-u_L}{(1+\beta)}\right)}\right) \tag{1.23}$$

We can then write

$$P^H = aP^L + b$$

with a, $b > 0$ constants independent of γ.

Then

$$\frac{\left(\frac{dP^H}{d\gamma}\right)}{P^H} = a\frac{\left(\frac{dP^L}{d\gamma}\right)}{P^H} = a\frac{P^L}{P^H}\frac{\left(\frac{dP^L}{d\gamma}\right)}{P^L} < 0$$

Where we've used that as the low tier buyer is constrained, $\frac{dP^L}{d\gamma} < 0$.

Thus

$$\begin{aligned} \frac{\left(\frac{dP^H}{d\gamma}\right)}{P^H} &> \frac{\left(\frac{dP^L}{d\gamma}\right)}{P^L} \text{ iff} \\ a\frac{P^L}{P^H}\frac{\left(\frac{dP^L}{d\gamma}\right)}{P^L} &> \frac{\left(\frac{dP^L}{d\gamma}\right)}{P^L} \text{ iff} \\ a\frac{P^L}{P^H} &< 1 \end{aligned}$$

The last line again uses $\frac{dP^L}{d\gamma} < 0$.

Now

$$a\frac{P^L}{P^H} = \frac{aP^L}{aP^L + b} < 1$$

This completes the proof of the proposition. ■

Remark 31 *This result is not affected by the probability of reaching old age for group A agents, q. All that is required is* $\frac{dP^L}{d\gamma} < 0$ *which holds for all positive discount factors, so for all* $q \in (0, 1)$.

1.B.4 Low vs High Tier Response to Change in Optimism: Proof of Proposition 11

We prove a more general proposition. Suppose the low tier buyers are constrained when buying and high tier buyers are unconstrained when buying and would be unconstrained if they didn't buy. Further, suppose the relative expected price increase for high tier prices *is at least that of low tier prices*:[40]

$$\frac{dP_{t+1}^H}{P_t^H} \geq \frac{dP_{t+1}^L}{P_t^L}$$

Then the relative price increase in greater for the high tier house:

$$0 < \frac{dP_t^L}{P_t^L} < \frac{dP_t^H}{P_t^H}$$

We establish the result in general, for any increasing concave utility function with w.d. derivatives.

To implement the proof mathematically, we consider P_{t+1}^H to be a function of P_{t+1}^L and consider the impact of changes in P_{t+1}^L. We first establish a lemma capturing the impact of these changes.

[40] This clearly incorporates the case covered in the text.

Lemma 32 *Consider P^H_{t+1} as a function of P^L_{t+1}. Then, when group B agents are unconstrained both when moving to H and staying in L :*

$$\frac{dP^H_t}{dP^L_{t+1}} = \frac{dP^L_t}{dP^L_{t+1}} + \frac{1}{1+r_t}\left(\frac{dP^H_{t+1}}{dP^L_{t+1}}\right) - \frac{u'(C^{B,L}_{1,t})}{(1+r_t)u'(C^{B,H}_{1,t})}$$

where $C^{B,i}_{1,t}$ represents the second period consumption for group B agents when they make housing choice $i \in \{L, H\}$.

Remark 33 *Note that we're assuming common expectations: both group A and group B agents expect the same change in P^L_{t+1}.*

Proof. Let

$$g(P^H_t(P^L_{t+1}), P^L_t) := u(C^{B,H}_{0,t}) + \beta u(C^{B,H}_{1,t}) + u_H - u_L - \left(u(C^{B,L}_{0,t}) + \beta u(C^{B,L}_{1,t})\right)$$

Then in equilibrium

$$g(P^H_t(P^L_{t+1}), P^L_{t+1}) \equiv 0$$

And so

$$\begin{aligned}
\frac{\partial g}{\partial P^H_t}\frac{dP^H_t}{dP^L_{t+1}} + \frac{\partial g}{\partial P^L_{t+1}} &= 0 \text{ so} \\
\frac{dP^H_t}{dP^L_{t+1}} &= \frac{-\frac{\partial g}{\partial P^L_{t+1}}}{\frac{\partial g}{\partial P^H_t}}
\end{aligned}$$

We calculate these in turn (using the fact that the group B agents are unconstrained):

$$\begin{aligned}
\frac{\partial g}{\partial P^H_t} &= u'(C^{B,H}_{0,t})\frac{\partial C^{B,H}_{0,t}}{\partial P^H_t} + \beta u'(C^{B,H}_{1,t})\frac{\partial C^{B,H}_{1,t}}{\partial P^H_t} \\
&= (1+r_t)\beta u'(C^{B,H}_{1,t})\frac{\partial C^{B,H}_{0,t}}{\partial P^H_t} + \beta u'(C^{B,H}_{1,t})\frac{\partial C^{B,H}_{1,t}}{\partial P^H_t} \\
&= \beta u'(C^{B,H}_{1,t})\left[(1+r_t)\frac{\partial C^{B,H}_{0,t}}{\partial P^H_t} + \frac{\partial C^{B,H}_{1,t}}{\partial P^H_t}\right]
\end{aligned}$$

Now

$$C^{B,H}_{0,t}(1+r_t) + C^{B,H}_{1,t} = (y_{0,t} + P^L_t)(1+r_t) - (1+r_t)P^H_t + y_{1,t} - R_{t+1} + P^H_{t+1} \tag{1.24}$$

Thus

$$\begin{aligned}(1+r_t)\frac{\partial C_{0,t}^{B,H}}{\partial P_t^H}+\frac{\partial C_{1,t}^{B,H}}{\partial P_t^H} &= -(1+r_t) \text{ and}\\ \frac{\partial g}{\partial P_t^H} &= -(1+r_t)\beta u'(C_{1,t}^{B,H})\end{aligned}$$

Turning to the other term, we have

$$\begin{aligned}\frac{\partial g}{\partial P_{t+1}^L} &= u'(C_{0,t}^{B,H})\frac{\partial C_{0,t}^{B,H}}{\partial P_{t+1}^L}+\beta u'(C_{1,t}^{B,H})\frac{\partial C_{1,t}^{B,H}}{\partial P_{t+1}^L}-\left[u'(C_{0,t}^{B,L})\frac{\partial C_{0,t}^{B,L}}{\partial P_{t+1}^L}+\beta u'(C_{1,t}^{B,L})\frac{\partial C_{1,t}^{B,L}}{\partial P_{t+1}^L}\right]\\ &= \beta u'(C_{1,t}^{B,H})\left[(1+r_t)\frac{\partial C_{0,t}^{B,H}}{\partial P_{t+1}^L}+\frac{\partial C_{1,t}^{B,H}}{\partial P_{t+1}^L}\right]-\beta u'(C_{1,t}^{B,L})\left[(1+r_t)\frac{\partial C_{0,t}^{B,L}}{\partial P_{t+1}^L}+\frac{\partial C_{1,t}^{B,L}}{\partial P_{t+1}^L}\right]\end{aligned}$$

From (1.24)[41]

$$(1+r_t)\frac{\partial C_{0,t}^{B,H}}{\partial P_{t+1}^L}+\frac{\partial C_{1,t}^{B,H}}{\partial P_{t+1}^L}=(1+r_t)\frac{dP_t^L}{dP_{t+1}^L}+\frac{dP_{t+1}^H}{dP_{t+1}^L}$$

Now

$$C_{0,t}^{B,L}(1+r_t)+C_{1,t}^{B,L}=y_{0,t}(1+r_t)+y_{1,t}-R_{t+1}+P_{t+1}^L$$

So

$$(1+r_t)\frac{\partial C_{0,t}^{B,L}}{\partial P_{t+1}^L}+\frac{\partial C_{1,t}^{B,L}}{\partial P_{t+1}^L}=1$$

Thus

$$\frac{\partial g}{\partial P_{t+1}^L}=\beta u'(C_{1,t}^{B,H})\left[(1+r_t)\frac{dP_t^L}{dP_{t+1}^L}+\frac{dP_{t+1}^H}{dP_{t+1}^L}\right]-\beta u'(C_{1,t}^{B,L})$$

Combining results

$$\begin{aligned}\frac{dP_t^H}{dP_{t+1}^L} &= \frac{-\frac{\partial g}{\partial P_{t+1}^L}}{\frac{\partial g}{\partial P_t^H}}\\ &= \frac{-\left[\beta u'(C_{1,t}^{B,H})\left[(1+r_t)\frac{dP_t^L}{dP_{t+1}^L}+\frac{dP_{t+1}^H}{dP_{t+1}^L}\right]-\beta u'(C_{1,t}^{B,L})\right]}{-(1+r_t)\beta u'(C_{1,t}^{B,H})}\\ &= \frac{u'(C_{1,t}^{B,H})\left[(1+r_t)\frac{dP_t^L}{dP_{t+1}^L}+\frac{dP_{t+1}^H}{dP_{t+1}^L}\right]-u'(C_{1,t}^{B,L})}{(1+r_t)u'(C_{1,t}^{B,H})}\\ &= \frac{dP_t^L}{dP_{t+1}^L}+\frac{1}{1+r_t}\frac{dP_{t+1}^H}{dP_{t+1}^L}-\frac{u'(C_{1,t}^{B,L})}{(1+r_t)u'(C_{1,t}^{B,H})}\end{aligned}$$

This completes the proof of the Lemma. ∎

[41] The derivative is partial in the sense that it it ingores the effect on the equilibriating variable P_t^H. The full derivative is taken w.r.t. the remaining variables.

Using the lemma we now prove the proposition.

Proof. We show that

$$0 < \frac{dP_t^L}{dP_{t+1}^L}\frac{1}{P_t^L} < \frac{dP_t^H}{dP_{t+1}^L}\frac{1}{P_t^H}$$

From Lemma 32

$$\begin{aligned}
\frac{dP_t^H}{dP_{t+1}^L}\frac{1}{P_t^H} &= \frac{dP_t^L}{dP_{t+1}^L}\frac{1}{P_t^H} + \frac{1}{1+r_t}\left(\frac{dP_{t+1}^H}{dP_{t+1}^L}\right)\frac{1}{P_t^H} - \frac{u'(C_{1,t}^{B,L})}{(1+r_t)P_t^H u'(C_{1,t}^{B,H})} \\
&\geq \frac{dP_t^L}{dP_{t+1}^L}\frac{1}{P_t^H} + \frac{1}{1+r_t}\left(\frac{P_t^H}{P_t^L}\right)\frac{1}{P_t^H} - \frac{u'(C_{1,t}^{B,L})}{(1+r_t)P_t^H u'(C_{1,t}^{B,H})} \\
&= \frac{dP_t^L}{dP_{t+1}^L}\frac{1}{P_t^H} + \frac{1}{1+r_t}\frac{1}{P_t^L} - \frac{u'(C_{1,t}^{B,L})}{(1+r_t)P_t^H u'(C_{1,t}^{B,H})}
\end{aligned}$$

Thus for $\frac{dP_t^L}{dP_{t+1}^L}\frac{1}{P_t^L} < \frac{dP_t^H}{dP_{t+1}^L}\frac{1}{P_t^H}$ it is sufficient that

$$\begin{aligned}
\frac{dP_t^L}{dP_{t+1}^L}\frac{1}{P_t^L} &< \frac{dP_t^L}{dP_{t+1}^L}\frac{1}{P_t^H} + \frac{1}{1+r_t}\frac{1}{P_t^L} - \frac{u'(C_{1,t}^{B,L})}{(1+r_t)P_t^H u'(C_{1,t}^{B,H})} \text{ iff} \\
\frac{dP_t^L}{dP_{t+1}^L}\left[\frac{1}{P_t^L} - \frac{1}{P_t^H}\right] &< \frac{1}{1+r_t}\frac{1}{P_t^L} - \frac{u'(C_{1,t}^{B,L})}{(1+r_t)P_t^H u'(C_{1,t}^{B,H})}
\end{aligned}$$

Now

$$\frac{dP_t^L}{dP_{t+1}^L} \leq \frac{1}{1+r_t}$$

(this holds in all cases-regardless of whether they are constrained or not).
So, given that $\frac{1}{P_t^L} - \frac{1}{P_t^H} > 0$

$$\frac{dP_t^L}{dP_{t+1}^L}\left[\frac{1}{P_t^L} - \frac{1}{P_t^H}\right] \leq \frac{1}{1+r_t}\left[\frac{1}{P_t^L} - \frac{1}{P_t^H}\right]$$

It's thus sufficient that

$$\begin{aligned}
\frac{1}{1+r_t}\left[\frac{1}{P_t^L} - \frac{1}{P_t^H}\right] &< \frac{1}{1+r_t}\frac{1}{P_t^L} - \frac{u'(C_{1,t}^{B,L})}{(1+r_t)P_t^H u'(C_{1,t}^{B,H})} \text{ iff} \\
0 &< \frac{1}{1+r_t}\frac{1}{P_t^H} - \frac{u'(C_{1,t}^{B,L})}{(1+r_t)P_t^H u'(C_{1,t}^{B,H})} \text{ iff} \\
\frac{u'(C_{1,t}^{B,L})}{u'(C_{1,t}^{B,H})} &< 1
\end{aligned}$$

Now as $u_H > u_L$ and group B agents are unconstrained in both cases, it follows that $C_{1,t}^{B,L} > C_{1,t}^{B,H}$ and so $u'(C_{1,t}^{B,L}) < u'(C_{1,t}^{B,H})$. To show the first point more formally, note that if $C_{1,t}^{B,L} \leq C_{1,t}^{B,H}$

then $u'(C_{1,t}^{B,L}) \geq u'(C_{1,t}^{B,H})$ and so $u'(C_{0,t}^{B,L}) = (1+r_t)\beta u'(C_{1,t}^{B,L}) \geq u'(C_{1,t}^{B,H})(1+r_t)\beta = u'(C_{0,t}^{B,H})$ so $C_{0,t}^{B,L} \leq C_{0,t}^{B,H}$. Then

$$u(C_{0,t}^{B,H}) + \beta u(C_{1,t}^{B,H}) + u_H - u_L > \left(u(C_{0,t}^{B,L}) + \beta u(C_{1,t}^{B,L})\right)$$

This is a contradiction as then the market is not in equilibrium. This completes the proof of the proposition. ■

Remark 34 *This result is not affected by the probability of reaching old age for group A agents, q. All that is required is that* $\frac{dP_t^L}{dP_{t+1}^L} \leq \frac{1}{1+r_t}$ *which holds for all positive discount factors, so for all* $q \in (0,1)$.

1.B.5 Low vs High Tier Response to Change in Interest Rate: Proof of Proposition 12

We restate the proposition.

Suppose low tier buyers are constrained when buying and high tier buyers are unconstrained when buying and would be unconstrained if they didn't buy. Further, suppose $u(C) = \log(C)$. Then the relative price change is greater for high tier housing:

$$0 > \left(\frac{dP_t^L}{dr_t}\right)\frac{1}{P_t^L} > \left(\frac{dP_t^H}{dr_t}\right)\frac{1}{P_t^H}$$

We prove this in a series of steps using several lemmas.

Lemma 35

$$\begin{aligned}\frac{dP^H}{dr_t} &= \frac{-1}{r_t^2}\left(P^L + y_{0,t} + y_{1,t} - R_{t+1}\right)\left(1 - \frac{1}{\exp\left(\frac{u_H - u_L}{1+\beta}\right)}\right) \\ &\quad + \frac{dP^L}{dr_t}\left(1 + \frac{1}{r_t}\left(1 - \frac{1}{\exp\left(\frac{u_H - u_L}{1+\beta}\right)}\right)\right)\end{aligned}$$

Proof. Given that the high tier buyer faces interest rate $1 + r_t$ and is unconstrained when buying and renting, it follows from (1.23) that

$$
\begin{aligned}
P^H &= \left(\frac{(1+r_t)\exp\left(\frac{u_H-u_L}{(1+\beta)}\right)-1}{r_t\exp\left(\frac{u_H-u_L}{(1+\beta)}\right)}\right)P^L+\left(\frac{1+r_t}{r_t}\right)\left(y_{0,t}+\frac{y_{1,t}-R_{t+1}}{(1+r_t)}\right)\left(1-\frac{1}{\exp\left(\frac{u_H-u_L}{(1+\beta)}\right)}\right) \\
&= \left(\frac{(1+r_t)}{r_t}-\frac{1}{r\exp\left(\frac{u_H-u_L}{(1+\beta)}\right)}\right)P^L+\left(y_{0,t}\left(\frac{1+r_t}{r_t}\right)+\frac{y_{1,t}-R_{t+1}}{r_t}\right)\left(1-\frac{1}{\exp\left(\frac{u_H-u_L}{(1+\beta)}\right)}\right) \\
&= \left(1+\frac{1}{r_t}-\frac{1}{r_t\exp\left(\frac{u_H-u_L}{(1+\beta)}\right)}\right)P^L+\left(y_{0,t}\left(\frac{1}{r_t}+1\right)+\frac{y_{1,t}-R_{t+1}}{r_t}\right)\left(1-\frac{1}{\exp\left(\frac{u_H-u_L}{(1+\beta)}\right)}\right) \\
&= \left(1-\frac{1}{\exp\left(\frac{u_H-u_L}{(1+\beta)}\right)}\right)\left[\frac{y_{0,t}}{r_t}+\frac{y_{1,t}-R_{t+1}}{r_t}+\frac{P^L}{r_t}\right]+P^L+y_{0,t}\left(1-\frac{1}{\exp\left(\frac{u_H-u_L}{(1+\beta)}\right)}\right)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{dP^H}{dr_t} &= \frac{-1}{r_t^2}\left(y_{0,t}+y_{1,t}-R_{t+1}\right)\left(1-\frac{1}{\exp\left(\frac{u_H-u_L}{1+\beta}\right)}\right) \\
&\quad +\frac{dP^L}{dr_t}+\left(\frac{d\left(\frac{P^L}{r_t}\right)}{dr_t}\right)\left(1-\frac{1}{\exp\left(\frac{u_H-u_L}{1+\beta}\right)}\right) \\
&= \frac{-1}{r_t^2}\left(y_{0,t}+y_{1,t}-R_{t+1}\right)\left(1-\frac{1}{\exp\left(\frac{u_H-u_L}{1+\beta}\right)}\right) \\
&\quad +\frac{dP^L}{dr_t} \\
&\quad +\left(\frac{dP^L}{dr_t}\frac{1}{r_t}-\frac{1}{r_t^2}P^L\right)\left(1-\frac{1}{\exp\left(\frac{u_H-u_L}{1+\beta}\right)}\right) \\
&= \frac{-1}{r_t^2}\left(P^L+y_{0,t}+y_{1,t}-R_{t+1}\right)\left(1-\frac{1}{\exp\left(\frac{u_H-u_L}{1+\beta}\right)}\right) \\
&\quad +\frac{dP^L}{dr_t}\left(1+\frac{1}{r_t}\left(1-\frac{1}{\exp\left(\frac{u_H-u_L}{1+\beta}\right)}\right)\right)
\end{aligned}
$$

This completes the proof of the lemma ■

Lemma 36 *Suppose the low tier buyer is constrained (by an LTV or LTI constraint), and we have* $u(C)=\log(C)$ *then the following inequality holds*

$$
\left(\frac{1}{P^L}\right)\frac{dP^L}{dr_t} > \frac{-1}{r_t}\frac{\left(P^L+y_{0,t}+y_{1,t}-R_{t+1}\right)}{\left[r_t y_{0,t}+y_{0,t}+y_{1,t}-R_{t+1}\right]}
$$

Proof. We establish the following:

$$\begin{aligned}
\left(\frac{1}{P^L}\right)\frac{dP^L}{dr_t} &> \frac{\frac{-1}{r_t}\left(R_t + (y_{0,t} - R_t + y_{1,t} - R_{t+1})\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)\right)}{(1+r_t)\,R_t + (1+r_t)\left(y_{0,t} - R_t + \frac{y_{1,t} - R_{t+1}}{1+r_t}\right)\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)} \\
&> \frac{-1}{r_t}\frac{\left(P^L + y_{0,t} + y_{1,t} - R_{t+1}\right)}{\left[r_t y_{0,t} + y_{0,t} + y_{1,t} - R_{t+1}\right]}
\end{aligned}$$

From Proposition 8 with the low tier buyer constrained,

$$0 < -\left(\frac{dP_t^L}{dr_t}\right)\frac{1}{P_t^L} < -\left(\frac{dP_t^{L,u}}{dr_t}\right)\frac{1}{P_t^{L,u}}$$

Where $P_t^{L,u}$ is the low tier price is the low tier buyer didn't face a credit constraint. From (1.4)

$$P_t^{L,u} = R_t + \frac{P_{t+1}^{L,u}}{1+r_t} + \left(y_{0,t} - R_t + \frac{y_{1,t} - R_{t+1}}{1+r_t}\right)\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)$$

In equilibrium we have

$$\begin{aligned}
P_t^{L,u} &= \frac{(1+r_t)}{r_t}R_t + \frac{(1+r_t)}{r_t}\left(y_{0,t} - R_t + \frac{y_{1,t} - R_{t+1}}{1+r_t}\right)\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right) \\
&= R_t\left(\frac{1}{r_t} + 1\right) + \left[(y_{0,t} - R_t)\left(\frac{1}{r_t} + 1\right) + \frac{y_{1,t} - R_{t+1}}{r_t}\right]\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right) \\
&= R_t + (y_{0,t} - R_t)\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right) \\
&\quad + \frac{R_t}{r_t} + \frac{1}{r_t}\left[y_{0,t} - R_t + y_{1,t} - R_{t+1}\right]\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)
\end{aligned}$$

Thus

$$\frac{dP_t^{L,u}}{dr}\frac{1}{P_t^{L,u}} = \frac{-1}{P^{L,u}r_t^2}\left(R_t + (y_{0,t} - R_t + y_{1,t} - R_{t+1})\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)\right)$$

So

$$
\begin{aligned}
& \frac{dP_t^L}{dr_t}\left(\frac{1}{P_t^L}\right) \\
> \; & \left(\frac{dP^{L,u}}{dr_t}\right)\frac{1}{P_t^{L,u}} \\
= \; & \frac{-1}{P_t^{L,u} r_t^2}\left(R_t + (y_{0,t} - R_t + y_{1,t} - R_{t+1})\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)\right) \\
= \; & \frac{-\frac{1}{r_t^2}\left(R_t + (y_{0,t} - R_t + y_{1,t} - R_{t+1})\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)\right)}{\frac{(1+r_t)}{r_t}R_t + \frac{(1+r_t)}{r_t}\left(y_{0,t} - R_t + \frac{y_{1,t} - R_{t+1}}{1+r_t}\right)\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)} \\
= \; & \frac{-\frac{1}{r_t}\left(R_t + (y_{0,t} - R_t + y_{1,t} - R_{t+1})\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)\right)}{(1+r_t)R_t + (1+r_t)\left(y_{0,t} - R_t + \frac{y_{1,t} - R_{t+1}}{1+r_t}\right)\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)}
\end{aligned}
$$

The second part of the inequality holds iff

$$
\frac{-\left(R_t + (y_{0,t} - R_t + y_{1,t} - R_{t+1})\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)\right)}{(1+r_t)\,R_t + (1+r_t)\left(y_{0,t} - R_t + \frac{y_{1,t} - R_{t+1}}{1+r_t}\right)\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)} > -\frac{\left(P^L + y_{0,t} + y_{1,t} - R_{t+1}\right)}{[r_t y_{0,t} + y_{0,t} + y_{1,t} - R_{t+1}]}
$$

iff

$$
\frac{\left(P^L + y_{0,t} + y_{1,t} - R_{t+1}\right)}{[r_t y_{0,t} + y_{0,t} + y_{1,t} - R_{t+1}]} > \frac{\left(R_t + (y_{0,t} - R_t + y_{1,t} - R_{t+1})\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)\right)}{(1+r_t)\,R_t + (1+r_t)\left(y_{0,t} - R_t + \frac{y_{1,t} - R_{t+1}}{1+r_t}\right)\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)}
$$

Cross multiplying, this reduces to

$$
\begin{aligned}
& \left(P^L + y_{0,t} + y_{1,t} - R_{t+1}\right) \\
& \cdot\left[(1+r_t)\,R_t + ((y_{0,t} - R_t)(1+r_t) + y_{1,t} - R_{t+1})\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)\right] \\
> \; & [r_t y_{0,t} + y_{0,t} + y_{1,t} - R_{t+1}]\left(R_t + (y_{0,t} - R_t + y_{1,t} - R_{t+1})\left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)\right)
\end{aligned}
$$

Expanding the LHS we have

$$
\begin{aligned}
& \left(P^L + y_{0,t} + y_{1,t} - R_{t+1}\right) \left[R_t + (y_{0,t} - R_t + y_{1,t} - R_{t+1}) \left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)\right] \\
& + \left(P^L + y_{0,t} + y_{1,t} - R_{t+1}\right) r_t \left[R_t + (y_{0,t} - R_t) \left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)\right] \\
> \quad & [r_t y_{0,t} + y_{0,t} + y_{1,t} - R_{t+1}] \left(R_t + (y_{0,t} - R_t + y_{1,t} - R_{t+1}) \left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)\right)
\end{aligned}
$$

Cancelling the common terms this reduces to

$$
\begin{aligned}
& P^L \left[R_t + (y_{0,t} - R_t + y_{1,t} - R_{t+1}) \left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)\right] \\
& + \left(P^L + y_{0,t} + y_{1,t} - R_{t+1}\right) r_t \left[R_t + (y_{0,t} - R_t) \left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)\right] \\
> \quad & r_t y_{0,t} \left(R_t + (y_{0,t} - R_t + y_{1,t} - R_{t+1}) \left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)\right)
\end{aligned}
$$

Cancelling further common terms gives

$$
\begin{aligned}
& P^L \left[R_t + (y_{0,t} - R_t + y_{1,t} - R_{t+1}) \left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)\right] \\
& + \left(P^L + y_{1,t} - R_{t+1}\right) r_t \left[R_t + (y_{0,t} - R_t) \left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)\right] \\
> \quad & r_t y_{0,t} \left((y_{1,t} - R_{t+1}) \left(1 - \frac{1}{\exp\left(\frac{u_L - u_R}{1+\beta}\right)}\right)\right)
\end{aligned}
$$

Rewriting the second term on the LHS we have that

$$\begin{aligned}
& P^L\left[R_t+(y_{0,t}-R_t+y_{1,t}-R_{t+1})\left(1-\frac{1}{\exp\left(\frac{u_L-u_R}{1+\beta}\right)}\right)\right] \\
& +\left(P^L+y_{1,t}-R_{t+1}\right)r_t\left[\frac{R_t}{\exp\left(\frac{u_L-u_R}{1+\beta}\right)}\right]+\left(P^L+y_{1,t}-R_{t+1}\right)r_t y_{0,t}\left(1-\frac{1}{\exp\left(\frac{u_L-u_R}{1+\beta}\right)}\right) \\
> \; & r_t y_{0,t}\left((y_{1,t}-R_{t+1})\left(1-\frac{1}{\exp\left(\frac{u_L-u_R}{1+\beta}\right)}\right)\right)
\end{aligned}$$

Cancelling further terms we're left with

$$\begin{aligned}
& P^L\left[R_t+(y_{0,t}-R_t+y_{1,t}-R_{t+1})\left(1-\frac{1}{\exp\left(\frac{u_L-u_R}{1+\beta}\right)}\right)\right] \\
& +\left(P^L+y_{1,t}-R_{t+1}\right)r_t\left[\frac{R_t}{\exp\left(\frac{u_L-u_R}{1+\beta}\right)}\right]+P^L r_t y_{0,t}\left(1-\frac{1}{\exp\left(\frac{u_L-u_R}{1+\beta}\right)}\right) \\
> \; & 0
\end{aligned}$$

This inequality is true because all the terms on the LHS are positive (given $r_t > 0$). This completes the proof of the lemma. ■

Using this lemma we now prove the main proposition.

Proof of Proposition 12. From Lemma 35

$$\frac{dP_t^H}{dr_t}=\frac{-1}{r_t^2}\left(P^L+y_{0,t}+y_{1,t}-R_{t+1}\right)\left(1-\frac{1}{\exp\left(\frac{u_H-u_L}{1+\beta}\right)}\right)+\frac{dP^L}{dr_t}\left(1+\frac{1}{r_t}\left(1-\frac{1}{\exp\left(\frac{u_H-u_L}{1+\beta}\right)}\right)\right)$$

Then we'll have

$$0>\frac{1}{P^L}\frac{dP^L}{dr_t}>\frac{1}{P^H}\frac{dP^H}{dr_t}$$

iff

$$\begin{aligned}
\frac{1}{P^L}\frac{dP^L}{dr_t}\left(P^H\right) \; > \; & \frac{-1}{r_t^2}\left(P^L+y_{0,t}+y_{1,t}-R_{t+1}\right)\left(1-\frac{1}{\exp\left(\frac{u_H-u_L}{1+\beta}\right)}\right) \\
& +\frac{dP^L}{dr_t}\left(1+\frac{1}{r_t}\left(1-\frac{1}{\exp\left(\frac{u_H-u_L}{1+\beta}\right)}\right)\right)
\end{aligned}$$

iff

$$\frac{1}{P^L}\frac{dP^L}{dr_t}\left[P^H - P^L\left(1+\frac{1}{r_t}\left(1-\frac{1}{\exp\left(\frac{u_H-u_L}{1+\beta}\right)}\right)\right)\right]$$
$$> \frac{-1}{r_t^2}\left(P^L + y_{0,t} + y_{1,t} - R_{t+1}\right)\left(1-\frac{1}{\exp\left(\frac{u_H-u_L}{1+\beta}\right)}\right)$$

Now, we have that

$$\begin{aligned} P_t^H &= P_t^L\left(\frac{1+r_t}{r_t}\right)\left(1-\left(\frac{1}{1+r_t}\right)\frac{1}{\exp\left(\frac{u_H-u_L}{1+\beta}\right)}\right) \\ &\quad + \left(\frac{1+r_t}{r_t}\right)\left(y_{0,t}+\frac{y_{1,t}-R_{t+1}}{1+r_t}\right)\left[1-\frac{1}{\exp\left(\frac{u_H-u_L}{1+\beta}\right)}\right] \end{aligned}$$

Hence the condition reduces to

$$\frac{1}{P^L}\frac{dP^L}{dr_t}\left(\frac{1+r_t}{r_t}\right)\left(y_{0,t}+\frac{y_{1,t}-R_{t+1}}{1+r_t}\right)\left[1-\frac{1}{\exp\left(\frac{u_H-u_L}{1+\beta}\right)}\right]$$
$$> \frac{-1}{r_t^2}\left(P^L + y_{0,t} + y_{1,t} - R_{t+1}\right)\left(1-\frac{1}{\exp\left(\frac{u_H-u_L}{1+\beta}\right)}\right)$$

Which gives a nice cancellation:

$$\frac{1}{P^L}\frac{dP^L}{dr_t}(1+r_t)\left(y_{0,t}+\frac{y_{1,t}-R_{t+1}}{1+r_t}\right) > \frac{-1}{r_t}\left(P^L + y_{0,t} + y_{1,t} - R_{t+1}\right)$$

Reducing to

$$\frac{1}{P^L}\frac{dP^L}{dr_t} > \frac{\frac{-1}{r_t}\left(P^L + y_{0,t} + y_{1,t} - R_{t+1}\right)}{y_{0,t} + r_t y_{0,t} + y_{1,t} - R_{t+1}}$$

But this conditions follows from Lemma 36.

This completes the proof of Proposition 12. ■

Remark 37 *This result is not affected by the probability of reaching old age for group A agents, q. All that is required is that Lemma 36 holds. It can be seen that this holds for all positive discount factors, so for all $q \in (0,1)$.*

1.C Proofs for Robustness Section

1.C.1 Discrete Changes in Expectations: Proof of Theorem 13

To implement this section mathematically, we assume the group B expectations for both the low and the high tier, $P_{t+1}^{L,B}, P_{t+1}^{H,B}$ are functions of the group A expectations for the low tier, $P_{t+1}^{L,A}$.

We assume that group B agents have proportional expectations for low and high tier growth. However, here we make it explicit that it's relative to the initial equilibrium prices:

$$\frac{dP_{t+1}^{H,B}}{dP_{t+1}^{L,A}}\frac{1}{\overline{P_t^H}} = \frac{dP_{t+1}^{L,B}}{dP_{t+1}^{L,A}}\frac{1}{\overline{P_t^L}}$$

Given this from Lemma 32 the following formula now describes the response of P_t^H to a change in expectations:

$$\frac{dP_t^H}{dP_{t+1}^{L,A}} = \frac{dP_t^L}{dP_{t+1}^{L,A}} + \frac{1}{1+r_t}\left(\frac{dP_{t+1}^{H,B}}{dP_{t+1}^{L,A}}\right)\left(1 - \frac{u'(C_{1,t}^{B,L})\overline{P}_t^L}{u'(C_{1,t}^{B,H})\overline{P}_t^H}\right) \tag{1.25}$$

In discrete form the expectations assumption for group B agents can be restated as

$$\Delta P_{t+1}^{H,B}\frac{1}{\overline{P_t^H}} = \Delta P_{t+1}^{L,B}\frac{1}{\overline{P_t^L}} \tag{1.26}$$

where

$$\begin{aligned}\Delta P_{t+1}^{i,B} \quad &: \quad = \widetilde{P}_{t+1}^{i,B} - \overline{P}_{t+1}^{i,B} \\ &\equiv \quad P_{t+1}^{i,B}(\widetilde{P}_{t+1}^{L,A}) - P_{t+1}^{i,B}(\overline{P}_{t+1}^{L,A})\end{aligned}$$

with $\overline{x}$ representing the initial values of the variables, prior to the change in expectations.

We establish the following lemma as a step towards the main result.

Lemma 38 *Suppose $\Delta P_{t+1}^{H,B}$ increases, with $\Delta P_{t+1}^{L,B}$ increasing according to (1.26). Then ΔP_t^H increases.*

Proof. Equilibrium for P_t^H comes from the following equality:

$$u(C_{0,t}^{H,B}) + \beta u(C_{1,t}^{H,B}) + u_H - u_L \equiv u(C_{0,t}^{L,B}) + \beta u(C_{1,t}^{L,B}) \tag{1.27}$$

With

$$\begin{aligned}C_{0,t}^{H,B} + \frac{C_{1,t}^{H,B}}{(1+r_t)} \quad &= \quad y_{0,t} + P_t^L - P_t^H + \frac{y_{1,t} - R_{t+1}}{1+r_t} + \frac{P_{t+1}^{H,B}}{1+r_t} \\ C_{0,t}^{L,B} + \frac{C_{1,t}^{L,B}}{(1+r_t)} \quad &= \quad y_{0,t} + \frac{y_{1,t} - R_{t+1}}{1+r_t} + \frac{P_{t+1}^{L,B}}{1+r_t}\end{aligned}$$

Given that $u_H > u_L$ and group B agents are unconstrained we have $C_{0,t}^{H,B} + \frac{C_{1,t}^{H,B}}{(1+r_t)} < C_{0,t}^{L,B} + \frac{C_{1,t}^{L,B}}{(1+r_t)}$. With the given parameter values, expectations and P_t^L, P_t^H adjusts to ensure that (1.27) always holds. Suppose now there is a discrete increase in $P_{t+1}^{H,B}, P_{t+1}^{L,B}$ with (1.26) holding: $\Delta P_{t+1}^{H,B} = \Delta P_{t+1}^{L,B} \frac{\overline{P_t^H}}{\overline{P_t^L}} > \Delta P_{t+1}^{L,B} > 0$.

We use this information after establishing a small result.

Let $V(a+x) := u(C_{0,t}) + \beta u(C_{1,t})$ where $C_{0,t} + \frac{C_{1,t}}{(1+r_t)} = a + x$ and where $u' > 0, u'' < 0$, with $u'(C_{0,t}) = \beta(1+r_t)u'(C_{1,t})$.

Suppose

$$V(a) + u_H - u_L = V(b)$$

with $a < b$ and $u_H - u_L > 0$. Suppose $c > d > 0$, then

$$V(a+c) + u_H - u_L > V(b+d)$$

To show this we first note that

$$\begin{aligned} V'(a+x) &= u'(C_{0,t})\frac{dC_{0,t}}{dx} + \beta u'(C_{1,t})\frac{dC_{1,t}}{dx} \\ &= \beta u'(C_{1,t})\left[(1+r_t)\frac{dC_{0,t}}{dx} + \frac{dC_{1,t}}{dx}\right] \\ &= \beta u'(C_{1,t})(1+r_t) > 0 \end{aligned}$$

And

$$V''(a+x) = \beta(1+r_t)u''(C_{1,t})\frac{dC_{1,t}}{dx} < 0$$

For the last inequality, note that $(1+r_t)\frac{dC_{0,t}}{dx} + \frac{dC_{1,t}}{dx} > 0$ and the Euler equation holds, so when x increases, $C_{0,t}$ and $C_{1,t}$ have to move in the same direction, which must be positive for both.

Given $V''(a+x) < 0$ and $a < b$ it follows that $V'(a+x) > V'(b+x)$. Thus

$$\frac{d\left[V(a+x) - V(b+x)\right]}{dx} > 0$$

Thus for $c > 0$

$$\begin{aligned} V(a+c) + u_H - u_L &> V(b+c) \\ &> V(b+d) \end{aligned}$$

where the last line uses $V' > 0$. This completes the proof of the small result.

We can now use this small result to complete the proof of the lemma. We apply the lemma with

$$
\begin{aligned}
a &: = y_{0,t} + P_t^L - P_t^H + \frac{y_{1,t} - R_{t+1}}{1+r_t} + \frac{P_{t+1}^{H,B}}{1+r_t} \\
b &: = y_{0,t} + \frac{y_{1,t} - R_{t+1}}{1+r_t} + \frac{P_{t+1}^{L,B}}{1+r_t} \\
c &: = \Delta P_{t+1}^{L,B} \frac{\overline{P_t^H}}{\overline{P_t^L}} - \Delta P_t^H \\
d &: = \Delta P_{t+1}^{L,B}
\end{aligned}
$$

Then $a < b$. If $\Delta P_t^H \leq 0$ then $c > d > 0$. From the small result, it then follows that the utility of group B agents is strictly greater when buying a high tier house, thus we cannot be in equilibrium. Thus, in equilibrium, we must have $\Delta P_t^H > 0$. This completes the proof of the lemma. ■

We can now state the main result:

Proposition 39 *Suppose* $\Delta P_{t+1}^{H,B} \frac{1}{\overline{P_t^H}} = \Delta P_{t+1}^{L,B} \frac{1}{\overline{P_t^L}}$ *and*

$$
\frac{\Delta P_{t+1}^{H,B}}{\overline{P}_t^H} \geq \frac{\Delta P_{t+1}^{L,A}}{\overline{P}_t^L} \left(\frac{(1+r_t)\Delta P_t^L}{\Delta P_{t+1}^{L,A}} \right)
$$

Then

$$
\frac{\Delta P_t^H}{\overline{P}_t^H} > \frac{\Delta P_t^L}{\overline{P}_t^L}
$$

Proof. Let

$$
\frac{dP_{t+1}^{H,B}}{dP_{t+1}^{L,A}} = (1+r_t) \frac{dP_t^L}{dP_{t+1}^{L,A}} \frac{\overline{P}_t^H}{\overline{P}_t^L}
$$

Then from (1.25) we have

$$
\begin{aligned}
\frac{dP_t^H}{dP_{t+1}^{L,A}} &= \frac{dP_t^L}{dP_{t+1}^{L,A}} + \frac{1}{1+r_t} \left((1+r_t) \frac{dP_t^L}{dP_{t+1}^{L,A}} \frac{\overline{P}_t^H}{\overline{P}_t^L} \right) \left(1 - \frac{u'(C_{1,t}^{B,L})\overline{P}_t^L}{u'(C_{1,t}^{B,H})\overline{P}_t^H} \right) \\
&= \frac{dP_t^L}{dP_{t+1}^{L,A}} + \frac{dP_t^L}{dP_{t+1}^{L,A}} \frac{\overline{P}_t^H}{\overline{P}_t^L} \left(1 - \frac{u'(C_{1,t}^{B,L})\overline{P}_t^L}{u'(C_{1,t}^{B,H})\overline{P}_t^H} \right) \\
&> \frac{dP_t^L}{dP_{t+1}^{L,A}} + \frac{dP_t^L}{dP_{t+1}^{L,A}} \frac{\overline{P}_t^H}{\overline{P}_t^L} \left(1 - \frac{\overline{P}_t^L}{\overline{P}_t^H} \right) \\
&= \frac{dP_t^L}{dP_{t+1}^{L,A}} \left[1 + \frac{\overline{P}_t^H}{\overline{P}_t^L} - 1 \right] \\
&= \frac{dP_t^L}{dP_{t+1}^{L,A}} \frac{\overline{P}_t^H}{\overline{P}_t^L}
\end{aligned}
$$

Where the inequality uses the fact that $u'(C_{1,t}^{B,L}) < u'(C_{1,t}^{B,H})$ as demonstrated before. Thus

$$\frac{dP_t^H}{dP_{t+1}^{L,A}} > \frac{dP_t^L}{dP_{t+1}^{L,A}} \frac{\overline{P}_t^H}{\overline{P}_t^L}$$

Therefore

$$\begin{aligned}
\int_{\overline{P}_{t+1}^{L,A}}^{\widetilde{P}_{t+1}^{L,A}} \left(\frac{dP_t^H}{dP_{t+1}^{L,A}} \right) dP_{t+1}^{L,A} &> \int_{\overline{P}_{t+1}^{L,A}}^{\widetilde{P}_{t+1}^{L,A}} \left(\frac{dP_t^L}{dP_{t+1}^{L,A}} \frac{\overline{P}_t^H}{\overline{P}_t^L} \right) dP_{t+1}^{L,A} \text{ so} \\
P_t^H \left(\widetilde{P}_{t+1}^{L,A} \right) - P_t^H \left(\overline{P}_{t+1}^{L,A} \right) &> \frac{\overline{P}_t^H}{\overline{P}_t^L} \int_{\overline{P}_{t+1}^{L,A}}^{\widetilde{P}_{t+1}^{L,A}} \left(\frac{dP_t^L}{dP_{t+1}^{L,A}} \right) dP_{t+1}^{L,A} \\
&= \frac{\overline{P}_t^H}{\overline{P}_t^L} \left(P_t^L \left(\widetilde{P}_{t+1}^{L,A} \right) - P_t^L \left(\overline{P}_{t+1}^{L,A} \right) \right) \text{ giving} \\
\Delta P_t^H &> \frac{\overline{P}_t^H}{\overline{P}_t^L} \Delta P_t^L
\end{aligned}$$

We thus have

$$\frac{\Delta P_t^H}{\overline{P}_t^H} > \frac{\Delta P_t^L}{\overline{P}_t^L}$$

Now, given

$$\begin{aligned}
\frac{dP_{t+1}^{H,B}}{dP_{t+1}^{L,A}} &= (1 + r_t) \frac{dP_t^L}{dP_{t+1}^{L,A}} \frac{\overline{P}_t^H}{\overline{P}_t^L} \text{ we have} \\
\int_{\overline{P}_{t+1}^{L,A}}^{\widetilde{P}_{t+1}^{L,A}} \frac{dP_{t+1}^{H,B}}{dP_{t+1}^{L,A}} dP_{t+1}^{L,A} &= (1 + r_t) \frac{\overline{P}_t^H}{\overline{P}_t^L} \int_{\overline{P}_{t+1}^{L,A}}^{\widetilde{P}_{t+1}^{L,A}} \frac{dP_t^L}{dP_{t+1}^{L,A}} dP_{t+1}^{L,A} \text{ so} \\
P_{t+1}^{H,B} \left(\widetilde{P}_{t+1}^{L,A} \right) - P_{t+1}^{H,B} \left(\overline{P}_{t+1}^{L,A} \right) &= (1 + r_t) \frac{\overline{P}_t^H}{\overline{P}_t^L} \left(P_t^L \left(\widetilde{P}_{t+1}^{L,A} \right) - P_t^L \left(\overline{P}_{t+1}^{L,A} \right) \right) \text{ giving} \\
\Delta P_{t+1}^{H,B} &= (1 + r_t) \frac{\overline{P}_t^H}{\overline{P}_t^L} \Delta P_t^L
\end{aligned}$$

This can be rewritten as

$$\frac{\Delta P_{t+1}^{H,B}}{\overline{P}_t^H} = \frac{\Delta P_{t+1}^{L,A}}{\overline{P}_t^L} \left(\frac{(1 + r_t) \Delta P_t^L}{\Delta P_{t+1}^{L,A}} \right)$$

From Lemma 38 if $\widehat{\Delta P_{t+1}^{H,B}} \geq \Delta P_{t+1}^{H,B}$ (and from (1.26) $\widehat{\Delta P_{t+1}^{L,B}} \geq \Delta P_{t+1}^{L,B}$), then $\widehat{\Delta P_t^H} \geq \Delta P_t^H$. It thus follows that if

$$\frac{\widehat{\Delta P_{t+1}^{H,B}}}{\overline{P}_t^H} \geq \frac{\Delta P_{t+1}^{L,A}}{\overline{P}_t^L} \left(\frac{(1 + r_t) \Delta P_t^L}{\Delta P_{t+1}^{L,A}} \right)$$

then

$$\frac{\widehat{\Delta P_t^H}}{\overline{P}_t^H} > \frac{\Delta P_t^L}{\overline{P}_t^L}$$

This completes the proof of the Theorem. ■

1.C.2 Calibration of $\frac{(1+r_t)\Delta P_t^L}{\Delta P_{t+1}^L}$ for Section 1.4.1

We first demonstrate that when the agent is initially unconstrained, the response of P_t^L to a change in P_{t+1}^L does not depend on $y_{0,t}, y_{1,t}, R_t, R_{t+1}$ but the ratio of these terms to $y_{0,t}$. We thus don't need to calibrate the level of these variables but only $\frac{y_{1,t}}{y_{0,t}}, \frac{R_t}{y_{0,t}}, \frac{R_{t+1}}{y_{0,t}}$.

Proposition 40 *Suppose the agent is unconstrained when buying, with $P_t^L = P_{t+1}^L = P^L$. Suppose $u(C) = \log(C)$. Then $\frac{(1+r_t)\Delta P_t^L}{\Delta P_{t+1}^L}$ does not depend on the levels of $y_{0,t}, y_{1,t}, R_t, R_{t+1}$, but only their size relative to each other: $\frac{y_{1,t}}{y_{0,t}}, \frac{R_t}{y_{0,t}}, \frac{R_{t+1}}{y_{0,t}}$*

Proof. In equilibrium, the utility from non-housing consumption must be the same for the agent before and after the change in P_{t+1}^L. Suppose P_{t+1}^L increases to $\widetilde{P}_{t+1}^L$, resulting in an equilibrium increase in P_t^L to $\widetilde{P}_t^L$. Then we must have

$$\begin{aligned} & \log\left(y_{0,t} - \gamma \widetilde{P}_t^L\right) + \beta \log\left(y_{1,t} - R_{t+1} + \widetilde{P}_{t+1}^L - (1-\gamma)(1+r_t)\widetilde{P}_t^L\right) \\ = \quad & \log\left(y_{0,t} - \gamma P^L\right) + \beta \log\left(y_{1,t} - R_{t+1} + P^L\left(1 - (1-\gamma)(1+r_t)\right)\right) \end{aligned}$$

Rearranging the first equation gives

$$\begin{aligned} & \log\left(y_{0,t} - \gamma P^L - \gamma \Delta P_t^L\right) \\ & +\beta \log\left(y_{1,t} - R_{t+1} + P^L\left(1-(1-\gamma)(1+r_t)\right) + \Delta P_{t+1}^L - (1-\gamma)(1+r_t)\Delta P_t^L\right) \\ = \quad & \log\left(y_{0,t} - \gamma P^L\right) + \beta \log\left(y_{1,t} - R_{t+1} + P^L\left(1-(1-\gamma)(1+r_t)\right)\right) \end{aligned}$$

Where $\Delta P_t^L := \widetilde{P}_t^L - P^L$ and $\Delta P_{t+1}^L := \widetilde{P}_{t+1}^L - P^L$. Let $\delta := \frac{\Delta P_{t+1}^L}{P^L}$, so δ gives the percentage increase in buyer expectations.

Then

$$\begin{aligned} & \log\left(y_{0,t}\left[1 - \frac{\gamma P^L}{y_{0,t}} - \frac{\gamma \Delta P_t^L}{y_{0,t}}\right]\right) \\ & +\beta \log\left(y_{0,t}\left[\frac{y_{1,t}}{y_{0,t}} - \frac{R_{t+1}}{y_{0,t}} + \frac{P^L}{y_{0,t}}\left(1-(1-\gamma)(1+r_t)\right) + \frac{\Delta P_{t+1}^L}{y_{0,t}} - (1-\gamma)(1+r_t)\frac{\Delta P_t^L}{y_{0,t}}\right]\right) \\ = \quad & \log\left(y_{0,t}\left[1 - \gamma\frac{P^L}{y_{0,t}}\right]\right) + \beta \log\left(y_{0,t}\left[\frac{y_{1,t}}{y_{0,t}} - \frac{R_{t+1}}{y_{0,t}} + \frac{P^L}{y_{0,t}}\left(1-(1-\gamma)(1+r_t)\right)\right]\right) \end{aligned}$$

Thus, cancelling $\log(y_{0,t}) + \beta\log(y_{0,t})$ from each side we have

$$\begin{aligned}
& \log\left(1 - \frac{\gamma P^L}{y_{0,t}} - \frac{\gamma\Delta P^L_t}{y_{0,t}}\right) \\
& +\beta\log\left(\frac{y_{1,t}}{y_{0,t}} - \frac{R_{t+1}}{y_{0,t}} + \frac{P^L}{y_{0,t}}\left(1-(1-\gamma)(1+r_t)\right) + \frac{\Delta P^L_{t+1}}{y_{0,t}} - (1-\gamma)(1+r_t)\frac{\Delta P^L_t}{y_{0,t}}\right) \\
= & \log\left(1 - \gamma\frac{P^L}{y_{0,t}}\right) + \beta\log\left(\frac{y_{1,t}}{y_{0,t}} - \frac{R_{t+1}}{y_{0,t}} + \frac{P^L}{y_{0,t}}\left(1-(1-\gamma)(1+r_t)\right)\right)
\end{aligned}$$

Now, with the agent unconstrained initially and $P^L_t = P^L_{t+1}$,

$$P^L = \frac{(1+r_t)}{r_t}R_t + \frac{(1+r_t)}{r_t}\left(y_{0,t} - R_t - \frac{y_{1,t} - R_{t+1}}{(1+r_t)}\right)\left(1 - \frac{1}{\exp\left(\frac{\Delta u}{1+\beta}\right)}\right)$$

Thus

$$\frac{P^L}{y_{0,t}} = \frac{(1+r_t)}{r_t}\frac{R_t}{y_{0,t}} + \frac{(1+r_t)}{r_t}\left(1 - \frac{R_t}{y_{0,t}} - \frac{\frac{y_{1,t}}{y_{0,t}} - \frac{R_{t+1}}{y_{0,t}}}{(1+r_t)}\right)\left(1 - \frac{1}{\exp\left(\frac{\Delta u}{1+\beta}\right)}\right)$$

Thus $y_{0,t} = P^L c$ where c is a constant depending only on the ratios $\frac{y_{1,t}}{y_{0,t}}, \frac{R_t}{y_{0,t}}, \frac{R_{t+1}}{y_{0,t}}$ and not on the levels of these variables.

Thus, we can write

$$\begin{aligned}
& \log\left(1 - \frac{\gamma}{c} - \frac{\gamma\Delta P^L_t}{cP^L}\right) \\
& +\beta\log\left(\frac{y_{1,t}}{y_{0,t}} - \frac{R_{t+1}}{y_{0,t}} + \frac{1}{c}\left(1-(1-\gamma)(1+r_t)\right) + \frac{\Delta P^L_{t+1}}{cP^L} - (1-\gamma)(1+r_t)\frac{\Delta P^L_t}{cP^L}\right) \\
= & \log\left(1 - \frac{\gamma}{c}\right) + \beta\log\left(\frac{y_{1,t}}{y_{0,t}} - \frac{R_{t+1}}{y_{0,t}} + \frac{1}{c}\left(1-(1-\gamma)(1+r_t)\right)\right)
\end{aligned}$$

With $\delta := \frac{\Delta P^L_{t+1}}{P^L}$, we can write $P^L = \frac{\Delta P^L_{t+1}}{\delta}$ and the equation becomes

$$\begin{aligned}
& \log\left(1 - \frac{\gamma}{c} - \frac{\gamma\delta}{c(1+r_t)}\left(\frac{(1+r_t)\Delta P^L_t}{\Delta P^L_{t+1}}\right)\right) \\
& +\beta\log\left(\frac{y_{1,t}}{y_{0,t}} - \frac{R_{t+1}}{y_{0,t}} + \frac{1}{c}\left(1-(1-\gamma)(1+r_t)\right) + \frac{\delta}{c} - (1-\gamma)\frac{\delta}{c}\left(\frac{(1+r_t)\Delta P^L_t}{\Delta P^L_{t+1}}\right)\right) \\
= & \log\left(1 - \frac{\gamma}{c}\right) + \beta\log\left(\frac{y_{1,t}}{y_{0,t}} - \frac{R_{t+1}}{y_{0,t}} + \frac{1}{c}\left(1-(1-\gamma)(1+r_t)\right)\right)
\end{aligned}$$

The equation shows that $\frac{(1+r_t)\Delta P^L_t}{\Delta P^L_{t+1}}$ only depends on the ratios $\frac{y_{1,t}}{y_{0,t}}, \frac{R_t}{y_{0,t}}, \frac{R_{t+1}}{y_{0,t}}$, not the levels of these variables (noting that c also only depends on the ratios).

This completes the proof. ■

Remark 41 *To calibrate* $\frac{(1+r_t)\Delta P_t^L}{\Delta P_{t+1}^L}$ *for a given percentage change in future price expectations (i.e., a given* δ*), we need to calibrate the following variables:* $\frac{y_{1,t}}{y_{0,t}}, \frac{R_t}{y_{0,t}}, \frac{R_{t+1}}{y_{0,t}}, r_t, \beta, u_L - u_R$. *The value for* γ *is chosen to ensure the agent is just unconstrained in the initial equilibrium (i.e. their unconstrained desired borrowing is exactly* 0*).*

Lemma 42 *The value of* γ *that leaves the agent initially just unconstrained is given by*

$$\gamma^* = \frac{\left(y_{0,t} - \frac{1}{1+\beta}\left(y_{0,t} - R_t + \frac{y_{1,t}-R_{t+1}}{1+r_t}\right)\frac{1}{\exp\left(\frac{\Delta u}{1+\beta}\right)}\right)\frac{r_t}{1+r_t}}{R_t + \left(y_{0,t} - R_t + \frac{y_{1,t}-R_{t+1}}{(1+r_t)}\right)\left(1 - \frac{1}{\exp\left(\frac{\Delta u}{1+\beta}\right)}\right)}$$

Proof. For an unconstrained buyer (with log utility),

$$C_{0,t} = \frac{1}{1+\beta}\left(y_{0,t} + \frac{y_{1,t}-R_{t+1}}{(1+r_t)} - \frac{r_t}{1+r_t}P^L\right)$$

But

$$P^L = \frac{(1+r_t)}{r_t}R_t + \frac{(1+r_t)}{r_t}\left(y_{0,t} - R_t - \frac{y_{1,t}-R_{t+1}}{(1+r_t)}\right)\left(1 - \frac{1}{\exp\left(\frac{\Delta u}{1+\beta}\right)}\right)$$

So

$$C_{0,t} = \frac{1}{(1+\beta)}\left(y_{0,t} - R_t + \frac{y_{1,t}-R_{t+1}}{(1+r_t)}\right)\frac{1}{\exp\left(\frac{\Delta u}{1+\beta}\right)}$$

But $C_{0,t} = y_{0,t} - \gamma P^L - b_0^*$. If the agent is just unconstrained (with desired saving of exactly 0), then $C_{0,t} = y_{0,t} - \gamma P^L$

Thus, we must have

$$y_{0,t} - \gamma\frac{(1+r_t)}{r_t}\left[R_t + \left(y_{0,t} - R_t - \frac{y_{1,t}-R_{t+1}}{(1+r_t)}\right)\left(1 - \frac{1}{\exp\left(\frac{\Delta u}{1+\beta}\right)}\right)\right]$$
$$= \frac{1}{(1+\beta)}\left(y_{0,t} - R_t + \frac{y_{1,t}-R_{t+1}}{(1+r_t)}\right)\frac{1}{\exp\left(\frac{\Delta u}{1+\beta}\right)}$$

This holds iff

$$\gamma = \frac{\left(y_{0,t} - \frac{1}{1+\beta}\left(y_{0,t} - R_t + \frac{y_{1,t}-R_{t+1}}{1+r_t}\right)\frac{1}{\exp\left(\frac{\Delta u}{1+\beta}\right)}\right)\frac{r_t}{1+r_t}}{R_t + \left(y_{0,t} - R_t + \frac{y_{1,t}-R_{t+1}}{(1+r_t)}\right)\left(1 - \frac{1}{\exp\left(\frac{\Delta u}{1+\beta}\right)}\right)}$$

This completes the proof of the lemma. ■

Table 1.4 gives the calibration for the model used in the text.

Table 1.4: Model Calibration for Optimism Calculation

Parameter	Value	Source
Length of Period	11 years	AHS
r_t	1.22	Mortgage-x.com (annual interest rate of 7.75% in 1997)
β	0.90	Standard (based on annual discount rate of 0.99)
$\frac{R}{y_{0,t}}$	0.3	AHS
$\frac{y_{1,t}}{y_{0.t}}$	1	Imposed
$u_L - u_R$	0	Imposed

Table 1.5: Optimism Calibration Robustness

		$\frac{(1+r_t)\Delta P_t^L}{\Delta P_{t+1}^L}$	
$\frac{\Delta P_{t+1}^L}{P_t^L}$	$\frac{y_{1,t}}{y_{0.t}} = 0.5$	$\frac{y_{1,t}}{y_{0.t}} = 1$	$\frac{y_{1,t}}{y_{0.t}} = 1.5$
100%	0.74	0.73	0.85
150%	0.49	0.64	0.79
200%	0.42	0.57	0.73

The first choice for the calibration is the choice of period length. We choose 11 years as that is the average length of time people live in a house they buy (AHS). We take a standard annual discount factor of 0.99, resulting in the discount factor over our period of 0.9. The quantity $\frac{R}{y_{0,t}}$ is matched to the average proportion of income spent on rent, with the rental price assumed constant over time. The mortgage rate is set to the average of the rate on 30 and 15-year fixed rate mortgages in 1997, prior to the beginning of the boom. This is then compounded over 11 years. The difference in utility between renting and buying a house is set conservatively to 0 : in calculations, a greater value for this results in a lower value for $\frac{(1+r_t)\Delta P_t^L}{\Delta P_{t+1}^L}$. The final choice of parameter is $\frac{y_{1,t}}{y_{0.t}}$. It is not clear whether this should be greater than 1 reflecting income growth over the life-cycle or less than 1 with the last period of life representing retirement. In the baseline calibration we set it to 1 and perform robustness to this parameter assumption in Table 1.5.

The higher $y_{1,t}$ is relative to $y_{0,t}$ the greater the value of $\frac{(1+r_t)\Delta P_t^L}{\Delta P_{t+1}^L}$. However, even with $\frac{y_{1,t}}{y_{0.t}} = 1.5$, this value is 0.73 when prices are expected to triple over the boom. Thus, even in this case, if the high tier expected growth is at least 73% of low tier expected growth, the high tier will grow relatively more over the boom than the low tier.

1.C.3 Proofs of Tiered Results with LTI Constraint

We split the proof of this into three subsections, one for the change in each variable. From the above proofs for the low versus the high tier when the low tier buyer is LTV constrained, we only need to show analogous proofs for LTI constrained vs unconstrained buyers, when there is only one type of house to buy. This follows as the high tier buyer who is not constrained by an LTI constraint behaves identically to a high tier buyer who is not constrained by an LTV constraint.

Constrained vs Unconstrained Response to Change in LTI Constraint δ

Proposition 43 *Suppose the home-buyer is constrained by the LTI constraint in equilibrium with price P_t^c. Then (noting that an increase in δ is an easing of the LTI constraint):*

$$\frac{dP_t^c}{d\delta} > 0$$

Suppose the home-buyer is unconstrained in equilibrium with price P_t^u. Then

$$\frac{dP_t^u}{d\delta} = 0$$

Proof. Similarly to the proof of Proposition 5 we'll have

$$\frac{dP_t}{d\delta} = \frac{-\left(u'(C_{0,t}^L)\frac{\partial C_{0,t}^L}{\partial \delta} + \beta u'(C_{1,t}^L)\frac{\partial C_{1,t}^L}{\partial \delta}\right)}{u'(C_{0,t}^L)\frac{\partial C_{0,t}^L}{\partial P_t^*} + \beta u'(C_{1,t}^L)\frac{\partial C_{1,t}^L}{\partial P_t^*}} \tag{1.28}$$

(i) Suppose the buyer is constrained.
Then

$$\begin{aligned} C_{0,t}^L &= y_{0,t}(1+\delta) - P_t^* \\ C_{1,t}^L &= y_{1,t} - R_{t+1} - (1+r_t)\delta y_{0,t} + P_t^* \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial C_{0,t}^L}{\partial \delta} &= y_{0,t} \\ \frac{\partial C_{1,t}^L}{\partial \delta} &= -y_{0,t}(1+r_t) \\ \frac{\partial C_{0,t}^L}{\partial P_t^*} &= -1 \\ \frac{\partial C_{1,t}^L}{\partial P_t^*} &= 1 \end{aligned}$$

So, from (1.28)

$$\begin{aligned} \frac{dP_t}{d\delta} &= \frac{-\left(u'(C_{0,t}^L)y_{0,t} - y_{0,t}\beta(1+r_t)u'(C_{1,t}^L)\right)}{-u'(C_{0,t}^L) + \beta u'(C_{1,t}^L)} \\ &= \frac{y_{0,t}\left(u'(C_{0,t}^L) - \beta(1+r_t)u'(C_{1,t}^L)\right)}{u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)} \end{aligned}$$

As the agent is constrained,

$$u'(C_{0,t}^L) > \beta(1+r_t)u'(C_{1,t}^L) > \beta u'(C_{1,t}^L)$$

Hence both numerator and denominator are positive, giving $\frac{dP_t^c}{d\delta} > 0$
(ii) Suppose the buyer is unconstrained.
Then $u'(C_{0,t}^L) = \beta(1+r_t)u'(C_{1,t}^L)$ and so from (1.28) we have

$$\begin{aligned}\frac{dP_t}{d\delta} &= \frac{-\left(\beta(1+r_t)u'(C_{1,t}^L)\frac{\partial C_{0,t}^L}{\partial\delta} + \beta u'(C_{1,t}^L)\frac{\partial C_{1,t}^L}{\partial\delta}\right)}{\beta(1+r_t)u'(C_{1,t}^L)\frac{\partial C_{0,t}^L}{\partial P_t^*} + \beta u'(C_{1,t}^L)\frac{\partial C_{1,t}^L}{\partial P_t^*}} \\ &= \frac{-\left((1+r_t)\frac{\partial C_{0,t}^L}{\partial\delta} + \frac{\partial C_{1,t}^L}{\partial\delta}\right)}{(1+r_t)\frac{\partial C_{0,t}^L}{\partial P_t^*} + \frac{\partial C_{1,t}^L}{\partial P_t^*}}\end{aligned}$$

For the unconstrained buyer

$$\begin{aligned}(1+r_t)C_{0,t}^L + C_{1,t}^L &\equiv (1+r_t)\left[y_{0,t}(1+\delta) - P_t^*\right] + y_{1,t} - R_{t+1} - (1+r_t)\delta y_{0,t} + P_t^* \\ &\equiv (1+r_t)\left[y_{0,t} - P_t^*\right] + y_{1,t} - R_{t+1} + P_t^* \\ &\equiv (1+r_t)y_{0,t} + y_{1,t} - R_{t+1} - r_tP_t^*\end{aligned}$$

Thus

$$\begin{aligned}(1+r_t)\frac{\partial C_{0,t}^L}{\partial\delta} + \frac{\partial C_{1,t}^L}{\partial\delta} &= 0 \\ (1+r_t)\frac{\partial C_{0,t}^L}{\partial P_t^*} + \frac{\partial C_{1,t}^L}{\partial P_t^*} &= -r_t\end{aligned}$$

Hence $\frac{dP_t^u}{d\delta} = 0$
This completes the unconstrained case and the proof of the proposition. ■

LTI: Constrained vs Unconstrained Response to Change in P_{t+1}

Proposition 44 *Suppose the home-buyer is constrained in equilibrium with price P_t^c. Then*

$$0 < \frac{dP_t^c}{dP_{t+1}^c} < \frac{1}{1+r_t}$$

Suppose the home-buyer is unconstrained in equilibrium with price P_t^u. Then

$$\frac{dP_t^u}{dP_{t+1}^u} = \frac{1}{1+r_t}$$

Proof. Similarly to the proof of Proposition 6 we'll have

$$\frac{dP_t}{dP_{t+1}} = \frac{-\left(u'(C^L_{0,t})\frac{\partial C^L_{0,t}}{\partial P_{t+1}} + \beta u'(C^L_{1,t})\frac{\partial C^L_{1,t}}{\partial P_{t+1}}\right)}{u'(C^L_{0,t})\frac{\partial C^L_{0,t}}{\partial P_t} + \beta u'(C^L_{1,t})\frac{\partial C^L_{1,t}}{\partial P_t}} \tag{1.29}$$

(i) Suppose the buyer is constrained.
Then

$$\begin{aligned} C^L_{0,t} &= y_{0,t}(1+\delta) - P_t \\ C^L_{1,t} &= y_{1,t} - R_{t+1} - (1+r_t)\delta y_{0,t} + P_{t+1} \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial C^L_{0,t}}{\partial P_{t+1}} &= 0 \\ \frac{\partial C^L_{1,t}}{\partial P_{t+1}} &= 1 \\ \frac{\partial C^L_{0,t}}{\partial P_t} &= -1 \\ \frac{\partial C^L_{1,t}}{\partial P_t} &= 0 \end{aligned}$$

So, from (1.29)

$$\begin{aligned} \frac{dP_t}{dP_{t+1}} &= \frac{-\beta u'(C^L_{1,t})}{-u'(C^L_{0,t})} \\ &= \frac{\left(\frac{1}{1+r_t}\right)}{\left(\frac{u'(C^L_{0,t})}{(1+r_t)\beta u'(C^L_{1,t})}\right)} \in \left(0, \frac{1}{(1+r_t)}\right) \end{aligned}$$

Where we have used the fact that the denominator is greater than 1 as the agent is constrained.
(ii) Suppose the buyer is unconstrained.
Then $u'(C^L_{0,t}) = \beta(1+r_t)u'(C^L_{1,t})$ and so from (1.29) we have

$$\frac{dP_t}{dP_{t+1}} = \frac{-\left((1+r_t)\frac{\partial C^L_{0,t}}{\partial P_{t+1}} + \frac{\partial C^L_{1,t}}{\partial P_{t+1}}\right)}{(1+r_t)\frac{\partial C^L_{0,t}}{\partial P_t} + \frac{\partial C^L_{1,t}}{\partial P_t}}$$

For the unconstrained buyer

$$\begin{aligned} (1+r_t)C^L_{0,t} + C^L_{1,t} &\equiv (1+r_t)\left[y_{0,t}(1+\delta) - P_t\right] + y_{1,t} - R_{t+1} - (1+r_t)\delta y_{0,t} + P_{t+1} \\ &\equiv (1+r_t)\left[y_{0,t} - P_t\right] + y_{1,t} - R_{t+1} + P_{t+1} \end{aligned}$$

Thus

$$\begin{aligned}(1+r_t)\frac{\partial C^L_{0,t}}{\partial P_{t+1}}+\frac{\partial C^L_{1,t}}{\partial P_{t+1}} &= 1\\ (1+r_t)\frac{\partial C^L_{0,t}}{\partial P_t}+\frac{\partial C^L_{1,t}}{\partial P_t} &= -(1+r_t)\end{aligned}$$

Hence $\frac{dP_t}{dP_{t+1}}=\frac{-1}{-(1+r_t)}=\frac{1}{(1+r_t)}$

This completes the unconstrained case and the proof of the proposition. ■

LTI: Constrained vs Unconstrained Response to Change in Interest Rate

Proposition 45 *Let P^c_t be the equilibrium house price with a buyer constrained by an LTI constraint, and P^u_t be the equilibrium house price when the buyer does not face a down-payment constraint. Then*

$$0>\left(\frac{dP^c_t}{dr_t}\right)\frac{1}{P^c_t}>\left(\frac{dP^u_t}{dr_t}\right)\frac{1}{P^u_t}$$

The proof is highly involved so we proceed via a number of lemmas. We first define the following functions.

For the constrained agent, let

$$g^c(P^{c,*}_t(r_t),r_t):=u(C^{L,c}_{1,t})+\beta u(C^{L,c}_{1,t})+u_L+\beta u_R-\big(u(C^R_{0,t})+\beta u(C^R_{1,t})+(1+\beta)u_R\big)$$

And for the unconstrained agent let

$$g^u(P^{u,*}_t(r_t),r_t):=u(C^{L,u}_{0,t})+\beta u(C^{L,u}_{1,t})+u_L+\beta u_R-\big(u(C^R_{0,t})+\beta u(C^R_{1,t})+(1+\beta)u_R\big)$$

In g^c when the agent is buying $b_{0,t}\equiv 0$ so the agent cannot borrow or save. In g^u, $b_{0,t}$ can take any value when the agent buys, thus they are unconstrained.

Then we have

$$\begin{aligned}\left(\frac{dP^{c,*}_t}{dr_t}\right)\frac{1}{P^{c,*}_t} &= \frac{-\frac{\partial g^c(P^{c,*}_t(r_t),r_t)}{\partial r_t}}{\frac{\partial g^c(P^{c,*}_t(r_t),r_t)}{\partial P^{c,*}_t}P^{c,*}_t}\\ \left(\frac{dP^{u,*}_t}{dr_t}\right)\frac{1}{P^{u,*}_t} &= \frac{-\frac{\partial g^u(P^{u,*}_t(r_t),r_t)}{\partial r_t}}{\frac{\partial g^u(P^{u,*}_t(r_t),r_t)}{\partial P^{u,*}_t}P^{u,*}_t}\end{aligned}$$

Lemma 46 *When $b^*_{0,t}=0$ in the 'constrained' model (i.e. desired borrowing is 0) then*

$$\left(\frac{dP^{c,*}_t}{dr_t}\right)\frac{1}{P^{c,*}_t}=\left(\frac{dP^{u,*}_t}{dr_t}\right)\frac{1}{P^{u,*}_t}$$

Proof. When $b^*_{0,t} = 0$ the agent is just unconstrained. Thus the constrained and unconstrained buyers would be solving an identical problem so $P_t^{c,*} \equiv P_t^{u,*}$. Thus

$$\frac{dP_t^{c,*}}{dr_t} = \frac{dP_t^{u,*}}{dr_t}$$

This completes the proof of the lemma. ■

Lemma 47 *In the constrained model with desired* $b_0 \leq 0$ *(i.e. the agent wishes to borrow unsecured but cannot):*

$$\frac{d\left[u'(C^L_{0,t}) - \beta u'(C^L_{1,t})(1+r_t)\right]}{d\delta} < 0$$

This says that as δ *is decreased (the LTI constraint is tightened), the agent becomes more constrained. In particular, it follows that if the agent is constrained and* δ *is decreased, then the agent will still be constrained.*

Proof. As the agent wishes to borrow unsecured but cannot

$$u'(C^L_{0,t}) > \beta u'(C^L_{1,t})(1+r_t)$$

Now

$$\begin{aligned} & \frac{d\left[u'(C^L_{0,t}) - \beta u'(C^L_{1,t})(1+r_t)\right]}{d\delta} \\ = \; & u''(C^L_{0,t})\frac{dC^L_{0,t}}{d\delta} - \beta(1+r_t)u''(C^L_{1,t})\frac{dC^L_{1,t}}{d\delta} \end{aligned}$$

As we're in the constrained model

$$\begin{aligned} C^L_{0,t} &= y_{0,t}(1+\delta) - P^*_t \\ C^L_{1,t} &= y_{1,t} - R_{t+1} - (1+r_t)\delta y_{0,t} + P^*_t \end{aligned}$$

Hence

$$\begin{aligned} \frac{dC^L_{0,t}}{d\delta} &= y_{0,t} - \frac{dP^*_t}{d\delta} \\ \frac{dC^L_{1,t}}{d\delta} &= -y_{0,t}(1+r_t) + \frac{dP^*_t}{d\delta} \end{aligned}$$

From the proof of Proposition 1.28

$$\frac{dP_t}{d\delta} = \frac{y_{0,t}\left(u'(C^L_{0,t}) - \beta(1+r_t)u'(C^L_{1,t})\right)}{u'(C^L_{0,t}) - \beta u'(C^L_{1,t})}$$

Thus

$$
\begin{aligned}
\frac{dC^L_{0,t}}{d\delta} &= y_{0,t} - \frac{y_{0,t}\left(u'(C^L_{0,t}) - \beta(1+r_t)u'(C^L_{1,t})\right)}{u'(C^L_{0,t}) - \beta u'(C^L_{1,t})} \\
&= \frac{y_{0,t}\left[u'(C^L_{0,t}) - \beta u'(C^L_{1,t}) - \left(u'(C^L_{0,t}) - \beta(1+r_t)u'(C^L_{1,t})\right)\right]}{u'(C^L_{0,t}) - \beta u'(C^L_{1,t})} \\
&= \frac{y_{0,t}u'(C^L_{1,t})\left[-\beta + \beta(1+r_t)\right]}{u'(C^L_{0,t}) - \beta u'(C^L_{1,t})} = \frac{y_{0,t}\beta r_t u'(C^L_{1,t})}{u'(C^L_{0,t}) - \beta u'(C^L_{1,t})} > 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{dC^L_{1,t}}{d\delta} &= -y_{0,t}(1+r_t) + \frac{y_{0,t}\left(u'(C^L_{0,t}) - \beta(1+r_t)u'(C^L_{1,t})\right)}{u'(C^L_{0,t}) - \beta u'(C^L_{1,t})} \\
&= \frac{y_{0,t}\left(u'(C^L_{0,t}) - \beta(1+r_t)u'(C^L_{1,t})\right) - y_{0,t}(1+r_t)\left(u'(C^L_{0,t}) - \beta u'(C^L_{1,t})\right)}{u'(C^L_{0,t}) - \beta u'(C^L_{1,t})} \\
&= \frac{y_{0,t}u'(C^L_{0,t}) - y_{0,t}(1+r_t)u'(C^L_{0,t})}{u'(C^L_{0,t}) - \beta u'(C^L_{1,t})} = \frac{-ry_{0,t}u'(C^L_{0,t})}{u'(C^L_{0,t}) - \beta u'(C^L_{1,t})} < 0
\end{aligned}
$$

Where for both we have used that as the agent wishes to borrow unsecured, but cannot, $u'(C^L_{0,t}) > \beta u'(C^L_{1,t})(1+r_t) > \beta u'(C^L_{1,t})$. We see that an increase in δ (an easing of the LTI constraint) thus shifts resources from old to young for a constrained buyer, making them less constrained.

To show this formally, putting the parts together we have

$$
\begin{aligned}
&\frac{d\left[u'(C^L_{0,t}) - \beta u'(C^L_{1,t})(1+r_t)\right]}{d\delta} \\
= \; & u''(C^L_{0,t})\frac{y_{0,t}\beta r_t u'(C^L_{1,t})}{u'(C^L_{0,t}) - \beta u'(C^L_{1,t})} + \beta(1+r_t)u''(C^L_{1,t})\frac{ry_{0,t}u'(C^L_{0,t})}{u'(C^L_{0,t}) - \beta u'(C^L_{1,t})} < 0
\end{aligned}
$$

Where we have used the fact that $u''(.) < 0$.

This completes the proof of the lemma. ■

Lemma 48

$$
\begin{aligned}
\frac{\partial g^c(P^{c,*}_t(r_t), r_t)}{\partial r_t} &= -\beta y_{0,t}\delta u'(C^L_{1,t}) - T(r_t) \\
\frac{d\left[\frac{\partial g^c(P^{c,*}_t(r_t), r_t)}{\partial r_t}\right]}{d\delta} &= -\beta y_{0,t}\left(u'(C^c_{1,t}) - \frac{r\delta y_{0,t}u''(C^L_{1,t})u'(C^L_{0,t})}{u'(C^L_{0,t}) - \beta u'(C^L_{1,t})}\right)
\end{aligned}
$$

Where (as with the proof in the LTV case)

$$
T(r_t) := u'(C^R_{0,t})\frac{\partial C^R_{0,t}}{\partial r_t} + \beta u'(C^R_{1,t})\frac{\partial C^R_{1,t}}{\partial r_t}
$$

Proof.

$$\frac{\partial g^c(P_t^{c,*}(r_t), r_t)}{\partial r_t} = u'(C_{0,t}^L)\frac{\partial C_{0,t}^L}{\partial r_t} + \beta u'(C_{1,t}^L)\frac{\partial C_{1,t}^L}{\partial r_t}$$

With a constrained buyer

$$\begin{aligned} C_{0,t}^L &= y_{0,t}(1+\delta) - P_t^* \\ C_{1,t}^L &= y_{1,t} - R_{t+1} - (1+r_t)\delta y_{0,t} + P_t^* \end{aligned}$$

So

$$\begin{aligned} \frac{\partial C_{0,t}^L}{\partial r_t} &= 0 \\ \frac{\partial C_{1,t}^L}{\partial r_t} &= -\delta y_{0,t} \end{aligned}$$

This gives

$$\frac{\partial g^c(P_t^{c,*}(r_t), r_t)}{\partial r_t} = -\beta y_{0,t}\delta u'(C_{1,t}^L) - T(r_t)$$

It follows that

$$\frac{d\left[\frac{\partial g^c(P_t^{c,*}(r_t), r_t)}{\partial r_t}\right]}{d\delta} = -\beta y_{0,t}u'(C_{1,t}^L) - \beta y_{0,t}\delta u''(C_{1,t}^L)\frac{dC_{1,t}^L}{d\delta}$$

But from Lemma 47

$$\frac{dC_{1,t}^L}{d\delta} = \frac{-ry_{0,t}u'(C_{0,t}^L)}{u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)}$$

Thus

$$\begin{aligned} \frac{d\left[\frac{\partial g^c(P_t^{c,*}(r_t), r_t)}{\partial r_t}\right]}{d\delta} &= -\beta y_{0,t}u'(C_{1,t}^L) + \beta y_{0,t}\delta u''(C_{1,t}^L)\frac{ry_{0,t}u'(C_{0,t}^L)}{u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)} \\ &= -\beta y_{0,t}\left[u'(C_{1,t}^L) - \frac{r\delta y_{0,t}u''(C_{1,t}^L)u'(C_{0,t}^L)}{u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)}\right] \end{aligned}$$

This completes the proof of the Lemma. ■

Lemma 49

$$\begin{aligned} -\frac{\partial g^c(P_t^{c,*}(r_t), r_t)}{\partial P_t^{c,*}}P_t^{c,*} &= P_t^{c,*}\left(u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)\right) \\ \frac{d\left[-\frac{\partial g^c(P_t^{c,*}(r_t), r_t)}{\partial P_t^{c,*}}P_t^{c,*}\right]}{d\delta} &= y_{0,t}\left(u'(C_{0,t}^L) - \beta(1+r_t)u'(C_{1,t}^L)\right) \\ &\quad + \frac{y_{0,t}\beta r P_t^{c,*}\left[u''(C_{0,t}^L)u'(C_{1,t}^L) + u''(C_{1,t}^L)u'(C_{0,t}^L)\right]}{u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)} \end{aligned}$$

Proof. As we have a constrained buyer

$$\begin{aligned}C^L_{0,t} &= y_{0,t}(1+\delta) - P^*_t \\ C^L_{1,t} &= y_{1,t} - R_{t+1} - (1+r_t)\delta y_{0,t} + P^*_t\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial C^L_{0,t}}{\partial P^{c,*}_t} &= -1 \\ \frac{\partial C^L_{1,t}}{\partial P^{c,*}_t} &= 1\end{aligned}$$

Now

$$\begin{aligned}-\frac{\partial g^c(P^{c,*}_t(r_t), r_t)}{\partial P^{c,*}_t} P^{c,*}_t &= -u'(C^L_{0,t})\frac{\partial C^L_{0,t}}{\partial P^{c,*}_t} P^{c,*}_t - \beta u'(C^L_{1,t})\frac{\partial C^L_{1,t}}{\partial P^{c,*}_t} P^{c,*}_t \\ &= u'(C^L_{0,t})P^{c,*}_t - \beta u'(C^L_{1,t})P^{c,*}_t > 0\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{d\left[-\frac{\partial g^c(P^{c,*}_t(r_t), r_t)}{\partial P^{c,*}_t} P^{c,*}_t\right]}{d\delta} \\ =\;& u'(C^L_{0,t})\frac{dP^{c,*}_t}{d\delta} + u''(C^L_{0,t})P^{c,*}_t\frac{dC^L_{0,t}}{d\delta} - \beta u'(C^L_{1,t})\frac{dP^{c,*}_t}{d\delta} - \beta u''(C^L_{1,t})P^{c,*}_t\frac{dC^L_{1,t}}{d\delta} \\ =\;& \left(u'(C^L_{0,t}) - \beta u'(C^L_{1,t})\right)\frac{dP^{c,*}_t}{d\delta} + u''(C^L_{0,t})P^{c,*}_t\frac{dC^L_{0,t}}{d\delta} - \beta u''(C^L_{1,t})P^{c,*}_t\frac{dC^L_{1,t}}{d\delta}\end{aligned}$$

So

$$\begin{aligned}&\frac{d\left[-\frac{\partial g^c(P^{c,*}_t(r_t), r_t)}{\partial P^{c,*}_t} P^{c,*}_t\right]}{d\delta} \\ =\;& \left(u'(C^L_{0,t}) - \beta u'(C^L_{1,t})\right)\frac{y_{0,t}\left(u'(C^L_{0,t}) - \beta(1+r_t)u'(C^L_{1,t})\right)}{u'(C^L_{0,t}) - \beta u'(C^L_{1,t})} + u''(C^L_{0,t})P^{c,*}_t\frac{y_{0,t}\beta r_t u'(C^L_{1,t})}{u'(C^L_{0,t}) - \beta u'(C^L_{1,t})} \\ & + \beta u''(C^L_{1,t})P^{c,*}_t\frac{r y_{0,t} u'(C^L_{0,t})}{u'(C^L_{0,t}) - \beta u'(C^L_{1,t})} \\ =\;& y_{0,t}\left(u'(C^L_{0,t}) - \beta(1+r_t)u'(C^L_{1,t})\right) + \frac{y_{0,t}\beta r_t P^{c,*}_t\left[u''(C^L_{0,t})u'(C^L_{1,t}) + u''(C^L_{1,t})u'(C^L_{0,t})\right]}{u'(C^L_{0,t}) - \beta u'(C^L_{1,t})}\end{aligned}$$

where we've used results from Lemma 47.

This completes the proof of the lemma. ■

Lemma 50

$$\frac{d\left[\left(\frac{dP_t^{c,*}}{dr_t}\right)\frac{1}{P_t^{c,*}}\right]}{d\delta} \equiv \frac{d\left[\frac{-\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial r_t}}{\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial P_t^{c,*}}P_t^{c,*}}\right]}{d\delta} < 0$$

Proof. Let

$$\begin{aligned} W(\delta) &: = \frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial r_t} \\ V(\delta) &: = -\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial P_t^{c,*}}P_t^{c,*} \end{aligned}$$

Then

$$\frac{d\left[\frac{-\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial r_t}}{\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial P_t^{c,*}}P_t^{c,*}}\right]}{d\delta} = \left(\frac{W(\delta)}{V(\delta)}\right)' = \frac{W'(\delta)V(\delta) - W(\delta)V'(\delta)}{V(\delta)^2}$$

We show that $W'(\delta)V(\delta) < W(\delta)V'(\delta)$

From Lemmas 48,49 we have

$$\begin{aligned} W(\delta) &= -\beta y_{0,t}\delta u'(C_{1,t}^L) - T(r_t) \\ W'(\delta) &= -\beta y_{0,t}u'(C_{1,t}^L) - \beta y_{0,t}\delta u''(C_{1,t}^L)\frac{dC_{1,t}^L}{d\delta} \\ V(\delta) &= P_t^{c,*}\left(u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)\right) \\ V'(\delta) &= \left(u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)\right)\frac{dP_t^{c,*}}{d\delta} + u''(C_{0,t}^L)P_t^{c,*}\frac{dC_{0,t}^L}{d\delta} - \beta u''(C_{1,t}^L)P_t^{c,*}\frac{dC_{1,t}^L}{d\delta} \end{aligned}$$

Thus $W(\delta)V'(\delta) > W'(\delta)V(\delta)$ holds iff

$$\begin{aligned} &\left[\left(u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)\right)\frac{dP_t^{c,*}}{d\delta} + u''(C_{0,t}^L)P_t^{c,*}\frac{dC_{0,t}^L}{d\delta} - \beta u''(C_{1,t}^L)P_t^{c,*}\frac{dC_{1,t}^L}{d\delta}\right] \\ &\cdot\left(-\beta y_{0,t}\delta u'(C_{1,t}^L) - T(r_t)\right) \\ > \; &\left(-\beta y_{0,t}u'(C_{1,t}^L) - \beta y_{0,t}\delta u''(C_{1,t}^L)\frac{dC_{1,t}^L}{d\delta}\right)P_t^{c,*}\left(u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)\right) \end{aligned}$$

We gather the terms in $u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)$ together on the RHS. Then $W(\delta)V'(\delta) > W'(\delta)V(\delta)$ iff

$$\begin{aligned} &\left[u''(C_{0,t}^L)P_t^{c,*}\frac{dC_{0,t}^L}{d\delta} - \beta u''(C_{1,t}^L)P_t^{c,*}\frac{dC_{1,t}^L}{d\delta}\right]\left(-\beta y_{0,t}\delta u'(C_{1,t}^L) - T(r_t)\right) \\ > \; &\left[\left(-\beta y_{0,t}u'(C_{1,t}^L) - \beta y_{0,t}\delta u''(C_{1,t}^L)\frac{dC_{1,t}^L}{d\delta}\right)P_t^{c,*} + \frac{dP_t^{c,*}}{d\delta}\left(\beta y_{0,t}\delta u'(C_{1,t}^L) + T(r_t)\right)\right] \\ &\cdot\left(u'(C_{0,t}^L) - \beta u'(C_{1,t}^L)\right) \end{aligned}$$

Now from the proof of Lemma 49

$$u''(C^L_{0,t})P^{c,*}_t\frac{dC^L_{0,t}}{d\delta}-\beta u''(C^L_{1,t})P^{c,*}_t\frac{dC^L_{1,t}}{d\delta}=\frac{y_{0,t}\beta r_t P^{c,*}_t\left[u''(C^L_{0,t})u'(C^L_{1,t})+u''(C^L_{1,t})u'(C^L_{0,t})\right]}{u'(C^L_{0,t})-\beta u'(C^L_{1,t})}<0$$

Hence, the LHS of the expression is positive (noting from the Lemma 26 that $T(r_t)\geq 0$). As usual, with the agent desiring to borrow unsecured, $u'(C^L_{0,t})>(1+r_t)\beta u'(C^L_{1,t})>\beta u'(C^L_{1,t})$. Thus, to prove the lemma it is sufficient to establish that

$$\left(-\beta y_{0,t}u'(C^L_{1,t})-\beta y_{0,t}\delta u''(C^L_{1,t})\frac{dC^L_{1,t}}{d\delta}\right)P^{c,*}_t+\frac{dP^{c,*}_t}{d\delta}\left(\beta y_{0,t}\delta u'(C^L_{1,t})+T(r_t)\right)\leq 0$$

But

$$\frac{dC^L_{1,t}}{d\delta}=-y_{0,t}(1+r_t)+\frac{dP^*_t}{d\delta}$$

So it's enough to establish that

$$\left(-\beta y_{0,t}u'(C^L_{1,t})-\beta y_{0,t}\delta u''(C^L_{1,t})\left(-y_{0,t}(1+r_t)+\frac{dP^*_t}{d\delta}\right)\right)P^{c,*}_t+\frac{dP^{c,*}_t}{d\delta}\left(\beta y_{0,t}\delta u'(C^L_{1,t})+T(r_t)\right)\leq 0$$

iff

$$\begin{aligned}&\frac{dP^{c,*}_t}{d\delta}\left[\left(\beta y_{0,t}\delta u'(C^L_{1,t})+T(r_t)\right)-\beta y_{0,t}\delta u''(C^L_{1,t})P^{c,*}_t\right]\\ \leq\ & P^{c,*}_t\left(\beta y_{0,t}u'(C^L_{1,t})-\beta y_{0,t}\delta u''(C^L_{1,t})y_{0,t}(1+r_t)\right)\end{aligned}$$

iff

$$\begin{aligned}&\frac{dP^{c,*}_t}{d\delta}\left[\left(\beta y_{0,t}\delta u'(C^L_{1,t})+T(r_t)\right)\right]-P^{c,*}_t\beta y_{0,t}u'(C^L_{1,t})\\ \leq\ & -P^{c,*}_t\beta y_{0,t}\delta u''(C^L_{1,t})\left(y_{0,t}\left(1+r_t\right)-\frac{dP^{c,*}_t}{d\delta}\right)\end{aligned}$$

Now, using from the proof of Lemma 47

$$\begin{aligned}&y_{0,t}\left(1+r_t\right)-\frac{dP^{c,*}_t}{d\delta}\\ =\ &\frac{y_{0,t}r_t u'(C^L_{0,t})}{u'(C^L_{0,t})-\beta u'(C^L_{1,t})}>0\end{aligned}$$

Given $u''(C^L_{1,t})<0$ the RHS is positive. Thus it's sufficient to show that the LHS ≤ 0 :

$$\begin{aligned}\frac{dP^{c,*}_t}{d\delta}\left[\left(\beta y_{0,t}\delta u'(C^L_{1,t})+T(r_t)\right)\right]-P^{c,*}_t\beta y_{0,t}u'(C^L_{1,t})&\leq 0\text{ iff}\\ \frac{\left(u'(C^L_{0,t})-\beta(1+r_t)u'(C^L_{1,t})\right)}{u'(C^L_{0,t})-\beta u'(C^L_{1,t})}\left[\left(\beta y_{0,t}\delta u'(C^L_{1,t})+T(r_t)\right)\right]&\leq P^{c,*}_t\beta u'(C^L_{1,t})\end{aligned}$$

As $\frac{\left(u'(C^L_{0,t})-\beta(1+r_t)u'(C^L_{1,t})\right)}{u'(C^L_{0,t})-\beta u'(C^L_{1,t})} \in (0,1)$ it's sufficient to show that

$$\left[\left(\beta y_{0,t}\delta u'(C^L_{1,t}) + T(r_t)\right)\right] \leq P^{c,*}_t \beta u'(C^L_{1,t})$$

We consider two cases.

(i) The renter is constrained/the interest rate they face doesn't move with the mortgage rate. Then $T(r_t) = 0$

Thus, we require that

$$\begin{aligned} \beta y_{0,t}\delta u'(C^L_{1,t}) &\leq P^{c,*}_t \beta u'(C^L_{1,t}) \text{ iff} \\ y_{0,t}\delta &\leq P^{c,*}_t \end{aligned}$$

This condition states that some deposit has to be made, and we maintain this assumption. The the lemma holds in this case.

(ii) The renter is unconstrained and their interest rate moves with the mortgage rate. Then from Lemma 26

$$T(r_t) = \beta u'\left(C^R_{1,t}\right)\left[(y_{0,t} - R_t) - C^R_{0,t}\right] \geq 0$$

It's enough to show that

$$T(r_t) \leq \left(P^{c,*}_t - y_{0,t}\delta\right)\beta u'(C^L_{1,t})$$

So, we want to show that

$$\left(P^{c,*}_t - y_{0,t}\delta\right) u'(C^L_{1,t}) \geq u'\left(C^R_{1,t}\right)\left[(y_{0,t} - R_t) - C^R_{0,t}\right]$$

With log utility, this condition becomes

$$\begin{aligned} \frac{\left(P^{c,*}_t - y_{0,t}\delta\right)}{C^L_{1,t}} &\geq \frac{\left[(y_{0,t} - R_t) - C^R_{0,t}\right]}{C^R_{1,t}} \text{iff} \\ \left(P^{c,*}_t - y_{0,t}\delta\right) C^R_{1,t} &\geq \left[(y_{0,t} - R_t) - C^R_{0,t}\right] C^L_{1,t} \text{ iff} \\ & \left(P^{c,*}_t - y_{0,t}\delta\right)\left(y_{1,t} - R_{t+1} + (1+r_t)\left(y_{0,t} - R_t - C^R_{0,t}\right)\right) \\ &\geq \left[(y_{0,t} - R_t) - C^R_{0,t}\right]\left(y_{1,t} - R_{t+1} - (1+r_t)\delta y_{0,t} + P^{c,*}_t\right) \end{aligned}$$

iff

$$\begin{aligned} & \left(P^{c,*}_t - y_{0,t}\delta\right)(y_{1,t} - R_{t+1}) \\ \geq\ & \left[(y_{0,t} - R_t) - C^R_{0,t}\right] \\ & \cdot\left[y_{1,t} - R_{t+1} - (1+r_t)\delta y_{0,t} + P^{c,*}_t - \left(P^{c,*}_t - y_{0,t}\delta\right)(1+r_t)\right] \\ =\ & \left[(y_{0,t} - R_t) - C^R_{0,t}\right]\left[y_{1,t} - R_{t+1} - r_t P^{c,*}_t\right] \end{aligned}$$

iff

$$\begin{aligned}(y_{1,t}-R_{t+1})\left[P_t^{c,*}-y_{0,t}\delta-\left(y_{0,t}-R_t-C_{0,t}^R\right)\right] &\geq \left[\left((y_{0,t}-R_t)-C_{0,t}^R\right)\right]\left(-r_tP_t^{c,*}\right) \text{ iff}\\ (y_{1,t}-R_{t+1})\left[P_t^{c,*}-y_{0,t}(1+\delta)+R_t+C_{0,t}^R\right] &\geq \left[\left((y_{0,t}-R_t)-C_{0,t}^R\right)\right]\left(-r_tP_t^{c,*}\right) \text{ iff}\\ (y_{1,t}-R_{t+1})\left[R_t+C_{0,t}^R-C_{0,t}^L\right] &\geq \left[\left((y_{0,t}-R_t)-C_{0,t}^R\right)\right]\left(-r_tP_t^{c,*}\right)\end{aligned}$$

As the renter is unconstrained $(y_{0,t}-R_t)-C_{0,t}^R \geq 0$ so the RHS ≤ 0. It's thus sufficient that $R_t+C_{0,t}^R \geq C_{0,t}^L$.

But, in the proof of Lemma 28 is was shown that with an unconstrained renter and a constrained buyer we have $C_{0,t}^L < C_{0,t}^R$, and this result also holds here. This thus completes the proof for the unconstrained renter case.

This completes the proof of the lemma. ■

Lemma 51

$$0 > \left(\frac{dP_t^{c,*}}{dr_t}\right)\frac{1}{P_t^{c,*}}$$

Proof.

$$\left(\frac{dP_t^{c,*}}{dr_t}\right)\frac{1}{P_t^{c,*}} = \frac{-\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial r_t}r_t}{\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial P_t^{c,*}}P_t^{c,*}}$$

From Lemmas 48,49

$$\begin{aligned}\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial r_t} &= -\beta y_{0,t}\delta u'(C_{1,t}^L)-T(r_t)<0\\ -\frac{\partial g^c(P_t^{c,*}(r_t),r_t)}{\partial P_t^{c,*}}P_t^{c,*} &= P_t^{c,*}\left(u'(C_{0,t}^L)-\beta u'(C_{1,t}^L)\right)>0\end{aligned}$$

To see the second inequality, note that as the agent is constrained when buying

$$u'(C_{0,t}^L) > \beta(1+r_t)u'(C_{1,t}^L) > \beta u'(C_{1,t}^L)$$

Thus $0 > \left(\frac{dP_t^{c,*}}{dr_t}\right)\frac{1}{P_t^{c,*}}$. This completes the proof of the lemma. ■

Proof of Proposition 45. From Lemma 46, when the buying agent is just unconstrained

$$\left(\frac{dP_t^{c,*}}{dr_t}\right)\frac{1}{P_t^{c,*}} = \left(\frac{dP_t^{u,*}}{dr_t}\right)\frac{1}{P_t^{u,*}}$$

Let this correspond to $\overline{\delta}$ i.e. $b_{0,t}(\overline{\delta})=0$ (i.e. for desired borrowing for the buying agent). From Lemma 47 as δ is decreased from this point, the agent becomes constrained when buying and is constrained for all lower δ. From Lemma 50 $\left(\frac{dP_t^{c,*}}{dr_t}\right)\frac{1}{P_t^{c,*}}$ increases as δ decreases. Further, $\left(\frac{dP_t^{u,*}}{dr_t}\right)\frac{1}{P_t^{u,*}}$ is clearly unaffected by changes in δ.

Thus for a constrained agent, $\delta < \overline{\delta}$ and

$$\left[\left(\frac{dP_t^{c,*}}{dr_t}\right)\frac{1}{P_t^{c,*}}\right]_{\delta} > \left[\left(\frac{dP_t^{c,*}}{dr_t}\right)\frac{1}{P_t^{c,*}}\right]_{\delta=\overline{\delta}} = \left(\frac{dP_t^{u,*}}{dr_t}\right)\frac{1}{P_t^{u,*}}$$

Finally, from Lemma 51

$$0 > \left(\frac{dP_t^{c,*}}{dr_t}\right)\frac{1}{P_t^{c,*}}$$

This completes the proof of the proposition. ■

1.D Welfare Proofs

1.D.1 Welfare Cost of Boom and Bust Cycle

Here we Prove Proposition 15

Proof. Lifetime utility is given by,

$$\begin{aligned} & V(P_{t+1}, P_{t+1}+x) \\ = \; & u\left(y_{0,t} - \gamma P_t(P_{t+1}) - b_{0,t}(P_{t+1})\right) \\ & +\beta u\left(y_{1,t} - R + (1+r_t)\left[b_{0,t}(P_{t+1}) - (1-\gamma)P_t(P_{t+1})\right] + P_{t+1} + x\right) + u_L + \beta u_R \end{aligned}$$

Where we note that the price they bought for, $P_t(P_{t+1})$ and the saving $b_{0,t}(P_{t+1})$ depends on the price they expected to sell for (with $b_{0,t}(P_{t+1}) = 0$ if they're constrained).

Then

$$\begin{aligned} & V(P_{t+1}, P_{t+1}+x) - V(P_{t+1}, P_{t+1}) \\ = \; & \beta u\left(y_{1,t} - R + (1+r_t)\left[b_{0,t}(P_{t+1}) - (1-\gamma)P_t(P_{t+1})\right] + P_{t+1} + x\right) \\ & -\beta u\left(y_{1,t} - R + (1+r_t)\left[b_{0,t}(P_{t+1}) - (1-\gamma)P_t(P_{t+1})\right] + P_{t+1}\right) \end{aligned}$$

This is positive as $x > 0$ (the resale price is greater than expected) and $u'(.) > 0$.

Similarly

$$\begin{aligned} & V(P_{t+1}+x, P_{t+1}) - V(P_{t+1}+x, P_{t+1}+x) \\ = \; & \beta u\left(y_{1,t} - R + (1+r_t)\left[b_{0,t}(P_{t+1}+x) - (1-\gamma)P_t(P_{t+1}+x)\right] + P_{t+1}\right) \\ & -\beta u\left(y_{1,t} - R + (1+r_t)\left[b_{0,t}(P_{t+1}+x) - (1-\gamma)P_t(P_{t+1}+x)\right] + P_{t+1} + x\right) \end{aligned}$$

This is negative as they sell for P_{t+1} having expected to sell at the boom price of $P_{t+1} + x$.

Let

$$f(x) := \left[V(P_{t+1}, P_{t+1}+x) - V(P_{t+1}, P_{t+1})\right] + \left[V(P_{t+1}+x, P_{t+1}) - V(P_{t+1}+x, P_{t+1}+x)\right]$$

We show that $f(x) < 0$ for $x > 0$ and $f'(x) < 0$, establishing the result.

We first show that $f(0) = 0:$

$$\begin{aligned} f(0) &= \beta u\left(y_{1,t} - R + (1+r_t)\left[b_{0,t}(P_{t+1}) - (1-\gamma)P_t(P_{t+1})\right] + P_{t+1}\right) \\ &\quad -\beta u\left(y_{1,t} - R + (1+r_t)\left[b_{0,t}(P_{t+1}) - (1-\gamma)P_t(P_{t+1})\right] + P_{t+1}\right) \\ &\quad +\beta u\left(y_{1,t} - R + (1+r_t)\left[b_{0,t}(P_{t+1}) - (1-\gamma)P_t(P_{t+1})\right] + P_{t+1}\right) \\ &\quad -\beta u\left(y_{1,t} - R + (1+r_t)\left[b_{0,t}(P_{t+1}) - (1-\gamma)P_t(P_{t+1})\right] + P_{t+1}\right) \\ &= 0 \end{aligned}$$

When there are no shocks, lifetime utility is unaffected for both.

We show that $f'(x) < 0$ for $x > 0$ which then also ensures that $f(x) < 0$ for $x > 0$.

$$\begin{aligned} f'(x) &= \beta u'\left(y_{1,t} - R + (1+r_t)\left[b_{0,t}(P_{t+1}) - (1-\gamma)P_t(P_{t+1})\right] + P_{t+1} + x\right) \\ &\quad +\beta u'\left(y_{1,t} - R + (1+r_t)\left[b_{0,t}(P_{t+1}+x) - (1-\gamma)P_t(P_{t+1}+x)\right] + P_{t+1}\right) \\ &\quad \cdot(1+r_t)\left[b'_{0,t}(P_{t+1}+x) - (1-\gamma)P'_t(P_{t+1}+x)\right] \\ &\quad -\beta u'\left(y_{1,t} - R + (1+r_t)\left[b_{0,t}(P_{t+1}+x) - (1-\gamma)P_t(P_{t+1}+x)\right] + P_{t+1} + x\right) \\ &\quad \cdot\left[(1+r_t)\left(b'_{0,t}(P_{t+1}+x) - (1-\gamma)P'_t(P_{t+1}+x)\right) + 1\right] \end{aligned}$$

So

$$\begin{aligned} f'(x) &= \left[\begin{array}{c} \beta u'\left(y_{1,t} - R + (1+r_t)\left[b_{0,t}(P_{t+1}) - (1-\gamma)P_t(P_{t+1})\right] + P_{t+1} + x\right) \\ -\beta u'\left(y_{1,t} - R + (1+r_t)\left[b_{0,t}(P_{t+1}+x) - (1-\gamma)P_t(P_{t+1}+x)\right] + P_{t+1} + x\right) \end{array}\right] \\ &\quad +(1+r_t)\left[(1-\gamma)P'_t(P_{t+1}+x) - b'_{0,t}(P_{t+1}+x)\right] \\ &\quad \cdot\left[\begin{array}{c} \beta u'\left(y_{1,t} - R + (1+r_t)\left[b_{0,t}(P_{t+1}+x) - (1-\gamma)P_t(P_{t+1}+x)\right] + P_{t+1} + x\right) \\ -\beta u'\left(y_{1,t} - R + (1+r_t)\left[b_{0,t}(P_{t+1}+x) - (1-\gamma)P_t(P_{t+1}+x)\right] + P_{t+1}\right) \end{array}\right] \end{aligned}$$

We now show that $b'_{0,t}(P_{t+1}+x) \leq 0$. First suppose the buyer is constrained. Then $b_{0,t}(P_{t+1}+x) \equiv 0, b'_{0,t}(P_{t+1}+x) = 0$.

Otherwise if they are unconstrained, we have $u'(C^L_{0,t}) = \beta(1+r_t)u'(C^L_{1,t})$. Suppose P_{t+1} increases. Then in equilibrium, non-housing consumption in both periods is unchanged. To see this, suppose $C^L_{1,t}$ increases, then from the Euler equation, $C^L_{0,t}$ must increase as well. But then lifetime utility has increased when buying, which is a contradiction as it is fixed at the same level as the lifetime utility from renting. Similarly, $C^L_{1,t}$ cannot decrease in equilibrium. Thus $C^L_{1,t}, C^L_{0,t}$ are unchanged in equilibrium following a change in P_{t+1}.

Now $C^L_{0,t} \equiv y_{0,t} - \gamma P_t(P_{t+1}) - b_{0,t}(P_{t+1})$. Hence, totally differentiating wrt $x : -\gamma P'_t(P_{t+1}+x) - b'_{0,t}(P_{t+1}+x) = 0$, giving

$$b'_{0,t}(P_{t+1}+x) = -\gamma P'_t(P_{t+1}+x) = \frac{-\gamma}{(1+r_t)} < 0$$

Where we have used that $P_t'(P_{t+1}+x) = \frac{1}{1+r_t}$ for the unconstrained buyer. So, in both cases $b_{0,t}'(P_{t+1}+x) \leq 0$.

Now, as shown in the appendix, for both constrained and unconstrained buyers, $P_t'(P_{t+1}+x) > 0$ and so $P_t(P_{t+1}+x) > P_t(P_{t+1})$. Then, noting that for $x > 0$ $b_{0,t}(P_{t+1}+x) \leq b_{0,t}(P_{t+1})$, the first [.] term is then negative as consumption is greater in the top expression (having paid a lower price for the house and sold at the same price) and $u''(.) < 0$.

Similarly the second [.] term is negative as consumption is higher in the top expression, due to the same price being paid and a higher resale price. Further $(1-\gamma)P_t'(P_{t+1}+x) - b_{0,t}'(P_{t+1}+x) > 0$ regardless of whether the buyer is constrained or not. Thus $f'(x) < 0$. This completes the proof of the proposition. ■

1.D.2 Benefit of Policy

Here we prove Proposition 16
Proof. Clearly, as $\overline{\gamma} := \gamma(P_{t+1})$,

$$[V(P_{t+1}, P_{t+1}+x, \gamma(P_{t+1})) - V(P_{t+1}, P_{t+1}, \gamma(P_{t+1}))] = [V(P_{t+1}, P_{t+1}+x, \overline{\gamma}) - V(P_{t+1}, P_{t+1}, \overline{\gamma})]$$

Thus it is sufficient to prove that the cost from the bust is worse without regulation:

$$\begin{aligned} & [V(P_{t+1}+x, P_{t+1}, \gamma(P_{t+1}+x)) - V(P_{t+1}+x, P_{t+1}+x, \gamma(P_{t+1}+x))] \\ < \; & [V(P_{t+1}+x, P_{t+1}, \overline{\gamma}) - V(P_{t+1}+x, P_{t+1}+x, \overline{\gamma})] \end{aligned}$$

Now, as with the prior proof (in constrained case)

$$\begin{aligned} & V(P_{t+1}+x, P_{t+1}, \gamma(P_{t+1}+x)) - V(P_{t+1}+x, P_{t+1}+x, \gamma(P_{t+1}+x)) \\ = \; & \beta u\left(y_{1,t} - R - (1+r_t)(1-\gamma(P_{t+1}+x))P_t(P_{t+1}+x, \gamma(P_{t+1}+x)) + P_{t+1}\right) \\ & -\beta u\left(y_{1,t} - R - (1+r_t)(1-\gamma(P_{t+1}+x))P_t(P_{t+1}+x, \gamma(P_{t+1}+x)) + P_{t+1}+x\right) \end{aligned}$$

where we have written $P_t(P_{t+1}+x, \gamma(P_{t+1}+x))$ to emphasise that P_{t+1} affects P_t not only directly through the usual channel, but also indirectly through changing γ.

When policy constrains $\gamma = \overline{\gamma} = \gamma(P_{t+1})$ the utility cost in the bust is given by

$$\begin{aligned} & V(P_{t+1}+x, P_{t+1}, \gamma(P_{t+1})) - V(P_{t+1}+x, P_{t+1}+x, \gamma(P_{t+1})) \\ = \; & \beta u\left(y_{1,t} - R - (1+r_t)(1-\gamma(P_{t+1}))P_t(P_{t+1}+x, \gamma(P_{t+1})) + P_{t+1}\right) \\ & -\beta u\left(y_{1,t} - R - (1+r_t)(1-\gamma(P_{t+1}))P_t(P_{t+1}+x, \gamma(P_{t+1})) + P_{t+1}+x\right) \end{aligned}$$

Now for $x > 0$,the total amount to be repaid on the mortgage is higher when γ is endogenous:

$$(1-\gamma(P_{t+1}))P_t(P_{t+1}+x, \gamma(P_{t+1}) < (1-\gamma(P_{t+1}+x))P_t(P_{t+1}+x, \gamma(P_{t+1}+x)) \tag{1.30}$$

This reflects γ being lower, which means a greater percentage of the purchase price needs to be repaid when old. Further, it reflects that with lower γ, the initial price paid was higher as $\frac{\partial P_t}{\partial \gamma} < 0$.

We obtain our result by combining (1.30) with the following claim.

Claim

Let

$$f(z) := u(a - z) - u(b - z)$$

With $u' > 0, u'' < 0$ and $a < b$. Then $z_1 > z_2 \Longrightarrow f(z_1) < f(z_2) < 0$

Proof of Claim

$f(z) < 0$ iff $u(a - z) < u(b - z)$ iff $a - z < b - z$ which is true.

We now establish that $f'(z) < 0$.

$$\begin{aligned} f'(z) &= -u'(a - z) + u'(b - z) \\ &< 0 \text{ iff} \\ u'(b - z) &< u'(a - z) \text{ iff} \\ a - z &< b - z \text{ iff} \\ a &< b \end{aligned}$$

This completes the proof of the claim.

Apply this with

$$\begin{aligned} a &: = y_{1,t} - R + P_{t+1} \\ b &: = y_{1,t} - R + P_{t+1} + x \\ z_1 &: = (1 + r_t)(1 - \gamma(P_{t+1} + x))P_t(P_{t+1} + x, \gamma(P_{t+1} + x)) \\ z_2 &: = (1 + r_t)(1 - \gamma(P_{t+1}))P_t(P_{t+1} + x, \gamma(P_{t+1})) \end{aligned}$$

Then $f(z)$ is the utility cost from the unexpected drop in price, z_1 is the case with endogenous γ, and z_2 is the case with policy preventing the fall in γ.

Then, by (1.30) $z_1 > z_2$ so by the claim $f(z_1) < f(z_2) < 0$ and the utility cost of the bust is lower with policy present. This completes the proof of the proposition. ■

Chapter 2

Empirical Analysis of Housing Model

This chapter introduces a new house price dataset for the US, with separate house price indices for low, middle and high tier houses across 52 cities during the recent boom and bust. Using this data we introduce several new facts about the boom, the bust and the link between them. These facts can be accounted for in a parsimonious way using the theoretical prediction of Chapter 1 that an easing of non-price credit terms will have a relatively greater impact on low tier prices. Using this data we test this implication of the theory, finding statistically and economically significant relationships between two separate measures of changes in credit availability, and relative changes in low and high tier prices, both during the boom and bust. Further, we then augment the analysis of Chapter 1 by examining alternative explanations for the tiered pattern, including changes in housing supply, speculators and differential income growth for low and high tier buyers. We show that these variables are not responsible for the pattern.

2.1 Introduction

Previous studies of the US housing boom have documented the wide dispersion in growth rates experienced across different regions (Davis et al 2007, Van Nieuwerburgh and Weill 2010). For example, many Californian cities experienced nominal growth of more than 200% during the housing boom, whilst in the major cities in Ohio, it was less than 50% (see Table 2.5 in the appendix).

However, the dispersion in growth rates within cities has received much less attention, despite striking variation. In Riverside, CA, the cheapest third of houses ("low-tier houses") grew 166 percentage points (pp.) more than the most expensive third ("high-tier houses") during the boom. By contrast, during the bust, the cheapest third of houses in Atlanta lost 59% of their value, over twice as much as the 22% lost by the most expensive houses. However, more striking than these large differences within cities is the systematic pattern in the within-city variation across cities. In the appendix we use a new housing dataset to provide real house price graphs from January 1997 until September 2012 for 52 metro areas that accounted for 41% of the US population in 1997.[1 2] In 51 of the 52 cities, the low-tier houses had greater price growth than the high tier during the boom[3] (growing 55pp. more on average), whilst the low-tier prices fell by even more in 46 of the 51 cities during the bust[4] (falling 14pp. more on average). The result of this collapse is that at the time of the bust in 2011, the cumulative growth since 1997 was greater for the low tier in only 26 of the 51 cities (down from 50 at the peak).

Even though low-tier house prices had higher growth in 51 of the 52 cities during the boom, there is large variation in the size of this difference, ranging from the 166pp. difference in Riverside, CA to 8pp. in Colorado Springs. Using this measure, we uncover further facts about the recent boom and bust. In the paper we show that the difference in growth rates between cheap and expensive houses increases with the size of the boom in a city. There is also mean-reversion in the tiered pattern, and we establish that cheaper houses have the largest collapse relative to expensive houses in the places where the reverse was true during the boom. Finally we show that, mirroring the pattern during the boom, the relative collapse in low-tier houses is particularly bad in cities with the largest housing bust.

The theory developed in the previous chapter provides a parsimonious explanation of these facts. Recall that the theory predicts that an easing of non-price credit terms will result in greater relative price growth for low rather than high-tier houses. This is because, unlike high tier buyers, low tier buyers are credit constrained, so a relaxation of these constraints only directly affects the price low tier buyers can pay for a house. The price paid by the high-tier buyer also increases due to capital gains being passed on by the housing ladder, though the relative increase is not as great. As we show in the text, from this prediction it follows that the difference between the low and high tier growth rates will increase in the extent of credit easing. Further, where there is a greater degree of credit easing, both low and high tier prices will grow by more, increasing the growth of the aggregate house price level. Thus, with credit easing, we'd expect the difference between low and high tier growth rates to be greater in places with larger booms. The same logic applies in reverse: all else equal, places with a greater degree of credit tightening will have larger collapses in the aggregate house price index, and will experience low tier prices falling even more than high tier ones. Finally, with a boom caused by credit easing and a bust following subsequent tightening back to pre-boom standards, the places with the biggest easing in credit will have the largest subsequent tightening. With this we'll see the largest differential between low and high tier price growth in the boom matched with the largest difference between their price collapses in the bust.[5]

In the paper we test the predicted relationship between relative house price movements and

[1] The house price data comes from S&P/Case-Shiller and Fiserv, Inc.

[2] The data for Cleveland finishes in 2008.

[3] From 1997-2006.

[4] Defined as 2006-2011.

[5] The role of credit tightening in the bust is likely to be reinforced in practice by an increase in foreclosures due to people on "teaser-rate" mortgages being unable to refinance them.

credit, comparing variation in credit measures with variation in the difference between low and high tier price growth across cities. We do this both with a proxy for changes in credit availability and the Loan-to-Income (LTI) ratio, and perform the analysis both during the housing boom and bust. Both unconditionally, and whilst controlling for other variables, we show statistically significant relationships between both credit measures and relative house price movements, in the direction predicted by theory. The estimates are also economically significant, predicting price changes reassuringly close to those in the data, at the mean. For example, the credit easing proxy predicts that in the average city the low tier will grow 62pp. more than the high tier during the boom, compared with the observed average of 55pp. The same variable predicts the low tier to fall by 13pp. more than the high tier in the bust in the typical city, compared to 14pp. observed in the data.

In the previous chapter we used a model to distinguish between three possible explanations of the pattern observed in tiered housing during the boom, arguing that without an easing of non-price credit terms, we would not have observed the low tier growing relatively more in 51 of the 52 cities. Here we rule out alternative explanations for the pattern not considered in the model, including greater income growth for low tier buyers, a surge in speculators buying low tier houses, and a surge in house building for high tier houses. This is done by comparing variation in these variables with variation in the difference between low and high tier price growth across the cities. We show that none of these alternative explanations can account for the observed pattern, either being statistically insignificant or pushing in the opposite direction, predicting the high tier should have grown more than the low tier during the boom. Ruling out these alternative explanations re-enforces the conclusions of the first chapter that the tiered pattern during the boom provides indirect evidence of a significant effect of credit easing on house prices during the boom.

The rest of this chapter is organised as follows. Section 2.2 explains the housing data in detail and details new facts about the housing boom and bust and their relation to the existing literature. Section 2.3 then introduces the data on other economic variables during the boom and their relation with the tiered pattern, with regression results in Section 2.4. Section 2.5 provides similar analysis for the housing bust and Section 2.6 concludes.

2.2 New Facts About the US Housing Boom and Bust

In this section we discuss new facts about the recent US experience, looking in detail at the variation *within* cities during the recent US housing boom and bust. We first explain the data used for this.

2.2.1 Case-Shiller Tiered House Price Data

A major difficulty with constructing accurate time series for house prices is how to control for the changing composition of houses sold over time. This is an acute issue because in the US in any given year, typically around only 5% of the housing stock is sold (Case et al 2012). Thus, a change in the average price of a house sold in a given place over time could reflect either a genuine change in house prices (i.e. the change in the price of a given standard house) or simply that a different type of house has been sold. One approach to this problem is to construct a *hedonic* index, which runs regressions to account for prices in terms of observable characteristics, such as the number of rooms a house has.

The major alternative approach, taken by the Case-Shiller index, is to use a *repeat-sales* index. This index examines the arms-length sale of the *same* single-family house at two points in time, thus largely controlling for changes in the composition of houses sold, picking up genuine changes in house prices. Transactions are excluded when the time between two sales is less than six months, which likely reflects substantial physical changes to the property by a developer, or a transaction that is non-arms-length, so not reflective of the true market price. Evidence of substantial physical changes from deed records are also used to rule out certain properties. This data cleaning typically removes less than 15% of total repeat sales transactions in a given metro area. If a long time passes between two sales of a property, the change in its sale price could reflect changes in the quality of the neighbourhood or improvements in the quality of the house, rather than changes in the price of a house with unchanging quality. To correct for this, a weighting procedure is applied that places less weight on properties with a longer period between the two transactions. Finally, each pair of sales is allocated a weight based on its first sale price. This is done to ensure that the constructed index is representative of the average home in the metro area (Standard and Poors 2009).

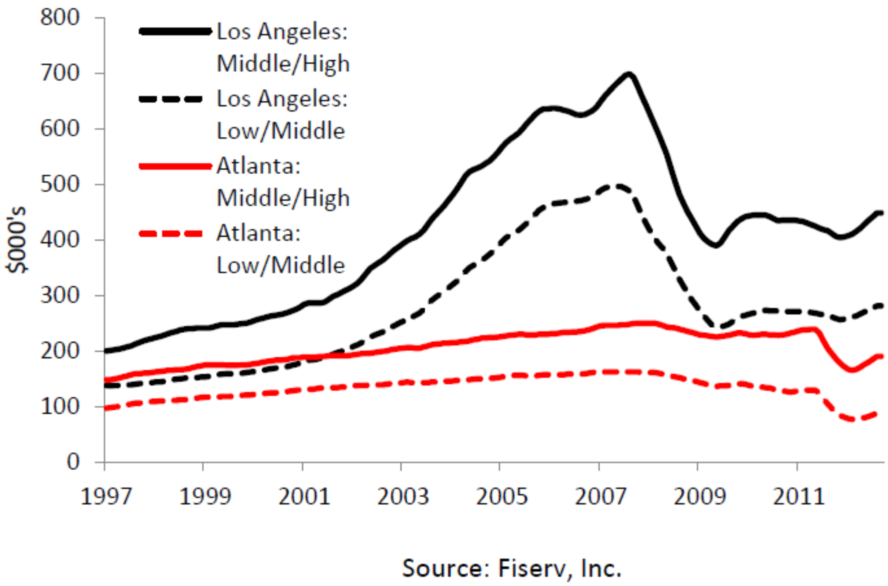
The above procedure is used to construct an aggregate house price index for a given metro area. Additionally, within each metro area, separate repeat-sales indices are constructed for three equal-sized price tiers: low, middle, and high. The low tier represents a price index for the cheapest third of houses in a metro area, the middle the middle third, and the high tier the index for the most expensive third of houses. To construct these, price breakpoints for low/middle and middle/high-tier houses are calculated through time to ensure equal numbers in each group. A given house is placed in the appropriate tier based on the first of its two sale prices, with these breakpoints smoothed through time, to rule out seasonal effects.[6] An example of these breakpoints for Los Angeles and Atlanta during the recent boom and bust is given in Figure 2.1. Due to the significant price increases experienced, the price of a low tier house in Los Angeles in 2007 would be sufficient to be a high tier house in 2001. Once repeat sales-pairs are placed into the appropriate tier, the same weighting and cleaning procedures are used as for the construction of the aggregate index.

It is important to emphasise that the breakpoints, and thus price tiers, are specific to each metro area. A low tier house is in the cheapest third in *its* metro area, not nationally. This is clearly demonstrated in Figure 2.1, which shows that during the peak of the boom years middle tier houses in Los Angeles (and likely several low tier houses) are expensive enough to be high tier houses in Atlanta.

As the price cutoffs for the different tiers differ across metro areas, so too will the income of those buying the houses. A typical low-tier buyer in an expensive housing market like San Francisco will

[6] It is possible for a given property to move between price tiers over time. For example, if a low-tier house experiences a sufficiently large increase in price it will be part of the middle tier index. However, this will be *when it sells again*, as the tier grouping is always based on the initial price of a repeat sales pair.

Figure 2.1: Tiered Break Points: Los Angeles, Atlanta



have a higher income than the typical low-tier buyer in a cheaper market like Tampa. To examine this we use data from the Home Mortgage Disclosure Act (HMDA). The HMDA was brought into effect in 1975 to identify the degree of discrimination within mortgage lending. It required most mortgage lenders to collect data on housing loan applications along with several other attributes of the applicants such as race and income. The coverage is near universal, being around 90 percent of the total market during the boom years (Dell'Ariccia et al 2008). With the HMDA data we can isolate the loans approved for home purchase, and the income of those who took the loans out for each of our metro areas. The HMDA data does not have information on the price paid for the home bought, so we cannot directly split people into different price tiers. Rather, we utilise the fact that people with higher incomes tend to buy more expensive homes.[7] We assume this association is 1-1, and within a given metro area, identify those in the 0-33 percentile of the income distribution *of those purchasing homes* with the low tier buyers, 33-67 with the middle tier, and 67-100 with the high tier. We take the median of each of these groups, identifying the typical low, middle and high tier buyers with the 16.7, 50 and 83.3 income percentiles of those buying houses. We calculate this for each metro area at the start of the boom, in 1997, with the results given in Table 2.7 in the appendix. We indeed see significant variation in the income of our typical buyers across metro areas. In Tampa, the typical *middle-tier* buyer has an annual income of $45,000, whilst in San Francisco, the typical *low-tier* buyer has an income of $58,000. If this person moved to Tampa with the same income they would likely own a middle-tier rather than low tier house.

[7] This is apparent in the American Housing Survey (AHS).

2.2.2 Tiered Housing Patterns During Boom and Bust

Tiered Case-Shiller indexes are publicly available for 17 metro areas in the United States. Fiserv, a data company, also compile tiered house price data using exactly the same method for a greater range of metro areas. Combining the two data sources, we have tiered monthly house price data for 52 US metro areas during the recent boom and bust, covering 26 states, and 41%[8] of the US population in 1997.[9] In the appendix we display graphs (in real terms) for these 52 tiered house price indexes from January 1997 (around the start of the boom) until September 2012 (2008 in Cleveland). Within each area the graphs are normalised to 100 in January 1997. As far as I am aware, this is the first extensive analysis of tiered house price data during the US housing boom and bust.

The first thing to note from the graphs is that there is significant dispersion in the house price movements *within* metro areas, with results in Table 2.5 in the appendix.[10] To take a few salient examples, during the boom, the high-tier in Riverside "only" grew 210% in nominal terms, whilst the low tier grew by 377%. During the bust in Atlanta, high tier homes lost 22% of their value, whilst low tier homes lost 59%, over twice as much in percentage terms. However, more striking than the within-city variation, is the pattern to the variation: in 51 of the 52 metro areas, the low-tier *grew more* in relative terms than the high-tier during the boom (from 1997-2006).[11] By contrast, in 46 of the cities, the low tier *fell by more* from the peak of the boom in 2006 to the general bottoming out of the market in 2011.[12] As a result of this, in 2011, the number of cities for which the growth from 1997 was greater for the low tier had dropped from 50 out of 51 to only 26.

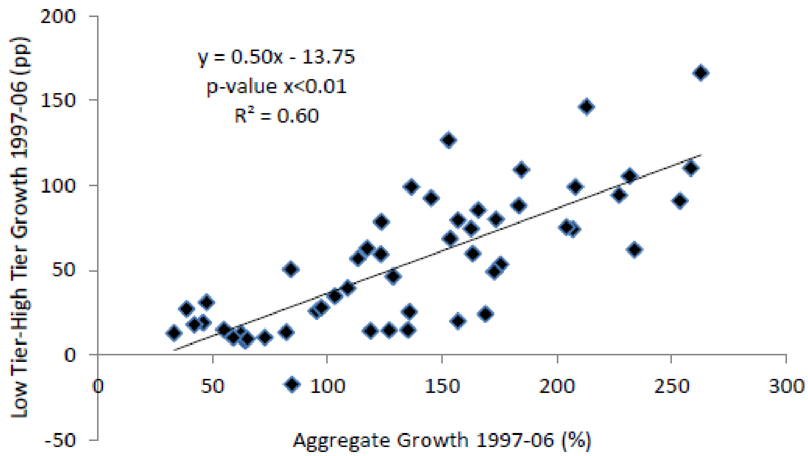
2.2.3 Quantitative Measure of Tiered Pattern

Whilst low tier prices grew more than high tier prices during the boom in 51 of the 52 metro areas, there is significant variation in the extent of this, from a small difference in Colorado Springs to a very large difference in many of the Californian cities. By exploiting this variation, we can uncover further facts about the housing boom, and in the next section, explore the link between the tiered pattern and other economic variables.

Here we construct a measure of the strength of tiered housing pattern to translate the qualitative pictures into quantitative measures. We use a simple measure that looks at the difference in the percentage growth rates between the low and high tiers during the boom, from 1997, before the start of the boom, to 2006, which is a good approximation to the peak of the housing market in

[8] Bureau of Economic Analysis, US Census Bureau.
[9] The Fiserv data is used for all but Cleveland and Las Vegas.
[10] This data is presented in nominal terms.
[11] The exception is Boulder, CO.
[12] The exceptions are Boulder, Fort Collins and Grand Junction in Colorado, and Rochester, Binghampton in New York. There is no data on the bust for Cleveland.

Figure 2.2: Tiered Pattern and Size of Boom



Source: S&P/Case-Shiller, Fiserv, Inc.

most of our metro areas.[13]

$$M := (\text{low tier \% price growth 1997-06}) - (\text{high tier \% price growth 1997-06}) \tag{2.1}$$

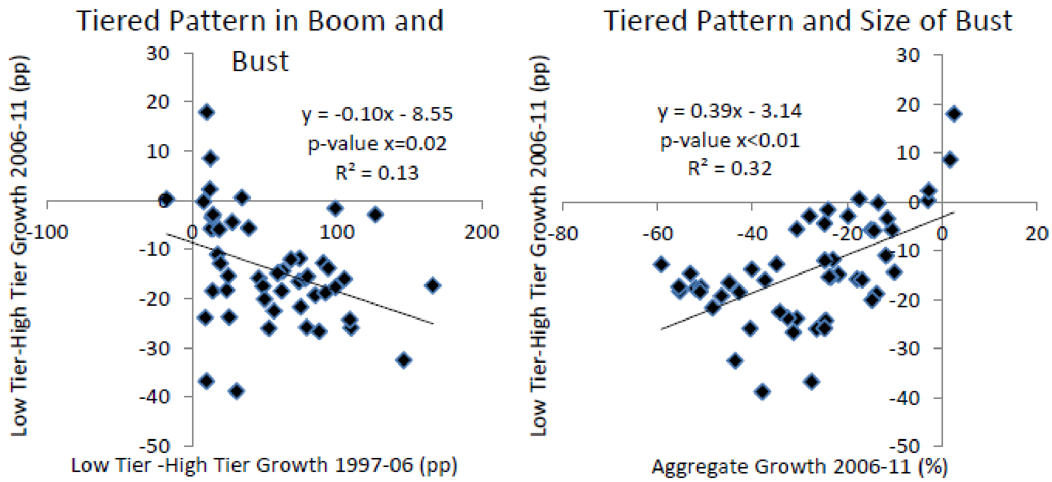
The average value of M across the cities is 55pp., and there is significant variation around this with a standard deviation of 41. The maximum value is in Riverside, CA, where the low tier grew 166pp. more than the high tier during the boom. We now use this measure to introduce new facts about the boom.

In Figure 2.2 we look at how the tiered pattern varies with the aggregate price growth in each metro area during the boom. There is a positive and statistically significant relationship between the two, showing that in metros that experienced a larger overall boom, the low tier price growth outstripped the high tier growth to a greater extent.

We now turn to tiered housing patterns in the bust, looking at the changes in prices from 2006 to 2011. In Figure 2.3 we compare this tiered measure during the bust to the tiered measure during the boom and the size of the local collapse in the housing market. In the first panel we see that there is a negative relationship between tiered growth in the boom and bust. In other words, in places where low tier growth greatly outstripped high tier growth during the boom, the low tier fell significantly more than the high tier in the bust. The second panel shows that in housing markets with a greater bust, the extent to which the low tier fell more than the high tier was greater.

[13] A justification for the use of this measure is given in Proposition 52 below.

Figure 2.3: Housing Patterns in Bust



Source: S&P/Case-Shiller. Fiserv. Inc.

2.2.4 Summary of Facts and Related Literature

Here we summarise the facts presented and discuss the prior literature looking at house price variation within cities.

Facts

1. In the recent boom from 1997-2006, low tier house prices grew more than high tier house prices in 51 of 52 metro areas.

2. In the recent bust from 2006-2011, the low tier fell by more than high tier in 46 of 51 metro areas.

3. The extent to which low tier growth outstripped high tier growth in the boom was greater in metros with greater housing booms.

4. The extent to which the collapse in the low tier was greater than the high tier was greater in places with worse housing busts.

5. Places with particularly strong relative low tier growth in the boom, had particularly larger relative low tier price collapses in the bust.

Overall, there was a larger boom and bust cycle in the cheapest houses in each metro area.

The pattern documented in the first fact is new and remarkable in light of previously published research on tiered housing. Using Case-Shiller data, Mayer (1993) documented that in the 1970s and 80s the opposite pattern occurred with high tier house prices growing relatively more in Atlanta, Chicago, Dallas and Oakland. Poterba (1991) has similar findings. Finally, Smith and Tesarek

(1991) show that in the 1970s boom in Houston, high tier houses had the greatest appreciation. Indeed such was the prevalence of this pattern, Mayer (1993) suggests a theory based on an extension of Stein (1995) showing why it's inevitable that high tier house prices will grow more than low tier prices during a boom. A similar result to Fact 1 has been documented across a smaller range of cities using data at the zip-code level within cities. Guerrieri et al (2011) document that during the recent housing boom, neighbourhoods in cities with initially cheaper housing experience larger relative booms. Landvoigt et al (2012) use repeat-sales data for San Diego from 2000-2005 and show that capital gains over this period were significantly greater for cheaper houses.[14]

The second fact presented here is entirely unique to this paper to my knowledge. The pattern observed in previous busts was not as uniform. Mayer (1993) finds greater price crashes in the high tier in Chicago and Oakland in the late 1970s. Smith and Tesarek (1991) find that high-quality properties fell at a greater rate in Houston during the 1980s following the oil bust. Case and Shiller (1994) find high tier properties falling at a greater rate in Los Angeles during the bust in the late 1980s, whilst low tier properties had a greater crash in Boston at the same time. Landvoigt et al (2012) find that from 2006-2008 in San Diego, less expensive houses depreciated relatively more.

An analogous result to the third fact is shown for their selection of cities by Guerrieri et al (2011). They show that the larger the city-wide housing boom, the greater the difference in growth rates between low and high price neighbourhoods. As with Fact 1, we complement this with analysis for a larger number of cities, at a more aggregated level.

To my knowledge, the fourth and fifth facts have not been discussed before for this or previous booms.

I view the new facts presented about the housing boom and bust to be of independent interest in their own right, providing new data about this historic episode, suggesting new areas to work on. As discussed in the introduction, the theoretical framework of the previous chapter provides a parsimonious explanation for these facts, based on the easing then tightening of non-price credit conditions. We next turn to testing this theory empirically.

2.3 Data and Empirical Approach

2.3.1 Credit Easing

In the previous chapter, we proved that when low tier buyers are constrained and high tier buyers are unconstrained, the marginal relative price response to an easing of the LTV ratio is greater for low tier prices:

$$\left(\frac{-dP_t^L(\gamma)}{d\gamma}\right)\frac{1}{P_t^L(\gamma)} > \left(\frac{-dP_t^H(\gamma)}{d\gamma}\right)\frac{1}{P_t^H(\gamma)} > 0$$

An implication of this theory is that the gap between low and high tier price growth is greater

[14] For example, they find that the average house that sold for \$200K in 2000 experienced average growth of 17% per year until 2005. By constrast, for a house that sold for \$500K in 2000, the average annual appreciation over this period was only 12%. Rather than just comparing low and high tier prices, they also show there is a decreasing monotonic relationship between the initial price and subsequent capital gains.

the greater the credit easing, as summarised in the following proposition.

Proposition 52 *Let $P_t^L(\gamma)$, $P_t^H(\gamma)$ be the low and high tier prices as functions of the minimum down payment γ. Let $\gamma_2 < \gamma_1 < \overline{\gamma}$ with $\overline{\gamma}$ the original level of credit standards and γ_2, γ_1 looser down payment requirements, with γ_2 the looser. Suppose the low tier buyer is constrained and the high tier buyer is unconstrained.*

Then

$$\frac{\frac{P_t^L(\gamma_2)}{P_t^L(\overline{\gamma})}}{\frac{P_t^H(\gamma_2)}{P_t^H(\overline{\gamma})}} > \frac{\frac{P_t^L(\gamma_1)}{P_t^L(\overline{\gamma})}}{\frac{P_t^H(\gamma_1)}{P_t^H(\overline{\gamma})}} > 1$$

And

$$\frac{P_t^L(\gamma_2) - P_t^L(\overline{\gamma})}{P_t^L(\overline{\gamma})} - \frac{\left(P_t^H(\gamma_2) - P_t^H(\overline{\gamma})\right)}{P_t^H(\overline{\gamma})} > \frac{P_t^L(\gamma_1) - P_t^L(\overline{\gamma})}{P_t^L(\overline{\gamma})} - \frac{\left(P_t^H(\gamma_1) - P_t^H(\overline{\gamma})\right)}{P_t^H(\overline{\gamma})} > 0$$

Remark 53 *An identical result can be shown when the maximum LTI ratio is loosened.*

The theory implies two different measures can be use to capture the difference between low and high tier price growth. The first calculates the ratio of low and high tier price changes, whilst the second examines the percentage point difference between low and high tier price growth. Either measure can be used to test the predicted relationship between relative price changes and credit easing. In the paper we use (2.1), the percentage point difference between growth rates, which is perhaps easier to interpret, but we obtain very similar results when the alternative measure is used instead.

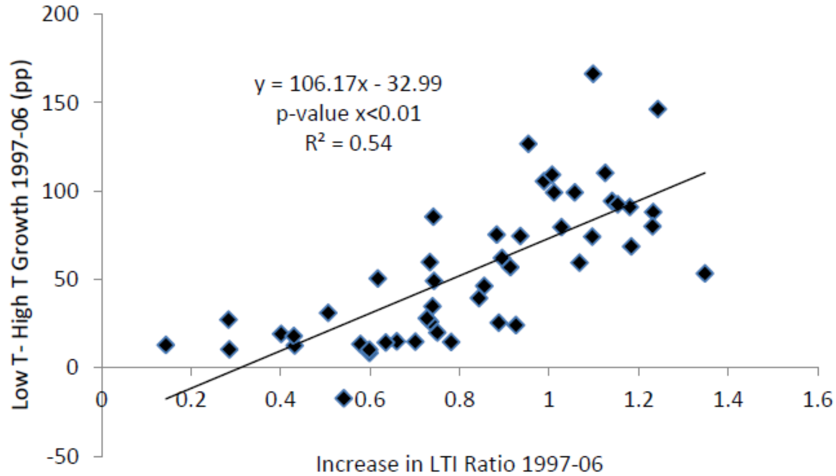
Ideally we'd like to test the link between relative price growth and both LTV and LTI easing, to examine which is more important. From the HMDA dataset we can construct LTI measures for each metro and changes in it over time. Unfortunately, it does not also record the value of the home purchased, so it cannot be used to calculate LTV ratios. Instead, we use a credit easing proxy for this, which arguably is more important for LTV than LTI easing. We begin with a discussion of the LTI data.

From the HMDA data we construct an average LTI ratio for each metro in 1997 and 2006, with details on this construction given in the appendix.[15] We plot (2.1) against the change in this in Figure 2.4. We see that in all 49 cities[16], there was an increase in the average LTI ratio, but also significant dispersion in this, from a low of 0.14 in Rochester, NY, to a maximum of 1.35 in Washington DC, where the average LTI ratio increased from 2.13 in 1997 to 3.48 in 2006. However, most strikingly, there is a positive statistically significant relationship between the two variables: *places with greater LTI easing experienced a greater difference between low and high tier price growth, consistent with the predictions of theory.* We now turn to our second measure of credit easing.

[15] The measure utilises the income distribution of those that bought a house in a given city in a given each year. The LTI measure is the average of the LTI ratios for the buyers between the 33rd and 66th percentiles of this income distribution. This measure is intended to rule out the impact of outliers on the calculated LTI ratio.

[16] There is no data on income changes in the period of interest for Gainesville, GA; Peabody, MA; Cambridge, MA. For consistency, for the remaining variables, we also restrict the sample to the 49 cities.

Figure 2.4: Tiered Measure and LTI Easing in Boom

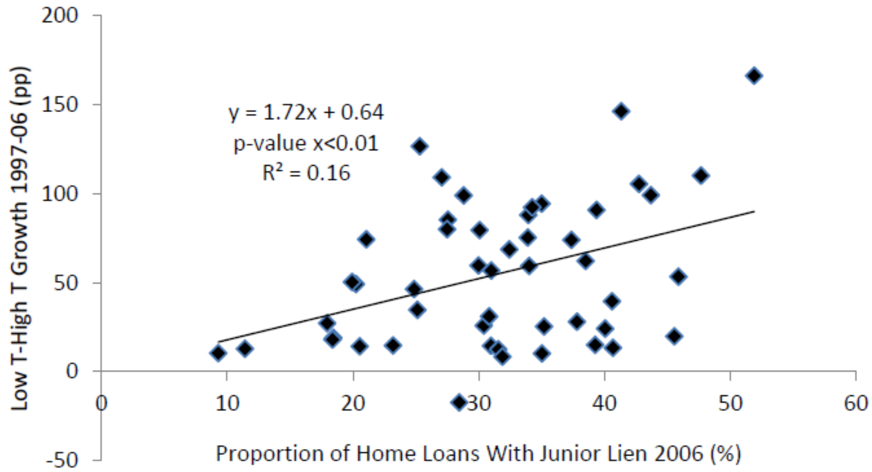


Source: S&P/Case-Shiller, Fiserv, Inc.; HMDA

As discussed, the HMDA dataset does not record the value of the home purchased so we cannot use it to construct LTV measures for each metro area. Instead we use a proxy based on the prevalence of junior liens at the peak of the boom in 2006. Traditional prime conforming mortgages in the US had an LTV ratio of 80%. If a borrower wanted to make a smaller down payment and have an LTV ratio over 80%, private mortgage insurance payments (PMI) had to be taken out (Calhoun 2005). This cost is substantial, with PMI on a typical 95-100% LTV loan costing 1% of the value of the loan annually (Credit Suisse 2007). Further, unlike mortgage payments, PMI was not tax-deductible. A popular alternative to PMI became taking out two mortgages, a conforming first mortgage for 80% of the value of the loan (which does not need PMI payments) and a second mortgage (or even third mortgage)-a junior lien-for a large part or all of the remainder of the value of the house.[17] Crucially, interest payments on this second mortgage were tax-deductible. Originally, second mortgages were primarily used to circumvent PMI, however their use exploded during the recent boom, particularly in areas with the largest house price increases (Credit Suisse 2007). Junior liens are particularly useful for borrowers constrained by the down payment requirement with many borrowers able to get a mortgage with *zero down payment* with the popular 80/20 combination. Given this, I view the increase in use of junior liens over the boom as a useful proxy for the easing of down payment constraints. Unfortunately, the HMDA data only separately identifies senior and junior liens from 2004 onwards so we cannot track the increase in each metro area. However, prior to the boom the use of second mortgages was small, with the fraction of housing transactions featuring a junior lien around 5 times lower in 1998 than in 2006 (Adelino et al 2012). I thus view the fraction of housing transactions in 2006 in each metro using a junior lien

[17] Popular options were 80/10/10 with an 80% LTV first mortgage, 10% LTV second mortgage, and 10% deposit, and the 80/20 with a 20% LTV second mortage and zero deposit.

Figure 2.5: Tiered Measure and Credit Easing Proxy in Boom



Source: S&P/Case-Shiller, Fiserv, Inc.; HMDA

as a useful proxy for the *increase* in the use of junior liens, and hence easing of LTV requirements.[18]

In Figure 2.5 we plot the difference in low and high tier growth rates during the boom against the fraction of home purchase loans with junior liens in 2006. The widespread use of these can be seen, with over 50% of home purchase loans using a junior lien in Riverside, CA. Furthermore, consistent with the theory, there is a statistically significant positive relationship between the prevalence of junior liens in 2006 and the difference between low and high tier price growth rates during the boom.

In summary, without other variables controlled for, we see positive significant relationships between both measures of credit easing and the dispersion in tiered price growth during the boom. We next consider whether there could be confounding variables responsible for this association.

2.3.2 Other Variables

In the previous chapter we theoretically examined the response of tiered housing under a fall in interest rates, an increase in buyer optimism and an easing of LTV and LTI ratios. We concluded that out of those explanations, the documented pattern of greater price growth for the low tier could not have occurred without an easing of non-price credit terms. There are other potential

[18] Of course, the addition of a junior lien will also enable an increased LTI ratio, by allowing the borrower a greater total loan against their income. However, the impact on the borrower and their buying power is likely significantly greater through the easing of the LTV ratio. For example, suppose a borrower has a prime conforming mortgage L and is simultaneously constrained by a LTI limit on this mortgage, and a LTV limit of 80%. With the addition of an unsecured 10% LTV second lien they can pay 100% more for a house and still satisfy the LTV limit. This is because their initial deposit D allowed them to pay $5D$ for a house at 80% LTV, but the same deposit can now pay up to $10D$ and satisfy the combined LTV limit of 90%. By contrast, they can only pay 11.1% more for the house and satisfy the combined LTI limit. To see this, with the LTI limit binding, initially the maximum house price they could afford is P. With the addition of a 2nd lien for up to 10% of the value of the house the maximum price they can now pay is $P' = P + 0.1P'$. Solving this gives $P' = 1.111P$, or an 11.1% increase in the price they can pay.

explanations for this pattern. Low tier buyers having greater relative income growth, investors primarily buying low tier houses, and builders primarily building high tier houses could all generate greater low tier growth, else equal. Further, these variables could be positively associated with credit easing, giving rise to the apparent association documented in the previous section. For example, places with greater relative income growth for low tier buyers may also experience an increase in average LTV and LTI ratios, as these borrowers are perceived to become relatively more credit worthy. Or, perhaps in places with greater mortgage credit easing, there was greater credit easing generally, making it easier for builders to obtain the finances to build more houses. If they primarily built houses for the high price tier, we could observe places with greater credit easing experiencing relatively greater low tier price increases, but this would be operating through supply and not demand. Finally, places with greater credit easing may draw in speculators, either because it is easier for them to obtain funding, or because they believe prices in those areas are particularly likely to rise. If they primarily bought low tier houses, the documented link between credit easing and greater relative low tier price growth could actually be operating through a different channel than the one proposed. It is important to address these other explanations to be sure that the policy of restricting the easing of non-price credit terms would actually be effective in attenuating the housing boom.

Our empirical approach to tackle these, and other alternative explanations, is to compare variation in the differential growth rates during the boom given by (2.1) with variation in these variables. As the data on the other variables is only available annually, for each city we compute (2.1), from January 1997 to the 2006 price averages. We compare this to variation in the other cities from 1997-2006 (with exceptions noted below). Before turning to regressions, we discuss three of the variables in detail.

2.3.3 Change in Housing Supply

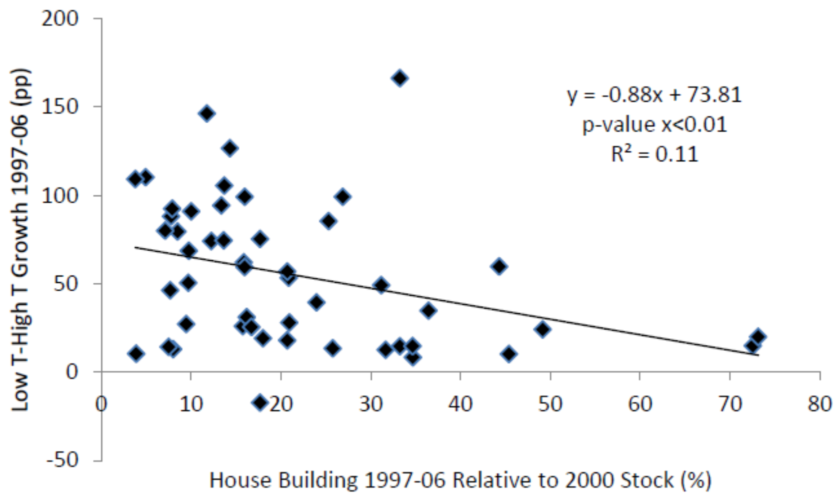
The model of the previous chapter abstracts from changes in the housing supply. With equal increases in demand for low and high tier houses, the low tier would experience a greater boom if more high tier houses were built in response. Here we examine the link between the tiered pattern and changes in housing supply.

Most papers looking at the housing boom have focused on the demand side and the role of changing financial variables in facilitating this. However, the boom was also a time of dramatic building in many metro areas, which may have exacerbated the bust and resulted in too many houses being built.[19] Glaeser et al (2008) have examined the role of supply in the evolution of booms. Using the Saiz (2008) proxy for local supply elasticities, they tend to find that areas with more elastic housing supply experienced weaker housing booms, with a few notable exceptions such as Phoenix.

To my knowledge, there have been no papers estimating the elasticity of supply for different tiers of housing within a metro area. Given this, my approach to test the link between the tiered pattern

[19] Haughwout et al (2012) calculate that the boom contributed an excess of over 3 million extra houses nationally.

Figure 2.6: Tiered Measure and House Building in Boom



Source: S&P/Case-Shiller, Fiserv, Inc.; HUD; Census Bureau

and housing supply is to compare the variation in house building during the boom with the tiered pattern. Taking data from The Department of Housing and Urban Development (HUD)[20], for each of our metro areas we compute the number of permits issued for single-family house building during the boom years 1997-2006. We then normalise this relative to the 2000 single-family housing stock as counted in the census, giving us a measure of the amount of house building that took place relative to the existing stock. There is significant variation in this measure from a low of 3.8% in San Francisco, to a high of 73.1% in Las Vegas (meaning that over 1997-2006 they built 73% of the 2000 stock). In Figure 2.6 we plot the difference between low and high tier growth during the boom against this building data.

We note a significant downwards sloping relationship between the two: in places with more house building, the tiered pattern is less pronounced.[21] This suggests that changes in housing supply are not responsible for the tiered pattern. If they were, and low tier house prices grew more than high tier house prices because more high tier houses were built, we'd expect to see the *opposite* pattern, namely that places with more building featured a more pronounced tiered pattern.

The above reasoning implicitly assumes that similar types of houses were built across all areas, relative to their MSA. As against this, the pattern could arguably be generated if cities with little building built high tier houses, whilst those with significant building built mainly low tier houses (both relative to their own metro). However, given that many places (several in California) experienced large low- relative to high-tier price growth and had very little building, it does not seem likely that their experience can be attributed to changes in supply along the lines of this

[20]Precisely, the data is from the State of the Cities Data Systems (SOCDS), Building Permits Database.

[21]If we remover the two outliers of Greeley, CO, and Las Vegas, in each of which over 70% of the 2000 stock was built during the boom, the relationship is still statistically significant at the 5% level.

argument. Further, in places with a lot of building, the low tier still grew more than the high tier, in spite of the hypothesised low-tier house building, suggesting that something else was driving this.[22]

2.3.4 Speculators

The key identification assumption in the model of the previous chapter is that house buyers are heterogeneous: low tier buyers are credit constrained while high tier buyers are unconstrained. All the predictions of the model follow from this critical assumption. Whilst we have shown evidence that typical people buying low tier houses are credit constrained, the clean separation between the two groups could be broken by the presence of speculators. Speculators may not be credit constrained, and if they primarily bought low tier houses, this could explain the observed pattern more so than our explication. It has been documented that there were large increases in the share of homes bought in many metro areas by investors and second home owners (Wheaton and Nechayev 2007; Haughwout et al 2011). Using HMDA data, we examine whether this was linked to the observed tiered pattern. With a mortgage application, the HMDA dataset requires identification of whether the applicant intends to live primarily in the property or not. If not, they are classified as a non-owner-occupier. Non-owner occupiers could either be speculators buying an investment property, or people buying a second home. Using this measure we also find a large increase in the percentage of home loans going to non-owner occupiers. For example, in 1997 in Las Vegas 9% fell into this category, but by the peak of the boom in 2006 this had risen to 26%.

Our approach here is to compare the strength of the tiered pattern with the percentage point increase in non-owner occupiers from 1997-2006. The results are displayed in Figure 2.7. The graph shows that unconditionally, there is a statistically significant negative relationship between the two: places with a greater increase in non-owner-occupiers saw a smaller excess of low-tier over high-tier growth during the boom. As with house building, if the tiered pattern was driven by an increase in speculation, we'd expect the *opposite* pattern, namely a stronger tiered pattern in places with more investors.

As with the housing supply case, an implicit assumption in this analysis is that investors are buying the same types of houses in each metro area. Perhaps this does not hold, and where there was a small increase in the number of investors they are primarily buying cheap houses, and where there was a larger increase, they are primarily buying expensive houses. As with house building it is hard to reconcile this explanation with the fact that most places with the strongest tiered pattern had a very small increase in the proportion of investors.

We can test this alternative explanation using further HMDA data. Whilst the HMDA data does

[22] It would of course be interesting to test this alternative hypothesis more directly. However, to my knowledge the data to do so does not exist. HUD have microdata on construction during the boom years, including the square footage of housing built. One could compare this with the average square footage of houses in the same areas. Unfortunately, the smallest geographical level the data is available for is for the 9 census divisions, not the MSA level. An alternative approach based on the change in the median number of rooms during the boom years in each county was considered, but there is a lack of significant variation in this over time (it's generally within the estimated margin of error).

Figure 2.7: Tiered Measure and Increase in Non-Owner Occupiers in Boom

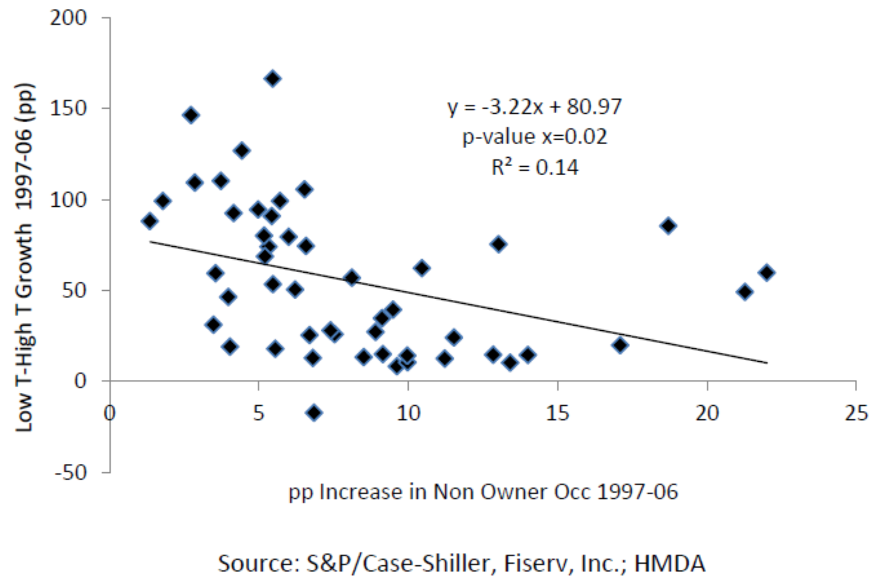
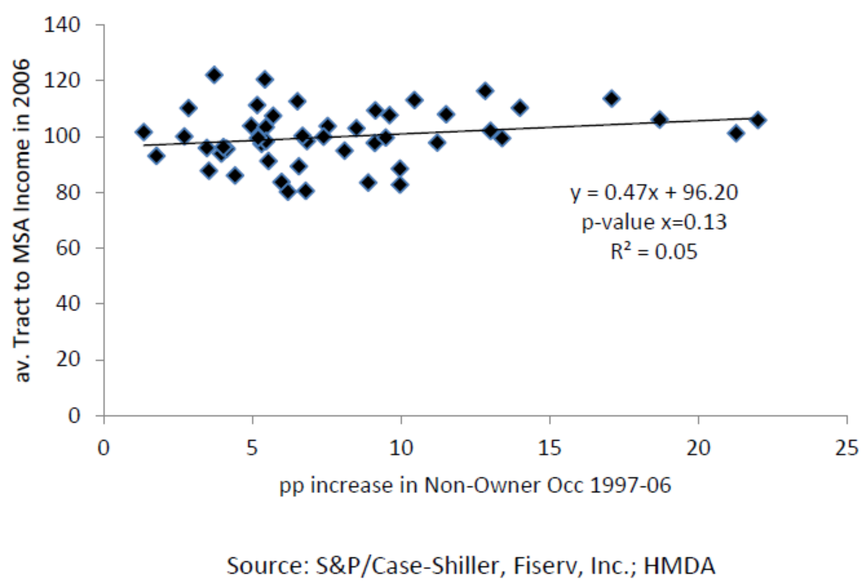


Figure 2.8: Income Tract and Nooc: Boom



not record the purchase price of houses, it does record the median income of the neighbourhood it is in relative to its MSA, and as there is a strong correlation between income and purchase price[23], neighbourhood income is a useful proxy for house value. To understand the measure, if the number is 120% for a neighbourhood in San Francisco, this means that the median income of that neighbourhood is 120% of the median income in the whole of San Francisco. Such a neighbourhood thus has higher average income than the whole of San Francisco. For each MSA we calculate the average neighbourhood to MSA income measure for non-owner-occupiers in 2006. In Figure 2.8 we plot this against the percentage point increase in non-owner-occupiers in each MSA from 1997-2006. The alternative explanation predicts a strong positive relationship between these two: where investors were more prevalent, they were investing in relatively more expensive houses. The chart shows a weak positive relationship that is not statistically significant at standard confidence levels. Even if the positive relationship were statistically significant, the predicted magnitude is not economically significant: if the increase in investors was 10 percentage points greater in a given metro, the average neighbourhood to MSA income only increases by 5 percentage points. We can thus reject the alternative explanation that the types of properties investors bought varied significantly between cities.

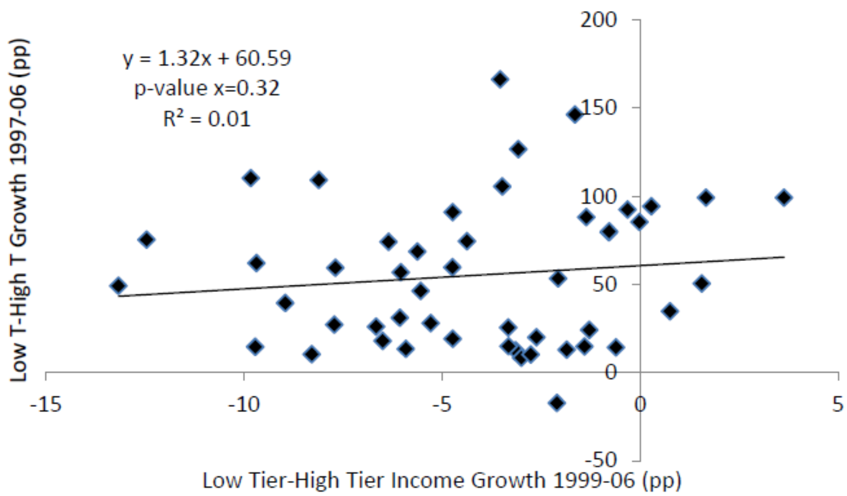
2.3.5 Relative Income Growth

An alternative explanation to the easing of non-price credit terms for the relative boom in low tier housing is that income grew relatively more for low tier than high tier home-buyers. To examine this we look at income growth over the housing boom for both groups of buyers. As discussed in Section 2.2, we can proxy the income of those who *did buy* different tiers of houses from HMDA data. We could do this every year but it is not the most appropriate measure as it could reflect a changing composition of those who buy, rather than a genuine change in income of the *would be* buyers of each type of house within each metro. Instead, we take data from the Bureau of Labor Statistics (BLS) Occupational Employment Statistics (OES). This has data on nominal income changes over time for all those in a city (not just home-buyers like HMDA) in 49 of our 52 MSAs at the 10th, 25th, 50th, 75th and 90th percentiles.[24] Within each metro area we identify the low tier buyers with the 50th percentile group and the high tier buyers with the 90th percentile group with the following rationale.[25] The US home-ownership rate is around 67%. We assume that all owners are those above the 33rd income percentile, with higher earners buying more expensive houses. With three equal sized tiers, the low tier buyers are between the 33rd and 56th income percentiles and the high tier buyers above the 78th income percentile. The median low tier buyer is then at the 44.5th percentile group and the median high tier buyer at the 89th percentile. Our identification of low tier buyers with the 50th percentile and high tier buyers with the 90th percentile is thus reasonably accurate.

For each metro area we calculate the percentage growth in income for the 5 percentile groups

[23] This emerges from the American Housing Survey.
[24] There is no data in the period of interest for Gainesville, GA; Peabody, MA; Cambridge, MA.
[25] This is similar to the rationale in Mayer (1993).

Figure 2.9: Tiered Measure and Relative Income Growth in Boom



from 1999 (the earliest available) to 2006.[26] We then look at the percentage point difference between the two, i.e. low tier income growth-high tier income growth. The tiered pattern is plotted against this in Figure 2.9. We see that unconditionally, there is a weak, statistically insignificant, positive relationship between the two. The sign of the slope is intuitive enough: in places in which the income growth of would-be low-tier buyers was relatively greater than for the high-tier, the gap between low- and high-tier house price growth is greater. However, note that in 44 of the 49 cities, income growth was *lower* for the low tier group. Thus, in all but 5 of the metros, far from accounting for the tiered pattern, relative income growth actually *goes against it*. In other words, had income growth over the boom been equal for would-be low and high tier buyers, the difference between low and high house price growth rates would have been *greater*.

2.4 Housing Boom: Results

In this section we run regressions linking the variation in low and high tier price growth during the boom to credit easing whilst controlling for these and other variables. In Table 2.1 we present summary statistics for the variables used in the regression.

Table 2.2 presents the results of regressing the difference between low and high tier price growth during the boom on a variety of explanatory variables. Throughout we drop the observations on Las Vegas and Greeley, CO, the two outliers for house building, leaving us with 47 observations (this does not significantly affect the regressions). The first column presents the simple univariate regression

[26] A detailed description of how this was calculated is given in the appendix.

Table 2.1: Summary Statistics for Boom Variables

	Mean	Median	St. Dev.	Min	Max
Low Tier-High Tier Price Growth 97-06 (pp.)	55.14	50.35	40.82	-17.37	166.32
Home Purchases Including Junior Lien 2006 (%)	31.60	31.58	9.33	9.30	51.90
Increase in LTI Ratio 97-06	0.83	0.86	0.28	0.14	1.35
Housing Building 97-06 as % of 2000 Stock	21.23	16.23	15.57	3.79	73.14
Increase in Non-Owner Occupiers (pp.)	8.03	6.70	4.71	1.35	22.01
Low Tier-High Tier Income Growth 99-06 (pp.)	-4.13	-3.48	3.68	-13.16	3.64
Increase in Unemployment Rate 97-06 (pp.)	0.09	0.20	1.21	-3.55	2.75
Immigration 2000-06 as % of 06 Population	4.02	3.91	2.09	1.21	11.28
Observations	49				

of (2.1) on the credit easing proxy, showing the positive relationship between the two, significant at the 1% level.[27] In the second column we add the three control variables discussed in the text, along with two additional regressors; the increase in unemployment during the boom years and migration from 2000-2006.[28] The coefficient on the credit easing proxy remains significant at the 1% level, and of similar magnitude. We note that house building, the increase in non-owner occupiers and the relative income changes all have the same signs as they did unconditionally, *pushing against the tiered pattern,* though none are significant at the 10% confidence level (though the house building coefficient almost is). The coefficient on immigration is not statistically significant, though the increase in unemployment is, and is negative, indicating that in places in which unemployment rose during the boom, low tier housing had relatively weaker price growth. The experience of unemployment varied across metros, with unemployment rising during the boom in 27 of the 47 cities, and falling in the remaining 20. In the majority then, the change in unemployment does not help explain the pattern of greater low tier growth, rather would predict greater price growth for the high tier.

To interpret the results of this regression, we compute the estimated contribution of selected variables to the tiered pattern. In the average city, low tier house prices grew by 55.1pp. *more* than high tier prices during the boom. At the same time, in the average city, the stock of houses built increased by 21.2%. With an estimated coefficient of -1.08, house building is predicted to result in the low tier growing by 22.9pp. *less* than the high tier during the boom. Similarly, for the average city, the change in relative income is predicted to make the low tier grow 7.2pp. *less*, whilst for the increase in unemployment it's 1.2pp. *less*. These predictions thus go in the *wrong* direction to explain the observed pattern. By contrast, credit easing is predicted to make low tier prices grow 61.6pp. *more* than the high tier, which is reassuringly close to the 55pp. difference observed on average.

[27] Note that the estimated coefficients differ from those on the graph in the text. This is due to the dropping of Las Vegas and Greeley.

[28] Higher unemployment may primarily affect potential low-tier buyers, as they will be in lower paying jobs which may suffer more during economic downturns, whilst higher immigration (from foreign countries) may primarily affect demand for low tier houses, if immigrants are poorer than the typical local resident. The unemployment data comes from the Bureau of Labor Statistics and the migration data from the US Census Bureau.

Table 2.2: Low Tier -High Tier Price Growth During Boom

	(1)	(2)	(3)	(4)
Home Purchases With Junior Liens 2006	2.05***	1.95***		
	(3.70)	(3.03)		
Increase in LTI Ratio 1997-06			104.56***	91.29***
			(7.04)	(5.46)
House Building 1997-06		−1.08		−0.36
		(−1.68)		(−0.54)
Increase in Non-Owner Occupiers 1997-06		−1.00		−0.97
		(−0.47)		(−0.46)
Low-High Tier Income Growth 1999-06		1.75		0.36
		(1.31)		(0.31)
Increase in Unemployment 1997-2006		−13.03**		−9.34
		(−2.57)		(−1.68)
Immigration 2000-2006		−1.85		−3.04
		(−0.59)		(−1.31)
R^2	0.21	0.49	0.54	0.61
Observations	47	47	47	47

t-statistics in parentheses (White heteroskedasticity-consistent standard errors used)
*p<0.1, **p<0.05, ***p<0.01

The next two columns repeat the analysis using the change in the LTI ratio instead of the credit easing proxy. The third column presents the univariate regression, with the fourth column adding in the control variables. The estimated coefficient on the LTI variable is similar across both specifications and significant at the 1% confidence level in both cases. The coefficients on the control variables retain their signs, but generally become smaller in absolute magnitude and less significant. The coefficient on the LTI variable in column four predicts that in the average city, low tier prices should grow by 76pp. more than high tier prices during the boom. It is again reassuring that this is of comparable magnitude to the actual observed change.

In the previous chapter, we theoretically considered three possible explanations for the observed pattern in tiered housing during the boom: an easing of non-price credit terms, an increase in buyer optimism, and a fall in interest rates, concluding in favour of the first explanation. The evidence presented here rules out leading alternative explanations outside of the three considered, backing up the results of the first chapter, and presents evidence consistent with the prediction of the theoretical model regarding the important role of the easing of non-price credit terms.

Of course, we have not shown that credit easing exogenously *caused* the changes in the tiered pattern, merely an association between the two. As discussed in the first chapter, many studies of the housing boom have not managed to tackle the likely endogeneity between house prices and credit easing during the boom, with both feeding off each other. Endogeneity may also be an issue here, though the exact mechanism would be different. Whereas in standard empirical housing papers, the worry is that rising house prices cause a further easing of credit standards, here endogeneity would arise if credit standards are eased more when the low tier grows relatively more than the high tier within a given metro. It's possible that a general link from aggregate house prices to credit standards exits without a link from tiered house price changes to easing credit standards, but we can not rule this out, and our results should be interpreted in light of that.

2.5 The Bust

We now turn to the housing bust as a second episode for testing the link between credit and relative tiered prices. As with the boom, our approach is to compare variation in low and high tier price growth across cities during the bust with variation in other variables. Specifically, for each city we calculate:

$$MB := (\text{low tier \% price growth 2006-11}) - (\text{high tier \% price growth 2006-11}) \tag{2.2}$$

As discussed, 2006 is a good approximation for the peak of the housing boom for the vast majority of our cities. Similarly, 2011 provides a good approximation for the trough of the subsequent bust in house prices, meaning (2.2) captures the relative price movements during the bust well. We similarly calculate changes in the other economic variables from 2006 to 2011. We briefly discuss these variables and the unconditional relationships in the next section before turning to econometric results.[29]

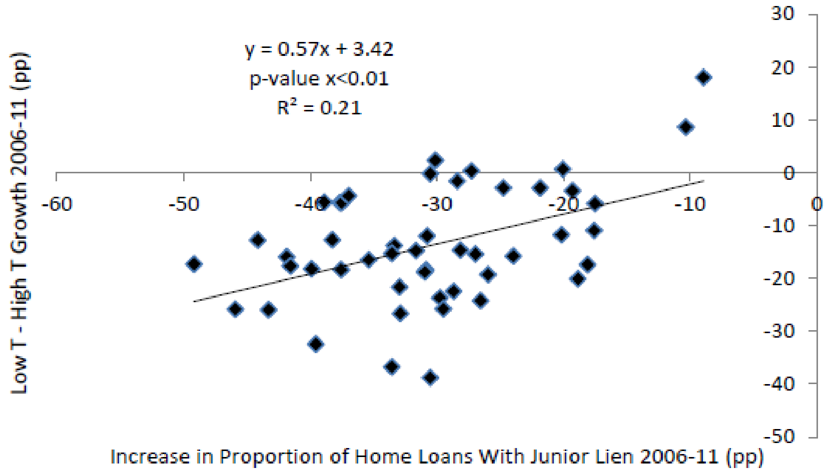
2.5.1 Tiered Pattern and Other Variables

There was a significant tightening of mortgage credit during the housing bust (La Cava 2013). As with the boom, we capture this with changes in the credit proxy and changes in the LTI ratio. The proxy for this change in credit is the percentage point increase in the proportion of home purchase loans including a junior lien, using HMDA data. The prevalence of junior mortgage liens decreased dramatically during the housing bust, with the maximum proportion during the bust years being 5.1%, with the average just 1.4%, compared with an average of 31.6% in the peak in 2006. The minimum value for our credit tightening proxy is -49.2pp. in Riverside, CA, where from a peak of 51.9% of home purchases involving a junior lien in 2006, in 2011 only 2.7% of loans did. The difference in low and high tier price growth during the bust is plotted against the change in this credit proxy in Figure 2.10, showing a statistically significant positive relationship. Thus, low tier housing did relatively worse than the high tier in places that experienced a greater tightening in credit availability. This is consistent with the theoretical predictions of the previous chapter, with the tightening of credit particularly affecting the credit-constrained would-be low tier house buyers.

The second measure of changes in credit is from changes in the LTI ratio, given in Figure 2.11. In all but 9 of the 48 metros, the average LTI ratio decreased during the housing bust, with the

[29] In this analysis we drop Cleveland as the data series finishes in 2008. This leaves us with observations on 48 metro areas.

Figure 2.10: Tiered Measure and Credit Tightening: Bust



Source: S&P/Case-Shiller, Fiserv, Inc.; HMDA

largest decrease in Phoenix, AZ, where the LTI ratio dropped 0.82 units from 2.95 in 2006 to 2.12 in 2011. As with the housing boom, we see that in the housing bust there is a statistically significant positive relationship between increase in the LTI ratio and the relative performance of low tier prices: places with greater tightening saw the low tier prices do relatively worse than high tier prices.

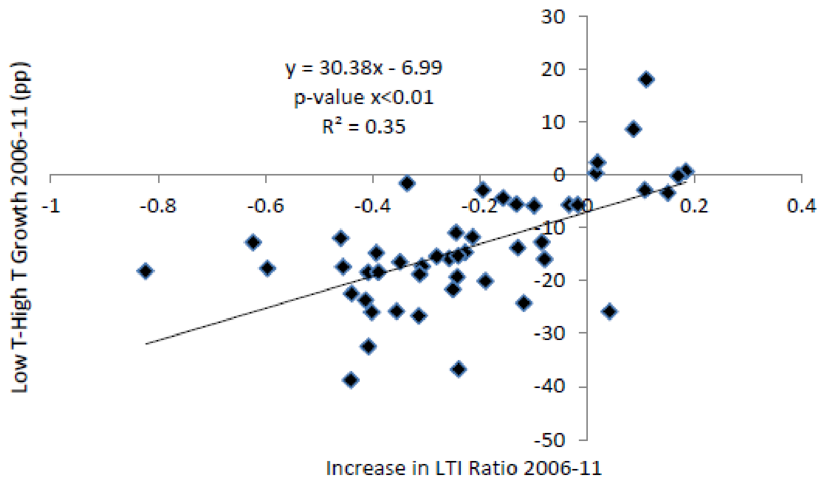
As with the housing boom, there could be confounding factors that account for these relationships between relative prices and credit. We consider a few of these variables next.

In Figure 2.12 we plot the tiered pattern during the bust against the relative income growth during the period, noting a statistically significant positive relationship between the two. Thus, as in the boom, places in which income growth was relatively greater for would-be low rather than high tier buyers, low tier prices grew relatively more than high tier ones. Further, as with the boom, in the vast majority of places (44 out of 48 here), low tier buyers had relatively lower income growth than for the high tier. Whereas during the boom income pushed against low tier growth, here it pushes in the same direction as it.

The bust in the US housing market was followed by a financial crisis and a recession, and reflecting this, unemployment rose substantially in all 48 metros during the bust, from a minimum increase of 2pp. in Manchester-Nashua, NH, to a maximum of 9.3pp. in Las Vegas, NV. As with the boom, in Figure 2.13 we see a negative relationship between the two, with low tier housing doing particularly bad in places with a large increase in unemployment. This may be because greater unemployment particularly affects the lower income workers-the natural buyers of low tier houses. Or it may reflect the local economy doing particularly badly in places in which the low tier housing market had a particularly large crash.

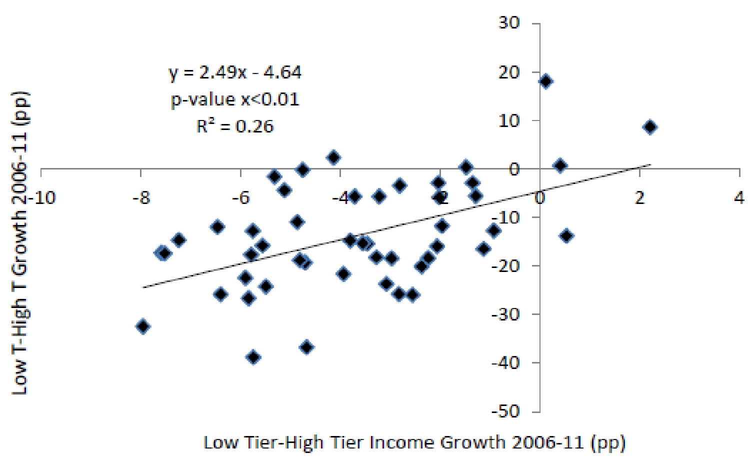
In Figure 2.14 we plot the differential low and high tier growth rates during the bust against

Figure 2.11: Tiered Measure and LTI Tightening: Bust



Source: S&P/Case-Shiller, Fiserv, Inc.; HMDA

Figure 2.12: Tiered Measure and Relative Income Growth: Bust



Source: S&P/Case-Shiller, Fiserv, Inc; BLS

Figure 2.13: Tiered Measure and Increase in Unemployment: Bust

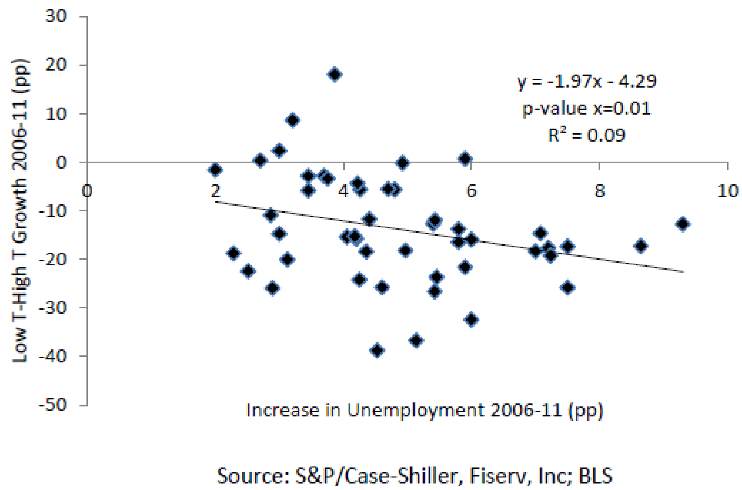


Table 2.3: Summary Statistics for Bust Variables: 2006-11

	Mean	Median	St. Dev.	Min	Max
Low Tier-High Tier Price Growth (pp.)	−13.79	−15.40	11.20	−38.2	17.95
Increase in use of Junior Liens (pp.)	−30.46	−30.64	9.00	−49.19	−8.93
Increase in LTI ratio	−0.22	−0.24	0.22	−0.82	0.18
Increase in Non-Owner Occupiers (pp.)	−2.04	−1.78	5.37	−15.04	8.64
Low Tier-High Tier Income Growth (pp.)	−3.68	−3.63	2.30	−7.97	2.21
Increase in Unemployment Rate (pp.)	4.83	4.57	1.68	2.00	9.30
Observations	48				

house building during the boom. This relationship can be used as a further test of the role of house building in generating the tiered pattern during the boom. If primarily high tier houses were built during the boom, we'd expect the low tier to do relatively better in the bust in places that had more building during the boom (due to less excess supply). The figure shows there is essentially no relationship between the two, with a p-value of 0.68. Further, if the two house-building outliers of Las Vegas, NV and Greeley, CO are removed (where over 70% of each city's 2000 housing stock was built during the boom), the p-value rises to 0.99.

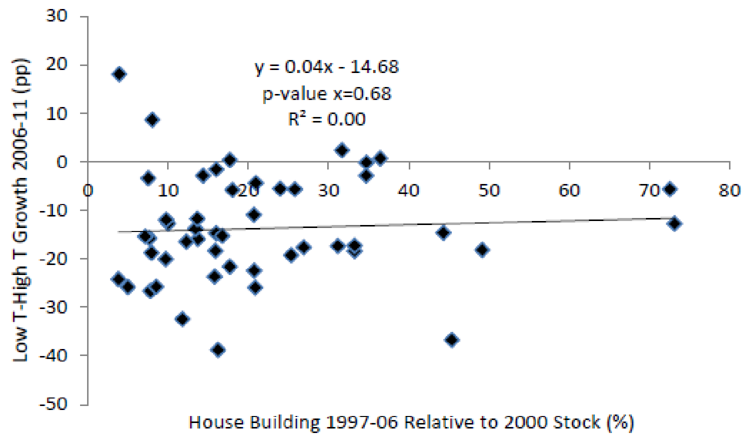
A summary table of the variables used in the regressions is presented in Table 2.3.

2.5.2 The Bust: Results

Table 2.4 presents the results of regressing (2.2) on various explanatory variables.[30] The first column shows that, without controlling for other variables, the coefficient on the change in the credit proxy is positive and statistically significant at the 1% level. In column 2 we add in the

[30] As with the boom regressions, we drop the two house building outliers, Las Vegas and Greeley. This leaves us with 46 observations.

Figure 2.14: Tiered Measure During Bust and Building During Boom



Source: S&P/Case-Shiller, Fiserv, Inc.; HUD; Census Bureau

other explanatory variables, and whilst the significance of the credit proxy drops slightly, it's still significant at the 5% level, and of comparable magnitude. The only other explanatory variable that's statistically significant is the relative income growth variable. Using the results from this regression we can calculate the relative contributions of the two significant explanatory variables. In the average city, during the bust low tier prices fell 13.8pp. more than high tier prices. Further, would-be low tier buyer income growth was on average -3.7pp. lower than for the high tier, so combined with an estimated coefficient of 1.93, low tier house prices would be predicted to fall by 7.1pp. more than high tier prices. The coefficient on credit tightening predicts low tier prices to fall by 13.1pp. more than high tier prices in the average city during the bust, close the observed mean of 13.8pp. The third and fourth column repeat the regressions with the alternative measure of the change in credit. In both cases, the coefficient on the change in average LTI ratios is positive and statistically significant at the 1% level. As with the prior regressions, the only other significant explanatory variable is the relative income change variable. At the mean, the results in column 4 predict low tier prices to fall 5.6pp. and 4.7pp. more than high tier prices, for credit and income respectively, during the bust.

In summary, from both sets of regression results, in places with greater credit easing during the housing boom, low tier prices grew more than high tier prices to a greater extent, with low tier prices crashing more during the bust where there was greater credit tightening. This is consistent with the theory developed in the last chapter regarding the tiered price responses to a change in non-price credit terms.

Table 2.4: Low Tier-High Tier Price Growth During Bust

	(1)	(2)	(3)	(4)
Increase in use of Junior Liens 2006-11	0.63***	0.43**		
	(3.66)	(2.11)		
Increase in LTI ratio 2006-11			32.73***	25.43***
			(4.30)	(4.15)
Low T-High T Price Growth 1997-06		−0.00		−0.01
		(−0.06)		(−0.12)
House Building 1997-06		0.14		0.14
		(0.82)		(0.80)
Increase in Non-Owner Occupiers 2006-11		0.01		−0.00
		(0.04)		(−0.02)
Low-High Tier Income Growth 2006-11		1.93***		1.28**
		(3.43)		(2.15)
Increase in Unemployment 2006-11		−0.65		−1.37
		(−0.59)		(−1.24)
R^2	0.24	0.40	0.38	0.50
Observations	46	46	46	46

t-statistics in parentheses (White heteroskedasticity-consistent standard errors used)
*p<0.1, **p<0.05, ***p<0.01

2.6 Conclusion

Despite the huge amount of research into the recent US housing boom and bust, the exact causes of it are still not well understood. An easing of non-price credit terms is generally thought to have been a key culprit, but formal econometric attempts to establish this have not succeeded, giving policy-makers less confidence in the efficacy of using macroprudential tools to attempt to dampen future booms. A key technical difficulty is the likely endogeneity between the rise in house prices and credit easing, with both likely feeding off each other. Whilst there have been some papers to approach this with instruments for credit changes (Adelino et al 2012, Favara and Imbs 2011), the specific changes in credit regulations they examine account for less than 3pp. of house price growth during the boom.

The last two chapters have tackled this problem in a new way using tiered housing data. In the first chapter we showed theoretically that if there had been no easing of credit standards during the boom, and only either a fall in interest rates or optimism on the part of house-buyers, we would observe high tier houses having had the greatest price growth during the boom. The fact that the opposite happened in 51 of 52 metros studied suggests there was an easing of credit standards.

The work of this chapter complements this by addressing alternative explanations of the observed tiered pattern beyond the three considered theoretically. These include an increase in the proportion of houses bought by non-owner occupiers, differential income growth, and changes in housing supply. The empirical results show that these alternative explanations cannot account for the observed pattern, *instead predicting greater growth for high tier prices.* Further, consistent with the theory of the first chapter, two separate measures of credit easing can, predicting the low tier to grow, respectively 62 and 76pp. more than the high tier in the average metro, compared to the 55pp. observed in the data. Taken together, the results strongly suggest that the tiered pattern observed *could not* have occurred without the easing of non-price credit terms. If correct, this implies substantial effectiveness for mortgage product regulation, such as LTV and LTI caps, reducing low tier growth by at least 55pp. in the average city during the US boom (the reduction in low tier price growth required for it to be less than high tier price growth).

This chapter also uncovered additional housing facts regarding tiered patterns during the boom and bust and performed empirical analysis of the housing bust, finding credit tightening largely responsible. I believe the use of tiered data can bring new insights to the housing market, and in future work it would be interesting to extend the empirical work here with more detailed loan level data. In particular, it would be interesting to use loan-level LTV data (instead of our credit easing proxy) and mortgage interest rate data, which would allow further testing of the theoretical predictions of the first chapter. Further, with data on both LTV and LTI easing, we could test which of the two contributed most to the tiered pattern during the boom, and thus which it would be most effective to target to attenuate future housing booms.

2.A Proof of Proposition 52

Here we prove the proposition linking the theoretical predictions of the model of Chapter 1 with empirically observable implications.

Proof. From the tiered model we have

$$\left(\frac{-dP_t^L(\gamma)}{d\gamma}\right)\frac{1}{P_t^L(\gamma)} > \left(\frac{-dP_t^H(\gamma)}{d\gamma}\right)\frac{1}{P_t^H(\gamma)} > 0$$

We first show that for any *discrete change* in γ, the response of the low tier is greater (so showing the first condition is greater than 1 and the second is greater than 0).

Let $\gamma_1 < \overline{\gamma}$ then

$$\int_{\gamma_1}^{\overline{\gamma}}\left(\frac{-dP_t^L(\gamma)}{d\gamma}\right)\frac{1}{P_t^L(\gamma)}d\gamma > \int_{\gamma_1}^{\overline{\gamma}}\left(\frac{-dP_t^H(\gamma)}{d\gamma}\right)\frac{1}{P_t^H(\gamma)}d\gamma$$

So

$$\begin{aligned}
\int_{\overline{\gamma}}^{\gamma_1}\frac{dP_t^L(\gamma)}{d\gamma}\frac{1}{P_t^L(\gamma)}d\gamma &> \int_{\overline{\gamma}}^{\gamma_1}\frac{dP_t^H(\gamma)}{d\gamma}\frac{1}{P_t^H(\gamma)}d\gamma \\
\left[\ln\left(P_t^L(\gamma)\right)\right]_{\overline{\gamma}}^{\gamma_1} &> \left[\ln\left(P_t^H(\gamma)\right)\right]_{\overline{\gamma}}^{\gamma_1} \\
\ln\left(\frac{P_t^L(\gamma_1)}{P_t^L(\overline{\gamma})}\right) &> \ln\left(\frac{P_t^H(\gamma_1)}{P_t^H(\overline{\gamma})}\right) \\
\frac{P_t^L(\gamma_1)}{P_t^L(\overline{\gamma})} &> \frac{P_t^H(\gamma_1)}{P_t^H(\overline{\gamma})}
\end{aligned}$$

From this it clearly follows that $\frac{\frac{P_t^L(\gamma_1)}{P_t^L(\overline{\gamma})}}{\frac{P_t^H(\gamma_1)}{P_t^H(\overline{\gamma})}} > 1$ and $\frac{P_t^L(\gamma_1)-P_t^L(\overline{\gamma})}{P_t^L(\overline{\gamma})} > \frac{\left(P_t^H(\gamma_1)-P_t^H(\overline{\gamma})\right)}{P_t^H(\overline{\gamma})}$.

We now show that under both measures, the gap between the tiers *increases* the greater is the credit easing.

Let $\gamma_2 < \gamma_1 < \gamma$. Then as $\left(\frac{-dP_t^L(\gamma)}{d\gamma}\right)\frac{1}{P_t^L(\gamma)} - \left(\frac{-dP_t^H(\gamma)}{d\gamma}\right)\frac{1}{P_t^H(\gamma)} > 0$ we have (as we're integrating a positive term over a greater distance):

$$\begin{aligned} &\int_{\gamma_2}^{\overline{\gamma}} \left[\left(\frac{-dP_t^L(\gamma)}{d\gamma}\right)\frac{1}{P_t^L(\gamma)} - \left(\frac{-dP_t^H(\gamma)}{d\gamma}\right)\frac{1}{P_t^H(\gamma)}\right] d\gamma \\ > \quad &\int_{\gamma_1}^{\overline{\gamma}} \left[\left(\frac{-dP_t^L(\gamma)}{d\gamma}\right)\frac{1}{P_t^L(\gamma)} - \left(\frac{-dP_t^H(\gamma)}{d\gamma}\right)\frac{1}{P_t^H(\gamma)}\right] d\gamma \end{aligned}$$

Hence

$$\begin{aligned} &\int_{\overline{\gamma}}^{\gamma_2} \left[\left(\frac{dP_t^L(\gamma)}{d\gamma}\right)\frac{1}{P_t^L(\gamma)} - \left(\frac{dP_t^H(\gamma)}{d\gamma}\right)\frac{1}{P_t^H(\gamma)}\right] d\gamma \\ > \quad &\int_{\overline{\gamma}}^{\gamma_1} \left[\left(\frac{dP_t^L(\gamma)}{d\gamma}\right)\frac{1}{P_t^L(\gamma)} - \left(\frac{dP_t^H(\gamma)}{d\gamma}\right)\frac{1}{P_t^H(\gamma)}\right] d\gamma \end{aligned}$$

And so

$$\begin{aligned} \left[\ln\left(P_t^L(\gamma)\right) - \ln\left(P_t^H(\gamma)\right)\right]_{\overline{\gamma}}^{\gamma_2} &> \left[\ln\left(P_t^L(\gamma)\right) - \ln\left(P_t^H(\gamma)\right)\right]_{\overline{\gamma}}^{\gamma_1} \\ \ln\frac{\left(\frac{P_t^L(\gamma_2)}{dP_t^L(\overline{\gamma})}\right)}{\left(\frac{P_t^H(\gamma_2)}{P_t^H(\overline{\gamma})}\right)} &> \ln\frac{\left(\frac{P_t^L(\gamma_1)}{P_t^L(\overline{\gamma})}\right)}{\left(\frac{P_t^H(\gamma_1)}{P_t^H(\overline{\gamma})}\right)} \\ \frac{\left(\frac{P_t^L(\gamma_2)}{P_t^L(\overline{\gamma})}\right)}{\left(\frac{P_t^H(\gamma_2)}{P_t^H(\overline{\gamma})}\right)} &> \frac{\left(\frac{P_t^L(\gamma_1)}{P_t^L(\overline{\gamma})}\right)}{\left(\frac{P_t^H(\gamma_1)}{P_t^H(\overline{\gamma})}\right)} \end{aligned}$$

This completes the result for the $\frac{L}{H}$ measure.

Further

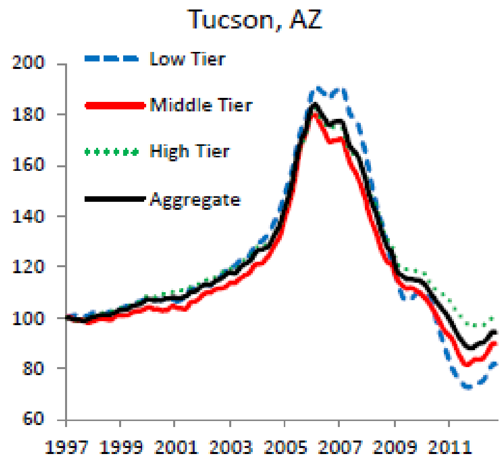
$$
\begin{aligned}
\left(\frac{P_t^L(\gamma_2)}{P_t^L(\overline{\gamma})}\right) &> \frac{\left(\frac{P_t^H(\gamma_2)}{P_t^H(\overline{\gamma})}\right)}{\left(\frac{P_t^H(\gamma_1)}{P_t^H(\overline{\gamma})}\right)}\left(\frac{P_t^L(\gamma_1)}{P_t^L(\overline{\gamma})}\right) \\
\left(\frac{P_t^L(\gamma_2)}{P_t^L(\overline{\gamma})}\right)-\left(\frac{P_t^H(\gamma_2)}{P_t^H(\overline{\gamma})}\right) &> \frac{\left(\frac{P_t^H(\gamma_2)}{P_t^H(\overline{\gamma})}\right)}{\left(\frac{P_t^H(\gamma_1)}{P_t^H(\overline{\gamma})}\right)}\left(\frac{P_t^L(\gamma_1)}{P_t^L(\overline{\gamma})}\right)-\left(\frac{P_t^H(\gamma_2)}{P_t^H(\overline{\gamma})}\right) \\
&= \frac{\left(\frac{P_t^H(\gamma_2)}{P_t^H(\overline{\gamma})}\right)}{\left(\frac{P_t^H(\gamma_1)}{P_t^H(\overline{\gamma})}\right)}\left(\frac{P_t^L(\gamma_1)}{P_t^L(\overline{\gamma})}-\frac{P_t^H(\gamma_1)}{P_t^H(\overline{\gamma})}\right) \\
&= \frac{P_t^H(\gamma_2)}{P_t^H(\gamma_1)}\left(\frac{P_t^L(\gamma_1)}{P_t^L(\overline{\gamma})}-\frac{P_t^H(\gamma_1)}{P_t^H(\overline{\gamma})}\right) \\
&> \left(\frac{P_t^L(\gamma_1)}{P_t^L(\overline{\gamma})}-\frac{P_t^H(\gamma_1)}{P_t^H(\overline{\gamma})}\right)
\end{aligned}
$$

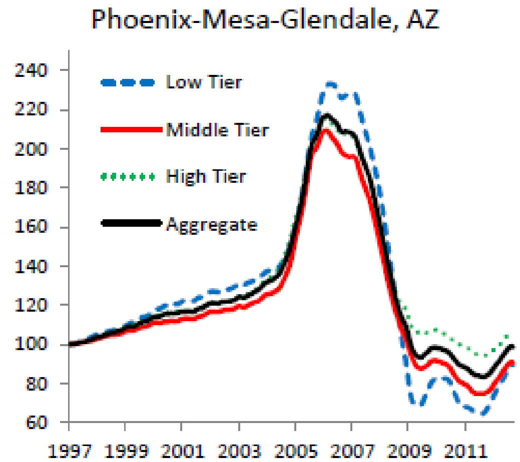
Where the last line follows as, $\gamma_2 < \gamma_1$ so $\frac{P_t^H(\gamma_2)}{P_t^H(\gamma_1)} > 1$ (high tier prices are higher with looser credit). This completes the proof for the $L\% - H\%$ measure, completing the proof. ■

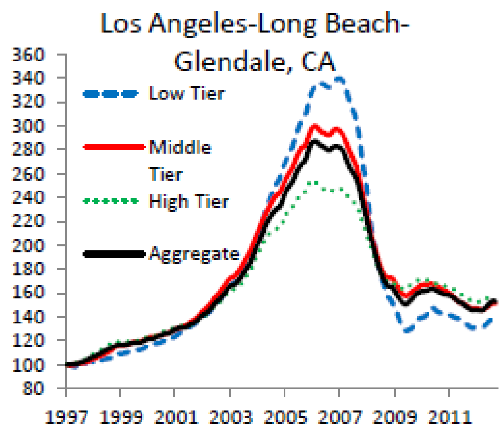
2.B Tiered Housing Pictures

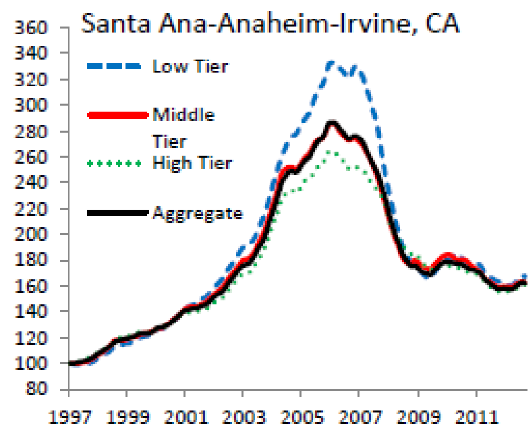
Source: S&P/Case-Shiller, Fiserv, Inc.

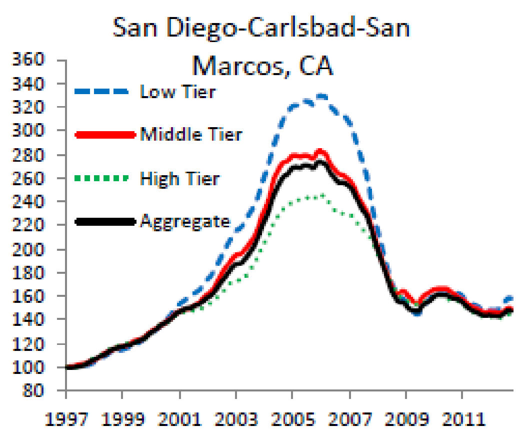
All the graphs are in real terms and have been normalised to 100 in Jan 1997

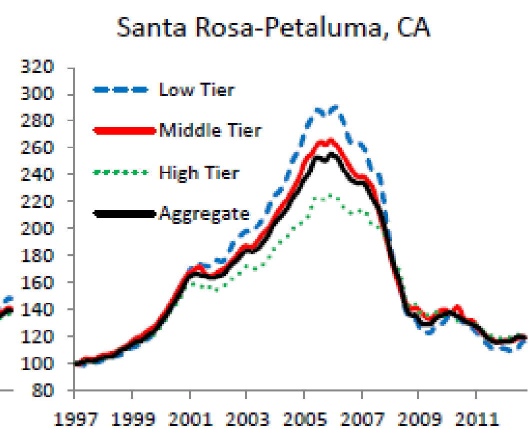


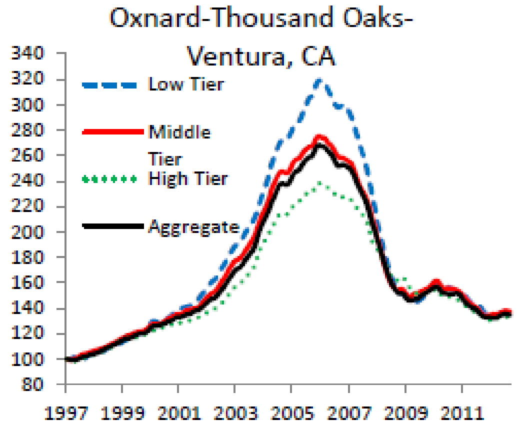


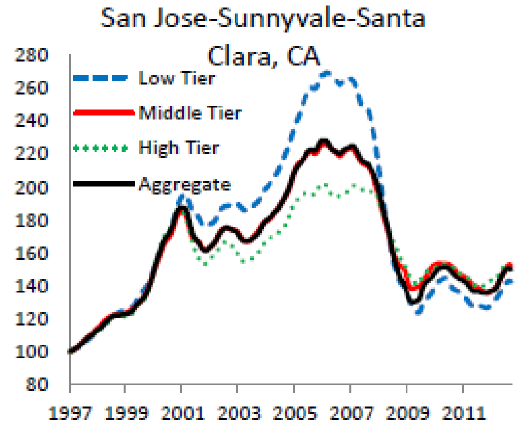


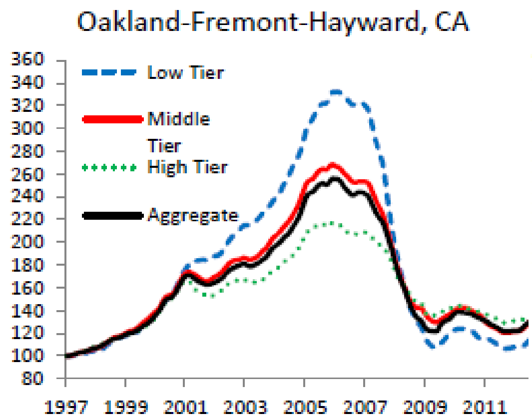


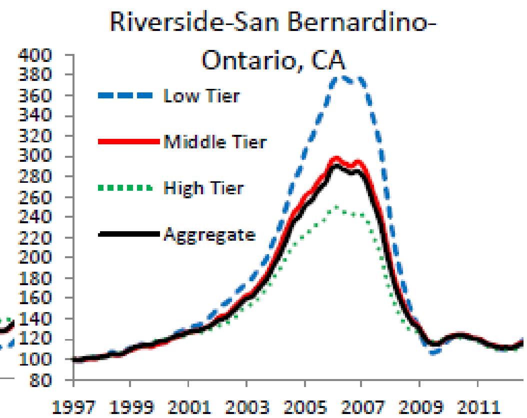


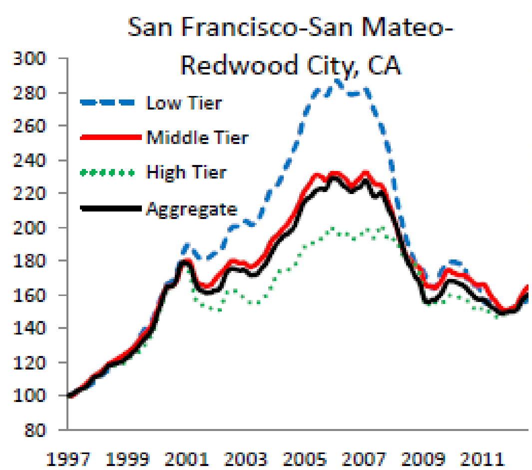


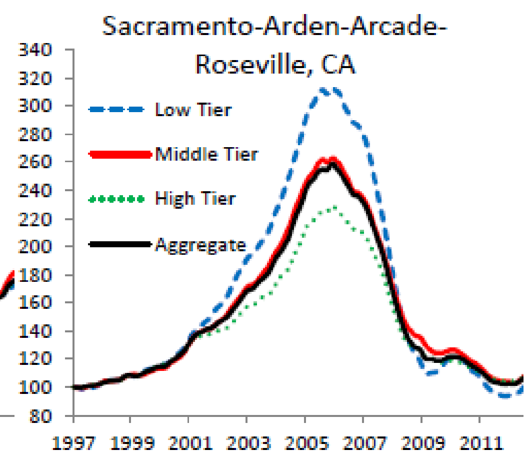

Oxnard-Thousand Oaks-Ventura, CA
Low Tier
Middle Tier
High Tier
Aggregate
340 320 300 280 260 240 220 200 180 160 140 120 100 80
1997 1999 2001 2003 2005 2007 2009 2011

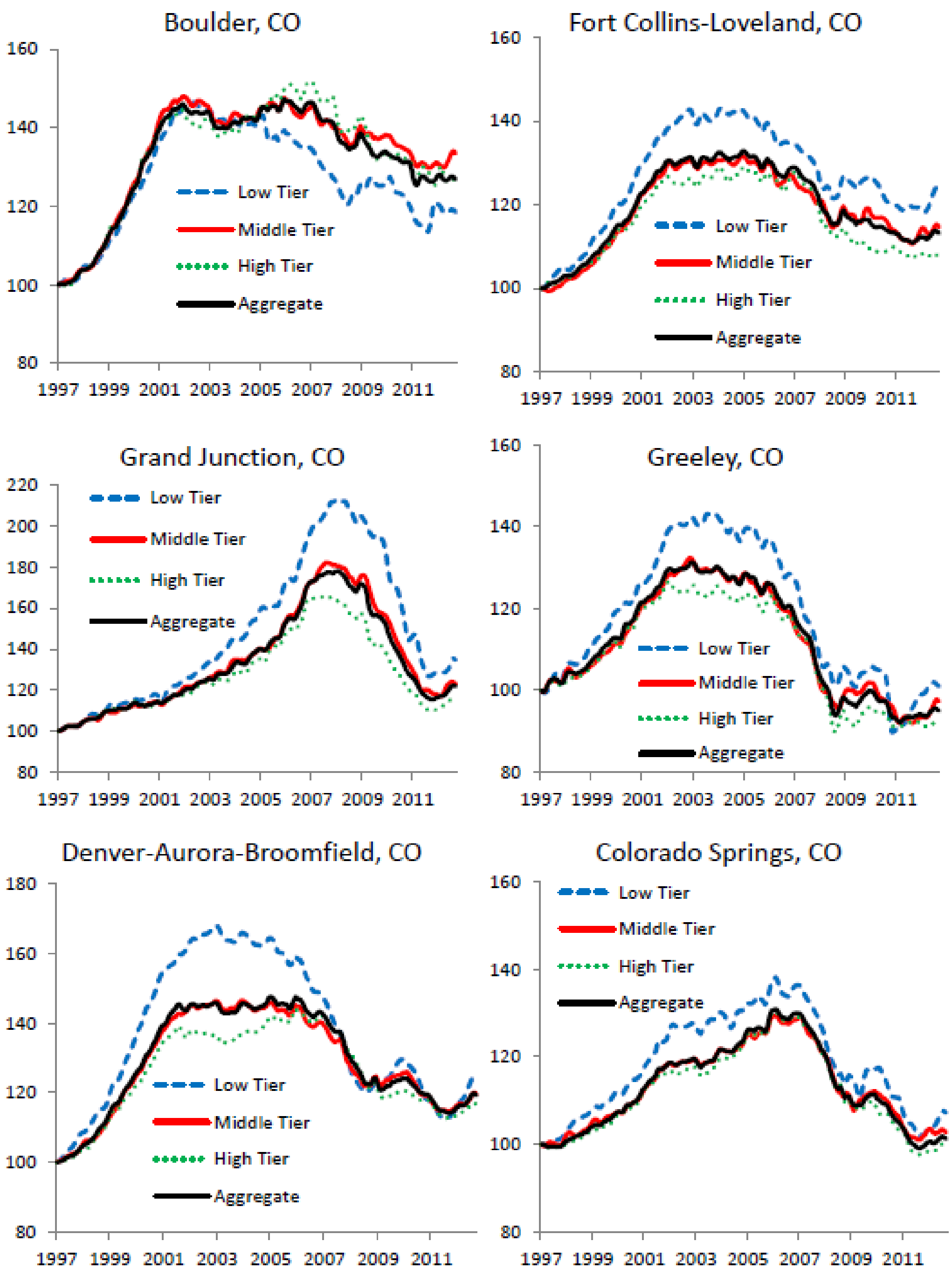

San Jose-Sunnyvale-Santa Clara, CA
Low Tier
Middle Tier
High Tier
Aggregate
280 260 240 220 200 180 160 140 120 100 80
1997 1999 2001 2003 2005 2007 2009 2011

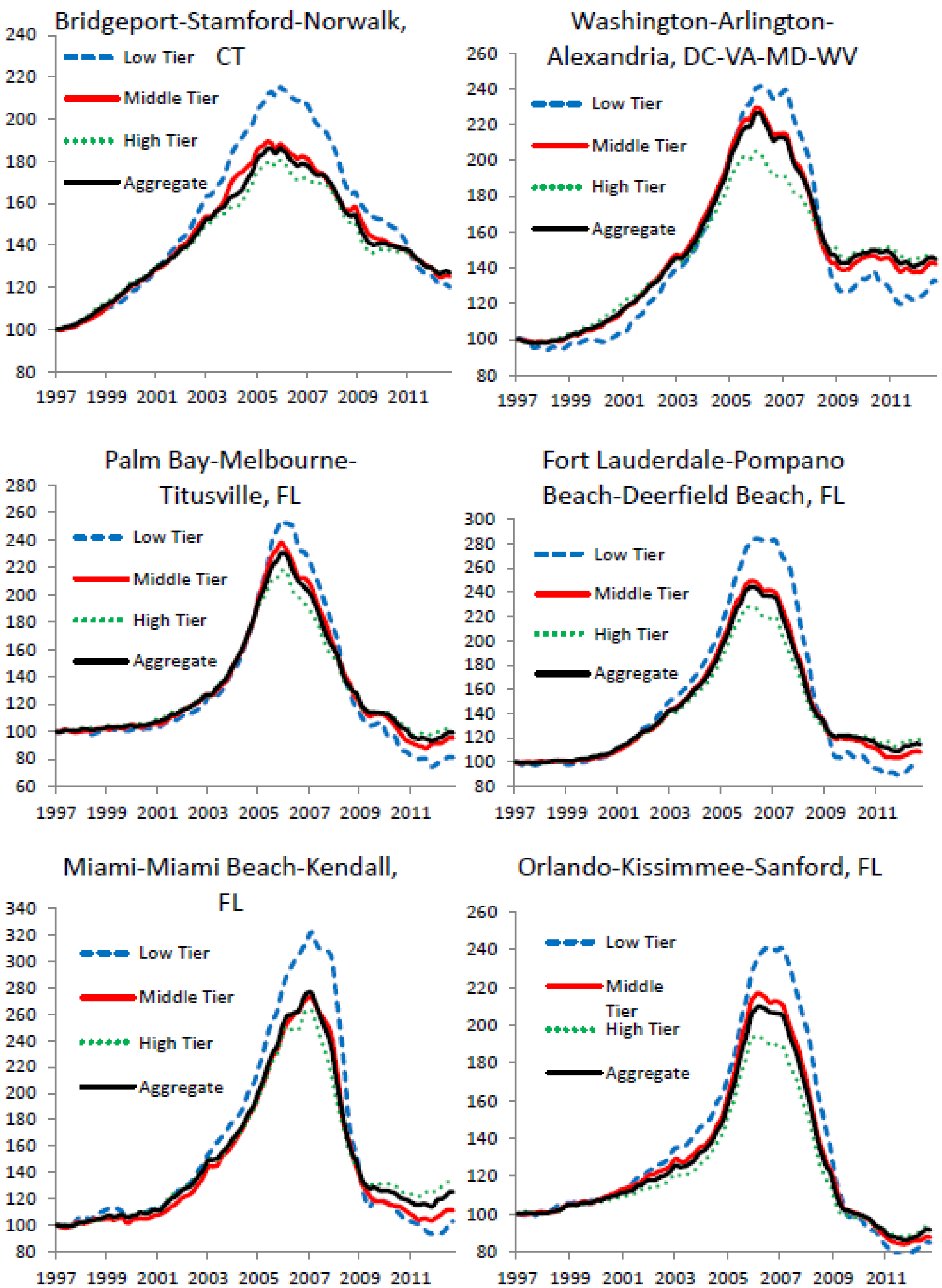

Oakland-Fremont-Hayward, CA
Low Tier
Middle Tier
High Tier
Aggregate
360 340 320 300 280 260 240 220 200 180 160 140 120 100 80
1997 1999 2001 2003 2005 2007 2009 2011

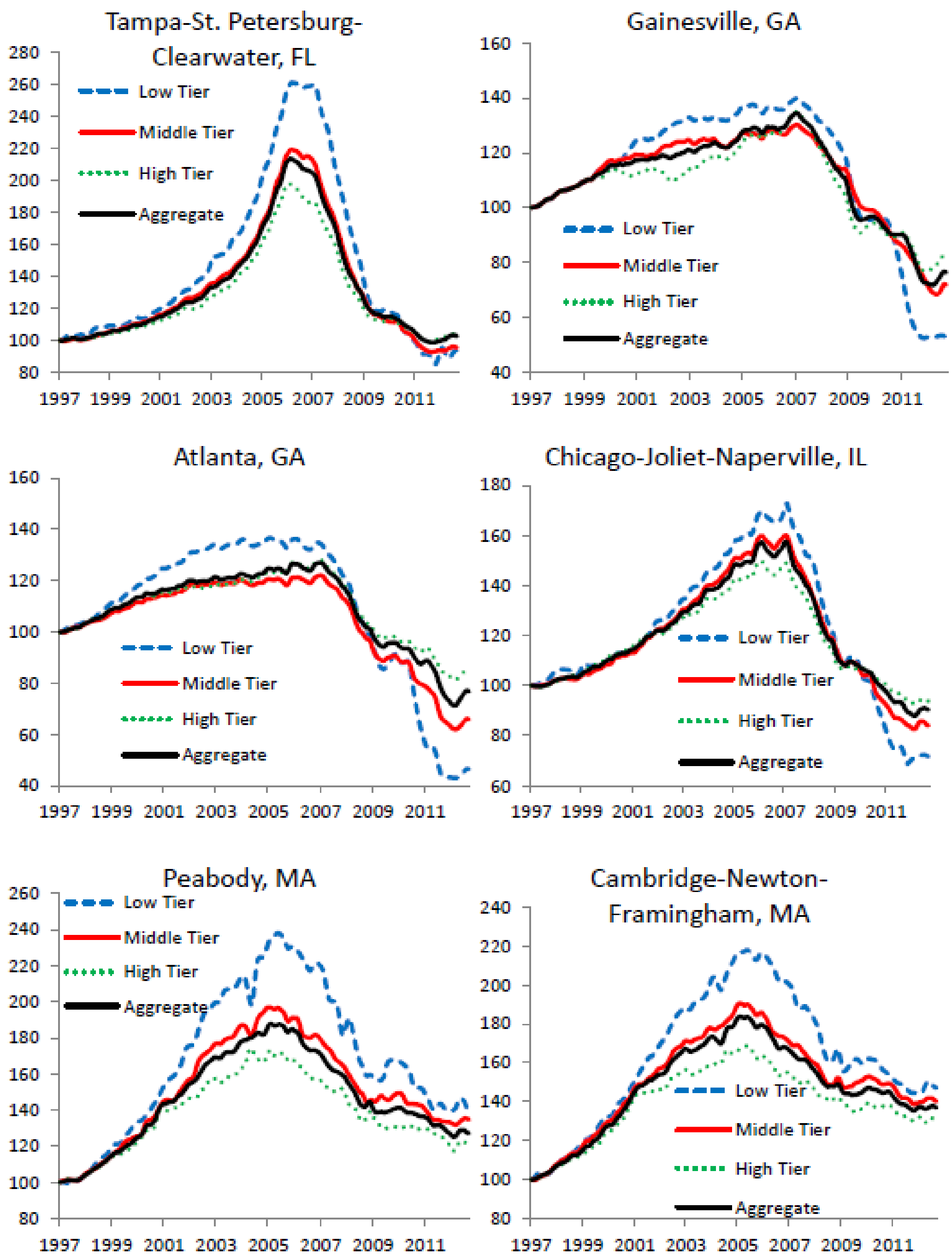

Riverside-San Bernardino-Ontario, CA
Low Tier
Middle Tier
High Tier
Aggregate
400 380 360 340 320 300 280 260 240 220 200 180 160 140 120 100 80
1997 1999 2001 2003 2005 2007 2009 2011

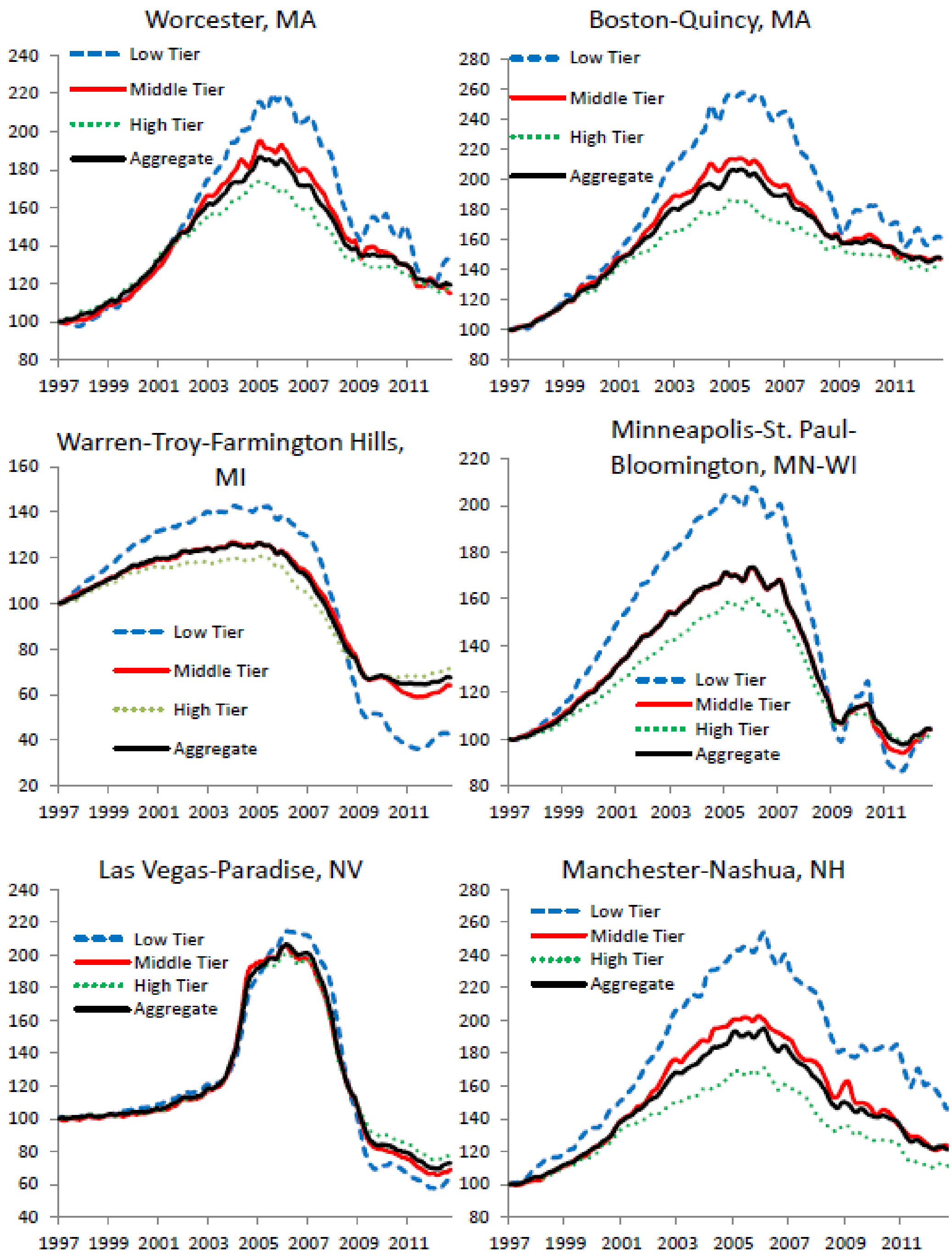

San Francisco-San Mateo-Redwood City, CA
Low Tier
Middle Tier
High Tier
Aggregate
300 280 260 240 220 200 180 160 140 120 100 80
1997 1999 2001 2003 2005 2007 2009 2011

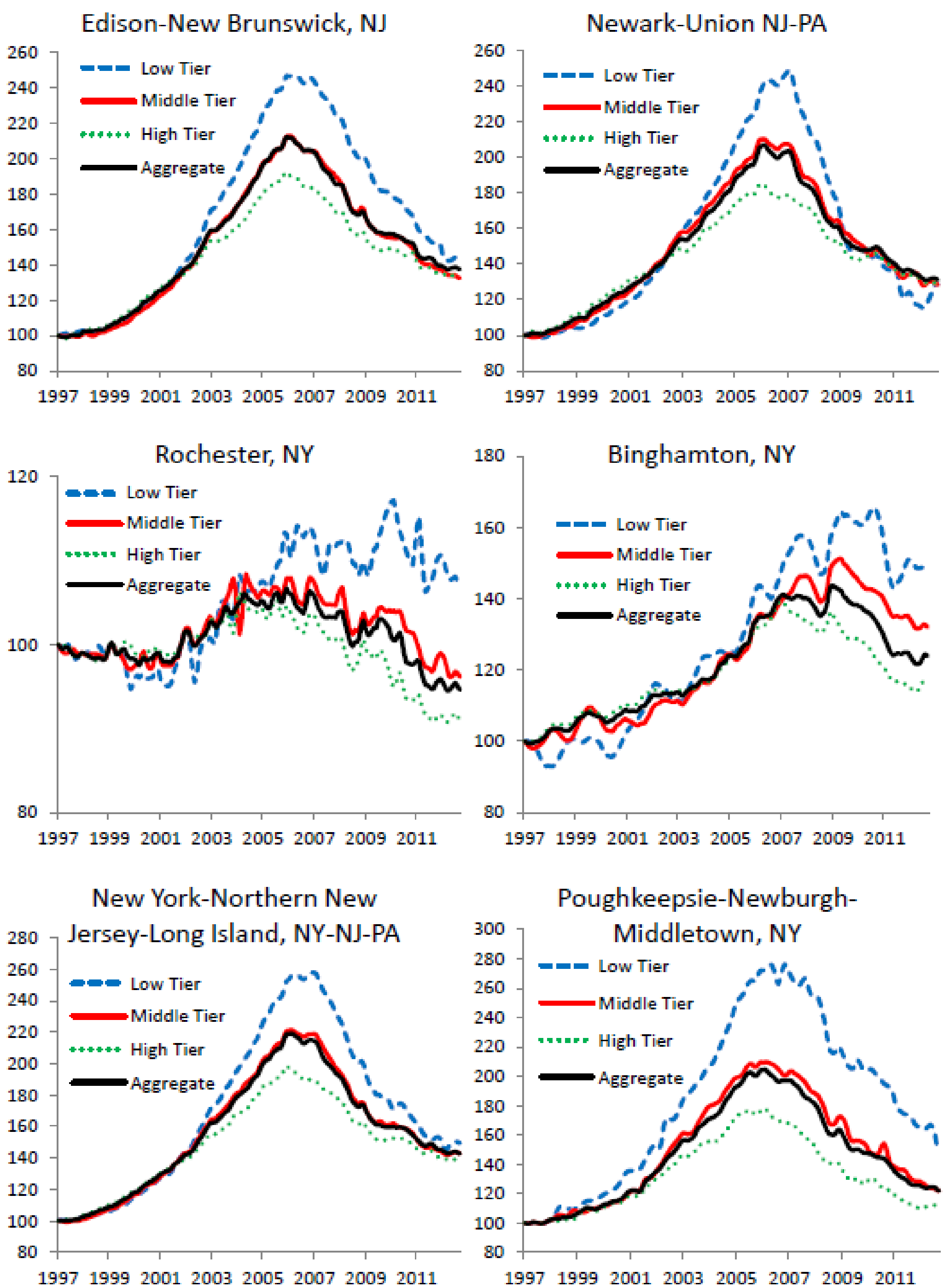

Sacramento-Arden-Arcade-Roseville, CA
Low Tier
Middle Tier
High Tier
Aggregate
340 320 300 280 260 240 220 200 180 160 140 120 100 80
1997 1999 2001 2003 2005 2007 2009 2011

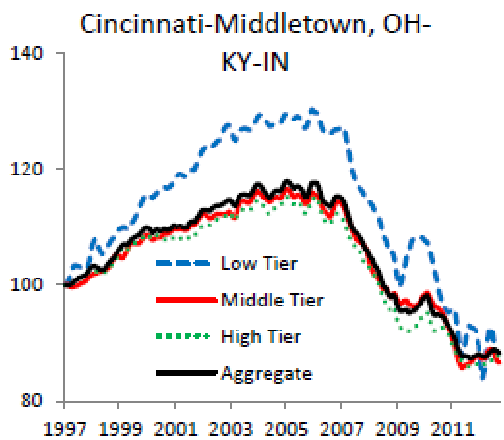

Boulder, CO
160
140
120
100
80
1997 1999 2001 2003 2005 2007 2009 2011
Low Tier
Middle Tier
High Tier
Aggregate
Fort Collins-Loveland, CO
160
140
120
100
80
1997 1999 2001 2003 2005 2007 2009 2011
Low Tier
Middle Tier
High Tier
Aggregate
Grand Junction, CO
220
200
180
160
140
120
100
80
1997 1999 2001 2003 2005 2007 2009 2011
Low Tier
Middle Tier
High Tier
Aggregate
Greeley, CO
160
140
120
100
80
1997 1999 2001 2003 2005 2007 2009 2011
Low Tier
Middle Tier
High Tier
Aggregate
Denver-Aurora-Broomfield, CO
180
160
140
120
100
80
1997 1999 2001 2003 2005 2007 2009 2011
Low Tier
Middle Tier
High Tier
Aggregate
Colorado Springs, CO
160
140
120
100
80
1997 1999 2001 2003 2005 2007 2009 2011
Low Tier
Middle Tier
High Tier
Aggregate

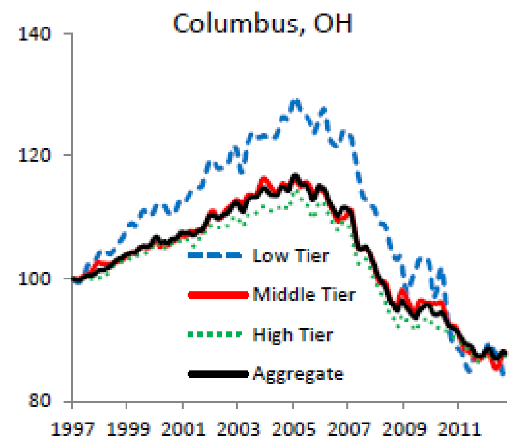

Bridgeport-Stamford-Norwalk, CT
Low Tier
Middle Tier
High Tier
Aggregate
240
220
200
180
160
140
120
100
80
1997 1999 2001 2003 2005 2007 2009 2011
Washington-Arlington-Alexandria, DC-VA-MD-WV
Low Tier
Middle Tier
High Tier
Aggregate
260
240
220
200
180
160
140
120
100
80
1997 1999 2001 2003 2005 2007 2009 2011
Palm Bay-Melbourne-Titusville, FL
Low Tier
Middle Tier
High Tier
Aggregate
280
260
240
220
200
180
160
140
120
100
80
60
1997 1999 2001 2003 2005 2007 2009 2011
Fort Lauderdale-Pompano Beach-Deerfield Beach, FL
Low Tier
Middle Tier
High Tier
Aggregate
300
280
260
240
220
200
180
160
140
120
100
80
1997 1999 2001 2003 2005 2007 2009 2011
Miami-Miami Beach-Kendall, FL
Low Tier
Middle Tier
High Tier
Aggregate
340
320
300
280
260
240
220
200
180
160
140
120
100
80
1997 1999 2001 2003 2005 2007 2009 2011
Orlando-Kissimmee-Sanford, FL
Low Tier
Middle Tier
High Tier
Aggregate
260
240
220
200
180
160
140
120
100
80
1997 1999 2001 2003 2005 2007 2009 2011

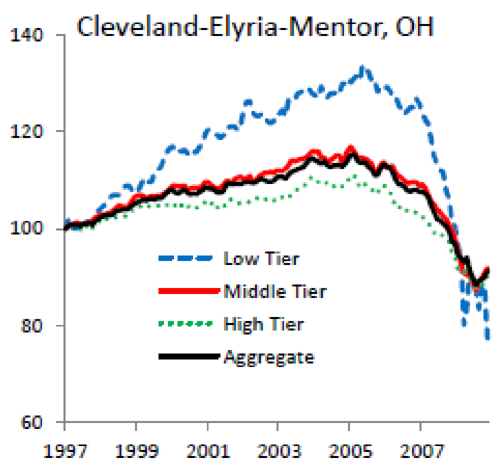

Tampa-St. Petersburg-Clearwater, FL
Low Tier
Middle Tier
High Tier
Aggregate
280
260
240
220
200
180
160
140
120
100
80
1997 1999 2001 2003 2005 2007 2009 2011
Gainesville, GA
Low Tier
Middle Tier
High Tier
Aggregate
160
140
120
100
80
60
40
1997 1999 2001 2003 2005 2007 2009 2011
Atlanta, GA
Low Tier
Middle Tier
High Tier
Aggregate
160
140
120
100
80
60
40
1997 1999 2001 2003 2005 2007 2009 2011
Chicago-Joliet-Naperville, IL
Low Tier
Middle Tier
High Tier
Aggregate
180
160
140
120
100
80
60
1997 1999 2001 2003 2005 2007 2009 2011
Peabody, MA
Low Tier
Middle Tier
High Tier
Aggregate
260
240
220
200
180
160
140
120
100
80
1997 1999 2001 2003 2005 2007 2009 2011
Cambridge-Newton-Framingham, MA
Low Tier
Middle Tier
High Tier
Aggregate
240
220
200
180
160
140
120
100
80
1997 1999 2001 2003 2005 2007 2009 2011

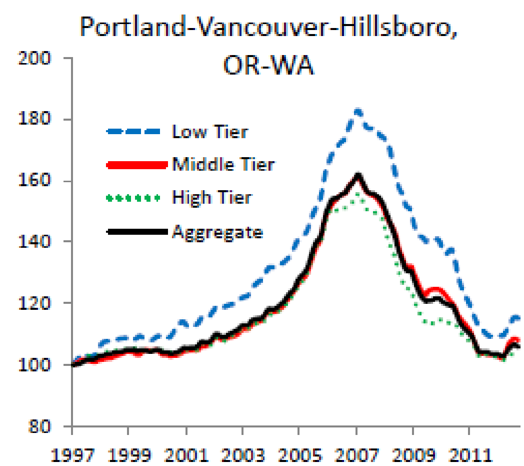

Worcester, MA
Low Tier
Middle Tier
High Tier
Aggregate
240
220
200
180
160
140
120
100
80
1997 1999 2001 2003 2005 2007 2009 2011
Boston-Quincy, MA
Low Tier
Middle Tier
High Tier
Aggregate
280
260
240
220
200
180
160
140
120
100
80
1997 1999 2001 2003 2005 2007 2009 2011
Warren-Troy-Farmington Hills, MI
Low Tier
Middle Tier
High Tier
Aggregate
160
140
120
100
80
60
40
20
1997 1999 2001 2003 2005 2007 2009 2011
Minneapolis-St. Paul-Bloomington, MN-WI
Low Tier
Middle Tier
High Tier
Aggregate
220
200
180
160
140
120
100
80
1997 1999 2001 2003 2005 2007 2009 2011
Las Vegas-Paradise, NV
Low Tier
Middle Tier
High Tier
Aggregate
240
220
200
180
160
140
120
100
80
60
40
1997 1999 2001 2003 2005 2007 2009 2011
Manchester-Nashua, NH
Low Tier
Middle Tier
High Tier
Aggregate
280
260
240
220
200
180
160
140
120
100
80
1997 1999 2001 2003 2005 2007 2009 2011

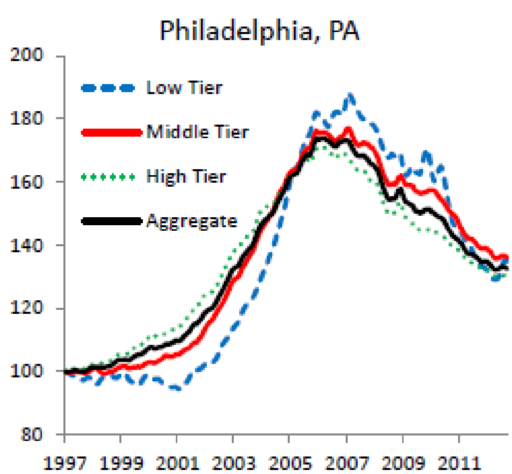

Edison-New Brunswick, NJ
Newark-Union NJ-PA
Rochester, NY
Binghamton, NY
New York-Northern New Jersey-Long Island, NY-NJ-PA
Poughkeepsie-Newburgh-Middletown, NY
Low Tier
Middle Tier
High Tier
Aggregate
1997 1999 2001 2003 2005 2007 2009 2011

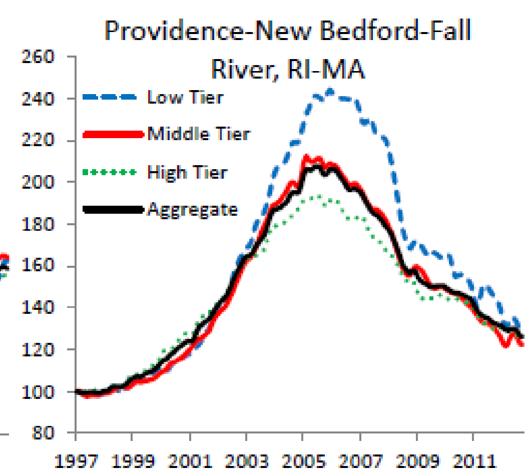

Cincinnati-Middletown, OH-KY-IN
140
120
100
80
Low Tier
Middle Tier
High Tier
Aggregate
1997 1999 2001 2003 2005 2007 2009 2011

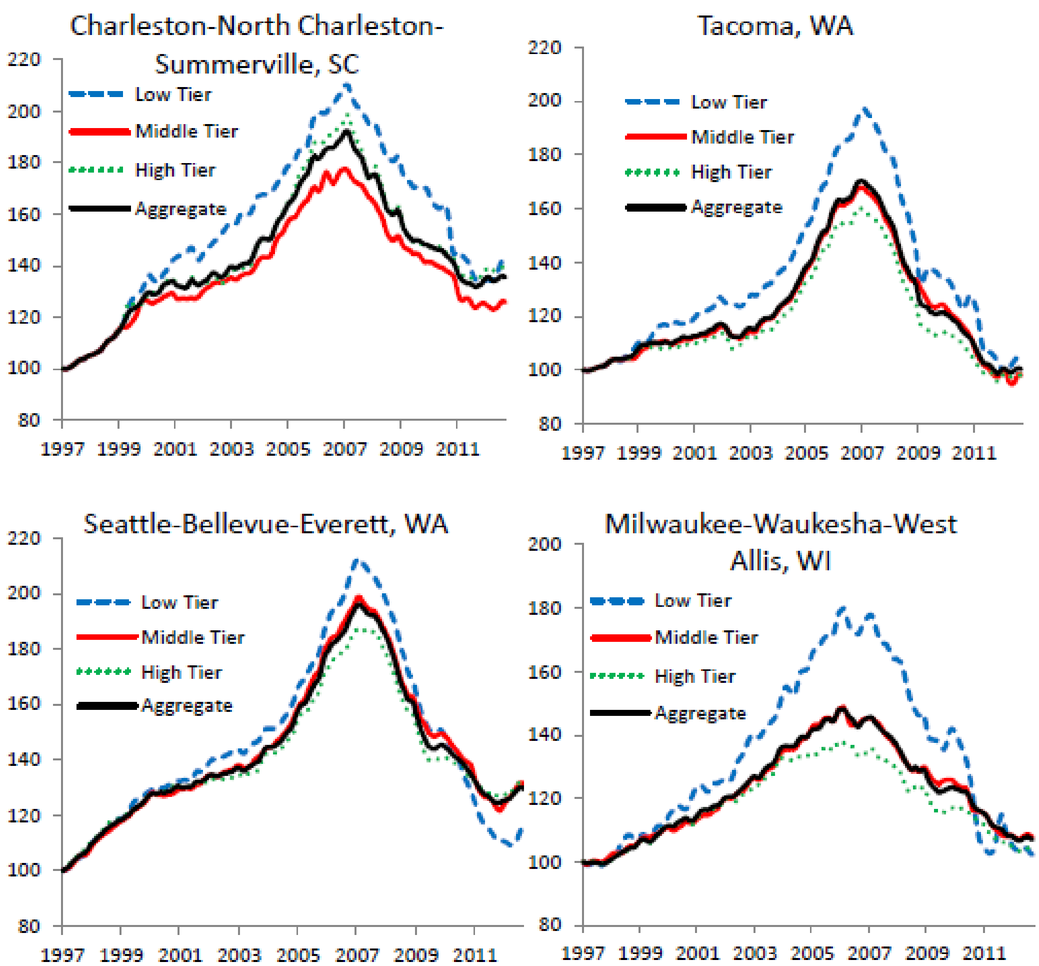

Columbus, OH
140
120
100
80
Low Tier
Middle Tier
High Tier
Aggregate
1997 1999 2001 2003 2005 2007 2009 2011


Cleveland-Elyria-Mentor, OH
140
120
100
80
60
Low Tier
Middle Tier
High Tier
Aggregate
1997 1999 2001 2003 2005 2007


Portland-Vancouver-Hillsboro, OR-WA
200
180
160
140
120
100
80
Low Tier
Middle Tier
High Tier
Aggregate
1997 1999 2001 2003 2005 2007 2009 2011


Philadelphia, PA
200
180
160
140
120
100
80
Low Tier
Middle Tier
High Tier
Aggregate
1997 1999 2001 2003 2005 2007 2009 2011


Providence-New Bedford-Fall River, RI-MA
260
240
220
200
180
160
140
120
100
80
Low Tier
Middle Tier
High Tier
Aggregate
1997 1999 2001 2003 2005 2007 2009 2011



2.C Variable Construction

The geographical regions covered by the house price data are based on MSA definitions after definitional changes that were brought in in 2003. The key challenge when constructing the other variables is to ensure a consistent geographical area is covered throughout time, so changes in the variables over time reflect genuine change, and not simply that the area covered has changed. For many of our MSAs, the geographical area changes, and for example the Atlanta, GA (0520) MSA pre- 2003 is not the same (although there is much overlap) as the Atlanta-Sandy Springs-Marietta, GA (12060) MSA which was created in 2003.[32] For the construction of variables covering the housing boom, we compare their value in 1997 and in the peak of the boom in 2006.[33] Except in the case of relative income data where this was not possible, our approach for the construction of

[32] Information on the MSA definitions comes from the Census Bureau.
[33] Geographical definitional changes did not affect the construction of variables for the housing bust.

Table 2.5: Facts About Tiered House Price Movements

	Nominal Price Growth During Boom and Bust (%)					
	Boom: 1997-2006			Bust: 2006-2011		
Metro Area	Low T.	High T.	Agg.	Low T.	High T.	Agg.
Tucson, AZ	139.13	124.68	126.88	−55.31	−36.87	−42.79
Phoenix, AZ	190.80	166.73	168.86	−67.57	−49.32	−55.32
Los Angeles, CA	324.38	214.22	258.40	−54.70	−28.80	−40.41
Santa Ana, CA	315.30	224.49	253.59	−42.73	−29.99	−34.87
San Diego, CA	303.38	197.91	231.69	−46.77	−30.77	−37.32
Santa Rosa, CA	248.07	174.04	206.97	−53.42	−36.89	−44.88
Oxnard, CA	287.28	192.97	227.17	−47.92	−34.11	−40.10
San Jose, CA	237.57	149.60	183.46	−46.20	−19.53	−31.32
Oakland, CA	312.45	166.10	212.91	−62.27	−29.76	−43.59
Riverside, CA	376.75	210.44	262.75	−66.36	−49.01	−55.48
San Francisco, CA	256.56	147.38	184.53	−39.09	−14.84	−24.49
Sacramento, CA	274.05	174.92	208.06	−63.17	−45.48	−51.92
Boulder, CO	72.71	90.08	84.59	−4.59	−4.88	−2.93
Fort Collins, CO	71.55	59.11	62.42	−1.64	−3.89	−2.72
Grand Junction, CO	130.01	95.37	103.33	−17.74	−18.34	−17.39
Greeley, CO	66.22	51.33	54.97	−19.98	−14.25	−14.92
Denver, CO	92.81	79.53	82.16	−16.14	−10.42	−10.53
Colorado Springs, CO	71.81	63.58	64.15	−14.56	−14.34	−13.39
Bridgeport, CT	166.72	120.47	128.63	−31.08	−15.26	−17.84
Washington, DC	201.87	148.53	175.44	−42.51	−16.52	−26.39
Palm Bay, FL	205.71	156.56	172.71	−63.43	−46.01	−50.80
Fort Lauderdale, FL	257.80	182.55	204.26	−64.02	−42.34	−48.37
Miami, FL	281.21	219.14	233.89	−63.61	−45.21	−51.03
Orlando, FL	202.47	142.76	163.39	−62.92	−48.22	−53.08
Tampa, FL	228.90	143.58	165.76	−60.70	−41.40	−46.49
Gainesville, GA	73.21	63.76	65.00	−51.69	−27.83	−30.62
Atlanta, GA	69.92	59.75	59.01	−58.84	−22.06	−27.51
Chicago, IL	111.76	85.88	95.31	−50.16	−26.42	−32.55
Peabody, MA	182.26	103.82	123.50	−28.58	−12.63	−16.82
Cambridge, MA	162.48	99.59	117.38	−20.53	−6.15	−10.01
Worcester, MA	165.95	106.60	123.31	−32.22	−17.40	−21.81
Boston, MA	212.56	120.18	145.16	−26.13	−7.30	−13.75

Table 2.6: Facts About Tiered House Price Movements Cntd.

	Nominal Price Growth During Boom and Bust (%)					
	Boom: 1997-2006			Bust: 2006-2011		
Metro Area	Low T.	High T.	Agg.	Low T.	High T.	Agg.
Warren, MI	69.07	38.12	47.39	−68.86	−30.04	−37.85
Minneapolis, MN-	154.44	97.59	113.29	−50.65	−28.15	−34.21
Manchester, NH	206.33	107.28	136.62	−23.51	−21.88	−24.00
Edison, NJ	210.81	136.42	162.64	−29.47	−17.67	−23.01
Newark, NJ-PA	207.41	127.98	156.96	−42.52	−16.69	−24.76
Rochester, NY	43.06	30.22	33.23	8.44	−0.15	1.77
Binghamton, NY	80.75	70.45	72.71	15.18	−2.77	2.58
New York, NY	223.79	143.76	173.46	−32.43	−16.98	−23.41
Poughkeepsie, NY	244.52	117.82	152.77	−29.01	−26.09	−27.90
Cincinnati, OH	61.26	42.31	45.93	−19.43	−13.54	−14.28
Columbus, OH	56.94	39.05	42.06	−21.68	−10.68	−11.77
Cleveland, OH	59.75	32.64	38.69	*NA*[31]	*NA*	*NA*
Portland, OR	119.61	91.70	97.39	−27.86	−23.43	−24.70
Philadelphia, PA	128.81	114.64	118.82	−14.85	−11.40	−11.47
Providence, RI	204.35	135.73	153.53	−32.46	−20.46	−24.67
Charleston, SC	156.04	141.46	135.14	−23.29	−20.38	−19.73
Tacoma, WA	136.82	97.42	108.93	−34.59	−28.99	−30.59
Seattle, WA	152.30	126.95	135.73	−35.25	−19.89	−23.79
Milwaukee, WI	122.15	71.80	84.15	−31.67	−11.57	−14.75
Las Vegas, NV	170.28	150.43	156.89	−67.80	−54.99	−59.21
Mean	175.04	120.18	136.85	−38.01	−23.96	−28.18
Median	168.50	120.33	135.44	−35.25	−20.46	−24.76
Standard Dev.	83.30	51.17	61.75	20.61	13.66	15.73
Min	43.06	30.22	33.23	−68.86	−54.99	−59.21
Max	376.75	224.49	262.75	15.18	−0.15	2.58

Source: Fiserv, Inc., S&P/Case-Shiller

Table 2.7: Income Estimates for Low and High Tier Buyers

	Est. Median Income 1997 ($000's)		
Metro Area	Low T.	Middle T.	High T.
Tucson, AZ	27	45	81
Phoenix, AZ	30	50	92
Los Angeles, CA	39	63	123
Santa Ana, CA	45	76	133
San Diego, CA	38	64	115
Santa Rosa, CA	42	67	112
Oxnard, CA	45	72	120
San Jose, CA	55	87	144
Oakland, CA	45	76	128
Riverside, CA	29	49	86
San Francisco, CA	58	97	180
Sacramento, CA	32	55	96
Boulder, CO	39	65	115
Fort Collins, CO	32	53	89
Grand Junction, CO	24	39	67
Greeley, CO	29	44	74
Denver, CO	32	53	91
Colorado Springs, CO	31	49	80
Bridgeport, CT	47	86	174
Washington, DC	40	67	115
Palm Bay, FL	26	45	79
Fort Lauderdale, FL	31	52	94
Miami, FL	31	49	96
Orlando, FL	28	48	86
Tampa, FL	26	45	84
Gainesville, GA	28	44	78
Atlanta, GA	33	54	95
Chicago, IL	36	57	97
Peabody, MA	37	62	110
Cambridge, MA	42	71	121
Worcester, MA	35	55	90
Boston, MA	38	63	111

Table 2.8: Income Estimates for Low and High Tier Buyers Cntd.

	Est. Median Income 1997 ($000's)		
Metro Area	Low T.	Middle T.	High T.
Warren, MI	35	57	95
Minneapolis, MN-	31	51	86
Manchester, NH	37	57	92
Edison, NJ	41	67	115
Newark, NJ-PA	43	71	125
Rochester, NY	30	49	82
Binghamton, NY	24	41	71
New York, NY	38	65	114
Poughkeepsie, NY	39	60	95
Cincinnati, OH	28	48	81
Columbus, OH	30	50	84
Cleveland, OH	30	49	83
Portland, OR	35	55	93
Philadelphia, PA	30	56	102
Providence, RI	30	49	82
Charleston, SC	25	46	91
Tacoma, WA	33	51	80
Seattle, WA	38	60	100
Milwaukee, WI	34	55	88
Las Vegas, NV	31	49	86
Mean	34.85	57.46	100.03
Median	33.00	55.00	93.50
Standard Dev.	7.33	12.20	23.12
Min	24	39	67
Max	58	97	180

Source: HMDA data

The estimated median income for the low, middle and high tiers are given by the 16.7th, 50th and 83.3 percentiles of income for those approved for a mortgage for home purchase in 1997 in each city. This is based on the assumption of a 1-1 mapping between income and the price of the house purchased in each metro.

these has been to construct 2006-consistent geographical areas in 1997 based on county-level data. For the relative income data we aimed for as close a geographical match as possible, discussed in detail below.

2.C.1 HMDA Variables

For the HMDA data in 2006 we compiled data for 49 MSAs. In 1997 we used MSA level data for the MSAs such as Phoenix, AZ where the definition was the same in 1997 and 2006. For the remainder, we constructed 1997 data based on the counties that comprised the 2006 MSA definition. For example the San Jose-Sunnyvale-Santa Clara, CA (41940) MSA in 2006 contains San Benito and Santa Clara counties. The San Jose, CA (7400) MSA that prevailed in 1997 only contains Santa Clara county. Thus, to ensure consistency over time, in 1997 we constructed HMDA data for San Benito and Santa Clara counties.

In both years we restricted attention to loans which were for home purchase (not for home improvement or refinancing), were for one to four-family housing (excluding manufactured housing), and where the loan was actually originated. For the 2006 data the non-owner occupancy percentage was based on first lien loans. This allows better comparison with 1997, where the use of junior liens was limited. First lien data was also used when looking at the census tract income that non-owner occupiers bought houses in in 2006.

Our measure of the LTI ratio is based on the buyers between the 33 and 66th percentiles of the income distribution of those that bought in each year, giving a measure of the average LTI ratio in each city in each year. In 2006 and 2011 we calculate the LTI ratio based on first and junior liens combined, thus arriving at a combined LTI ratio. On the assumption that everyone buying a house using a junior lien also has a senior lien on the same property, we calculate the average LTI ratio as the sum of the value of senior and junior liens divided by the total income of those associated with first liens only. This avoids double counting, which would lead to an artificially low LTI ratio. In 1997 we do not have separate data for senior and junior liens, so calculate the LTI ratio in the same way based on all home purchase loans. As the use of junior liens was limited during this period, this calculation should be reasonably accurate. To calculate the average low tier and high tier LTI ratios for Table 1.3 we separately calculate LTI ratios based on the different percentiles of income. The high tier LTI ratio is based on the buyers in the 67-100 income percentile bracket, whilst the low tier is based on the 0-33 bracket, for those with an income of at least $10,000 (this reduces the impact of outliers and buyers who may not be low tier buyers, having low income but high wealth. An example of this would be retirees).

2.C.2 Relative Income Changes

The construction of this variable for 1999-2006 presented two challenges.[34] The first was the MSA definitional changes in 2003, and the fact that the data was only available at the MSA (and not also

[34] These challenges were not an issue for constructing this variable during the bust years.

the county) level. The second was that the percentile data was only available across all industries for each MSA from 2001 onwards. Percentile data for each major industry group was however available from 1999 onwards.

Regarding the first challenge, we tried to match the geographical regions at the MSA level as well as possible. In 26 of the 49 cases, the MSA definition did not change. In many other cases, whilst there was a change, it was small so should not lead to much error. For example, the Atlanta MSA in 2001 was a subset of the Atlanta-Sandy Springs-Marietta MSA used for data in 2006, however, based on county-level data, in 2001, its population was 96.8% of the larger area so any errors should be small. A few remaining cases required something different. For example, the Edison-New Brunswick, NJ MSA from 2006 is comprised of Middlesex, Monmouth, Ocean and Somerset counties from New Jersey. In 2001 this is split between Monmouth-Ocean, NJ and Middlesex-Somerest-Hunterdon, NJ, the latter also containing Hunterdon county. Thus, combining the two MSAs in 2001 leads to a superset of the 2006 definition, but a close match with 105.6% of the population in 2001. To combine the wage data in 2001, we weight the date for each MSA based on the population of each in the 2006-consistent geographical definition. Using the process, we obtain wage growth rates from 2001-2006 for our 49 MSAs for each of the 10th,25th,50th,75th and 90th income percentiles.

To extend the series back to 1999, we require estimates from 1999-2001. Fortunately there were no geographical changes during this period, however wage percentile data was only available for 23 broad industry groups, and not for an average across all industries (it is available for both from 2001 onwards). Our approach here was to combine these industry groups together with weighting based on the number of people employed in each group in each MSA. We did this for both years to estimate income percentiles for the whole MSA, then took the change in these from 1999-2001. In many MSAs however, the 90th percentile data was censored for Management and Legal Occupations. To ensure consistency across MSAs we removed these occupations from the estimated data for both 1999 and 2001. Having obtained estimated percentile growth rates at the MSA level for 1999-2001 and 2001-2006, we combined them to produce an estimate from 1999-2006. The estimated data from 2001-2006 is likely of higher quality, and a as a robustness check we re-ran our regressions using this income measure instead. This produced similar results to those in the text.

2.C.3 Other Variables

The remaining variables could all be constructed at the county level, so ensuring geographical consistency over time was straightforward. Two variables deserve further mention. To calculate the migration data, we first took the estimated number of residents living in the MSA that were foreign born and moved in the year 2000 or later. We then expressed this as a percentage of the population in 2006. For the house permit data, we focused on permits issued for single-family house building as our price series are for single-family housing. As a robustness check we also ran the regressions looking at permits issued for all residential house building during the period and obtained similar results.

Chapter 3

Credit Traps[1]

This paper develops an overlapping generations model with credit frictions that can be used to analyse macroprudential policy options for avoiding and getting out of a credit trap. The model has multiple steady states, and following a negative shock to the financial sector, it can fall into a 'credit trap': a steady state featuring permanently low output, bank lending, and financial sector net worth. In our model, banks' borrowing constraints depend on the health of the whole banking system. A large, unexpected negative shock to banks' net worth makes them unable to finance productive investments, which in turn causes the economy to become stuck in a 'bad' equilibrium characterised by low investment and output. We show that a leverage ratio cap can reduce the risk of an economy falling into a credit trap, and that countercyclical leverage policy can facilitate a faster recovery after small negative shocks. Once the economy is in a credit trap, however, relaxing the leverage ratio cap is ineffective. Here we consider the unconventional credit policies of direct lending, discount window lending, and an equity injection, obtaining clear predictions about their relative efficacy.

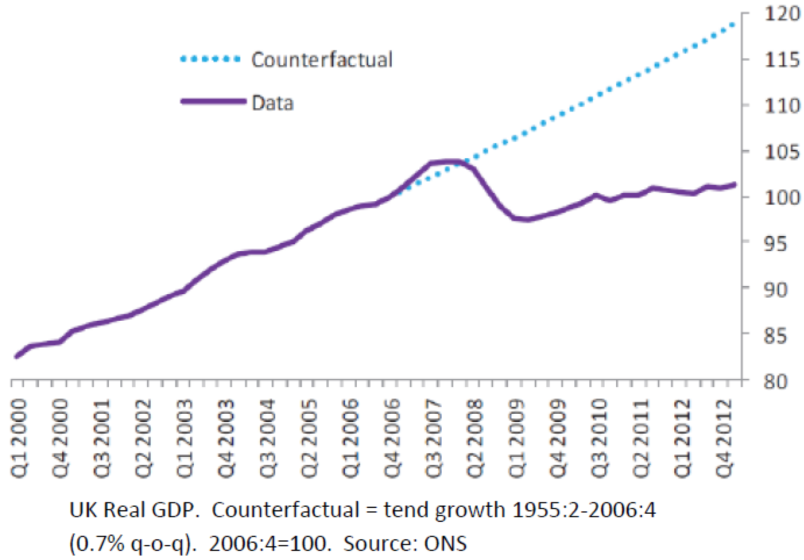
3.1 Introduction

After an initial recovery following the financial crisis, the UK economy has stagnated, with little growth in real GDP in recent years, leaving the economy far below its pre-crisis growth trend (Figure 3.1). At the same time, there has been a significant contraction in lending to the real economy, with net nominal lending to non-financial firms shrinking every year since the crisis (Figure 3.2). This occurrence of a stagnant economy with significantly reduced lending has led to real concerns that the recovery could be significantly different this time, due to the possibility of a *credit trap*. In this paper we explore the idea that the recovery from a financial crisis can be significantly different from a normal recovery.[2] In particular, we consider how credit traps can arise; that is, how an economy

[1] This chapter is joint work with Benjamin Nelson and Misa Tanaka from the Bank of England.

[2] As Claessens et al (2008) show, with data on 21 OECD countries from 1960-2007, recessions tend to be longer and deeper when accompanied by a credit crunch.

Figure 3.1: UK Real GDP: Actual vs Prior Trend



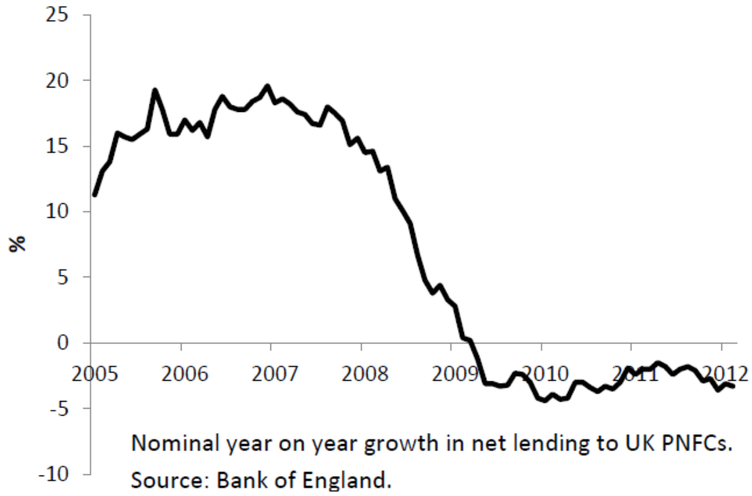
can become “trapped" following a financial crisis, with a prolonged period of weak levels of real activity, low bank lending, and an impaired financial sector. This characterizes the experience of Japan during its “lost decade”, and economists are beginning to worry that several advanced economies today could be in a similar situation for years to come.[3]

The possibility of a credit trap can have profound implications for policy. If the economy can become trapped at a permanently lower level of output following a financial crisis, there is a strong argument for using macroprudential policy to attenuate financial booms, limiting the fallout from any bust. Further, the appropriate policy response to a recession could be significantly different if it was or wasn't preceded by a severe financial crisis. Policies that work well in the former case could be ineffective if the economy has fallen into a trap. Here instead a range of unconventional policies may be required.

The main contribution of this paper is to build a tractable overlapping generation model (OLG) with credit frictions that can be used to analyse policy options for avoiding and getting out of credit traps. In our model, banks combine their net worth and deposits collected from households to invest in one of two projects. The amount that households are willing to deposit at banks depends both on the type of project that banks invest in as well as on the amount of equity capital that banks hold. There are two types of projects that banks can invest in – Project A (‘corporate loans’) and Project B (‘government bonds’) – these differ in their returns and their pledgeability to creditors. Project A yields higher total returns than Project B, but creditors' willingness to lend against Project A depends positively on the health of the banking system. This is because the liquidation value of Project A depends on the financial capacity of other banks to purchase and

[3] We have, of course, not established that the UK and other countries are currently in a credit trap. Rather, here we consider the appropriateness of various policy tools *if credit traps do occur*.

Figure 3.2: Collaspe in UK Credit



operate it, as they are the only institutions capable of managing productive projects. Project B yields lower total returns, but creditors' willingness to finance it is independent of financial system health. Project B could be interpreted as 'liquid assets' or government bonds that could be easily seized by creditors so that their returns cannot easily be diverted by bankers.

We show that such an economy is characterized by two steady states: 'good' and 'bad'. When banks' net worth is high, creditors are willing to finance productive projects, such that the economy converges to a good equilibrium characterized by high levels of output, physical capital, bank net worth and credit. However, a large, unexpected negative shock to banks' net worth can make them unable to finance productive investments. As poorly capitalized banks are forced by their creditors to invest in low-return, highly pledgeable assets, the economy can become stuck in a bad equilibrium – a credit trap – characterized by low output, low physical capital, impaired bank net worth, and low credit. Thus, even a *temporary* negative shock to banks' net worth can *permanently* trap the economy in a bad equilibrium.

Given the negative consequences of a credit trap, we first consider what policy can do to mitigate the chances of the economy falling into one. We focus on regulatory leverage policy, a new macro-prudential policy tool that many central banks are due to implement in future.[4] This tool leans against financial booms, reducing permitted leverage, first with the goal of reducing the magnitude of the boom, and second with the aim of making banks more resilient in the face of a negative shock. One key challenge for the policymaker using this tool is the trade-off between output and resilience. Reducing leverage may increase the resilience of the financial sector at the cost of reducing the level of output, with less lending to the real economy. We show in our model that under mild conditions there is no trade-off between the two for low levels of leverage: resilience against falling into the

[4] The UK government intends to provide the Bank of England with a time-varying leverage ratio tool some time from 2018 onwards (Bank of England 2013).

trap is maximised for a level of leverage greater than 1. The intuition for this is that at very low levels of leverage the financial system will be repressed with low banking system net worth, bringing banks closer to the critical level of net worth at which their creditors force them to invest in the highly pledgeable unproductive sector. A greater weight will then be placed upon improving the health of the banking system by allowing more leverage. Any policymaker focused on both output and resilience will then allow at least a moderate level of leverage.

It has been suggested that, in addition to leaning against banking sector booms, leverage limits should be relaxed after a crash, enabling a swifter recovery of the financial sector and the economy. In other words, leverage policy should be countercyclical. We show that the level of leverage that maximises the resilience of the economy is countercyclical, being higher following a negative shock. Further following a "small" shock (one for which the economy does not fall into the trap), countercyclical leverage policy facilitates a faster recovery. This contrasts with the response following a "large" shock when the economy has fallen into a trap: here relaxing the leverage ratio will be ineffective and alternative policies are needed.

Relaxing the leverage cap does not help the economy escape a credit trap, as the leverage on loans to the unproductive sector must be greater in the trap (this has to be the case for it to offer higher returns). Consequently, relaxing permitted leverage either does nothing (if it does not bind), or only makes unproductive loans relatively more attractive. A necessary condition for escaping the trap must then involve changing the relative attractiveness of investment in the two sectors, directing investment back to the productive sector A. This could be achieved by altering macroprudential sectoral risk weights, either making sector A more attractive or sector B less attractive. Whilst necessary, this may or may not be sufficient for the economy to recover to its good steady state. This depends on the strength of the feedback between the health of the economy and the banking system. Intuitively, the question is, if we force banks to make loans to the real economy, will the economy recover sufficiently, paying high returns to the banking system, helping them repair their balance sheets and extending more loans? If there is a virtuous feedback loop here, addressing the sectoral misallocation will be enough. Otherwise, if this is not sufficient, more direct action needs to be taken.

Following Gertler and Kiyotaki (2011) we consider three unconventional credit policies for use in extreme times: direct lending by the government, discount window lending, and an equity injection. Whilst these policies can be used to escape a credit trap, they can also help the economy recover faster in the absence of a trap, so the results we present on these apply in a more general setting.[5] In our simple setting we obtain clear predictions about the efficacy of the policies in raising future output. When the three policies face similar inefficiency costs of implementation, direct lending is more effective than discount window lending, as the funds raised are directly invested in the economy, and don't have to pass through the banking system, subject to its friction. Further, an equity injection is more effective than direct lending, as it has the additional positive effect of

[5] These policies are not a panacea and we show that all three can be effective when the banking system is weak, but detrimental, reducing future output, when the banking system is healthy. Thus, the model still captures genuine trade-offs when these policies are available.

relaxing the financial friction, *crowding in* depositors. By contrast, if discount window lending is inherently more efficient than the other two policies (with a lower inefficiency cost of implementation reflecting this being closer to the core activities of a central bank), it can be more effective following a milder banking crisis, but less effective than the other two policies in a severe banking collapse. These results are helpful for thinking about the most appropriate policy to employ following a financial crisis.

This paper is most closely related to Matsuyama (2007), which develops an OLG model with multiple steady states. As in Matsuyama (2007), a credit trap in our model arises when investment starts flowing into unproductive projects. Our paper introduces a banking sector into this framework to enable us to analyse the impact of regulatory policies in avoiding and getting out of a credit trap. This paper is also related to a range of papers that use Dynamic Stochastic General Equilibrium (DSGE) models to analyse the macroeconomic impact of bank capital requirements and leverage ratio caps, such as Angelini et al (2011) and Christensen et al. (2011). Contrary to these papers, which focus on the role of capital requirements in reducing macroeconomic volatility, our work can explicitly analyse the role that these policy instruments could play in preventing financial crises. Our paper is also related to Benmelech and Bergman (2012), which considers the role of monetary policy in stimulating the economy out of a credit trap. Contrary to their analysis, our focus is on macroprudential policy.

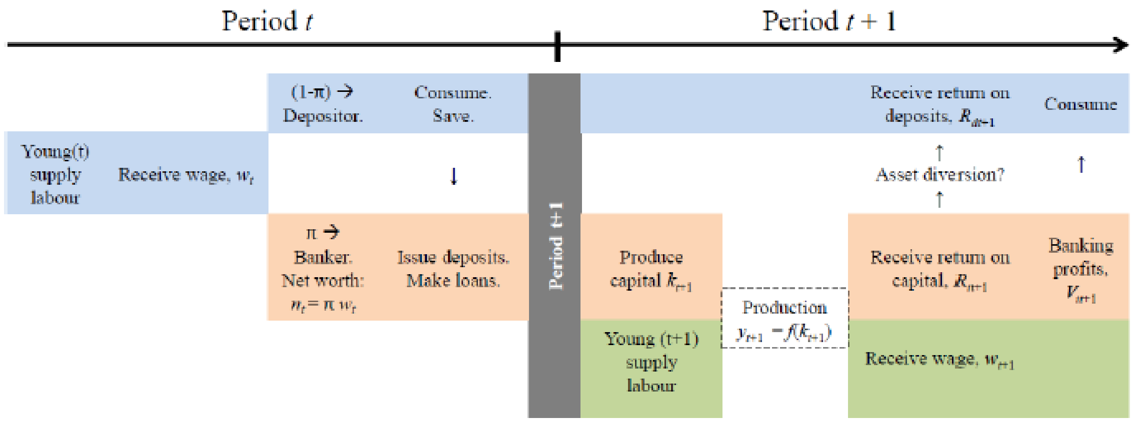
The rest of the paper is organised as follows. Section 3.2 sets up the OLG model and shows why multiple steady states may arise. Section 3.3 shows how an unexpected hit to bank capital – induced by a negative productivity shock – can tip the economy into a credit trap. Section 3.4 considers policy options for avoiding credit traps, with Section 3.5 considering the effectiveness of relaxing leverage limits and altering sectoral risk weights for getting out of a trap. Section 3.6 considers the use of unconventional credit policies, and section 3.7 concludes.

3.2 Model

3.2.1 Introduction

We begin with a brief overview of the model, with a timeline of the economy shown in Figure 3.3. Mass 1 of identical households are born each period. The life of a household is divided into two subperiods: 1, when the household is young (in period t), and 2, when the household is old (in period t+1). In the first period, each 'young' household receives a labour endowment of unity, which they sell in return for wage income w_t denominated in final consumption goods. At the end of period 1, fraction $1-\pi$ of households become depositors, whilst exogenous fraction π become bankers. Thus, households divide $(1-\pi)\, w_t$ between period 1 consumption and saving via deposits, whereas $n_t \equiv \pi w_t$ is used as bank equity to start a household bank. Banks combine their net worth with deposits, taking these output goods and invest in one of two physical capital producing technologies. In the following period, the physical capital the banks hold is combined with the labour endowment of the *next* generation, producing output goods. Out of their return on this, the banks pay back

Figure 3.3: Timeline of events: benchmark model



depositors before returning any profits lump-sum to the now old households and then die. The new young workers, having received their wage, form their own set of banks (which have no direct link to the previous banks) and the whole process repeats itself.

The use of the OLG structure is done purely for tractability, helping us obtain analytic expressions throughout. It should not be inferred that the intended model period is thus a generation, or around 30 years as is often the case with OLG models. Whilst we do not match this model to the data, the intended model length throughout is of the order of one year.

In the following sub-sections we describe the model in more detail.

3.2.2 Households

Lifetime utility for households is given by

$$U_t = \log c_{1t} + \beta \log c_{2t}, \tag{3.1}$$

where $\beta \leq 1$ is the household's discount factor, and c_{jt} denotes consumption in period $j = 1, 2$ of the household born in period t. The budget constraints facing the household in each period are

$$c_{1t} + d_{i,t} \leq (1 - \pi)\, w_t, \quad c_{2t} \leq R^d_{i,t+1} d_{i,t} + V_{i,t+1} \tag{3.2}$$

where $d_{i,t}$ denotes the household's saving via bank deposits, $R^d_{i,t+1}$denotes gross return on deposits[6], and $V_{i,t+1}$ denotes the profits obtained from banking activities. The subscript $i = \{A, B\}$ represents the sector the bank invests in.

[6] The rate paid on deposits, $R^d_{i,t+1}$, is agreed at time t, and is not state contingent. This rate is dated $t + 1$ to reflect when deposits are repaid.

3.2.3 Banking and output production

Part of the household's initial wealth is used to capitalise a bank, with net worth $n_t \equiv \pi w_t$. The bank takes deposits from households and combines these with its own net worth to invest in capital-producing projects. There are two sectors that banks can invest in, $i = \{A, B\}$. The differences between these two sectors are described in detail in section 3.2.4 below.

If the bank invests in sector i, then its balance sheet reads:

$$s_{i,t} = n_t + d_{i,t},$$

where $s_{i,t}$ denotes the stock of loans in sector i. If $n_t + d_{i,t}$ final goods are invested in period t, physical capital produced in period $t+1$ is

$$k_{t+1} = x_i \left(n_t + d_{i,t}\right), \quad i = \{A, B\}, \tag{3.3}$$

where x_i denotes the productivity of investment in sector i.[7]

In each period, final goods are produced using physical capital (financed by bank capital and deposits of the 'old') and labour provided by the 'young', using Cobb-Douglas production technology:

$$y_{t+1} = f(l_{t+1}, k_{t+1}) = l_{t+1}^{1-\alpha} k_{t+1}^{\alpha} = k_{t+1}^{\alpha}, \quad 0 \leq \alpha < 1. \tag{3.4}$$

Labour and capital receive their respective marginal product, such that the wage of the 'young' is given by $w_{t+1} = (1-\alpha)k_{t+1}^{\alpha}$ while the marginal product of capital is given by $f'(k_{t+1}) = \alpha k_{t+1}^{\alpha-1}$. This implies that the bank's net worth in $t+1$ is given by:

$$n_{t+1} = \pi(1-\alpha)k_{t+1}^{\alpha} \tag{3.5}$$

For simplicity, we assume that capital stock depreciates fully after each period. Thus, banks' investment in sector i at t generates gross return $R_{i,t+1}$in terms of final output. This is expected to yield:

$$R_{i,t+1} = x_i f'(k_{t+1}) = x_i \alpha k_{t+1}^{\alpha-1}$$

Bank profits from investing in sector i, after repaying depositors gross interest rate $R_{i,t+1}^{d}$, are expected to be:

$$V_{i,t+1} = R_{i,t+1}\left(n_t + d_{i,t}\right) - R_{i,t+1}^{d} d_{i,t} \tag{3.6}$$

3.2.4 Credit market frictions

Banks are subject to a borrowing constraint that depends on the project they invest in. This constraint arises because bankers can abscond with a fraction $1 - \lambda_i$ of gross project returns (e.g. by paying an unwarranted bonus to themselves). As a result, only a fraction λ_i of the gross return from investment in sector i is pledgeable to creditors. Thus, λ_i can be interpreted as a borrowing

[7] There is only one *type* of capital, but there are two *technologies*, A and B, for producing it from output goods.

constraint imposed by the market (with lower λ_i implying a tighter borrowing constraint). In order to guarantee repayment of $R^d_{i,t+1}d_{i,t}$, depositors demand that:

$$\lambda_i R_{i,t+1}\left(n_t + d_{i,t}\right) \geq R^d_{i,t+1}d_{i,t} \tag{3.7}$$

so that total pledgeable returns are at least the amount owed to depositors.

We can also interpret λ_i in terms of leverage. Leverage L (mark-to-market) is given by

$$L = \frac{R_{i,t+1}\left(n_t + d_{i,t}\right)}{R_{i,t+1}\left(n_t + d_{i,t}\right) - R^d_{i,t+1}d_{i,t}}$$

From (3.7),

$$L \leq \frac{R_{i,t+1}\left(n_t + d_{i,t}\right)}{R_{i,t+1}\left(n_t + d_{i,t}\right)\left(1-\lambda_i\right)} = \frac{1}{1-\lambda_i} \tag{3.8}$$

Thus, $\frac{1}{1-\lambda_i}$ is the maximum leverage the market allows when investing in sector i.

The two sectors that banks can invest in differ in both their productivity and pledgeability. We make the following assumptions:

Assumption 1 (Project Productivity): $x_A > x_B$[8]

Assumption 2 (Pledgeability): $\lambda_A = \lambda_A(n_t)$, $\lambda'_A > 0$, $\lim_{n_t\to\infty}\lambda_A = \bar{\lambda}_A$, $\lim_{n_t\to 0}\lambda_A = \underline{\lambda}_A$, $\lambda'_B(n_t) = 0$, $\lambda_B > \underline{\lambda}_A$.

Assumption 1 says that for a given input of final goods in period t, more capital is produced in period $t+1$ from investing in sector A than in sector B. Sector A is thus more productive than sector B. We interpret sector A as loans to the real economy, whilst sector B is an alternative use of bank funds such as holding cash, or buying government bonds, which does not contribute as much to output.

Assumption 2 says that whilst the market leverage limit permitted in sector B is independent of the net worth of the banking system, the leverage permitted when investing in sector A *increases* in banking system health. This matches the procyclicality of bank leverage documented by Adrian et al (2012). Our intuition for this assumption is based on an application of Shleifer and Vishny (1992) to the financial system, as has recently also been done by Benmelech & Bergman (2012). Shleifer and Vishny (1992) show how the collateral value of assets can depend on the health of the entire sector being invested in. For example, a loan to an airline company may be secured against an aircraft to protect the creditor in case of default. If default occurs, the creditor seizes the aircraft and sells it to cover their losses. The key insight is that the natural buyers of the aircraft are other airline companies, so the price it will sell for depends on the health of the whole airline industry. In particular, the collateral value of the aircraft would be greatly different if

[8] In this section, deposit contracts are signed with both banks and depositors assuming that x_A and x_B are non-stochastic. We will later consider in Section 3.3 what happens when the economy is hit by an *unanticipated* productivity shock.

the airline defaults because of an idiosyncratic shock or an aggregate negative shock to the airline industry. We apply this insight to financial assets: the natural buyers of these are other banks, so their collateral value depends on the health of the banking system. When the banking system is healthy and liquid, these assets can be resold easily, allowing banks to take on greater leverage, with the reverse being true in a crisis. Investments in sector A depend on the health of the banking system in this way.[9] By contrast, the low return sector B assets are unaffected by the health of the banking system. This will be true for highly liquid claims such as cash or government bonds, which have many buyers beyond the banking system.

In summary, loans to sector A are more productive but sensitive to the state of the financial system, whilst loans to sector B are less productive but resilient to financial system stress.[10]

3.2.5 Credit market equilibrium

To derive the credit market equilibrium, we first derive households' supply of deposits. The household's optimal consumption-saving decision is governed by the first-order condition using (3.1) and (3.2):

$$\beta \frac{R^d_{i,t+1}}{c_{2t}} = \frac{1}{c_{1t}},$$

which gives optimal saving:

$$d_t = \frac{\beta}{1+\beta} (1-\pi) w_t - \frac{1}{1+\beta} \frac{V_{i,t+1}}{R^d_{i,t+1}}. \tag{3.9}$$

The following series of events determines deposit market equilibrium. First, depositors determine their deposit supply schedules, taking into account the different levels of pledgeable returns delivered by banks' portfolios. Second, conditional on these deposit supply schedules, banks choose their debt issuance and total asset holdings.

We begin with the banks' optimisation problem. For banks that invest in sector i, raising deposits to invest will be profitable as long as

$$R_{i,t+1} > R^d_{i,t+1} \tag{3.10}$$

When this is the case, the bank will borrow up until the point at which its borrowing constraint (3.7) binds. Thus, banks' demand for funds for investing in sector i are given by:

$$d_{i,t} = \frac{\lambda_i R_{i,t+1}}{R_{d,t+1} - \lambda_i R_{i,t+1}} n_t \tag{3.11}$$

[9] An example would be Mortgage Backed Securities. The collateral value of these will be greater when the natural buyers of these-financial institutions-are healthy.

[10] For our results it's not crucial that $\lambda'_B(n_t) = 0$. Rather, all we need is $\lambda'_A(n_t) > \lambda'_B(n_t)$ so the leverage permitted for investing in sector A is more sensitive to the health of the banking system.

Using (3.6) and (3.11), bank profit from investing in sector i is given by:

$$\begin{aligned} V_{i,t+1} &= \left(R_{i,t+1} - R^d_{i,t+1}\right) d_{i,t} + R_{i,t+1} n_t \\ &= \frac{1-\lambda_i}{R^d_{i,t+1} - \lambda_i R_{i,t+1}} R^d_{i,t+1} R_{i,t+1} n_t. \end{aligned} \tag{3.12}$$

These profits are returned lump-sum to households at the end of their life. Thus, the deposit supply of households to sector i is given by:

$$d_{i,t} = \frac{\beta}{1+\beta}\left(1-\pi\right) w_t - \frac{1}{1+\beta} \frac{1-\lambda_i}{R^d_{i,t+1} - \lambda_i R_{i,t+1}} R_{i,t+1} n_t. \tag{3.13}$$

In equilibrium, deposit supply (3.13) must equal deposit demand (3.11). Using $n_t = \pi w_t$, the equilibrium deposit quantity when the bank invests in sector i is given by:

$$d^*_{i,t} = \frac{\lambda_i \beta}{1+\lambda_i \beta}\left(1-\pi\right) w_t \tag{3.14}$$

Equilibrium deposits are increasing in λ_i, the pledgeability of bank returns. Alleviating the financial friction then raises the amount of saving and investment in the economy. The reason for this is that a greater degree of asset pledgeability reassures bank creditors that their deposits will be safe, so they are willing to expand the equilibrium quantity of saving.

Given (3.3) and (3.14), capital produced at $t+1$ when the bank invests in sector i is given by:

$$\begin{aligned} k^*_{i,t+1} &= x_i \left[\pi w_t + \frac{\lambda_i \beta}{1+\lambda_i \beta}\left(1-\pi\right) w_t\right] \\ &= x_i \frac{(1-\alpha) k_t^\alpha}{1+\lambda_i \beta}\left(\pi + \lambda_i \beta\right) \end{aligned} \tag{3.15}$$

In equilibrium, the (expected) return on the bank's investment in sector i is given by (see appendix):

$$R^*_{i,t+1} = \frac{\alpha x_i^\alpha}{\left[\frac{\pi+\lambda_i\beta}{1+\lambda_i\beta}(1-\alpha)k_t^\alpha\right]^{1-\alpha}} \tag{3.16}$$

Using the equilibrium condition that deposit supply (3.13) must equal deposit demand (3.11), the equilibrium deposit rate[11], given that the bank invests in sector i, is given by:

$$\begin{aligned} R^{d*}_{i,t+1} &= \lambda_i R_{i,t+1}\left(1 + \frac{n_t}{d^*_{i,t}}\right) \\ &= \frac{\alpha x_i^\alpha (1+\lambda_i\beta)^{1-\alpha} (\pi + \lambda_i \beta)^\alpha}{\beta(1-\pi)\left[(1-\alpha)k_t^\alpha\right]^{1-\alpha}} \end{aligned} \tag{3.17}$$

[11] We note again that the deposit rate is agreed at time t and is not state contingent.

The deposit rate paid within a given sector is increasing in the productivity of that sector and its pledgeability. The pledgeability and productivity of the two sectors are thus crucial in determining the sector that yields better returns for depositors. It can be shown that condition (3.10) holds in equilibrium as long as:

$$\beta(1-\pi) > \pi + \lambda_i \beta \tag{3.18}$$

In what follows, we assume that (3.18) holds for both sectors. It can also been shown that $R^{d*}_{i,t+1} > \lambda_i R^*_{i,t+1}$ so the financial constraint binds in equilibrium.

3.2.6 Credit trap

Given that the borrowing constraint (3.7) binds on banks in equilibrium and banks compete with each other for deposits, households choose to deposit in banks that can offer the highest deposit rate. In the appendix, we show that, given (3.18) holds, banks will invest in the sector that pays depositors the highest return as long as:

$$x_B(1-\lambda_B)\frac{\pi + \lambda_B \beta}{1 + \lambda_B \beta} \geq x_A \pi \tag{3.19}$$

Given this, from (3.17), $R^{d*}_{A,t+1} < R^{d*}_{B,t+1}$ and banks invest in sector B when:

$$x_A^\alpha (1 + \lambda_A(n)\beta)^{1-\alpha}(\pi + \lambda_A(n)\beta)^\alpha \leq x_B^\alpha (1 + \lambda_B \beta)^{1-\alpha}(\pi + \lambda_B \beta)^\alpha$$

From this it follows that investment will flow to sector B when the net worth of the banking system falls below a critical threshold.

Lemma 54 *Under conditions 3.18 and 3.19, banks invest in sector B at time t when $n_t < \tilde{n}$, where $\tilde{n}$ solves:*

$$x_A^\alpha (1 + \lambda_A(\tilde{n})\beta)^{1-\alpha}(\pi + \lambda_A(\tilde{n})\beta)^\alpha = x_B^\alpha (1 + \lambda_B \beta)^{1-\alpha}(\pi + \lambda_B \beta)^\alpha \tag{3.20}$$

Thus, banks invest in sector A and the credit market equilibrium is given by $(d^*_{A,t}, R^{d*}_{A.t+1})$ when $n_t > \tilde{n}$; they invest in sector B and the credit market equilibrium is given by $(d^*_{B.t}, R^{d*}_{B.t+1})$ when $n_t \leq \tilde{n}$.

Proof. See appendix. ■

This establishes that the supply of deposits to the banking system features a critical threshold at which point creditors become unwilling and banks become unable to invest in one sector in favour of another. This is because as banking sector net worth changes, so does the pledgeability of investments in sector A relative to sector B. In particular, for sufficiently low banking system net worth, the pledgeability of A is so low that creditors demand that investment be channelled to B. Put differently, because sector A is inherently more productive than sector B, a higher return on sector B can only arise if there is more investment in it, i.e. a greater amount of leverage. When the banking system is healthy, high leverage when investing in sector A will be possible, making it more attractive. Only when the banking system-and then this leverage-is sufficiently impaired will investment flow to B. We next establish the aggregate consequences of these investment decisions.

In the general equilibrium of the economy, the capital stock evolves according to equation (3.3) in the absence of any unanticipated shock to the capital producing technology. Using equilibrium deposits and bank capital, the law of motion for physical capital can be expressed as:

$$k_{t+1} = x_i \frac{\pi + \lambda_i \beta}{1 + \lambda_i \beta} (1 - \alpha) k_t^{\alpha}, \quad i = \{A, B\}. \tag{3.21}$$

Tomorrow's capital stock will be larger the less severe the financial friction (higher λ_i), and larger bank capital is relative to debt (higher π). We can then establish:

Lemma 55 *Conditional on bank portfolios being allocated to Sector B, the steady state level of physical capital converges to*

$$k_B^* = \left(x_B \frac{\pi + \lambda_B \beta}{1 + \lambda_B \beta} (1 - \alpha) \right)^{\frac{1}{1-\alpha}} \tag{3.22}$$

which is the unique, stable steady state under investment in sector B. Conditional on bank portfolios being allocated to sector A, the steady states of A (possibly multiple) satisfy

$$k_A^* = \left(x_A \frac{\pi + \lambda_A \left(\pi(1-\alpha) k_A^{*\alpha}\right) \beta}{1 + \lambda_A \left(\pi(1-\alpha) k_A^{*\alpha}\right) \beta} (1 - \alpha) \right)^{\frac{1}{1-\alpha}} \tag{3.23}$$

Proof. It is straightforward to demonstrate this using (3.5) and (3.21). ■

In the following analysis we assume that sector A has a unique stable steady state when $n_t > \tilde{n}$.[12] This ensures that if banks invest in sector A, the economy will converge to k_A^* absent any shocks.

We now establish a proposition under which the economy features a credit trap.

Proposition 56 *Suppose (3.18), (3.19) hold. Let n_B^* be the steady state level of banker net worth when sector B is invested in:*

$$n_B^* = \pi(1 - \alpha) \left(x_B \frac{\pi + \lambda_B \beta}{1 + \lambda_B \beta} (1 - \alpha) \right)^{\frac{\alpha}{1-\alpha}}$$

Then the economy features a credit trap if

$$x_A^{\alpha} (1 + \lambda_A(n_B^*)\beta)^{1-\alpha} (\pi + \lambda_A(n_B^*)\beta)^{\alpha} < x_B^{\alpha} (1 + \lambda_B \beta)^{1-\alpha} (\pi + \lambda_B \beta)^{\alpha} \tag{3.24}$$

Proof. Given (3.18) and (3.19), banks invest in sector B rather than sector A iff

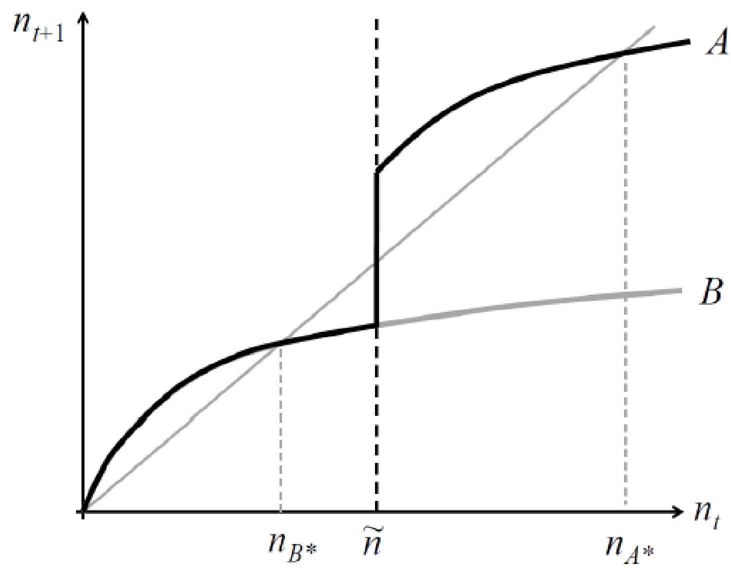
$$x_A^{\alpha} (1 + \lambda_A(n_t)\beta)^{1-\alpha} (\pi + \lambda_A(n_t)\beta)^{\alpha} < x_B^{\alpha} (1 + \lambda_B \beta)^{1-\alpha} (\pi + \lambda_B \beta)^{\alpha}$$

Hence if

$$x_A^{\alpha} (1 + \lambda_A(n_B^*)\beta)^{1-\alpha} (\pi + \lambda_A(n_B^*)\beta)^{\alpha} < x_B^{\alpha} (1 + \lambda_B \beta)^{1-\alpha} (\pi + \lambda_B \beta)^{\alpha}$$

[12] The shape of $\lambda_A(n_t)$ is relevant for this. Conditions on this functional form can be given that ensure there is a unique stable state in sector A for $n_t > \widetilde{n}$.

Figure 3.4: Aggregate law of motion in an economy with a credit trap



the banks will invest in sector B when $n_t = n_B^*$. i.e. they invest in sector B in the steady state of B. This is thus a steady state equilibrium: without shocks the economy will invest in sector B for the rest of time, so is stuck in a credit trap. ■

An economy with a credit trap is shown in Figure 3.4.[13] The critical value of banking system net worth at which investment flows to A is given by $\widetilde{n}$. Above this level of banking system health, the economy invests exclusively in sector A, and the economy converges to the 'good' steady state (n_A^*), featuring high levels of capital, output and income. If the banking system is sufficiently impaired with $n_t < \widetilde{n}$, sector B is invested in, and the economy converges to the 'bad' steady state (n_B^*), featuring low levels of capital, output and bank lending. This is indeed a steady state when banks invest in sector B when $n_t = n_B^*$, for which we require $n_B^* < \widetilde{n}$, which is ensured by (3.24).

When the banking system is healthy, the collateral value of financial assets is high, allowing banks high leverage when investing in sector A, making it more attractive than sector B (by allowing them to pay higher returns to depositors). A is productive and so delivers high returns ensuring high banking system net worth in the next period, which keeps investment flowing to A. Conversely, when the financial system is severely impaired, sector B is more attractive than sector

[13] Note that there will always be a jump in the law of motion at $\widetilde{n}$. To see this, at the trap threshold, the return paid on deposits in A and B is the same, so after rearranging

$$\begin{aligned} x_A^\alpha \left(\frac{\pi + \lambda_A(\widetilde{n})}{1 + \lambda_A(\widetilde{n})}\right)^\alpha &= x_B^\alpha \left(\frac{\pi + \lambda_B}{1 + \lambda_B}\right)^\alpha \left(\frac{1 + \lambda_B \beta}{1 + \beta \lambda_A(\widetilde{n})}\right) \\ &> x_B^\alpha \left(\frac{\pi + \lambda_B}{1 + \lambda_B}\right)^\alpha \end{aligned}$$

The last part follows as we must have $\lambda_B > \lambda_A(\widetilde{n})$, given $x_A > x_B$. Applying (3.21) its clear that at $\widetilde{n}$, k_{t+1} (and so n_{t+1}) is greater when A is invested in.

A due to the low leverage permitted on financial assets. Crucially, because the banks invest in the unproductive sector B, bank net worth remains low in future periods, keeping them investing in B.

As the economy enters the credit trap there is a discrete decrease in the gross rate of return banks receive on their investments, $R_{i,t+1}$, as they switch from investing in the productive sector A to the unproductive sector B. Recall their return from investing in sector i is given by $R_{i,t+1} = x_i \alpha k_{t+1}^{\alpha-1}$. Whilst the decrease in output[14] the economy experiences as investment is switched to sector B decreases k_{t+1}, pushing up the return, this effect is dominated by the reduction in productivity, x_i.[15] By contrast, there is no change in the interest rate paid on deposits as the credit trap is entered. This is because at the trap threshold, $\widetilde{n}$, the deposit rate is the same regardless of the sector the banks invest in (as given by (3.20)). Thus, on entering the credit trap, the spread between $R_{i,t+1}$ and $R^d_{i,t+1}$ narrows.

3.3 A Financial Crisis

We now illustrate how a large negative shock to banks' net worth can send the economy from the good to the bad steady state. A revised timeline for the economy is shown is Figure 3.5.

Suppose that in period t, the economy is in the good equilibrium in which banks invest in sector A. Suppose that, after deposits have been collected and investment in sector A is made, an *unexpected* negative productivity shock hits at the start of period $t+1$, such that the realised productivity, $\hat{x}_A$, is less than what was initially expected: $\hat{x}_A < x_A$, where $\hat{x}_A \in [\underline{x}_A, \bar{x}_A]$. Given the realised shock, the actual capital produced is less than the initially expected amount (3.3), and is given by:

$$\hat{k}_{t+1} = \hat{x}_A \left(n_t + d^*_{A,t} \right)$$

This implies that bankers will default on deposits at the end of period $t+1$ if left to themselves,

[14] See footnote 13 above.

[15] Formally we can write

$$R^*_{i,t+1} = \frac{\alpha x_i^\alpha}{\left[\frac{\pi+\lambda_i\beta}{1+\lambda_i\beta}(1-\alpha)k_t^\alpha\right]^{1-\alpha}} = \frac{\alpha x_i^\alpha (1+\lambda_i\beta)^{1-\alpha}(\pi+\lambda_i\beta)^\alpha}{(\pi+\lambda_i\beta)((1-\alpha)k_t^\alpha)^{1-\alpha}}$$

Thus, at the trap threshold $\widetilde{n}$, $R^*_{B,t+1}(\widetilde{n}) < R^*_{A,t+1}(\widetilde{n})$ iff

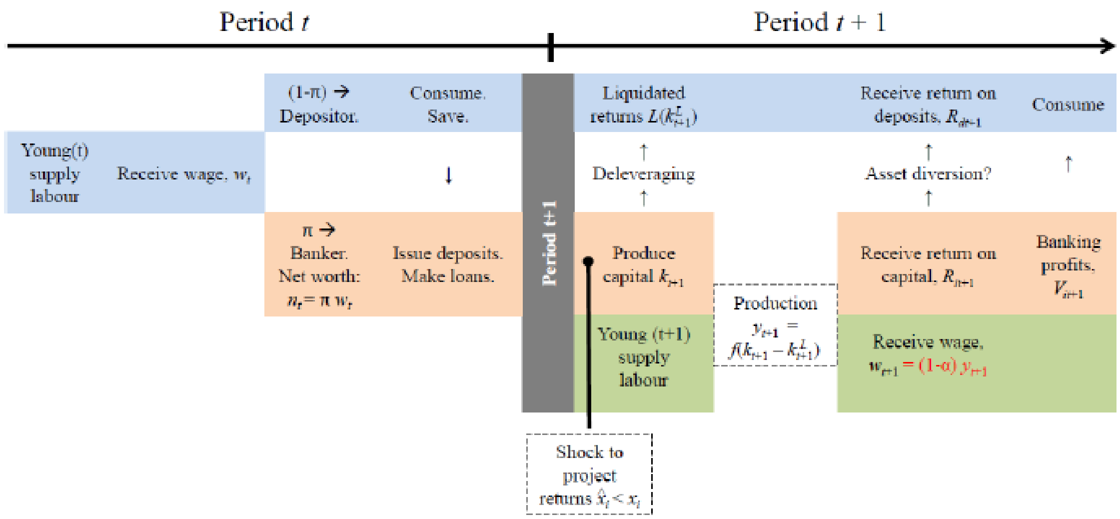
$$\frac{\alpha x_B^\alpha (1+\lambda_B\beta)^{1-\alpha}(\pi+\lambda_B\beta)^\alpha}{(\pi+\lambda_B\beta)((1-\alpha)k_t^\alpha)^{1-\alpha}} < \frac{\alpha x_A^\alpha (1+\lambda_A(\widetilde{n})\beta)^{1-\alpha}(\pi+\lambda_A(\widetilde{n})\beta)^\alpha}{(\pi+\lambda_A(\widetilde{n})\beta)((1-\alpha)k_t^\alpha)^{1-\alpha}}$$

Applying (3.20) this holds iff

$$\frac{1}{(\pi+\lambda_B\beta)} < \frac{1}{(\pi+\lambda_A(\widetilde{n})\beta)}$$

This follows as given (3.20) and $x_A > x_B$ we must have $\lambda_A(\widetilde{n}) < \lambda_B$.

Figure 3.5: Timeline of events: case of financial shock



since (3.7) no longer holds under the realised return

$$\hat{R}_{A,t+1} = \frac{\alpha \hat{x}_A^{\alpha}}{\left[\frac{\pi+\lambda_A(n_t)\beta}{1+\lambda_A(n_t)\beta}(1-\alpha)k_t^{\alpha}\right]^{1-\alpha}} < R^*_{A,t+1}$$

When deposit contracts were signed, households did not think bank asset returns $R_{A,t+1}$ were stochastic.[16] When asset returns are at the level households expected, banks have exactly[17] the required level of pledgeable assets to repay depositors fully. However, following the reduction in the value of their assets, banks no longer have enough pledgeable assets to do this and (3.7) is violated. Realising this, depositors will withdraw their funds until (3.7) holds again, as we discuss in the next sub-section.

Intuitively, following the shock, the value of the banks' assets has dropped, but their liabilities (what they promised to depositors) are unchanged. Without an adjustment to their balance sheet, their leverage will then increase. However, at the expected level of asset returns, (3.8) holds with equality and bank leverage is just low enough that they can pledge the required amount to depositors. Thus, following the negative shock, bank leverage is too high to fully repay depositors.

3.3.1 Depositor run and asset liquidation

Realising that they will not be repaid fully if they wait till the end of period $t+1$, depositors withdraw their funds, forcing partial liquidation of the project, by seizing capital $k^L_{t+1} \leq \hat{k}_{t+1}$ from banks at the start of $t+1$. Here we are simply capturing the idea that following a negative shock

[16] See footnote 8.

[17] Given our assumption (3.18), the pledgeability constraint (3.7) binds.

to asset values, deleveraging is required to bring leverage back to its original level. Unlike the standard output-producing technology (3.4), the interim liquidation technology uses *only capital* to produce output: 'old' households (depositors) seize physical capital from banks before banks can use it to produce final output, but since 'old' households do not have labour endowment, they use their own unproductive 'cottage' technology to turn the capital seized from banks into final output goods. The liquidation technology has the following form:

$$\hat{y}^L_{t+1} = L(\hat{k}_{t+1}, k^L_{t+1}) \tag{3.25}$$

where k^L_{t+1} is the amount of capital being liquidated by the depositors and $L(\hat{k}_{t+1}, 0) = 0$. We allow that the technology may depend on the aggregate amount of capital in the economy, $\hat{k}_{t+1}$. The aggregate output produced after the negative productivity shock and liquidation, $\hat{y}_{t+1}$, is given by the sum of the output produced by 'old' households using liquidation technology (3.25), $\hat{y}^L_{t+1}$, and the output produced by bankers with the remaining capital using the standard technology (3.4), $\hat{y}^P_{t+1}$:

$$\hat{y}_{t+1} = \hat{y}^P_{t+1} + \hat{y}^L_{t+1} = (\hat{k}_{t+1} - k^L_{t+1})^\alpha + L(\hat{k}_{t+1}, k^L_{t+1})$$

Once the unexpected productivity shock is realised, depositors will withdraw capital from the bank and invest the proceeds into the liquidation technology until bank leverage falls to the point where they can credibly promise to repay the remaining deposit liabilities. Thus, the equilibrium liquidation k^{L*}_{t+1} following a negative shock $\hat{x}_A$ is given by the solution to the following equality:

$$R^{d*}_{A,t+1} d^*_{A,t} - L(\hat{k}_{t+1}, k^{L*}_{t+1}) = \lambda_A(n_t)\alpha(\hat{k}_{t+1} - k^{L*}_{t+1})^\alpha$$

The above expression can be rewritten as:

$$\lambda_A(n_t)\alpha(k_{t+1})^\alpha - L(\hat{k}_{t+1}, k^{L*}_{t+1}) = \lambda_A(n_t)\alpha(\hat{k}_{t+1} - k^{L*}_{t+1})^\alpha \tag{3.26}$$

where k_{t+1} is the level of capital that was expected to be produced before the shock took place (given by (3.21), where $x_i = x_A$).

3.3.2 Benchmark case

Consider now a benchmark case in which the total final output available for consumption of the 'old' is invariant to the size of liquidation, k^L_{t+1}.[18] It can be shown that the liquidation technology that ensures this has the following form (see appendix):

$$L(\hat{k}_{t+1}, k^L_{t+1}) = \alpha\hat{k}^\alpha_{t+1} - \alpha(\hat{k}_{t+1} - k^L_{t+1})^\alpha \tag{3.27}$$

[18] Alternatively we can interpret this as the absence of firesale costs: under this specification, the *current* banks' profits are invariant to the amount of deleveraging done. The benchmark case then provides a conservative estimate of the damage done to the economy by deleveraging.

Note that even in this benchmark case, liquidation by the 'old' depositors imposes costs on the young, who faces lower wages as they have less physical capital to work with and hence see their marginal product of labour reduced:

$$\hat{w}_{t+1} = (1-\alpha)\hat{y}^P_{t+1} = (1-\alpha)(\hat{k}_{t+1} - k^L_{t+1})^\alpha$$

This in turn implies that liquidation by the 'old' depositors also reduces bank capital in the next period:

$$\hat{n}_{t+1} = \pi(1-\alpha)\hat{y}^P_{t+1} = \pi(1-\alpha)(\hat{k}_{t+1} - k^L_{t+1})^\alpha \tag{3.28}$$

Thus, in this benchmark case, the burden of liquidation by the 'old' is imposed entirely on the 'young' and the subsequent generations, who need to work with less capital and thus face lower wages and consumption. Thus, liquidation gives rise to negative intergenerational externalities.

Substituting (3.27) into (3.26), we can derive the equilibrium output produced using the standard technology:

$$\hat{y}^{P*}_{t+1} = (\hat{k}_{t+1} - k^{L*}_{t+1})^\alpha = \frac{\hat{k}^\alpha_{t+1} - \lambda_A(n_t)(k_{t+1})^\alpha}{(1-\lambda_A(n_t))}$$

From (3.21), we know that $\hat{k}_{t+1} = \hat{x}_A \frac{\pi+\lambda_i\beta}{1+\lambda_i\beta}(1-\alpha)\, k^\alpha_t$, so that

$$\hat{y}^{P*}_{t+1} = \frac{(\widehat{x}^\alpha_A - \lambda_A(n_t)x^\alpha_A)}{(1-\lambda_A(n_t))}\left[\frac{\pi+\lambda_A(n_t)\beta}{1+\lambda_A(n_t)\beta}(1-\alpha)k^\alpha_t\right]^\alpha \tag{3.29}$$

Clearly, $\hat{y}^{P*}_{t+1} < y_{t+1}$, where y_{t+1}(given by (3.4)) is the level of output that was originally expected before the negative productivity shock took place.

A crucial question is whether an economy falls into a credit trap following a negative productivity shock. We know from Lemma 54 that this crucially depends on the size of the reduction in bank capital following the shock. Specifically, if bank capital only experiences a relatively small shock, such that $\hat{n}_{t+1}$ remains above $\tilde{n}$ (given by (3.20)), then the economy converges back to the 'good' steady state k^*_A following a one-off negative productivity shock. However, if the shock to bank capital is sufficiently large such that $\hat{n}_{t+1} \leq \tilde{n}$, then the economy will converge to the credit trap equilibrium and remain stuck at k^*_B. In the next section we consider what leverage policy can do to help banks avoid falling into credit traps.

3.4 Policy Options To Avoid Credit Traps

3.4.1 Leverage ratio cap

We consider how a leverage ratio cap could be set to reduce the probability of the economy falling into a credit trap. Consider a leverage ratio cap, λ_r[19], which limits the amount of bank borrowing

[19] In equilibrium bank leverage $= \frac{1}{1-\lambda}$, so by choosing λ, the regulator also chooses the banking leverage ratio.

as follows:

$$\lambda_r R_{i,t+1}\left(n_t + d_{i,t}\right) \geq R^d_{i,t+1} d_{i,t}$$

Assume that the economy at t starts with physical capital $k_t > \tilde{k}$[20], such that banks invest in sector A. Suppose that the regulator imposes a leverage cap, $\lambda_r < \lambda_A(n^*_A)$, where n^*_A is the level of bank capital in a 'good' steady state. We assume that the regulatory leverage ratio does not bind on sector B: $\lambda_r > \lambda_B$. This ensures that the leverage requirement does not alter the threshold $\tilde{n}$ for bank capital below which the economy falls into a credit trap.

Define $x^T_A(\lambda_r)$ to be the threshold productivity realisation that results in banks investing in sector B next period, sending the economy into a credit trap. This threshold is a function of the regulatory leverage ratio cap (see appendix for derivation):

$$x^T_A(\lambda_r) := \left[\lambda_r x^\alpha_A + \frac{\widetilde{n}(1-\lambda_r)}{(1-\alpha)\pi\left[\frac{\pi+\lambda_r\beta}{1+\lambda_r\beta}(1-\alpha)k^\alpha_t\right]^\alpha}\right]^{\frac{1}{\alpha}} \tag{3.30}$$

The economy falls into a credit trap whenever $\hat{x}_A \leq x^T_A(\lambda_r)$. Thus, $x^T_A(\lambda_r)$ is a measure of the *resilience* of the financial system: the lower $x^T_A(\lambda_r)$, the more resilient the financial system, in the sense that the economy avoids the credit trap for a larger range of low productivity realisations.

It can be shown that, under certain conditions, $x^T_A(\lambda_r)$ is U-shaped, reaching its minimum at $\lambda_r = \hat{\lambda} \in (0,1)$. This is demonstrated in Figure 3.6. Formally, we have the following proposition.

Proposition 57 *Suppose*

$$(1-\alpha)\pi x^\alpha_A\left[\frac{\pi+\beta}{1+\beta}(1-\alpha)k^\alpha_t\right]^\alpha > \widetilde{n}$$

And

$$x^\alpha_A < \frac{\widetilde{n}}{(1-\alpha)\pi((1-\alpha)k^\alpha_t)^\alpha}\left(\frac{\alpha\beta(1-\pi)+\pi}{\pi^{1+\alpha}}\right)$$

Where $\widetilde{n}$ is given by (3.20)

Then

$$\exists\widehat{\lambda} \in (0,1) : \frac{dx^T_A(\lambda_r)}{d\lambda_r}\left\{\begin{array}{c} <0 \text{ for } \lambda_r \in [0,\widehat{\lambda}) \\ =0 \text{ for } \lambda_r = \widehat{\lambda} \\ >0 \text{ for } \lambda_r \in (\widehat{\lambda},1] \end{array}\right\}$$

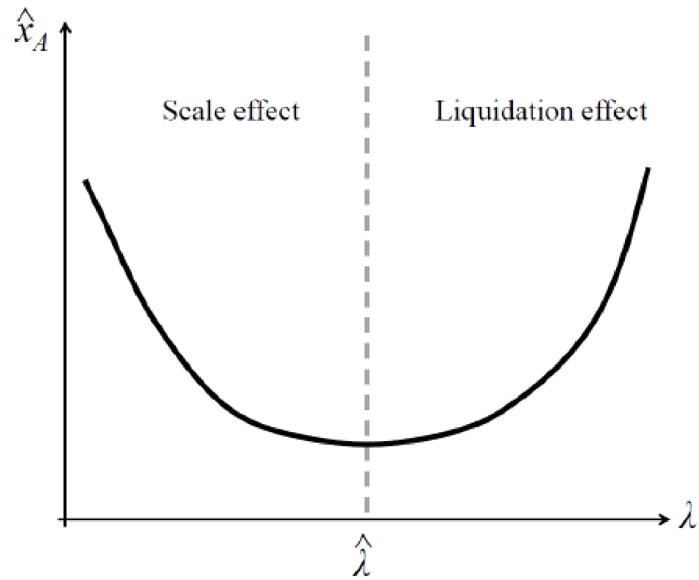
Further, $\widehat{\lambda}$ is unique and $x^T_A(\lambda_r)$ reaches a unique minimum at $\lambda_r = \widehat{\lambda}$

Proof. See appendix. ■

Remark 58 *The first condition states that when there are no shocks ($\widehat{x}_A = x_A$) and $\lambda_r = 1$, the economy avoids the credit trap. The second condition ensures that $\frac{dx^T_A(0)}{d\lambda_r} < 0$, that is, when $\lambda_r = 0$, increasing leverage* increases *resilience.*

[20] $\widetilde{k}$ corresponds to $\widetilde{n}$, the threshold above which banks invest in sector A. Specifically, $\widetilde{n} = \pi(1-\alpha)\widetilde{k}^\alpha$.

Figure 3.6: Resilience and leverage: the scale effect and the liquidation effect



The U-shape reflects the two opposing effects of leverage on resilience. On the one hand, for any productivity realisation $\hat{x}_A$, more capital is produced at $t+1$ the higher leverage was at t (λ_r is high). This puts the economy farther away from the credit trap threshold $\tilde{k}$, increasing resilience (*scale effect*). On the other hand, for any given negative shock to asset returns, the reduction in net worth is greater when leverage is high. Thus, depositors liquidate a greater proportion of the capital produced following the shock at $t+1$ the greater leverage at t. This makes it more likely that the economy falls into a credit trap, reducing resilience (*liquidation effect*). When leverage is low ($\lambda_r < \hat{\lambda}$), the scale effect dominates, and allowing banks to increase leverage will increase resilience. Over this range, *there is no trade-off between expected output and resilience*: increasing leverage increases both. However, when leverage is high ($\lambda_r > \hat{\lambda}$), the liquidation effect dominates, and allowing banks to increase leverage will reduce resilience. Under the conditions given, $\hat{\lambda} > 0$, implying that the leverage ratio that maximises resilience is greater than 1. Due to the scale effect, even a policy-maker who focused only on the resilience of the financial system would allow *some* leverage.

It is interesting to examine how the desirability of leverage policy varies with the state of the economy. First, it is clear that following a small negative shock to the financial system, the economy will recover to its steady state faster if leverage policy is relaxed (it can then be tightened again once the steady state is reached). This is because doing so allows more deposits to flow into the banking system, raising the amount of investment and future output. It may be thought that this comes at the cost of lowering resilience, by letting weaker banks take on higher leverage. However, the proposition below shows that, on the contrary, the leverage ratio that maximises resilience is *countercyclical.*

Proposition 59 *Suppose the conditions of Proposition (57) hold.*

Then

$$\frac{d\widehat{\lambda}}{dk_t} < 0$$

Proof. See appendix. ■

The proposition shows that when the state of the economy becomes worse-a decrease in k_t-the λ_r that maximises resilience *increases.* Thus, the policy-maker who only cares about resilience would allow greater leverage in a downturn. This is because the scale effect becomes relatively more important when k_t is lower. With n_t closer to the trap threshold $\widetilde{n}$, it is desirable to allow more investment to help banks improve their balance sheets. Thus, if a policymaker cared only about resilience, they would conduct counter-cyclical leverage policy.

3.4.2 Summary of Policies for Avoiding the Trap

In summary, leverage policy can be effective in reducing the chance of the economy falling into a credit trap. In particular, if the privately determined leverage ratio is greater than $\widehat{\lambda}$, resilience could be improved by implementing this as a leverage cap (and in this case it would bind too). After a small negative shock that does not result in the economy falling into the trap, and at which the original leverage ratio still binds, relaxing the leverage limit would be desirable. Doing so helps the economy recover faster and will increase the economy's resilience against falling into the trap following a further negative shock.

3.5 Policies to Get Out of the Credit Trap

We now consider what policy can do to get the economy out of a trap, first showing that counter-cyclical leverage policy will be ineffective, in contrast to the case of a small shock.

3.5.1 Relaxing the leverage ratio cap

Proposition 60 *Suppose (3.18), (3.19) hold. Suppose with regulatory leverage ratio λ_r in place the economy is stuck investing in sector B. Then relaxing λ_r will not help the economy escape from the credit trap.*

Proof. Given (3.18) and (3.19), banks invest in sector B rather than sector A iff

$$x_A^{\alpha}(1+\lambda_A(n_t)\beta)^{1-\alpha}(\pi+\lambda_A(n_t)\beta)^{\alpha} < x_B^{\alpha}(1+\lambda\beta)^{1-\alpha}(\pi+\lambda\beta)^{\alpha} \tag{3.31}$$

Where $\lambda = \min\{\lambda_r, \lambda_B\}$. As we're in the trap, with banks investing in sector B, (3.31) must hold. As $x_A > x_B$, it must be that $\lambda > \lambda_A(n_t)$. In other words, permitted leverage when investing in B must exceed permitted leverage when investing in A. If $\lambda_r \geq \lambda_B$, the regulator permits higher leverage than the market, thus relaxing the regulatory constraint will not alter equilibrium. If

$\lambda_B > \lambda_r$, the regulatory constraint binds, and relaxing it permits higher leverage in B. But this only enhances the attractiveness of investing in B rather then A. Thus, in both cases, relaxing λ_r will not direct investment towards A. ■

The logic of the proof is intuitive. As sector A is inherently more productive, a higher rate on deposits can only be paid when investing in B (making it more attractive) if the volume of lending in B is greater. Thus, with policy in place, more leverage is possible in sector B than in A, and relaxing the policy constraint either has no effect (if not binding) or allows an even greater volume of investment in B, thereby making it more attractive. Neither of these help with reallocation towards the more productive sector.

Thus, whilst countercyclical leverage policy can be beneficial in facilitating recovery after a small shock, it is not helpful if the shock is sufficiently large to result in a credit trap.

3.5.2 Policies that change the relative attractiveness of A and B

We now consider policies that can direct investment to sector A. In a credit trap, banks invest in sector B rather than sector A with the following inequality holding:

$$x_A^\alpha(1+\lambda_A(n_t)\beta)^{1-\alpha}(\pi+\lambda_A(n_t)\beta)^\alpha < x_B^\alpha(1+\lambda_B\beta)^{1-\alpha}(\pi+\lambda_B\beta)^\alpha$$

As the economy features a credit trap, when this holds, banks will invest in B forevermore and n_t will converge to n_B^*. Thus, a necessary condition for getting the economy out of the credit trap is to redirect investment to sector A. One way of doing this is by altering the regulatory risk weight on each sector, a macroprudential tool that some central banks will have in the future. In particular, suppose sectoral risk weights $\tau_A\lambda_A(n_t)$, $\tau_B\lambda_B$ are in place. Then, as the economy is in a credit trap in this position:

$$x_A^\alpha(1+\tau_A\lambda_A(n_t)\beta)^{1-\alpha}(\pi+\tau_A\lambda_A(n_t)\beta)^\alpha < x_B^\alpha(1+\tau_B\lambda_B\beta)^{1-\alpha}(\pi+\tau_B\lambda_B\beta)^\alpha$$

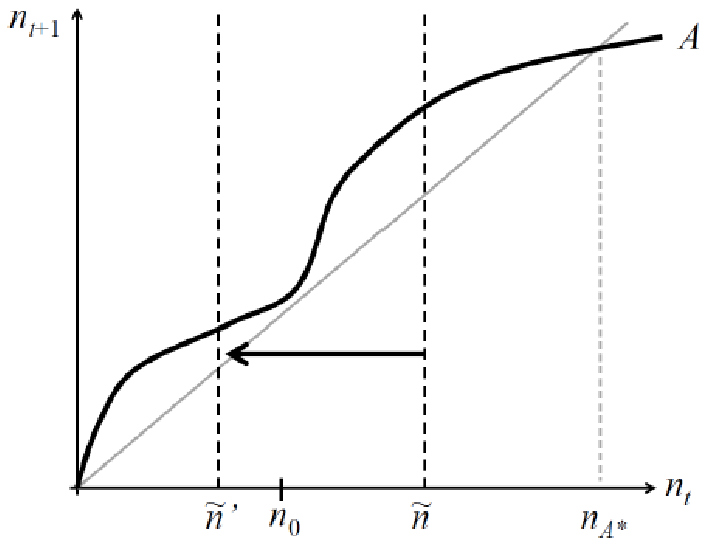
By changing these risk weights, say relaxing the risk weight on A (increasing τ_A) the critical net worth threshold required for investment in A decreases. If this decrease is sufficient, investment will be directed towards A.[21]

This is only a necessary condition, and we must consider when it is also sufficient for the economy to escape the credit trap. The key to this is whether the law of motion for sector A has multiple positive steady states. Until now, all we have assumed about sector A is that it has a unique steady state above the trap threshold $\widetilde{n}$. Here we consider the law of motion for *the whole of sector A*. Figure 3.7 shows an example in which A, considered in isolation, has a unique positive steady state.

Following the negative shock, the net worth of the banking system is given by n_0. As this is

[21] Under the interpretation of sector A being real economy lending and sector B being government bonds, quantitative easing has a similar effect, by altering the relative attractiveness of the two.

Figure 3.7: Unique steady state conditional on investing in sector A



less than the critical threshold $\widetilde{n}$, the economy invests in sector B. Following policy action such as the change in sectoral risk weights, the critical net worth threshold decreases to $\widetilde{n}'$. This is less than n_0 so the economy invests in A. Because A has a unique positive steady state, the economy converges to the good equilibrium, n^*_A. Thus, under these circumstances the policy is sufficient to lift the economy out of the credit trap.

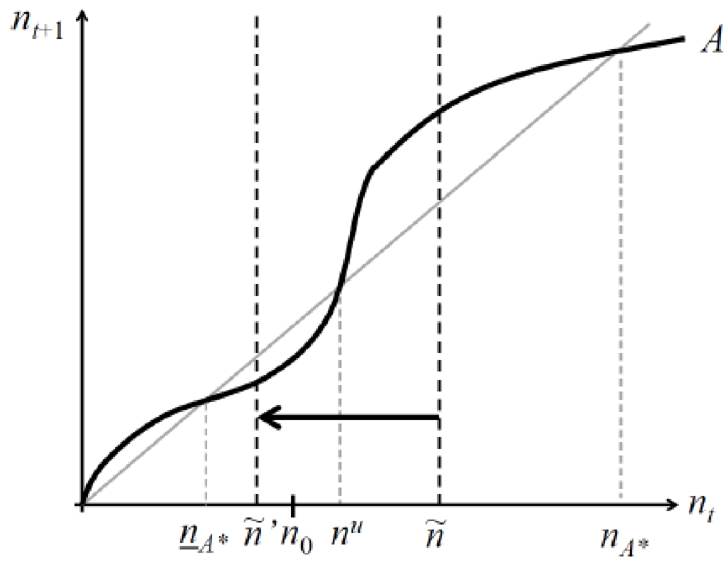
3.5.3 Credit trap in sector A alone

An alternative case is shown in Figure 3.8. Here sector A has multiple positive steady states. As in the prior case, the policy action decreases $\widetilde{n}$ to $\widetilde{n}'$ resulting in investment flowing to sector A. However, there is no virtuous feedback loop between the real economy and the financial sector, and the economy will not recover to n^*_A. Rather, n_t will decrease, and left alone the economy would converge to $\underline{n}_{A^*}$. In fact, as drawn, after a few periods $n_t < \widetilde{n}'$ and the economy will start investing in sector B again.

Form of $\lambda_A(n_t)$

To understand how sector A can have multiple positive steady states, as shown in Figure 3.8, we need to consider the shape of $\lambda_A(n_t)$. Our key assumption throughout has been that banking sector leverage is higher when the banking system is healthier: $\lambda'_A(n_t) > 0$. However, several paths for $\lambda_A(n_t)$ can match this broad pattern. It may be that the leverage permitted increases in n_t at a decreasing rate, or it may be that leverage is relatively unresponsive to banking system health until some minimum level is reached, after which it becomes very responsive, increasing at a high rate. This latter case may be due to increasing returns to scale for the banking system as a whole, over

Figure 3.8: Multiple steady states conditional on investing in sector A



some critical range.

We now introduce a parsimonious functional form to illustrate how a convexity can be generated in the law of motion for sector A.

Lemma 61 *Let*

$$\lambda_A(n_t) := \left(\overline{\lambda}_A - \underline{\lambda}_A\right)\left(\frac{n_t^\delta}{n_t^\delta + c}\right) + \underline{\lambda}_A$$

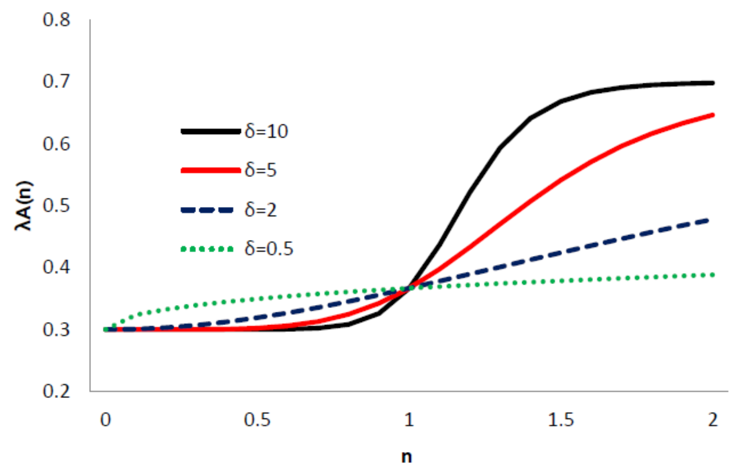
Where $\delta, c > 0$ *and* $\overline{\lambda}_A > \underline{\lambda}_A$.

Then

$$\begin{aligned}
\lim_{n_t \to 0} \lambda_A(n_t) &= \underline{\lambda}_A \\
\lim_{n_t \to \infty} \lambda_A(n_t) &= \overline{\lambda}_A \\
\lambda_A'(n_t) &> 0 \\
\lambda_A''(n_t) &> 0 \text{ iff } n_t^\delta < c\frac{(\delta - 1)}{(1 + \delta)}
\end{aligned}$$

The lemma shows that $\lambda_A(n_t)$ is always increasing in banker net worth, hits its upper and lower values for very low and very high net worth, and may or may not have a convex region depending on the size of δ. On the point regarding convexity, if $\delta < 1$ then $\lambda_A''(n_t) < 0 \ \forall n_t \geq 0$. Whilst if $\delta > 1$, then $\lambda_A''(n_t) > 0$ for $n_t \in \left[0, \left(c\frac{(\delta-1)}{(1+\delta)}\right)^{\frac{1}{\delta}}\right)$. In this case, λ_A has a convex region for small n_t and is concave thereafter. This functional form is thus very general and captures a wide range of possible shapes for $\lambda_A(n_t)$. A selection of these and how they vary with δ is shown in Figure

Figure 3.9: Functional Form for $\lambda_A(n_t)$



3.9.[22] We see that for low δ the function is concave throughout, whilst when $\delta > 1$, the function has a convex region followed by a concave one.

The convexity in $\lambda_A(n_t)$ is the key to the potential convexity in k^A_{t+1}. Indeed, it can be shown that if δ is sufficiently large, then $k^A_{t+1}(k_t)$ will have a convex region. This is quite intuitive as when $\delta \to \infty$, $\lambda_A(n_t)$ tends to a step function where at a crucial tipping point of banking system net worth, the leverage the private sector permits jumps from $\underline{\lambda}_A$ to $\overline{\lambda}_A$. Then, at this tipping point, there will be a large increase in the deposits taken, and so also the amount of capital produced in the next period.

Credit trap in sector A alone: summary

The credit trap in this model is based on banks having two sectors to invest in and only relies on $\lambda'_A(n_t) > 0$. Further properties of $\lambda_A(n_t)$ beyond this are important when considering whether addressing sector misallocation will be sufficient to escape a credit trap. Policies that shift the relative demand between investment in sector A and B will be enough if sector A has a unique positive steady state. Intuitively, the difference turns on whether when in a credit trap the economy would recover to the good steady state n_{A^*} if investment was channelled to the productive sector. This depends on whether lending to the real economy is sufficient to repair the banks' balance sheets, leading to a virtuous feedback loop between n_t and $\lambda_A(n_t)$. If it is, and we simply have sectoral misallocation, policies that alter the relative attractiveness of the two sectors will be sufficient. However, if it is not, different policies are needed, and we consider these next.

[22] In the picture we have set $\overline{\lambda}_A = 0.7$, $\underline{\lambda}_A = 0.3$ and $c = 5$.

3.6 Unconventional Credit Policies

In this section we suppose another policy has been successful in directing investment to sector A (such as sectoral risk weights). However, there is a credit trap in sector A alone, so alternative policies are required.[23] Following Gertler and Kiyotaki (2011) we consider three *unconventional credit policies*: direct lending by the government; discount window lending; an equity injection to the banking system. All three policies were employed during the financial crisis in the US.

The government's source of funding in each case is provided by issuing government bonds, which are perfect substitutes for bank deposits, paying the same return. Thus with $d_{g,t}$ government bonds issued, the household supply of funds for deposits is given by

$$d_{i,t} = \frac{\beta}{1+\beta}(1-\pi)w_t - \frac{(1-x_g)}{1+\beta}\frac{V_{i,t+1}}{R^d_{i,t+1}} - d_{g,t} \tag{3.32}$$

where $x_g \in [0,1]$ represents the equity stake in banks following any equity injection ($x_g = 0$ if there is no equity injection). We note how this this contrasts to (3.9), the case of no policy intervention.

In each case we assume the total extent of implementation of policy j is given by

$$(1+\tau_j)s_{j,t} = d_{g,t} - R^d_t d_{g,t-1} + R_j s_{g,t-1}$$

where $\tau_j > 0$ represents the government's inefficiency cost of implementing the policy, $R^d_t d_{g,t-1}$ is the total paid out on government bonds issued in the previous period and $R_j s_{g,t-1}$ is the return made on implementing the policy in the previous period. For simplicity we assume that the government has no outstanding debt, and did not conduct any policies previously, reducing the budget constraint to

$$s_{j,t} = \frac{d_{g,t}}{(1+\tau_j)} \tag{3.33}$$

Equation (3.33) demonstrates clearly the impact of the inefficiency cost of policy: the greater τ_j, the less policy can be implemented for a given amount of bonds raised.

We now consider implementing each policy separately.

3.6.1 Direct lending

In the case of direct lending, the funds the government raised are invested directly into sector A, contributing directly to the capital stock in the following period:

$$k_{t+1} = x_A(n_t + d_{A,t}) + x_A(s_{g,t}) \tag{3.34}$$

The amount of output goods the government invests in capital production, $s_{g,t}$, augments the

[23] Alternatively, if there is no credit trap in sector A alone, these policies could still be beneficial in speeding up the recovery of the economy back to the high output steady state. Thus, nothing in this section relies on any assumption about λ_A beyond $\lambda'_A(n_t) > 0$.

amount invested by the banking sector, $n_t + d_{A,t}$. However, the supply of deposits, $d_{A,t}$ is affected by the amount of government bonds issued, from (3.32). Following similar analysis to the basic model, it can be shown that the equilibrium amount of deposits supplied is given by

$$d^*_{A,t} = \frac{\beta\lambda_A(n_t)}{1+\beta\lambda_A(n_t)}(1-\pi)w_t - d_{g,t}\frac{(1+\beta)\lambda_A(n_t)}{1+\beta\lambda_A(n_t)} \tag{3.35}$$

Comparing (3.35) with (3.14) we see that government policy partially crowds-out private sector deposits (i.e. deposits are smaller with policy). However, the crowding out is not full, and the total level of bonds and deposits rises following policy:

$$d_{g,t}\left[1-\frac{(1+\beta)\lambda_A(n_t)}{1+\beta\lambda_A(n_t)}\right] = d_{g,t}\left[\frac{1-\lambda_A(n_t)}{1+\beta\lambda_A(n_t)}\right] > 0$$

It is then possible for policy to have a positive effect on k_{t+1}. To derive the law of motion for k_{t+1} we combine (3.34) and (3.35), giving (where τ_g is in the inefficiency cost on direct government lending)

$$k_{t+1} = x_A\left(\left(\frac{\pi+\lambda_A(n_t)\beta}{1+\lambda_A(n_t)\beta}\right)(1-\alpha)k_t^\alpha\right) + x_A d_{g,t}\left[\frac{1}{1+\tau_g} - \frac{\lambda_A(n_t)(1+\beta)}{1+\beta\lambda_A(n_t)}\right] \tag{3.36}$$

This clearly reduces to (3.21), the case of no policy, when $d_{g,t} = 0$. The second term represents the impact of policy, and direct lending is effective in raising k_{t+1} iff

$$\tau_g < \frac{1-\lambda_A(n_t)}{\lambda_A(n_t)(1+\beta)} \tag{3.37}$$

We note that the RHS of (3.37) is decreasing in λ_A : that is, direct lending is less effective when the economy is healthier. Further, it can be that direct lending raises k_{t+1} following a financial crash, but *lowers* it when the economy is healthy. These points are formalised in the following lemma.

Lemma 62 *The effectiveness of the direct lending policy is decreasing in λ_A :*

$$\frac{\partial^2 k_{t+1}}{\partial\lambda_A \partial d_{g,t}} < 0$$

Further, suppose that following a crash, $n_t = \underline{n}$ whilst, in the high output steady state of A $n_t = \overline{n} > \underline{n}$. Suppose further that

$$\frac{1-\lambda_A(\overline{n})}{(1+\beta)\lambda_A(\overline{n})} < \tau_g < \frac{1-\lambda_A(\underline{n})}{(1+\beta)\lambda_A(\underline{n})}$$

Then policy is effective in raising k_{t+1} following the crash, but lowers k_{t+1} in the good state of the economy.

Proof. The proof of the second part is immediate from (3.37). For the first part note that

$$\frac{\partial k_{t+1}}{\partial d_{g,t}} = x_A \left[\frac{1}{1+\tau_g} - \frac{\lambda_A(1+\beta)}{1+\beta\lambda_A} \right]$$

So

$$\begin{aligned} \frac{\partial^2 k_{t+1}}{\partial \lambda_A \partial d_{g,t}} &= -x_A(1+\beta) \left[\frac{(1+\beta\lambda_A) - \lambda_A\beta}{(1+\beta\lambda_A)^2} \right] \\ &= -x_A \left[\frac{1}{(1+\beta\lambda_A)^2} \right] < 0 \end{aligned}$$

■

Direct government intervention always has a positive impact on the economy, directly boosting k_{t+1}. However, this is paid for by government bonds which displace deposits (the 'crowding out' effect), thereby reducing the funding of the banking system. This is further exacerbated by the inefficiency of government intervention ($\tau_g > 0$), requiring extra deposits to be displaced to fund a given level of direct lending. When the financial friction is very tight (λ_A low), deposit levels are low,[24] thus there is little deposit displacement, and the direct benefit to the economy outweighs the negative crowding out effect. However, with a looser financial friction in a stronger economy (λ_A high), deposit levels are higher and there is a larger cost from crowding out, which can then dominate the positive effect (whose size does not change with λ_A). Thus, whilst this policy may be very effective during a credit-crunch, it does not follow that it would be desirable for the government to entirely displace the financial sector when the economy is healthy.

3.6.2 Discount window lending

With discount window lending, the government instead lends directly to the banks. Let m_t be the amount lent to the banking sector (where with inefficiency cost of τ_m we have $m_t = \frac{d_{g,t}}{(1+\tau_m)}$), then the total amount invested by the banking system is given by

$$n_t + d_{A,t} + m_t$$

The government can enforce repayment of its loans better than the private sector, so with discount window lending m_t, the credit constraint facing the banking system is

$$\lambda_A R_{A,t+1}(n_t + d_{A,t} + \omega m_t) \geq R^d_{A,t+1} d_{A,t} + R^m_{t+1} m_t$$

where R^m_{t+1} is the rate paid on loans from the government and $\omega > 1$ represents the greater *pledgeability* of these loans. The total pledgeability on loans from the government is given by $\omega\lambda_A$, which as it is a fraction of total project returns must be less than 1. A parsimonious way of

[24]Without policy, $d^*_{A,t} = \frac{\lambda_A \beta(1-\pi)w_t}{1+\beta\lambda_A}$, which is increasing in λ_A.

ensuring this is given by Gertler and Kiyotaki (2011) where the fraction that banks can divert on government lending is given by

$$(1-\lambda_A)(1-\omega_g)$$

with the constant $\omega_g \in (0,1)$. When $\omega_g = 0$ the government faces no advantage over the private sector in the pledgeability of its loans, whilst when $\omega_g = 1$, the lending friction disappears. With this specification, $\omega = 1 + \frac{\omega_g(1-\lambda_A)}{\lambda_A} > 1$.

Faced with two sources of funding (deposits and the discount window), the banks have a portfolio choice problem, maximising profits

$$V_{A,t+1} = R_{A,t+1}(n_t + d_{A,t} + \omega m_t) - R^d_{A,t+1} d_{A,t} - R^m_{t+1} m_t$$

subject to the leverage constraint.[25] As greater leverage is allowed when borrowing from the government, an endogenous 'penalty wedge' arises on discount window lending: $R^m_{t+1} > R^d_{t+1}$. The result of this portfolio choice problem is the following wedge (with a positive spread when $\omega > 1$).

$$R^m_{t+1} = R^d_{A,t+1} + \frac{\lambda_A}{1-\lambda_A}(\omega - 1)(R_{A,t+1} - R^d_{A,t+1})$$

Following the usual steps in the derivation, equilibrium deposit supply is given by

$$d^*_{A,t} = \frac{\lambda_A \beta}{1+\lambda_A\beta}(1-\pi)w_t - \frac{(1-\lambda_A)}{1+\lambda_A\beta}\left(\frac{R_{A,t+1} - R^m_{t+1}}{R_{A,t+1} - R^d_{A,t+1}}\right) m_t - \frac{\lambda_A(1+\beta)d_{g,t}}{1+\lambda_A\beta}$$

We show in the appendix that in equilibrium $\frac{R_{A,t+1}-R^m_{t+1}}{R_{A,t+1}-R^d_{A,t+1}} = \frac{1-\omega\lambda_A}{1-\lambda_A}$, which combined with the law of motion for capital $k_{t+1} = x_A(n_t + d_{A,t} + m_t)$ gives

$$k_{t+1} = x_A\left(\left(\frac{\pi + \lambda_A(n_t)\beta}{1+\lambda_A(n_t)\beta}\right)(1-\alpha)k_t^\alpha\right) + x_A d_{g,t}\left[\frac{1-(1-\omega_g)\left(\frac{1-\lambda_A(n_t)}{1+\lambda_A(n_t)\beta}\right)}{1+\tau_m} - \frac{\lambda_A(n_t)(1+\beta)}{1+\lambda_A(n_t)\beta}\right] \tag{3.38}$$

Policy is effective in raising k_{t+1} in the discount window case iff

$$\tau_m < w_g \frac{(1-\lambda_A(n_t))}{\lambda_A(n_t)(1+\beta)} \tag{3.39}$$

On comparison with (3.37) we see that this expression is identical, save for the inefficiency τ_m and the $w_g \in (0,1)$ term, representing the financial friction the central bank faces on its loans to the private sector banks. Thus, as with direct lending, policy can be effective when the economy is in a credit crunch, but ineffective, reducing k_{t+1} when the economy is healthy. We introduce an analogous lemma.

[25] A full derivation of the results in this section is given in the appendix.

Lemma 63 *The effectiveness of the policy is decreasing in* λ_A :

$$\frac{\partial^2 k_{t+1}}{\partial \lambda_A \partial d_{g,t}} < 0$$

Further, suppose that following a crash, $n_t = \underline{n}$ *whilst, in the high output steady state of A* $n_t = \overline{n} > \underline{n}$*. Suppose further that*

$$\omega_g \frac{(1-\lambda_A(\overline{n}))}{(1+\beta)\lambda_A(\overline{n})} < \tau_m < \omega_g \frac{(1-\lambda_A(\underline{n}))}{(1+\beta)\lambda_A(\underline{n})}$$

Then policy is effective in raising k_{t+1} *following the crash, but lowers* k_{t+1} *in the good state of the economy.*

Proof. The proof of the second part is immediate from (3.39). For the first part note that

$$\frac{\partial k_{t+1}}{\partial d_{g,t}} = x_A \left[\frac{1-(1-\omega_g)\left(\frac{(1-\lambda_A)}{(1+\lambda_A\beta)}\right)}{(1+\tau_m)} - \frac{\lambda_A(1+\beta)}{1+\lambda_A\beta} \right]$$

So

$$\begin{aligned}
\frac{\partial^2 k_{t+1}}{\partial \lambda_A \partial d_{g,t}} &= x_A \left[-\frac{(1-\omega_g)}{(1+\tau_m)} \left[\frac{-(1+\lambda_A\beta)-(1-\lambda_A)\beta}{(1+\lambda_A\beta)^2} \right] - (1+\beta)\frac{[(1+\lambda_A\beta)-\lambda_A\beta]}{(1+\lambda_A\beta)^2} \right] \\
&= x_A \left[-\frac{(1-\omega_g)}{(1+\tau_m)} \left[\frac{-(1+\beta)}{(1+\lambda_A\beta)^2} \right] - \frac{[(1+\beta)]}{(1+\lambda_A\beta)^2} \right] \\
&= x_A \left[\frac{(1+\beta)}{(1+\lambda_A\beta)^2} \left[\frac{(1-\omega_g)}{(1+\tau_m)} - 1 \right] \right] < 0
\end{aligned}$$

■

As with the direct lending case, when λ_A is higher, the negative effect on k_{t+1} from the crowding out of deposits becomes larger, as there are more deposits made when the banking system is healthier. With direct lending, the positive effect on k_{t+1} is independent of λ_A. In contrast, with discount window lending, the positive effect of policy is *increasing* in the health of the banking system (this can be seen in the first term of the derivative in the proof immediately above). Unlike direct lending, which goes round the banking system, discount window lending has to *work through the banking system.* Thus, after a severe banking crisis, offering a different source of funding to the banking system won't be very effective as the banks ability to borrow is still greatly reduced.[26] As the banks recover, the benefit from an alternative source of funding that allows greater leverage increases. Whilst-as with the lemma-the overall effectiveness of policy decreases as the economy recovers, this decrease can happen at a different rate as with the direct lending policy. As we

[26] In terms of the model, this is because the central bank has a constant *relative* advantage over the private sector in preventing loans being diverted. Thus, when the leverage prevailing in the banking sector is low, the leverage the central bank permits will also be low.

discuss below, this can result in direct lending being more effective following a very severe credit crunch, with discount window lending more effective for less severe crunches.

We now outline the equity injection policy before comparing all three.

3.6.3 Equity injection

As with the other two policy interventions, the government is inefficient in investing in equity, with inefficiency cost τ_{gn}. Thus the amount of equity invested by the government, $n_{g,t}$, satisfies:

$$n_{g,t} = \frac{d_{g,t}}{(1+\tau_{gn})}$$

In return for its injection of resources to the banking system, the government obtains x_g fraction of bank equity, resulting in optimal household saving given by (3.32), with the equity share in the bank watered down to $1 - x_g$.

An important direct effect of the equity injection is that λ_A increases, as it is now based on $n_t + n_{g,t} : \lambda_A(n_t + n_{g,t}) > \lambda_A(n_t)$. This direct effect of the injection, else equal, *crowds in depositors*: with the financial friction reduced, they're willing to supply more deposits, raising investment. This goes beyond the usual effect of higher net worth allowing more deposits to be taken at a fixed leverage ratio. Here the leverage ratio rises too.

To derive the equilibrium law of motion for k_{t+1} we follow the usual steps, first determining equilibrium in the banking sector.

With the banks' leverage constraints binding they demand deposits,[27]

$$d_{i,t} = \frac{[\lambda_A(n_t + n_{g,t})] R_{A,t+1}(n_t + n_{g,t})}{R^d_{A,t+1} - [\lambda_A(n_t + n_{g,t})] R_{A,t+1}}$$

Bank profits are given by[28]

$$V_{A,t+1} = \left(R_{A,t+1} - R^d_{A,t+1}\right) d_{A,t} + R_{A,t+1}(n_t + n_{g,t})$$

Following the usual steps, equilibrium deposits are given by

$$\begin{aligned} d^*_{A,t} \quad = \quad & \frac{\lambda_A(n_t + n_{g,t})\beta(1-\pi)w_t}{(1+\beta)\lambda_A(n_t + n_{g,t}) + (1-x_g)(1-\lambda_A(n_t + n_{g,t}))} \\ & - \frac{(1+\beta)d_{g,t}\lambda_A(n_t + n_{g,t})}{(1+\beta)\lambda_A(n_t + n_{g,t}) + (1-x_g)(1-\lambda_A(n_t + n_{g,t}))} \end{aligned}$$

The impact of policy is notably different to the other two cases, as the rise in λ_A, and watering down through the $x_g > 0$ term, increase the fraction of first period resources saved, $\frac{\beta(1-\pi)w_t}{1+\beta}$.[29]

[27] Note the addition of $n_{g,t}$ which is absent with no equity injection.

[28] The formula (save for the $n_{g,t}$ term) for bank profits has not changed here. What changes is *who* gets them once they're realised, i.e. the split between households and the government.

[29] The watering down effect occurs through households anticipating lower dividends from the banking system when

To determine the overall effect of an equity injection on $d^*_{A,t}$ we need to specify the relationship between x_g and $n_{g,t}$, i.e. how much equity the government gets in return for its investment. We consider the general form weighting the banks' current equity with factor $\gamma > 0$:

$$x_g = \frac{n_{g,t}}{n_{g,t} + \gamma n_t} \tag{3.40}$$

We give two examples of γ.

1. The fraction the government obtains reflects the banks' current equity ($\gamma = 1$)

$$x_g = \frac{n_{gt}}{n_{gt} + n_t}$$

For example, if the net worth of the banking system at time t is 100 units of output goods and the government invests 100 units, it ends up owning half the equity of the banking system.

2. The fraction the government obtains reflects the pdv of the banking system

$$x_g = \frac{n_{gt}}{n_{gt} + \frac{V_{A,t+1}}{R^d_{A,t+1}}}$$

From (3.49) in the appendix without government intervention, $\frac{V_{A,t+1}}{R^d_{A,t+1}} = \frac{n_t(1-\lambda_A)\beta(1-\pi)}{(1+\beta\lambda_A)\pi}$

So

$$x_g = \frac{n_{gt}}{n_{gt} + n_t\left[\frac{(1-\lambda_A)\beta(1-\pi)}{(1+\beta\lambda_A)\pi}\right]}$$

and $\gamma = \frac{(1-\lambda_A(n_t))\beta(1-\pi)}{(1+\beta\lambda_A(n_t))\pi}$.

In this case, the share is not based on the net worth the bank currently has, but the discounted value of what their lifetime profits. This is the value households place on the bank. Under this scheme, if the bank has current net worth of 100, but discounted profits of 400, and the government invests 100, they end up owning 20% of the banking system.

With this general form (3.40), we can re-write equilibrium deposits (with details in the appendix) in a way to make the effect of policy comparable to direct and discount window lending. Combined

old, inducing them to save more to better spread consumption.

with the law of motion for capital $k_{t+1} = x_A(n_t + n_{g,t} + d_{A,t})$, this gives

$$
\begin{aligned}
k_{t+1} &= x_A\left(\frac{\pi+\lambda_A(n_t)\beta}{1+\lambda_A(n_t)\beta}w_t\right) + x_A d_{g,t}\left[\frac{1}{(1+\tau_{gn})} - \frac{\lambda_A(n_t)(1+\beta)}{1+\lambda_A(n_t)\beta}\right] \\
&+ x_A \frac{\left[\lambda_A\left(n_t + \frac{d_{g,t}}{1+\tau_{gn}}\right) - \lambda_A(n_t)\right]}{\left(1+\lambda_A\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\beta\right)(1+\lambda_A(n_t)\beta)}\left[w_t\beta(1-\pi) - d_{g,t}(1+\beta)\right] \\
&+ \frac{x_A d_{g,t}\lambda_A\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\left(1-\lambda_A\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\right)}{(1+\tau_{gn})\left(1+\beta\lambda_A\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\right)} \\
&\cdot \frac{[\beta(1-\pi)w_t - (1+\beta)d_{g,t}]}{\left[\frac{d_{g,t}}{(1+\tau_{gn})}(1+\beta)\lambda_A\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right) + \left(1+\beta\lambda_A\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\right)\gamma n_t\right]}
\end{aligned} \tag{3.41}
$$

Written in this form, we can see the separate effects of the equity injection. As usual, the first term captures what k_{t+1} would have been absent policy, with the second term capturing the trade off between the crowding out effect and direct investment in the economy (the extra equity is automatically invested). The third term is new, capturing the 'crowding in of depositors', representing the fact that the equity injection increases λ_A, which induces more deposits to flow into the banking system. As $w_t\beta(1-\pi) - d_{g,t}(1+\beta) > 0$ this term is positive. Finally, the fourth term captures the impact of watering down depositors, which also draws resources into the banking system.

With direct and discount window lending, the effect of policy is linear in the amount of government borrowing $d_{g,t}$. This is not the case here, making it more difficult to establish when policy is effective. Rather, we focus on the marginal impact when $d_{g,t} = 0$, i.e. $\left\{\frac{dk_{t+1}}{d(d_{g,t})}\right\}_{d_{g,t}=0}$. We have the following lemma (with proof in the appendix).

Lemma 64 *With an equity injection, the marginal effect of policy at $d_{g,t} = 0$ is positive (i.e. $\left\{\frac{dk_{t+1}}{d(d_{g,t})}\right\}_{d_{g,t}=0} > 0$) iff*

$$\tau_{gn} < \frac{1-\lambda_A(n_t)}{\lambda_A(n_t)(1+\beta)}\left[1 + \frac{\left[n_t\lambda_A'(n_t) + \frac{\lambda_A(n_t)(1-\lambda_A(n_t))}{\gamma}\right][w_t\beta(1-\pi)]}{(1-\lambda_A(n_t))(1+\lambda_A(n_t)\beta)n_t}\right]$$

We note this is of a similar form to (3.37) and (3.39) with the addition of two positive terms, the first due to $\lambda_A'(n_t) > 0$, representing the crowding in of depositors, the second the watering down of shareholders (this second effect disappears when $x_g = 0$ (which can be seen as $\gamma \to \infty$), in which case households are not watered down).

In contrast to the prior two policies, the effectiveness of an equity injection need not be uniformly decreasing in λ_A. In particular, if $\lambda_A(.)$ has a steep convex region-for example a large δ in Figure 3.9-an equity injection will be particularly effective in this region, resulting in a large increase in bank leverage. However from Lemma 61, for a sufficiently healthy economy (large enough n_t) $\lambda_A''(n_t) < 0$ and $\lambda_A'(n_t)$ decreases as the economy recovers further. It can be shown that if $\lambda_A(n_t)$

is sufficiently large, $\frac{dk_{t+1}}{d(d_{g,t})}$ is decreasing in $d_{g,t}$. Hence, if the marginal impact is negative when $d_{g,t} = 0$, policy will reduce k_{t+1} for all positive $d_{g,t}$. This is summarised in the following lemma (with proof in the appendix).

Lemma 65 *Let net worth in the good steady state of the economy be $\overline{n}$. Suppose*

$$\lambda_A''(\overline{n}) < \frac{2\beta\left[\lambda_A'(\overline{n})\right]^2}{(1+\lambda_A(\overline{n})\beta)} \tag{3.42}$$

and

$$\lambda_A(\overline{n}) > \frac{-1+\sqrt{1+\beta(2+\beta)}}{\beta(2+\beta)} \tag{3.43}$$

Then $\frac{dk_{t+1}}{d(d_{g,t})}$ *is maximised at $d_{g,t} = 0$.*
Further, if

$$\tau_{gn} > \frac{1-\lambda_A(\overline{n})}{\lambda_A(\overline{n})(1+\beta)}\left[1+\frac{\left[n_t\lambda_A'(\overline{n})+\frac{\lambda_A(\overline{n})(1-\lambda_A(\overline{n}))}{\gamma}\right][w_t\beta(1-\pi)]}{(1-\lambda_A(\overline{n}))(1+\lambda_A(\overline{n})\beta)\overline{n}}\right]$$

Then, in the good steady state, an equity injection lowers k_{t+1} for all $d_{g,t} > 0$.

Remark 66 $\frac{-1+\sqrt{1+(\beta(2+\beta))}}{\beta(2+\beta)} < \frac{1}{2}$

Remark 67 *A sufficient condition for 3.42 holding is $\lambda_A''(\overline{n}) < 0$, that is, in the good steady state of the economy, the increase of λ_A in banking system net worth happens at a decreasing rate, as seems likely.*

In summary, for the equity injection, as with the other two policies, it can be effective in raising k_{t+1} when the economy is in bad health, but ineffective (lowering k_{t+1}) when the economy recovers.

3.6.4 Comparison of Policies

We have shown that all three policies can be effective in raising k_{t+1} during a banking crisis. Here we compare the effectiveness of these, questioning which deliveries the largest increase in k_{t+1} for a given amount of spending $d_{g,t}$.[30]

Case (i) $\tau_m \geq \tau_g \geq \tau_{gn}$

We first suppose that the inefficiencies in direct lending are at least as great as those with an equity injection, and those with discount window lending are at least as great as those with direct lending. Here we have a clear prediction about the relative effectiveness of the policies.

[30] In doing so we abstract from other relevant features such as which policy pays the highest return to the goverment in the future. We focus on the increase in k_{t+1} as raising this is the most important thing during a banking crisis, either helping escape from a trap, or speeding up the recovery.

Proposition 68 *Suppose $\tau_m \geq \tau_g \geq \tau_{gn}$, then for common ${d_{g,t}}$*[31]

$$k_{t+1}^{equity} > k_{t+1}^{direct} > k_{t+1}^{discount}$$

Further, if discount window lending raises k_{t+1} then so does direct lending, though the reverse is not true. If direct lending raises k_{t+1}, then so too does an equity injection, though the reverse is not true.

Proof. For the first part of the proof, from the above formulas it's clear we need to establish that

$$\frac{1}{(1+\tau_{gn})} + \frac{\lambda_A(1-\lambda_A)\left[\beta(1-\pi)w_t - (1+\beta)d_{g,t}\right]}{(1+\tau_{gn})(1+\beta\lambda_A)\left[\frac{d_{g,t}}{(1+\tau_{gn})}(1+\beta)\lambda_A + (1+\beta\lambda_A)\gamma n_t\right]}$$
$$+\frac{\left[\lambda_A\left(n_t + \frac{d_{g,t}}{1+\tau_{gn}}\right) - \lambda_A(n_t)\right]}{\left(1+\lambda_A\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\beta\right)\left(1+\lambda_A\left(n_t\right)\beta\right)}\left[w_t\beta(1-\pi) - d_{g,t}(1+\beta)\right]$$
$$> \quad \frac{1}{1+\tau_g} > \frac{1-(1-\omega_g)\left(\frac{(1-\lambda_i)}{(1+\lambda_i\beta)}\right)}{(1+\tau_m)}$$

The first inequality clearly follows from $\beta(1-\pi)w_t > (1+\beta)d_{g,t}$, $\lambda_A\left(n_t + \frac{d_{g,t}}{1+\tau_{gn}}\right) > \lambda_A(n_t)$ and $\tau_g \geq \tau_{gn}$. The second inequality follows from $\tau_m \geq \tau_g$ and $\omega_g < 1$.

For the second part of the proof, we first need to establish that

$$\tau_m < w_g\frac{(1-\lambda_A)}{\lambda_A(1+\beta)} \Rightarrow \tau_g < \frac{1-\lambda_A}{\lambda_A(1+\beta)}$$

This is clear as then $\tau_g \leq \tau_m < w_g\frac{(1-\lambda_A)}{\lambda_A(1+\beta)} < \frac{(1-\lambda_A)}{\lambda_A(1+\beta)}$. It's clear that the reverse implication does not hold as $w_g < 1$.

For the second, suppose that direct lending is effective:

$$\tau_g < \frac{1-\lambda_A}{\lambda_A(1+\beta)}$$

Then $\tau_{gn} \leq \tau_g < \frac{1-\lambda_A}{\lambda_A(1+\beta)}$ so $\left[\frac{1}{(1+\tau_{gn})} - \frac{\lambda_A(n_t)(1+\beta)}{1+\lambda_A(n_t)\beta}\right] > 0$. From (3.41) its clear that, as the other two terms are positive, k_{t+1} is raised with an equity injection. It's clear that the reverse implication does not hold. This completes the proof. ■

We've shown that if the inefficiencies are the same for the three policies, an equity injection will raise k_{t+1} the most, with discount window lending raising it the least. Further, the equity injection will be effective in raising k_{t+1} for the largest range of states of the economy (i.e. the largest range of λ_A) and discount window lending the smallest range of states of the economy. Thus, in a mild banking crisis, it may be that discount window and direct lending are ineffective, but the equity

[31] This is for feasible $d_{g,t}$ i.e. those less than the total amount households want save via deposits and government bonds. Note that the result does not depend on the specific γ used in the equity pricing rule.

injection is still effective.

The reason for these differences is intuitive. All three policies crowd out deposits in a similar way through the issuance of government bonds. With direct lending, the money raised is invested directly into the economy without any frictions. This is more effective than discount window lending when $\omega_g < 1$ because then the central bank still faces a friction when lending to banks, resulting in a smaller increase in investment than the amount lent. Thus, if discount window lending is at least as inefficient as direct lending ($\tau_m \geq \tau_g$), direct lending will be more effective. The equity injection resembles direct lending in that the amount invested directly adds to the capital stock. This is because it shows up as bank equity, so unlike with discount window lending, no financial friction is faced by the government. In addition, by raising λ_A directly, depositors are crowded in. A further positive impact from the equity injection arises from the watering down of households' bank equity. These last two effects both result in more deposits and a higher k_{t+1}. Thus, when direct lending is at least as inefficient as an equity injection ($\tau_g \geq \tau_{gn}$) k_{t+1} will be higher with the equity injection.

We next show that when the inefficiencies do not follow the order $\tau_m \geq \tau_g \geq \tau_{gn}$, the most effective policy can depend on the state of the economy.

Case (ii) Discount Window Lending Most Efficient $\tau_g, \tau_{gn} > \tau_m$

We first consider an interesting trade-off when discount window lending is more efficient than direct lending, i.e. $\tau_m < \tau_g$. This could be the case because this is closer in line with the specialities of a central bank/government.

Proposition 69 *Suppose*

$$\tau_m < \tau_g - (1+\tau_g)(1-\omega_g)\frac{(1-\lambda_A)}{(1+\lambda_A\beta)}$$

Then discount window lending is more effective in raising k_{t+1} than direct lending.

Proof. Discount window lending is more effective in raising k_{t+1} than direct lending when

$$\frac{1-(1-\omega_g)\left(\frac{(1-\lambda_A)}{(1+\lambda_A\beta)}\right)}{(1+\tau_m)} > \frac{1}{1+\tau_g} \text{ iff}$$

$$\tau_g - (1+\tau_g)(1-\omega_g)\left(\frac{(1-\lambda_A)}{(1+\lambda_A\beta)}\right) > \tau_m$$

■

We note that the LHS of this is increasing in λ_A, so this could hold for a large λ_A and fail for a small λ_A. We thus have a corollary.

Corollary 70 *Consider two credit crunches with associated banking system net worth n_1, n_2 with*

$n_1 > n_2$, *so* n_2 *is the more severe credit crunch. Suppose*

$$\begin{aligned} \tau_g - (1+\tau_g)(1-\omega_g)\left(\frac{(1-\lambda_A(n_1))}{(1+\lambda_A(n_1)\beta)}\right) &> \tau_m \\ \tau_g - (1+\tau_g)(1-\omega_g)\left(\frac{(1-\lambda_A(n_2))}{(1+\lambda_A(n_2)\beta)}\right) &< \tau_m \end{aligned}$$

Then direct lending is more effective in raising k_{t+1} *in the more severe credit crunch* (n_2)*, whilst discount window lending is more effective in the milder credit event* (n_1)*.*

Proof. Immediate. ■

The corollary highlights an interesting trade-off that can arise. With a mild shock to the banking system, discount window lending can be more effective due to the lower inherent inefficiency it involves (resulting in fewer crowded-out deposits). However, with a sufficiently severe shock to the banking system, λ_A will be sufficiently low that this policy will be less effective. This is because discount window lending must work *through* the banking system, and when the banks are severely impaired, the central bank also faces a large credit friction when lending to them. Here, circumventing the banking system, and lending directly to the economy can be more effective.

We now consider a similar case in which discount window lending is inherently more efficient than equity injections, i.e. $\tau_m < \tau_{gn}$. Here we also note that discount window lending can be more effective in a mild downturn, while an equity injection is more effective in a more severe banking crisis.

Proposition 71 *Suppose* $\omega_g < \frac{1+\beta}{2+\beta}$ *and we have the second equity pricing rule.*[32] *Consider two credit crunches with associated banking system net worth* n_1, n_2 *with* $n_1 > n_2$*, so* n_2 *is the more severe crunch. Suppose (3.42) and (3.43) hold for* n_1 *and it's sufficiently large that* $\lambda_A''(n_1) < 0$ *and further that*

$$\begin{aligned} \frac{(1+\tau_{gn})\left[(1+\beta\lambda_A(n_1)) - (1-\omega_g)(1-\lambda_A(n_1))\right]}{\left[1+(1+\beta)\lambda_A(n_1) + \frac{\lambda_A'(n_1)w_t\beta(1-\pi)}{(1+\lambda_A(n_1)\beta)}\right]} - 1 &> \tau_m \\ \frac{(1+\tau_{gn})\left[(1+\beta\lambda_A(n_2)) - (1-\omega_g)(1-\lambda_A(n_2))\right]}{\left[1+(1+\beta)\lambda_A(n_2) + \frac{\lambda_A'(n_2)w_t\beta(1-\pi)}{(1+\lambda_A(n_2)\beta)}\right]} - 1 &< \tau_m \end{aligned}$$

Then an equity injection is more effective in raising k_{t+1} *in the more severe credit crunch* (n_2)*, for a range of* $d_{g,t} > 0$*, whilst discount window lending is more effective in the milder credit event* (n_1) *for all* $d_{g,t} > 0$*.*

Remark 72 *The* $\omega_g < \frac{1+\beta}{2+\beta}$ *condition is required so that the impact of discount window lending closely follows the health of the economy.*

[32] That is, $\gamma = \frac{(1-\lambda_A)\beta(1-\pi)}{(1+\beta\lambda_A)\pi}$. The exact form of γ does not matter for the result, only simplifies the exposition.

Proof. See appendix. ■

In the more severe crunch, discount window lending is less effective as it has to work through the banking system, and with low λ_A, the fraction of government lending that makes it through to the real economy is limited. By contrast, the equity injection directly boosts output as the equity is directly invested in sector A. Further, the increase in λ_A can have a large positive impact on k_{t+1}, crowding in depositors. This effect is particularly near any convex region of λ_A. These large positive benefits outweigh the greater inherent inefficiency associated with an equity injection. In a less severe crunch, the benefit from increasing λ_A will not be as large, and with higher λ_A, discount window lending will become relatively more effective. Consequently, the lower inefficiency of this policy can result in it being more effective overall.

3.6.5 Summary

The work here shows how to compare the three considered unconventional credit policies. We have seen that for all of them, effectiveness depends on the state of the economy. Whilst they can be highly effective in a credit crunch, under given parameter restrictions they will actually make the economy worse if applied when the economy is healthy. We can thus resist the conclusion that it is always desirable for the government to fully replace the banking sector in this model.

When the inefficiencies of the three policies are equal, we have a clear ranking in terms of the effectiveness of raising k_{t+1}, with equity injections being the most effective and discount window lending the least effective. We have also seen that when discount window lending is more efficient than the other two policies, it can be more effective in a milder banking crisis but less effective than the other two policies in a severe banking crisis.

3.7 Conclusions

In this paper we have developed a simple, tractable OLG model for analysing credit traps. We have analysed the effectiveness of policy both at preventing the occurrence of a credit trap as well as in helping the economy to escape a trap if it falls into one (which becomes necessary as it will not recover without intervention). Our analysis shows that a leverage ratio cap is effective in increasing the resilience of the economy against shocks and reducing the probability of a financial crisis. Further, relaxing the cap is effective in encouraging faster recovery after a negative productivity shock, provided that the shock is sufficiently small. However, if the shock is large enough to tip the economy into a credit trap, then relaxing the leverage cap will not help the economy get out of it. Policies that affect the relative attractiveness of investment in sectors A and B, such as changing sectoral risk weights, will work if there is pure sectoral misallocation. If there is not, other policies are needed and we consider direct lending, equity injections, and discount window lending. These policies present rich, realistic trade-offs, and their effectiveness depends on the state of the economy, with each one being more effective when the economy is weaker.

In future work, it would be interesting to analyse the optimal leverage ratio that would be set by a policymaker in advance of a trap. The optimal leverage ratio would have to address the trade-off between resilience and output: in the absence of shocks, output will be higher when leverage is higher. We have shown that the level of leverage that maximises resilience is countercyclical: it would be interesting to assess numerically if the same holds true for the optimal level of leverage, and whether this would vary with the state of the economy in a non-linear way. This would be particularly interesting when the economy is just at the trap threshold, and the policymaker has to trade-off rebuilding the health of the banking system and the economy against the possibility of further negative shocks.

3.A Proof from Section 3.2: Model

3.A.1 Households

Lemma 73 *Households optimal saving is given by*

$$d_t = \frac{\beta}{1+\beta}(1-\pi)w_t - \frac{1}{1+\beta}\frac{V_{t+1}}{R_{d,t+1}} \tag{3.44}$$

Proof. The household problem is

$$\begin{aligned} \max_{c_{1t},c_{2t}} \log c_{1t} + \beta \log c_{2t} \quad &: \quad c_{1t} + d_t \leq (1-\pi)w_t \\ c_{2t} \quad &\leq \quad R_{d,t+1}d_t + V_{t+1} \end{aligned}$$

Optimally both constraints will bind so the problem can be rewritten as

$$\max_{d_t} \log((1-\pi)w_t - d_t) + \beta \log \left(R_{d,t+1}d_t + V_{t+1}\right)$$

With a strictly concave objective function, the FOC is sufficient for a global maximum.

$$FOC : \frac{-1}{(1-\pi)w_t - d_t} + \frac{\beta R_{d,t+1}}{R_{d,t+1}d_t + V_{t+1}} = 0$$

This is just the standard Euler equation:

$$\beta\frac{R_{d,t+1}}{c_{2t}} = \frac{1}{c_{1t}}$$

From the FOC we isolate the optimal deposits d_t :

$$\begin{aligned}
\frac{\beta R_{d,t+1}}{R_{d,t+1}d_t + V_{t+1}} &= \frac{1}{(1-\pi)w_t - d_t} \text{ so} \\
\beta R_{d,t+1}\left((1-\pi)w_t - d_t\right) &= R_{d,t+1}d_t + V_{t+1} \text{ so} \\
R_{d,t+1}d_t\,(1+\beta) &= \beta R_{d,t+1}(1-\pi)w_t - V_{t+1} \text{ so} \\
d_t &= \frac{\beta}{1+\beta}(1-\pi)w_t - \frac{1}{1+\beta}\frac{V_{t+1}}{R_{d,t+1}}
\end{aligned}$$

This completes the proof of the lemma. ■

3.A.2 Deposit market equilibrium

We consider different cases here, beginning with a positive spread in equilibrium followed by zero spread. We then summarise the results.

Positive Spread: $R_{i,t+1} > R_{d,t+1} > \lambda_i R_{i,t+1}$

Lemma 74 *Suppose sector i is invested in. If $R_{i,t+1} > R_{d,t+1} > \lambda_i R_{i,t+1}$ then the equilibrium supply of deposits from households is given by*

$$d^*_{i,t} = \frac{\lambda_i \beta}{1+\lambda_i\beta}(1-\pi)w_t$$

Proof. When $R_{i,t+1} > R_{d,t+1} > \lambda_i R_{i,t+1}$ the pledgeability constraint holds with equality.[33] Thus

$$\lambda_i R_{i,t+1}(n_t + d_{i,t}) = R_{d,t+1}d_{i,t} \tag{3.45}$$

Rearranging this gives

$$\begin{aligned}
d_{i,t}(R_{d,t+1} - \lambda_i R_{i,t+1}) &= \lambda_i R_{i,t+1} n_t \text{ and so} \\
d_{i,t} &= \frac{\lambda_i R_{i,t+1} n_t}{R_{d,t+1} - \lambda_i R_{i,t+1}}
\end{aligned} \tag{3.46}$$

This gives the deposit demand of banks.

[33] $R_{i,t+1} > R_{d,t+1}$ ensures the bank takes as many deposits as they can. $R_{d\,,t+1} > \lambda_i R_{i,t+1}$ ensures that they are constrained by the pledgability constraint.

To calculate the deposit supply of households we must look at the lump sum transfer households receive from banks:

$$
\begin{aligned}
V_{i,t+1} \quad &: \quad = (R_{i,t+1} - R_{d\,t+1})d_{i,t} + R_{i,t+1}n_t \\
&= \quad (R_{i,t+1} - R_{d\,t+1})\frac{\lambda_i R_{i,t+1}n_t}{R_{d,t+1} - \lambda_i R_{i,t+1}} + R_{i,t+1}n_t \\
&= \quad \frac{R_{i,t+1}n_t}{R_{d,t+1} - \lambda_i R_{i,t+1}}\left(\lambda_i(R_{i,t+1} - R_{d\,t+1}) + R_{d,t+1} - \lambda_i R_{i,t+1}\right) \\
&= \quad \frac{R_{i,t+1}n_t}{R_{d,t+1} - \lambda_i R_{i,t+1}} R_{d,t+1}(1-\lambda_i)
\end{aligned}
$$

Thus, from (3.44) deposit supply is given by

$$
\begin{aligned}
d_{i,t} &= \frac{\beta}{1+\beta}(1-\pi)w_t - \frac{1}{1+\beta}\frac{V_{i,t+1}}{R_{d,t+1}} \qquad (3.47)\\
&= \frac{\beta}{1+\beta}(1-\pi)w_t - \frac{1}{1+\beta}\frac{(1-\lambda_i)R_{i,t+1}n_t}{R_{d,t+1} - \lambda_i R_{i,t+1}}
\end{aligned}
$$

In equilibrium of the deposit market, deposit supply (3.46) equals deposit demand (3.47), so

$$
\frac{\lambda_i R_{i,t+1}n_t}{R_{d,t+1} - \lambda_i R_{i,t+1}} = \frac{\beta}{1+\beta}(1-\pi)w_t - \frac{1}{1+\beta}\frac{(1-\lambda_i)R_{i,t+1}n_t}{R_{d,t+1} - \lambda_i R_{i,t+1}}
$$

Now $n_t = \pi w_t$ so

$$
\begin{aligned}
\frac{\lambda_i R_{i,t+1}\pi w_t}{R_{d,t+1} - \lambda_i R_{i,t+1}} &= \frac{\beta}{1+\beta}(1-\pi)w_t - \frac{1}{1+\beta}\frac{(1-\lambda_i)R_{i,t+1}\pi w_t}{R_{d,t+1} - \lambda_i R_{i,t+1}} \text{ iff} \\
\frac{\lambda_i R_{i,t+1}\pi}{R_{d,t+1} - \lambda_i R_{i,t+1}} &= \frac{\beta}{1+\beta}(1-\pi) - \frac{1}{1+\beta}\frac{(1-\lambda_i)R_{i,t+1}\pi}{R_{d,t+1} - \lambda_i R_{i,t+1}} \text{ iff} \\
\beta(1-\pi) &= \frac{(1+\beta)\lambda_i R_{i,t+1}\pi}{R_{d,t+1} - \lambda_i R_{i,t+1}} + \frac{(1-\lambda_i)R_{i,t+1}\pi}{R_{d,t+1} - \lambda_i R_{i,t+1}} \\
&= \frac{R_{i,t+1}\pi}{R_{d,t+1} - \lambda_i R_{i,t+1}}\left((1+\beta)\lambda_i + (1-\lambda_i)\right) \\
&= \frac{R_{i,t+1}\pi}{R_{d,t+1} - \lambda_i R_{i,t+1}}\left(\beta\lambda_i + 1\right)
\end{aligned}
$$

Thus

$$
\frac{R_{i,t+1}}{R_{d,t+1} - \lambda_i R_{i,t+1}} = \frac{\beta(1-\pi)}{\pi\left(\beta\lambda_i + 1\right)} \qquad (3.48)
$$

Hence, in equilibrium, when sector i is invested in

$$\begin{aligned} V_{i,t+1} &= \frac{R_{i,t+1}n_t}{R_{d,t+1}-\lambda_i R_{i,t+1}} R_{d,t+1}(1-\lambda_i) \\ &= R_{d,t+1}(1-\lambda_i)n_t \frac{\beta(1-\pi)}{\pi\left(\beta\lambda_i+1\right)} \\ &= R_{d,t+1}(1-\lambda_i)w_t \frac{\beta(1-\pi)}{(\beta\lambda_i+1)} \end{aligned} \tag{3.49}$$

The equilibrium amount of deposits can be found by substituting (3.48) into (3.47):

$$\begin{aligned} d^*_{i,t} &= \frac{\beta}{1+\beta}(1-\pi)w_t - \frac{1}{1+\beta}\frac{(1-\lambda_i)R_{i,t+1}n_t}{R_{d,t+1}-\lambda_i R_{i,t+1}} \\ &= \frac{\beta}{1+\beta}(1-\pi)w_t - \frac{1}{1+\beta}(1-\lambda_i)\pi w_t \frac{\beta(1-\pi)}{\pi\left(\beta\lambda_i+1\right)} \\ &= \frac{\beta}{1+\beta}(1-\pi)w_t\left[1-\frac{(1-\lambda_i)}{(\beta\lambda_i+1)}\right] \\ &= \frac{\beta}{1+\beta}\frac{(1-\pi)w_t}{(\beta\lambda_i+1)}\left[(\beta\lambda_i+1)-(1-\lambda_i)\right] \\ &= \frac{\beta}{1+\beta}\frac{(1-\pi)w_t}{(\beta\lambda_i+1)}\left[\lambda_i(1+\beta)\right] \\ &= \frac{\lambda_i\beta w_t(1-\pi)}{1+\beta\lambda_i} \end{aligned}$$

This completes the proof of the lemma. ■

Lemma 75 *In equilibrium with sector i invested in and $R_{i,t+1} > R_{d,t+1} > \lambda_i R_{i,t+1}$*

$$R^*_{d,t+1} = R_{i,t+1}\frac{\pi+\lambda_i\beta}{\beta(1-\pi)}$$

Proof. Given $R_{i,t+1} > R_{d,t+1} > \lambda_i R_{i,t+1}$,(3.45) holds so:

$$R_{d,t+1} = \lambda_i R_{i,t+1}\left(\frac{n_t}{d_{i,t}}+1\right)$$

From the prior lemma, using $n_t = \pi w_t$:

$$\begin{aligned} d^*_{i,t} &= \frac{\lambda_i\beta}{1+\lambda_i\beta}(1-\pi)\frac{n_t}{\pi} \text{ so} \\ \frac{n_t}{d^*_{i,t}} &= \frac{(1+\lambda_i\beta)\pi}{\lambda_i\beta(1-\pi)} \end{aligned}$$

Thus

$$
\begin{aligned}
R^*_{d,t+1} &= \lambda_i R_{i,t+1}\left(\frac{n_t}{d^*_{i,t}}+1\right) \\
&= \lambda_i R_{i,t+1}\left(\frac{(1+\lambda_i\beta)\pi}{\lambda_i\beta(1-\pi)}+1\right) \\
&= \frac{R_{i,t+1}}{\beta(1-\pi)}\left((1+\lambda_i\beta)\pi+\lambda_i\beta(1-\pi)\right) \\
&= \frac{R_{i,t+1}}{\beta(1-\pi)}\left(\pi+\lambda_i\beta\right)
\end{aligned}
$$

This completes the proof of the lemma. ■

Corollary 76 *The above equilibrium indeed satisfies $R_{i,t+1} > R_{d,t+1} > \lambda_i R_{i,t+1}$ (so is consistent) if $\beta(1-\pi) > \pi + \lambda_i\beta$*

Proof. The condition ensures that $\frac{\pi+\lambda i\beta}{\beta(1-\pi)} < 1$ and so $R^*_{d,t+1} < R_{i,t+1}$.

For the second inequality

$$
\begin{aligned}
R_{d,t+1} &> \lambda_i R_{i,t+1} \text{ iff} \\
\frac{R_{i,t+1}}{\beta(1-\pi)}(\pi+\lambda_i\beta) &> \lambda_i R_{i,t+1} \text{ iff} \\
\frac{(\pi+\lambda_i\beta)}{\beta(1-\pi)} &> \lambda_i \text{ iff} \\
(\pi+\lambda_i\beta) &> \beta(1-\pi)\lambda_i \text{ iff} \\
\pi(1+\beta\lambda_i) &> \beta\lambda_i - \beta\lambda_i = 0
\end{aligned}
$$

This clearly holds regardless of the condition. This completes the proof of the corollary. ■

Lemma 77 *If $R_{i,t+1} > R_{d,t+1} > \lambda_i R_{i,t+1}$, and sector i is invested in, then*

$$k_{t+1} = \widehat{x}_i \frac{\pi+\lambda_i\beta}{1+\lambda_i\beta}(1-\alpha)k_t^\alpha$$

where $\widehat{x}_i$ represents the realised (as opposed to expected) level of capital produced per unit of output goods invested.

Proof. The amount of capital produced next period is given by the product of the amount of output goods invested and the realised level of technology $\widehat{x}_i$:

$$k_{t+1} = \widehat{x}_i(n_t + d^*_{i,t})$$

When $R_{i,t+1} > R_{d,t+1} > \lambda_i R_{i,t+1}$,

$$d^*_{i,t} = \frac{\lambda_i\beta}{1+\lambda_i\beta}(1-\pi)w_t$$

Thus

$$
\begin{aligned}
k_{t+1} &= \widehat{x}_i(\pi w_t + \frac{\lambda_i\beta}{1+\lambda_i\beta}(1-\pi)w_t) \\
&= \widehat{x}_i\frac{w_t}{1+\lambda_i\beta}\left(\pi\left(1+\lambda_i\beta\right)+\lambda_i\beta(1-\pi)\right) \\
&= \widehat{x}_i\frac{(1-\alpha)k_t^{\alpha}}{1+\lambda_i\beta}\left(\pi+\lambda_i\beta\right)
\end{aligned}
$$

Where we have used $w_t = (1-\alpha)k_t^{\alpha}$ which follows from the use of Cobb-Douglas technology. This completes the proof of the lemma. ■

Lemma 78 *If $R_{i,t+1} > R_{d,t+1} > \lambda_i R_{i,t+1}$ and sector i is invested in then*

$$
\begin{aligned}
\widehat{R}_{i,t+1} &= \frac{\alpha\widehat{x}_i^{\alpha}}{\left[\frac{\pi+\lambda_i\beta}{1+\lambda_i\beta}(1-\alpha)k_t^{\alpha}\right]^{1-\alpha}} \\
R_{d,t+1} &= \frac{\alpha x_i^{\alpha}(1+\lambda_i\beta)^{1-\alpha}(\pi+\lambda_i\beta)^{\alpha}}{\beta(1-\pi)\left[(1-\alpha)k_t^{\alpha}\right]^{1-\alpha}}
\end{aligned}
$$

where x_i is the expected level of capital produced per unit of output goods invested. Note that $\widehat{R}_{i,t+1}$ is the actual realised return on investment in sector i. The above deposit market clearing conditions are all based on the expected realised return $R_{i,t+1}$, that is, the return when $\widehat{x}_i = x_i$.

Proof. We assume full depreciation of capital during output production for tractability so

$$
\begin{aligned}
\widehat{R}_{i,t+1} &= \widehat{x}_i f'(k_{t+1}) \\
&= \frac{\alpha\widehat{x}_i}{k_{t+1}^{1-\alpha}}
\end{aligned}
$$

This is expression gives the gross return on output goods invested in sector i. Each unit of output goods invested produces $\widehat{x}_i$ units of capital goods next period, each of which earns the return to capital from output, which is the marginal product of capital.

Using the prior lemma:

$$
\begin{aligned}
\widehat{R}_{i,t+1} &= \frac{\alpha\widehat{x}_i}{\left[\widehat{x}_i\frac{(1-\alpha)k_t^{\alpha}}{1+\lambda_i\beta}\left(\pi+\lambda_i\beta\right)\right]^{1-\alpha}} \\
&= \frac{\alpha\widehat{x}_i^{\,\alpha}}{\left[\frac{(\pi+\lambda_i\beta)}{1+\lambda_i\beta}(1-\alpha)k_t^{\alpha}\right]^{1-\alpha}}
\end{aligned}
$$

For the deposit rate expression, note that from a prior lemma, given $R_{i,t+1} > R_{d,t+1} > \lambda_i R_{i,t+1}$ we have[34]

$$\begin{aligned} R^*_{d,t+1} &= R_{i,t+1}\frac{\pi+\lambda_i\beta}{\beta(1-\pi)} \\ &= \frac{\alpha x_i^\alpha}{\left[\frac{(\pi+\lambda_i\beta)}{1+\lambda_i\beta}(1-\alpha)k_t^\alpha\right]^{1-\alpha}}\left(\frac{\pi+\lambda_i\beta}{\beta(1-\pi)}\right) \\ &= \frac{\alpha x_i^\alpha}{\beta(1-\pi)}\frac{(\pi+\lambda_i\beta)(\pi+\lambda_i\beta)^{\alpha-1}(1+\lambda_i\beta)^{1-\alpha}}{[(1-\alpha)k_t^\alpha]^{1-\alpha}} \\ &= \frac{\alpha x_i^\alpha(\pi+\lambda_i\beta)^\alpha(1+\lambda_i\beta)^{1-\alpha}}{\beta(1-\pi)[(1-\alpha)k_t^\alpha]^{1-\alpha}} \end{aligned}$$

This completes the proof of the lemma. ■

Zero Spread: $R_{i,t+1} = R_{d,t+1} > \lambda_i R_{i,t+1}$

We establish an analogous series of results to the positive spread case.

Lemma 79 *Suppose sector i is invested in. If $R_{i,t+1} = R_{d,t+1} > \lambda_i R_{i,t+1}$ then the equilibrium supply of deposits from households is given by*

$$d^*_{i,t} = \frac{w_t}{1+\beta}(\beta(1-\pi)-\pi)$$

Thus in this case, $d^*_{i,t} > 0$ iff $\beta(1-\pi) > \pi$. We note that in this case the financial friction λ_i has no effect on the level of deposits and the pledgeability constraint does not bind.

Proof. *When $R_{i,t+1} = R_{d,t+1}$, $V_{i,t+1} = R_{i,t+1}n_t = R_{d,t+1}n_t$. Thus, from (3.44)*

$$\begin{aligned} d_{i,t} &= \frac{\beta}{1+\beta}(1-\pi)w_t - \frac{1}{1+\beta}\frac{R_{d,t+1}n_t}{R_{d,t+1}} \\ &= \frac{\beta}{1+\beta}(1-\pi)w_t - \frac{\pi w_t}{1+\beta} \\ &= \frac{w_t}{1+\beta}(\beta(1-\pi)-\pi) \end{aligned}$$

This completes the proof. ■

Lemma 80 *The above equilibrium indeed satisfies $R_{i,t+1} = R_{d,t+1} > \lambda_i R_{i,t+1}$ (so is consistent) if $\beta(1-\pi) \leq \pi + \beta\lambda_i$. We note that the positive spread and zero spread equilibria can not both occur at once.*

Proof. The pledgeability constraint requires that

$$\lambda_i R_{i,t+1}(n_t + d_{i,t}) \geq R_{d,t+1}d_{i,t}$$

[34] Note that this is based on the expected return from investment in sector i.

When there is zero spread, this equation simplifies to

$$\begin{aligned}\lambda_i(n_t + d_{i,t}) &\geq d_{i,t} \text{ or} \\ \lambda_i n_t &\geq d_{i,t}(1-\lambda_i)\end{aligned}$$

From the prior lemma, we must have

$$\begin{aligned}\lambda_i n_t &\geq \frac{w_t}{1+\beta}\left(\beta(1-\pi)-\pi\right)(1-\lambda_i) \text{ this holds iff} \\ \lambda_i \pi &\geq \frac{1}{1+\beta}\left(\beta(1-\pi)-\pi\right)(1-\lambda_i) \text{ iff} \\ \lambda_i\pi\,(1+\beta)+\pi(1-\lambda_i) &\geq \beta(1-\pi)(1-\lambda_i) \text{ iff} \\ \pi\left(\lambda_i\,(1+\beta)+(1-\lambda_i)\right) &\geq \beta(1-\pi)(1-\lambda_i) \text{ iff} \\ \pi\left(\lambda_i\beta+1\right) &\geq \beta\left(1+\pi\lambda_i-\pi-\lambda_i\right) \text{ iff} \\ \pi &\geq \beta\left(1-\pi-\lambda_i\right) \text{ iff} \\ \pi+\beta\lambda_i &\geq \beta\left(1-\pi\right)\end{aligned}$$

This completes the proof of the lemma. ■

Lemma 81 *If $R_{i,t+1} = R_{d,t+1} > \lambda_i R_{i,t+1}$, and sector i is invested in, then*

$$k_{t+1} = \frac{\widehat{x}_i\beta}{1+\beta}(1-\alpha)k_t^{\alpha}$$

where $\widehat{x}_i$ represents the realised (as opposed to expected) level of capital produced per unit of output goods invested.

Proof. The amount of capital produced next period is given by the product of the amount of output goods invested and the realised level of technology $\widehat{x}_i$:

$$k_{t+1} = \widehat{x}_i(n_t + d^*_{i,t})$$

When $R_{i,t+1} = R_{d,t+1} > \lambda_i R_{i,t+1}$,

$$d^*_{i,t} = \frac{w_t}{1+\beta}\left(\beta(1-\pi)-\pi\right)$$

Thus

$$\begin{aligned}
k_{t+1} &= \widehat{x}_i(\pi w_t + \frac{w_t}{1+\beta}(\beta(1-\pi)-\pi)) \\
&= \frac{\widehat{x}_i w_t}{1+\beta}(\pi(1+\beta)+(\beta(1-\pi)-\pi)) \\
&= \frac{\widehat{x}_i w_t}{1+\beta}(\pi+\pi\beta+\beta-\beta\pi-\pi) \\
&= \frac{\widehat{x}_i w_t \beta}{1+\beta} \\
&= \frac{\widehat{x}_i \beta}{1+\beta}(1-\alpha)k_t^\alpha
\end{aligned}$$

Where we have used $w_t = (1-\alpha)k_t^\alpha$ which follows from the use of Cobb-Douglas technology. This completes the proof of the lemma. ■

Lemma 82 *If $R_{i,t+1} = R_{d,t+1} > \lambda_i R_{i,t+1}$ and sector i is invested in then*

$$\begin{aligned}
\widehat{R}_{i,t+1} &= \frac{\alpha \widehat{x}_i^\alpha}{\left[\frac{\beta}{1+\beta}(1-\alpha)k_t^\alpha\right]^{1-\alpha}} \\
R_{d,t+1} &= R_{i,t+1} = \frac{\alpha x_i^\alpha}{\left[\frac{\beta}{1+\beta}(1-\alpha)k_t^\alpha\right]^{1-\alpha}}
\end{aligned}$$

where x_i is the expected level of capital produced per unit of output goods invested.

Proof. We assume full depreciation of capital during output production for tractability so

$$\begin{aligned}
\widehat{R}_{i,t+1} &= \widehat{x}_i f'(k_{t+1}) \\
&= \frac{\alpha \widehat{x}_i}{k_{t+1}^{1-\alpha}}
\end{aligned}$$

Using the prior lemma:

$$\begin{aligned}
\widehat{R}_{i,t+1} &= \frac{\alpha \widehat{x}_i}{\left[\frac{\widehat{x}_i\beta}{1+\beta}(1-\alpha)k_t^\alpha\right]^{1-\alpha}} \\
&= \frac{\alpha \widehat{x}_i{}^\alpha}{\left[\frac{\beta}{1+\beta}(1-\alpha)k_t^\alpha\right]^{1-\alpha}}
\end{aligned}$$

This is the actual realised gross return from investment in sector i. As there is zero spread, the *expected* gross return from investment in sector i is equal to the deposit rate and so

$$R_{d,t+1} = R_{i,t+1} = \frac{\alpha x_i^\alpha}{\left[\frac{\beta}{1+\beta}(1-\alpha)k_t^\alpha\right]^{1-\alpha}}$$

This completes the proof of the lemma. ■

Other Potential Cases

So far we have considered two mutually exclusive cases

$$\begin{aligned} R_{i,t+1} &> R_{d,t+1} > \lambda_i R_{i,t+1} \\ R_{i,t+1} &= R_{d,t+1} > \lambda_i R_{i,t+1} \end{aligned}$$

We now consider other possible cases.

Lemma 83 *In any equilibrium we must have* $R_{d,t+1} > \lambda_i R_{i,t+1}$

Proof. Suppose this doesn't hold, then we have

$$R_{i,t+1} > \lambda_i R_{i,t+1} \geq R_{d,t+1}$$

The pledgeability constraint requires that

$$\lambda_i R_{i,t+1}(n_t + d_{i,t}) \geq R_{d,t+1} d_{i,t}$$

This always holds here as

$$\lambda_i R_{i,t+1}(n_t + d_{i,t}) \geq R_{d,t+1}(n_t + d_{i,t}) \geq R_{d,t+1} d_{i,t}$$

This follows as $R_{d,t+1} > 0$ and $n_t \geq 0$.

Hence, in this case the constraint is satisfied for all $d_{i,t}$. Further, as $0 < \lambda_i < 1$ there is a positive spread and so the bank wants to take as many deposits as possible. Thus, optimally it sets $d_{i,t} = \infty$, which cannot be an equilibrium as there is a finite amount of potential deposits from households. ■

Lemma 84 *Suppose* $\beta(1-\pi) \geq \pi$. *Then in any equilibrium we must have*

$$R_{i,t+1} \geq R_{d,t+1}$$

The condition is the same condition that ensures that in the case of zero spreads, the households want to make non-negative deposits. This is not trivial in the model as the households can consume in the second period even if they don't make deposits, due to their equity stake in the bank which is paid out in the second period of their life.

Proof. Suppose this condition does not hold. Then $R_{i,t+1} < R_{d,t+1}$ and the banks lose money on every unit of deposits taken. Optimally they thus set $d_{i,t} = 0$. This fails to be an equilibrium if the households want to make deposits at these prices.

Given the banks set $d_{i,t} = 0$, it follows that $V_{i,t} = n_t R_{i,t+1}$ with bank returns just coming from them trading on their own account.

From (3.44) we then have

$$\begin{aligned} d_{i,t} &= \frac{\beta}{1+\beta}(1-\pi)w_t - \frac{1}{1+\beta}\frac{\pi w_t R_{i,t+1}}{R_{d,t+1}} \\ &= \frac{w_t}{1+\beta}\left(\beta(1-\pi) - \frac{\pi R_{i,t+1}}{R_{d,t+1}}\right) \\ &> \frac{w_t}{1+\beta}\left(\beta(1-\pi) - \pi\right) \\ &\geq 0 \text{ so} \\ d_{i,t} &> 0 \end{aligned}$$

In the derivation we used:

$$\begin{aligned} R_{i,t+1} &< R_{d,t+1} \text{ so} \\ \frac{R_{i,t+1}}{R_{d,t+1}} &< 1 \text{ so} \\ -\frac{R_{i,t+1}}{R_{d,t+1}} &> -1 \end{aligned}$$

Under these conditions we do not have an equilibrium as deposit supply is greater than deposit demand. This completes the proof of the lemma. ■

Summary For Sector i

We now establish a summary proposition for the deposit market equilibrium.

Proposition 85 *Suppose $\beta(1-\pi) \geq \pi$. Then in equilibrium in the deposit market we have*

$$R_{i,t+1} \geq R_{d,t+1} > \lambda_i R_{i,t+1}$$

There are two cases:

(i) If $\beta(1-\pi) > \pi + \lambda_i\beta$ then $R_{i,t+1} > R_{d,t+1} > \lambda_i R_{i,t+1}$ and the unique equilibrium is given by

$$\begin{aligned} d^*_{i,t} &= \frac{\lambda_i\beta}{1+\lambda_i\beta}(1-\pi)w_t \\ R_{i,t+1} &= \frac{\alpha x_i^\alpha}{\left[\frac{\pi+\lambda_i\beta}{1+\lambda_i\beta}(1-\alpha)k_t^\alpha\right]^{1-\alpha}} \\ R_{d,t+1} &= \frac{\alpha x_i^\alpha(1+\lambda_i\beta)^{1-\alpha}(\pi+\lambda_i\beta)^\alpha}{\beta(1-\pi)\left[(1-\alpha)k_t^\alpha\right]^{1-\alpha}} \\ k_{t+1} &= \widehat{x}_i\frac{\pi+\lambda_i\beta}{1+\lambda_i\beta}(1-\alpha)k_t^\alpha \end{aligned}$$

(ii) If $\beta(1-\pi) \leq \pi + \lambda_i \beta$ then $R_{i,t+1} = R_{d,t+1} > \lambda_i R_{i,t+1}$ and the unique equilibrium is given by

$$\begin{aligned} d^*_{i,t} &= \frac{w_t}{1+\beta}\left(\beta(1-\pi) - \pi\right) \\ R_{d,t+1} &= R_{i,t+1} = \frac{\alpha x_i^\alpha}{\left[\frac{\beta}{1+\beta}(1-\alpha)k_t^\alpha\right]^{1-\alpha}} \\ k_{t+1} &= \frac{\widehat{x}_i \beta}{1+\beta}(1-\alpha)k_t^\alpha \end{aligned}$$

Proof. From the prior lemmas with $\beta(1-\pi) \geq \pi$ we have $R_{i,t+1} \geq R_{d,t+1} > \lambda_i R_{i,t+1}$. As shown above, when $\beta(1-\pi) > \pi + \lambda_i \beta$ we have an equilibrium with a positive spread and no equilibrium with a zero spread. Further, when $\beta(1-\pi) \leq \pi + \lambda_i \beta$ we have an equilibrium with zero spread and no equilibrium with a positive spread. This completes the proof. ■

Corollary 86 *We have a positive spread in sector i if*

$$\lambda_i < \frac{\beta(1-\pi) - \pi}{\beta}$$

In particular, we are guaranteed a positive spread in both sectors in all states of the economy if

$$\begin{aligned} \lambda_B &< \frac{\beta(1-\pi) - \pi}{\beta} \\ \overline{\lambda}_A &< \frac{\beta(1-\pi) - \pi}{\beta} \end{aligned}$$

where $\overline{\lambda}_A$ is the maximum value $\lambda_A(n_t)$ takes.

Proof. This is immediate from the previous proposition. ■

3.A.3 Sector Invested In

In our specification, depositors dictate the sector that is invested in, based on which will pay a higher return to them. For this to be an equilibrium we require that bankers prefer to do this than take no deposits and invest in the other sector. Here we examine conditions that ensure the banks have no incentive to deviate from the derived equilibrium.[35]

Lemma 87 *Suppose $\beta(1-\pi) > \pi + \lambda_i \beta$ $(i = A, B)$ so that there would be positive spreads in both sectors were they invested in. Further, suppose that*

$$x_B(1-\lambda_B)\left(\frac{\pi + \lambda_B}{1+\beta\lambda_B}\right) \geq \pi x_A$$

[35] Note: given $(1-\pi)\beta > \pi$ so that households always wish to make deposits, the cases of banks not taking deposits are not equilibria. The work here verifies that our proposed equilibria are indeed equilibria.

Then the banks invest in sector A iff $R^A_{d,t+1} > R^B_{d,t+1}$.

Here the banks always take deposits and invest in the sector the depositors want rather than taking no deposits and investing by themselves.

Proof. Under the given conditions, there is a positive spread when both sectors are invested in.

Thus

$$\begin{aligned} V_{i,t+1} &= R_{d,t+1}(1-\lambda_i)w_t\frac{\beta(1-\pi)}{(\beta\lambda_i+1)} \text{ and} \\ R_{d,t+1} &= \frac{\alpha x_i^\alpha(1+\lambda_i\beta)^{1-\alpha}(\pi+\lambda_i\beta)^\alpha}{\beta(1-\pi)\left[(1-\alpha)k_t^\alpha\right]^{1-\alpha}} \end{aligned}$$

Combining these gives equilibrium bank profits when sector i is invested in *and deposits are taken:*

$$\begin{aligned} V^*_{i,t+1} &= \frac{\alpha x_i^\alpha(1+\lambda_i\beta)^{1-\alpha}(\pi+\lambda_i\beta)^\alpha}{\beta(1-\pi)\left[(1-\alpha)k_t^\alpha\right]^{1-\alpha}}(1-\lambda_i)w_t\frac{\beta(1-\pi)}{(\beta\lambda_i+1)} \\ V^*_{i,t+1} &= \frac{\alpha x_i^\alpha(1+\lambda_i\beta)^{1-\alpha}(\pi+\lambda_i\beta)^\alpha}{\left[(1-\alpha)k_t^\alpha\right]^{1-\alpha}}(1-\lambda_i)\frac{(1-\alpha)k_t^\alpha}{(\beta\lambda_i+1)} \\ V^*_{i,t+1} &= \alpha x_i^\alpha(1-\lambda_i)\left(\frac{(\pi+\lambda_i\beta)}{(1+\beta\lambda_i)}\right)^\alpha((1-\alpha)k_t^\alpha)^\alpha \end{aligned}$$

Consider the bank profits that one deviating bank would make if they switched to investment in sector $j \neq i$, taking no deposits:

$$\begin{aligned} V^{*nd}_{j,t+1} &= R_{j,t+1}n_t \\ &= \frac{\alpha x_j n_t}{k^{1-\alpha}_{t+1}} \end{aligned}$$

Crucially as the deviating bank is infinitesimal, the total capital next period is unaltered: it is the level of investment in capital from sector i that determines returns next period. Now expected capital next period is given by

$$k_{t+1} = x_i\frac{\pi+\lambda_i\beta}{1+\lambda_i\beta}(1-\alpha)k_t^\alpha$$

Thus

$$\begin{aligned} V^{*nd}_{j,t+1} &= \frac{\alpha x_j\pi(1-\alpha)k_t^\alpha}{\left(x_i\frac{\pi+\lambda_i\beta}{1+\lambda_i\beta}(1-\alpha)k_t^\alpha\right)^{1-\alpha}} \\ V^{*nd}_{j,t+1} &= \frac{\alpha x_j\pi\left((1-\alpha)k_t^\alpha\right)^\alpha}{\left(x_i\frac{\pi+\lambda_i\beta}{1+\lambda_i\beta}\right)^{1-\alpha}} \end{aligned}$$

We consider two cases:

(i) $R^A_{d,t+1} > R^B_{d,t+1}$ Then a potential deviating bank chooses not to deviate iff

$$\begin{aligned}
V^*_{A,t+1} &\geq V^{*nd}_{B,t+1} \text{ iff} \\
\alpha x_A^\alpha (1-\lambda_A) \left(\frac{(\pi + \lambda_A \beta}{1+\lambda_A \beta} \right)^\alpha ((1-\alpha)k_t^\alpha)^\alpha &\geq \frac{\alpha x_B \pi \left((1-\alpha)k_t^\alpha\right)^\alpha}{\left(x_A \frac{\pi + \lambda_A \beta}{1+\lambda_A \beta}\right)^{1-\alpha}} \text{ iff} \\
x_A^\alpha (1-\lambda_A) \left(\frac{(\pi + \lambda_A \beta}{1+\lambda_A \beta} \right)^\alpha \left(x_A \frac{\pi + \lambda_A \beta}{1+\lambda_A \beta}\right)^{1-\alpha} &\geq x_B \pi \text{ iff} \\
x_A (1-\lambda_A) \frac{\pi + \lambda_A \beta}{1+\lambda_A \beta} &\geq x_B \pi
\end{aligned}$$

Now as $\beta(1-\pi) > \pi + \lambda_A \beta$ we have

$$(1-\lambda_A) \frac{\pi + \lambda_A \beta}{1+\lambda_A \beta} > \pi$$

To see this:

$$\begin{aligned}
(1-\lambda_A) \frac{\pi + \lambda_A \beta}{1+\lambda_A \beta} &> \pi \text{ iff} \\
(1-\lambda_A)(\pi + \lambda_A \beta) &> \pi (1+\lambda_A \beta) \text{ iff} \\
\pi - \lambda_A \pi + \lambda_A \beta - \lambda_A^2 \beta &> \pi + \pi \lambda_A \beta \text{ iff} \\
-\lambda_A \pi + \lambda_A \beta - \lambda_A^2 \beta &> \pi \lambda_A \beta \text{ iff} \\
-\pi + \beta &> \pi \beta + \lambda_A \beta \text{ iff} \\
\beta(1-\pi) &> \pi + \lambda_A \beta
\end{aligned}$$

Thus we have that

$$x_A (1-\lambda_A) \frac{\pi + \lambda_A \beta}{1+\lambda_A \beta} > x_A \pi > x_B \pi$$

Hence $V^*_{A,t+1} > V^{*nd}_{B,t+1}$

Note we do not need the condition for this to hold. The intuition in this case is simple: when investing in sector A there are higher gross returns on each unit (given that $x_A > x_B$ and *there is the same amount of capital next period in both cases*) and more units are invested as deposits are taken. Further, there is a positive spread, so profit is made on each extra deposit taken and invested.

We now consider the other case:

(ii) $R^A_{d,t+1} < R^B_{d,t+1}$ Here depositors want the bank to invest in sector B. It is optimal for a bank to not deviate from this iff

$$\begin{aligned}
V^*_{B,t+1} &\geq V^{*nd}_{A,t+1} \text{ iff} \\
x_B (1-\lambda_B) \frac{\pi + \lambda_B \beta}{1+\lambda_B \beta} &\geq x_A \pi
\end{aligned}$$

which holds given the condition in the lemma.

In this case we need a condition as there is a trade off for the banks: they get a higher gross return on each unit when investing in A, but they invest a greater volume when investing in B. If this volume is great enough and the profit margin is too, it is optimal for the bank to take deposits and invest in B.

This completes the proof. ■

Lemma 88 *Suppose $\beta(1-\pi) > \pi + \lambda_i\beta$ $(i = A, B)$, then $R^A_{d,t+1} > R^B_{d,t+1}$ iff*

$$x_A^\alpha(1+\lambda_A\beta)^{1-\alpha}(\pi+\lambda_A\beta)^\alpha > x_B^\alpha(1+\lambda_B\beta)^{1-\alpha}(\pi+\lambda_B\beta)^\alpha$$

Proof. Given the above conditions both sectors will have positive spreads were they invested in. Thus

$$R^i_{d,t+1} = \frac{\alpha x_i^\alpha(1+\lambda_i\beta)^{1-\alpha}(\pi+\lambda_i\beta)^\alpha}{\beta(1-\pi)\left[(1-\alpha)k_t^\alpha\right]^{1-\alpha}}$$

Hence

$$\begin{aligned} R^A_{d,t+1} &> R^B_{d,t+1} \text{ iff} \\ \frac{\alpha x_A^\alpha(1+\lambda_A\beta)^{1-\alpha}(\pi+\lambda_A\beta)^\alpha}{\beta(1-\pi)\left[(1-\alpha)k_t^\alpha\right]^{1-\alpha}} &> \frac{\alpha x_B^\alpha(1+\lambda_B\beta)^{1-\alpha}(\pi+\lambda_B\beta)^\alpha}{\beta(1-\pi)\left[(1-\alpha)k_t^\alpha\right]^{1-\alpha}} \text{ iff} \\ x_A^\alpha(1+\lambda_A\beta)^{1-\alpha}(\pi+\lambda_A\beta)^\alpha &> x_B^\alpha(1+\lambda_B\beta)^{1-\alpha}(\pi+\lambda_B\beta)^\alpha \end{aligned}$$

This completes the proof. ■

Lemma 89 *Suppose $x_B(1-\lambda_B)\left(\frac{\pi+\lambda_B}{1+\beta\lambda_B}\right) \geq \pi x_A$ and $\beta(1-\pi) > \pi + \lambda_i\beta$ $(i = A, B)$. Further, suppose that*

$$\begin{aligned} \lambda_A'(n_t) &> 0 \;\forall n_t \geq 0; \\ x_A &> x_B; \\ \lambda_A(0) &= \underline{\lambda}_A \in [0, \lambda_B); \\ \lim_{n_t\to\infty} \lambda_A(n_t) &= \overline{\lambda}_A \in (\underline{\lambda}_A, 1); \\ x_A^\alpha(1+\underline{\lambda}_A\beta)^{1-\alpha}(\pi+\underline{\lambda}_A\beta)^\alpha &< x_B^\alpha(1+\lambda_B\beta)^{1-\alpha}(\pi+\lambda_B\beta)^\alpha; \\ x_A^\alpha(1+\overline{\lambda}_A\beta)^{1-\alpha}(\pi+\overline{\lambda}_A\beta)^\alpha &> x_B^\alpha(1+\lambda_B\beta)^{1-\alpha}(\pi+\lambda_B\beta)^\alpha \end{aligned}$$

Then there exists a unique level of banker net worth $\widetilde{n}$:bankers invest in A iff $n_t > \widetilde{n}$[36]*. This is defined implicitly by*

$$x_A^\alpha(1+\lambda_A(\widetilde{n})\beta)^{1-\alpha}(\pi+\lambda_A(\widetilde{n})\beta)^\alpha = x_B^\alpha(1+\lambda_B\beta)^{1-\alpha}(\pi+\lambda_B\beta)^\alpha$$

[36] This $\widetilde{n}$ is time-invariant so long as the expected level of technology in sector A is constant:x_A is constant over time.

Proof. With the given conditions $R^i_{d,t+1} = \frac{\alpha x_i^\alpha (1+\lambda_i\beta)^{1-\alpha}(\pi+\lambda_i\beta)^\alpha}{\beta(1-\pi)[(1-\alpha)k_t^\alpha]^{1-\alpha}}$, and banks invest in sector A rather than sector B iff $R^A_{d,t+1} > R^B_{d,t+1}$.

Let

$$g(n_t) := x_A^\alpha(1+\lambda_A(n_t)\beta)^{1-\alpha}(\pi+\lambda_A(n_t)\beta)^\alpha - x_B^\alpha(1+\lambda_B\beta)^{1-\alpha}(\pi+\lambda_B\beta)^\alpha$$

Then banks invest in sector A iff $g(n_t) > 0$.

By the above conditions, $g(0) < 0$. Further, $\lim_{n_t\to\infty} g(n_t) > 0$. Thus, for sufficiently large n_t, $g(n_t) > 0$. As $\lambda_A(.)$ is differentiable on $[0,\infty)$, it is continuous on the same interval and hence so too is $g(.)$. Thus, by the Intermediate Value Theorem, $\exists\widetilde{n} : g(\widetilde{n}) = 0$. Further, as $\lambda'_A(n_t) > 0$ $\forall n_t \geq 0$, $g'(n_t) > 0$ $\forall n_t \geq 0$. Hence, $\widetilde{n}$ is unique, and $g(n_t) > 0$ iff $n_t > \widetilde{n}$. This completes the proof. ■

Corollary 90 *Suppose* $x_B(1-\lambda_B)\left(\frac{\pi+\lambda_B}{1+\beta\lambda_B}\right) \geq \pi x_A$ *and* $\beta(1-\pi) > \pi + \lambda_i\beta$ $(i = A, B)$. *Let* n_B^* *be the steady state value of banker net worth when sector* B *is invested in. The economy features a credit trap if*

$$x_A^\alpha(1+\lambda_A(n_B^*)\beta)^{1-\alpha}(\pi+\lambda_A(n_B^*)\beta)^\alpha < x_B^\alpha(1+\lambda_B\beta)^{1-\alpha}(\pi+\lambda_B\beta)^\alpha$$

Proof. From the above lemmas, banks invest in sector B rather than sector A iff

$$x_A^\alpha(1+\lambda_A(n_t)\beta)^{1-\alpha}(\pi+\lambda_A(n_t)\beta)^\alpha < x_B^\alpha(1+\lambda_B\beta)^{1-\alpha}(\pi+\lambda_B\beta)^\alpha$$

Hence if

$$x_A^\alpha(1+\lambda_A(n_B^*)\beta)^{1-\alpha}(\pi+\lambda_A(n_B^*)\beta)^\alpha < x_B^\alpha(1+\lambda_B\beta)^{1-\alpha}(\pi+\lambda_B\beta)^\alpha$$

the banks will invest in sector B when $n_t = n_B^*$. i.e. they invest in the B in the steady state of B. This is thus a steady state equilibrium and without shocks the economy will invest in sector B for the rest of time, so is stuck in a credit trap. This completes the proof of the corollary. ■

3.B Proofs from Section 3.3: Financial Crisis

3.B.1 Deleveraging

In terms of the impact on the macroeconomy of deleveraging, we are interested in the output produced by the standard productive technology as this links to the wages of the next generation. This is given by

$$y^P_{t+1} = (k_{t+1} - k^L_{t+1})^\alpha$$

This is the quantity we focus on when examining how the leverage of the banking sector affects the resilience of the economy: the resilience is higher the higher this quantity.

The leverage limit will hold for the expected level of capital next period, $k^e_{t+1},$. Capital k_{t+1}

has value $V(k_{t+1})$ in terms of output where $V(k_{t+1}) = \alpha k_{t+1}^{\alpha}$. Given that they owe depositors $R_{d,t+1}\overline{d}$ units of output goods, their net worth, in terms of output goods, is $\alpha k_{t+1}^{\alpha} - R_{d,t+1}\overline{d}$. We thus have the following relationship holding for the expected amount of capital next period[37]:

$$\frac{\alpha(k_{t+1}^{e})^{\alpha}}{\alpha(k_{t+1}^{e})^{\alpha} - R_{d,t+1}\overline{d}} = \frac{1}{1-\lambda_A} \tag{3.50}$$

If $k_{t+1} < k_{t+1}^{e}$ then the leverage limit will be exceeded and depositors will withdraw deposits until it holds. To consider how much the bank may have to deleverage, it is useful to consider their net worth as a function of initial capital holdings k_{t+1} and the amount of capital liquidation they do k_{t+1}^{L} :

$$\begin{aligned} NW(k_{t+1}, k_{t+1}^{L}) &= \alpha(k_{t+1} - k_{t+1}^{L})^{\alpha} + L(k_{t+1}, k_{t+1}^{L}) - R_{d,t+1}\overline{d} \\ with\ L(k_{t+1}, 0) &= 0 \end{aligned}$$

To emphasise, if the bankers initially hold k_{t+1} units of capital and liquidate k_{t+1}^{L} units, then the value of their remaining capital holdings in terms of output is $\alpha(k_{t+1} - k_{t+1}^{L})^{\alpha}$.

In general, if less capital is produced than expected, we require that

$$\frac{\alpha(k_{t+1} - k_{t+1}^{L})^{\alpha}}{\alpha(k_{t+1} - k_{t+1}^{L})^{\alpha} + L(k_{t+1}, k_{t+1}^{L}) - R_{d,t+1}\overline{d}} = \frac{1}{1-\lambda_A}$$

This implies that

$$\begin{aligned} \alpha(k_{t+1} - k_{t+1}^{L})^{\alpha}(1-\lambda_A) &= \alpha(k_{t+1} - k_{t+1}^{L})^{\alpha} + L(k_{t+1}, k_{t+1}^{L}) - R_{d,t+1}\overline{d} \text{ so} \\ R_{d,t+1}\overline{d} - L(k_{t+1}, k_{t+1}^{L}) &= \lambda_A \alpha(k_{t+1} - k_{t+1}^{L})^{\alpha} \end{aligned}$$

This condition states that the amount owed to depositors after deleveraging is equal to the pledgeable return bankers can promise with their remaining capital.

To further analyse this expression, we note that from (3.50) we have that

$$\begin{aligned} \alpha(k_{t+1}^{e})^{\alpha}(1-\lambda_A) &= \alpha(k_{t+1}^{e})^{\alpha} - R_{d,t+1}\overline{d} \text{ so} \\ R_{d,t+1}\overline{d} &= \lambda_A \alpha(k_{t+1}^{e})^{\alpha} \end{aligned}$$

Substituting this into the above expression gives

$$\lambda_A \alpha(k_{t+1}^{e})^{\alpha} - L(k_{t+1}, k_{t+1}^{L}) = \lambda_A \alpha(k_{t+1} - k_{t+1}^{L})^{\alpha} \tag{3.51}$$

We note that, of course, if $k_{t+1} = k_{t+1}^{e}$ then this has solution $k_{t+1}^{L} = 0$, i.e. no deleveraging. For general $L(.,.)$ there will be no analytic solution to this. Below we consider a special case in which net worth is constant as the bank deleverages.

[37] That is the amount produced when the capital producing technology has its expected value: $\widehat{x}_A = x_A$

3.B.2 Benchmark case: net worth constant with deleveraging

This is a natural benchmark as it isolates the impact of deleveraging per se, without 'fire sale' costs. Using the above expressions

$$NW(k_{t+1}, k^L_{t+1}) = \alpha(k_{t+1} - k^L_{t+1})^\alpha + L(k_{t+1}, k^L_{t+1}) - \lambda_A \alpha (k^e_{t+1})^\alpha$$

Net worth is constant with deleveraging iff

$$\frac{\partial L(k_{t+1}, k^L_{t+1})}{\partial k^L_{t+1}} = \frac{\alpha^2}{(k_{t+1} - k^L_{t+1})^{1-\alpha}}$$

This requires that

$$L(k_{t+1}, k^L_{t+1}) = -\alpha(k_{t+1} - k^L_{t+1})^\alpha + C$$

where C is a constant. Given $L(k_{t+1}, 0) = 0$, $C = \alpha k^\alpha_{t+1}$. Thus our liquidation technology that gives constant net worth is given by

$$\widetilde{L}(k_{t+1}, k^L_{t+1}) = \alpha k^\alpha_{t+1} - \alpha(k_{t+1} - k^L_{t+1})^\alpha$$

With this,

$$NW(k_{t+1}, k^L_{t+1}) = \alpha k^\alpha_{t+1} - \lambda_A \alpha (k^e_{t+1})^\alpha$$

Further, (3.51) becomes

$$\begin{aligned}
\lambda_A \alpha (k^e_{t+1})^\alpha - \left(\alpha k^\alpha_{t+1} - \alpha(k_{t+1} - k^L_{t+1})^\alpha\right) &= \lambda_A \alpha (k_{t+1} - k^L_{t+1})^\alpha \text{ so} \\
(1-\lambda_A)\alpha(k_{t+1} - k^L_{t+1})^\alpha &= \alpha k^\alpha_{t+1} - \lambda_A \alpha (k^e_{t+1})^\alpha \text{ so} \\
(k_{t+1} - k^L_{t+1})^\alpha &= \frac{k^\alpha_{t+1} - \lambda_A (k^e_{t+1})^\alpha}{(1-\lambda_A)}
\end{aligned}$$

Now

$$k_{t+1} = \widehat{x}_A \frac{\pi + \lambda_A \beta}{1 + \lambda_A \beta}(1-\alpha)k^\alpha_t$$

Hence the output of productive technology, y^P_{t+1} is given by:

$$y^P_{t+1} = \frac{(\widehat{x}^\alpha_A - \lambda_A x^\alpha_A)}{(1-\lambda_A)} \left[\frac{\pi + \lambda_A \beta}{1 + \lambda_A \beta}(1-\alpha)k^\alpha_t\right]^\alpha$$

Proposition 91 *With the benchmark liquidation technology and sector A invested in*

$$\begin{aligned}
n_{t+1} &= (1-\alpha)\pi \frac{(\widehat{x}^\alpha_A - \lambda_A(n_t) x^\alpha_A)}{(1-\lambda_A(n_t))} \left[\frac{\pi + \lambda_A(n_t)\beta}{1 + \lambda_A(n_t)\beta}(1-\alpha)k^\alpha_t\right]^\alpha \quad \textit{if } \widehat{x}_A < x_A \\
n_{t+1} &= (1-\alpha)\pi \widehat{x}^\alpha_A \left[\frac{\pi + \lambda_A(n_t)\beta}{1 + \lambda_A(n_t)\beta}(1-\alpha)k^\alpha_t\right]^\alpha \quad \textit{if } \widehat{x}_A \geq x_A
\end{aligned}$$

Proof. The next generation wages are based on the amount of productive output:

$$n_{t+1} = (1-\alpha)\pi y^P_{t+1}$$

If $\widehat{x}_A < x_A$ then liquidation takes place and $y^P_{t+1} = \frac{(\widehat{x}^\alpha_A - \lambda_A x^\alpha_A)}{(1-\lambda_A)}\left[\frac{\pi+\lambda_A\beta}{1+\lambda_A\beta}(1-\alpha)k^\alpha_t\right]^\alpha$

If $\widehat{x}_A \geq x_A$ then no liquidation takes place (as the leverage limit is not violated) and $y^P_{t+1} = \widehat{x}^\alpha_A\left[\frac{\pi+\lambda_A\beta}{1+\lambda_A\beta}(1-\alpha)k^\alpha_t\right]^\alpha$

This completes the proof of the proposition. ■

3.C Proofs from Section 3.4: Policy Options to Avoid Credit Traps

Derivation of $x^T_A(\lambda)$

When the regulatory requirement λ is imposed, we know that the economy will fall into a credit trap whenever bank equity falls below $\tilde{n}$. This condition is given by:

$$\hat{n}_{t+1} = \pi(1-\alpha)\frac{(\widehat{x}^\alpha_A - \lambda x^\alpha_A)}{(1-\lambda)}\left[\frac{\pi+\lambda\beta}{1+\lambda\beta}(1-\alpha)k^\alpha_t\right]^\alpha \leq \tilde{n}$$

We now solve the above for $\hat{x}_A$.

$$\widehat{x}^\alpha_A - \lambda x^\alpha_A \leq \tilde{n}\frac{1-\lambda}{(1-\alpha)\pi}\left[\frac{\pi+\lambda\beta}{1+\lambda\beta}(1-\alpha)k^\alpha_t\right]^{-\alpha}$$

$$\widehat{x}^\alpha_A \leq \lambda x^\alpha_A + \frac{\tilde{n}\,(1-\lambda)}{(1-\alpha)\pi\left[\frac{\pi+\lambda\beta}{1+\lambda\beta}(1-\alpha)k^\alpha_t\right]^\alpha}$$

Hence the threshold productivity shock below which the economy falls into a credit trap in the next period is given by:

$$x^T_A(\lambda) := \left[\lambda x^\alpha_A + \frac{\widetilde{n}(1-\lambda)}{(1-\alpha)\pi\left[\frac{\pi+\lambda\beta}{1+\lambda\beta}(1-\alpha)k^\alpha_t\right]^\alpha}\right]^{\frac{1}{\alpha}}$$

We now demonstrate the "u-shaped" resilience proposition from the text.

Proof of Propostion 57. We first introduce some notation to simplify the exposition of the proof.

Let

$$z(\lambda) := \lambda x^\alpha_A + \frac{\widetilde{n}(1-\lambda)}{(1-\alpha)\pi\left[\frac{\pi+\lambda\beta}{1+\lambda\beta}(1-\alpha)k^\alpha_t\right]^\alpha}$$

Then

$$x^T_A(\lambda) \equiv (z(\lambda))^{\frac{1}{\alpha}}$$

Now

$$\frac{dx_A^T(\lambda)}{d\lambda} = \frac{1}{\alpha}\left(z(\lambda)\right)^{\frac{1}{\alpha}-1} z'(\lambda)$$
$$> 0 \text{ iff } z'(\lambda) > 0$$

Further,

$$\frac{d^2 x_A^T(\lambda)}{d\lambda^2} = \frac{1}{\alpha}(\frac{1}{\alpha} - 1)\left(z(\lambda)\right)^{\frac{1}{\alpha}-2}\left(z'(\lambda)\right)^2 + \frac{1}{\alpha}\left(z(\lambda)\right)^{\frac{1}{\alpha}-1} z''(\lambda)$$

Hence, if $z''(\lambda) > 0$ then $\frac{d^2 x_A^T(\lambda)}{d\lambda^2} > 0$.
Given these results, in the following steps of the proof we can work with $z(\lambda)$.
We introduce further notation: let

$$h(\lambda) := \frac{(1-\lambda)}{\left[\frac{\pi+\lambda\beta}{1+\lambda\beta}\right]^{\alpha}}$$

Then

$$z(\lambda) = \lambda x_A^{\alpha} + \frac{\widetilde{n} h(\lambda)}{(1-\alpha)\pi\left[(1-\alpha)k_t^{\alpha}\right]^{\alpha}}$$

The proof now proceeds via a series of steps.
(i) $\frac{dx_A^T(\lambda)}{d\lambda} > 0$ for λ close to 1.
We show $z'(\lambda) > 0$ for λ close to 1.

$$z'(\lambda) = x_A^{\alpha} + \frac{\widetilde{n} h'(\lambda)}{(1-\alpha)\pi\left[(1-\alpha)k_t^{\alpha}\right]^{\alpha}}$$

We turn to $h'(\lambda)$:

$$h(\lambda) = (1-\lambda)(1+\lambda\beta)^{\alpha}(\pi+\lambda\beta)^{-\alpha}$$

Thus

$$\begin{aligned}
&h'(\lambda)\\
&= -(1+\lambda\beta)^{\alpha}(\pi+\lambda\beta)^{-\alpha} + \alpha\beta(1-\lambda)(1+\lambda\beta)^{\alpha-1}(\pi+\lambda\beta)^{-\alpha} - \alpha\beta(1-\lambda)(1+\lambda\beta)^{\alpha}(\pi+\lambda\beta)^{-\alpha-1}\\
&= -(1+\lambda\beta)^{\alpha}(\pi+\lambda\beta)^{-\alpha} - \alpha\beta(1-\pi)(1-\lambda)(1+\lambda\beta)^{\alpha-1}(\pi+\lambda\beta)^{-\alpha-1}
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{\lambda\to 1} z'(\lambda) &= x_A^\alpha - \frac{\widetilde{n}}{(1-\alpha)\pi\left[(1-\alpha)k_t^\alpha\right]^\alpha}\left(\frac{1+\beta}{\pi+\beta}\right)^\alpha \\
&> 0 \; iff \\
x_A^\alpha &> \frac{\widetilde{n}}{(1-\alpha)\pi\left[(1-\alpha)k_t^\alpha\right]^\alpha}\left(\frac{1+\beta}{\pi+\beta}\right)^\alpha iff \\
x_A^\alpha(1-\alpha)\pi\left[(1-\alpha)k_t^\alpha\right]^\alpha\left(\frac{\pi+\beta}{1+\beta}\right)^\alpha &> \widetilde{n}
\end{aligned}$$

Thus, given our assumed condition $\lim_{\lambda\to 1} z'(\lambda) > 0$
However, $z'(\lambda)$ is continuous so $\exists \lambda^* < 1 : z'(\lambda) > 0 \; \forall \lambda \in [\lambda^*, 1)$.
Thus $\frac{dx_A^T(\lambda)}{d\lambda} > 0 \; \forall \lambda \in [\lambda^*, 1)$.
(ii) $\frac{d^2 x_A^T(\lambda)}{d\lambda^2} > 0 \; \forall \lambda \in [0,1]$
It is sufficient to show that

$$z''(\lambda) > 0 \; \forall \lambda \in [0,1]$$

$$z''(\lambda) = \frac{\widetilde{n} h''(\lambda)}{(1-\alpha)\pi\left[(1-\alpha)k_t^\alpha\right]^\alpha}$$

From step (i)

$$h'(\lambda) = -(1+\lambda\beta)^\alpha(\pi+\lambda\beta)^{-\alpha} - \alpha\beta(1-\pi)(1-\lambda)(1+\lambda\beta)^{\alpha-1}(\pi+\lambda\beta)^{-\alpha-1}$$

Thus

$$\begin{aligned}
h''(\lambda) &= -\alpha\beta(1+\lambda\beta)^{\alpha-1}(\pi+\lambda\beta)^{-\alpha} + \alpha\beta(1+\lambda\beta)^\alpha(\pi+\lambda\beta)^{-\alpha-1} \\
&- \alpha\beta(1-\pi)\left[\begin{array}{c} -(1+\lambda\beta)^{\alpha-1}(\pi+\lambda\beta)^{-\alpha-1} + (1-\lambda)\beta(\alpha-1)(1+\lambda\beta)^{\alpha-2}(\pi+\lambda\beta)^{-\alpha-1} \\ -\beta(\alpha+1)(1-\lambda)(1+\lambda\beta)^{\alpha-1}(\pi+\lambda\beta)^{-\alpha-2} \end{array}\right]
\end{aligned}$$

So

$$\begin{aligned}
\frac{h''(\lambda)}{\alpha\beta} &= \left(\frac{1+\lambda\beta}{\pi+\lambda\beta}\right)^\alpha\left[\frac{-1}{1+\lambda\beta} + \frac{1}{\pi+\lambda\beta}\right] \\
&+ (1-\pi)\left(\frac{1+\lambda\beta}{\pi+\lambda\beta}\right)\left[\frac{1}{(1+\lambda\beta)(\pi+\lambda\beta)} + \frac{(1-\lambda)\beta(1-\alpha)}{(1+\lambda\beta)^2(\pi+\lambda\beta)} + \frac{(1-\lambda)\beta(1+\alpha)}{(1+\lambda\beta)(\pi+\lambda\beta)^2}\right] \\
&> 0
\end{aligned}$$

Where we note that the first term is positive as $1 > \pi$
Hence

$$z''(\lambda) > 0 \; \forall \lambda \in [0,1]$$

(iii) We now use steps (i), (ii) to prove the proposition.
The second condition in the proposition gives $\frac{dx_A^T(0)}{d\lambda} < 0$. From step (i) $\exists \lambda^* < 1 : \frac{dx_A^T(\lambda^*)}{d\lambda} > 0$.

Now we must have $\lambda^* > 0$, for otherwise, given $\frac{d^2 x_A^T(\lambda)}{d\lambda^2} > 0$,we'd have $\frac{dx_A^T(0)}{d\lambda} > 0$, a contradiction. As $\frac{dx_A^T(\lambda)}{d\lambda}$ is continuous, by the Intermediate Value Theorem, $\exists\widehat{\lambda} : \frac{dx_A^T(\widehat{\lambda})}{d\lambda} = 0$. Further, as $\frac{d^2 x_A^T(\lambda)}{d\lambda^2} > 0$ $\widehat{\lambda}$ is unique. The following then holds

$$\frac{dx_A^T(\lambda)}{d\lambda} \left\{ \begin{array}{c} < 0 \text{ for } \lambda \in [0, \widehat{\lambda}) \\ = 0 \text{ for } \lambda = \widehat{\lambda} \\ > 0 \text{ for } \lambda \in (\widehat{\lambda}, 1] \end{array} \right\}$$

And so $\frac{dx_A^T(\lambda)}{d\lambda}$ reaches a unique minimum at $\lambda = \widehat{\lambda}$.

This completes the proof of the proposition. ■

3.C.1 Countercyclical Maximum Resilience Policy

Proof of Proposition 59.

Using the above notation:

$$\begin{aligned}
\frac{dx_A^T(\lambda)}{d\lambda} &= 0 \text{ iff } z'(\lambda) = 0 \text{ iff} \\
x_A^\alpha + \frac{\widetilde{n}h'(\lambda)}{(1-\alpha)\pi\left[(1-\alpha)k_t^\alpha\right]^\alpha} &= 0 \text{ iff} \\
x_A^\alpha &= -\frac{\widetilde{n}h'(\lambda)}{(1-\alpha)\pi\left[(1-\alpha)k_t^\alpha\right]^\alpha} \text{ iff} \\
x_A^\alpha &= \frac{\widetilde{n}\left(\frac{1+\lambda\beta}{\pi+\lambda\beta}\right)^\alpha\left[(1+\lambda\beta)(\pi+\lambda\beta)+\alpha\beta(1-\lambda)(1-\pi)\right]}{(1-\alpha)\pi\left[(1-\alpha)k_t^\alpha\right]^\alpha(1+\lambda\beta)(\pi+\lambda\beta)} \text{ iff} \\
\frac{x_A^\alpha(1-\alpha)\pi\left[(1-\alpha)k_t^\alpha\right]^\alpha}{\widetilde{n}} &= \frac{\left(\frac{1+\lambda\beta}{\pi+\lambda\beta}\right)^\alpha\left[(1+\lambda\beta)(\pi+\lambda\beta)+\alpha\beta(1-\lambda)(1-\pi)\right]}{(1+\lambda\beta)(\pi+\lambda\beta)} \text{ iff} \\
\frac{x_A^\alpha(1-\alpha)\pi\left[(1-\alpha)k_t^\alpha\right]^\alpha}{\widetilde{n}} &= \left(\frac{1+\lambda\beta}{\pi+\lambda\beta}\right)^\alpha\left[1+\frac{\alpha\beta(1-\lambda)(1-\pi)}{(1+\lambda\beta)(\pi+\lambda\beta)}\right]
\end{aligned}$$

This equation implicitly defines $\widehat{\lambda}$. The RHS is decreasing in λ. Increasing $\widetilde{n}$ decreases the LHS, so decreases the RHS, so increases $\widetilde{\lambda}$ (which maintains equality between the two sides of the the equation). Thus $\frac{d\widehat{\lambda}}{d\widetilde{n}} > 0$. By a similar argument $\frac{d\widehat{\lambda}}{dk_t} < 0$. This completes the proof of the proposition. ■

3.D Proofs from Section 3.6: Unconventional Credit Policy

We derive the laws of motion for k_{t+1} for each of the three policies separately.

3.D.1 Direct Lending

With $d_{g,t}$ government bonds issued, households' saving is given by

$$d_{i,t} = \frac{\beta}{1+\beta}\left[(1-\pi)(1-\alpha)k_t^{\alpha}\right] - \frac{1}{1+\beta}\frac{V_{i,t+1}}{R^d_{i,t+1}} - d_{g,t} \tag{3.52}$$

With a positive spread, the banks' borrowing constraint will bind giving $d_{i,t} = \frac{\lambda_i R_{i,t+1} n_t}{R^d_{i,t+1} - \lambda_i R_{i,t+1}}$ and following the prior proofs in the appendix, we have

$$\frac{V_{i,t+1}}{R_{d,t+1}} = \frac{R_{i,t+1} n_t}{R_{d,t+1} - \lambda_i R_{i,t+1}}(1-\lambda_i)$$

In banking system equilibrium, deposit demand is equal to deposit supply giving

$$\frac{\lambda_i R_{i,t+1} n_t}{R^d_{i,t+1} - \lambda_i R_{i,t+1}} = \frac{\beta}{1+\beta}\left[(1-\pi)(1-\alpha)k_t^{\alpha}\right] - \frac{1}{1+\beta}\frac{R_{i,t+1} n_t (1-\lambda_i)}{R_{d,t+1} - \lambda_i R_{i,t+1}} - d_{g,t}$$

After rearranging,this gives (3.35) in the text. Following the steps given there results in (3.36).

3.D.2 Discount Window Lending

The bank has two sources of funding: deposits and government loans, and maximises its profits with respect to these subject to its combined leverage constraint. We have the following Lagrangian:

$$\begin{aligned} L &= R_{A,t+1}(n_t + d_{A,t} + m_t) - R^d_{A,t+1} d_{A,t} - R^m_{t+1} m_t \\ &\quad + \mu\left[\lambda_A R_{A,t+1}(n_t + d_{A,t} + \omega m_t) - R^d_{A,t+1} d_{A,t} - R^m_{t+1} m_t\right] \end{aligned}$$

FOCs:

$$\begin{aligned} d_{A,t} &: \mu = \frac{R_{A,t+1} - R^d_{A,t+1}}{R^d_{A,t+1} - \lambda_A R_{A,t+1}} \\ m_t &: \mu = \frac{R_{A,t+1} - R^m_{A,t+1}}{R^m_{A,t+1} - \omega \lambda_A R_{A,t+1}} \end{aligned}$$

Combining the two gives

$$\frac{R_{A,t+1} - R^d_{A,t+1}}{R^d_{A,t+1} - \lambda_A R_{A,t+1}} = \frac{R_{A,t+1} - R^m_{A,t+1}}{R^m_{A,t+1} - \omega \lambda_A R_{A,t+1}} \tag{3.53}$$

We proceed to derive equilibrium through the usual series of steps.

Banks' Demand for Deposits

With a binding borrowing constraint, we have, after rearranging

$$d_{A,t} = \frac{\lambda_A R_{A,t+1}}{R^d_{A,t+1} - \lambda_A R_{A,t+1}} n_t - \frac{(R^m_{t+1} - \omega\lambda_A R_{A,t+1})}{(R^d_{A,t+1} - \lambda_A R_{A,t+1})} m_t$$

Applying (3.53) we have

$$d_{A,t} = \frac{\lambda_A R_{A,t+1}}{R^d_{A,t+1} - \lambda_A R_{A,t+1}} n_t - \frac{(R_{A,t+1} - R^m_{A,t+1})}{(R_{A,t+1} - R^d_{A,t+1})} m_t$$

Bank Profits

The profits for the bank are given by

$$V_{A,t+1} = (R_{A,t+1} - R^d_{A,t+1})d_{A,t} + R_{A,t+1}n_t + (R_{A,t+1} - R^m_{A,t+1})m_t$$

Substituting in the expression for deposits and rearranging gives

$$V_{A,t+1} = \frac{R_{A,t+1}R^d_{A,t+1}(1-\lambda_A)n_t}{R^d_{A,t+1} - \lambda_A R_{A,t+1}}$$

Household Deposit Demand

The equation for this is also given by (3.52), thus substituting in bank profits, we have household deposit demand given by

$$d_{A,t} = \frac{\beta}{1+\beta}\left[(1-\pi)w_t\right] - \frac{1}{1+\beta}\frac{R_{A,t+1}(1-\lambda_A)n_t}{R^d_{A,t+1} - \lambda_A R_{A,t+1}} - d_{g,t}$$

Deposit Market Equilibrium

To determine we equate the supply and demand for deposits:

$$\begin{aligned} & \frac{\lambda_A R_{A,t+1}}{R^d_{A,t+1} - \lambda_A R_{A,t+1}} n_t - \frac{(R_{A,t+1} - R^m_{A,t+1})}{(R_{A,t+1} - R^d_{A,t+1})} m_t \\ = \quad & \frac{\beta}{1+\beta}\left[(1-\pi)w_t\right] - \frac{1}{1+\beta}\frac{R_{A,t+1}(1-\lambda_A)n_t}{R^d_{A,t+1} - \lambda_A R_{A,t+1}} - d_{g,t} \end{aligned}$$

Solving, and rearranging gives

$$d^*_{A,t} = \frac{\lambda_A\beta}{1+\lambda_A\beta}(1-\pi)w_t - \frac{\lambda_A(1+\beta)}{1+\lambda_A\beta}d_{g,t} - \frac{(1-\lambda_A)}{1+\lambda_A\beta}\left(\frac{R_{A,t+1} - R^m_{t+1}}{R_{A,t+1} - R^d_{A,t+1}}\right)m_t \qquad (3.54)$$

Lemma 92 *In equilibrium*

$$\frac{R_{A,t+1} - R^m_{t+1}}{R_{A,t+1} - R^d_{A,t+1}} = \frac{1-\omega\lambda_A}{1-\lambda_A}$$

Proof. We first show that, in equilibrium,

$$\begin{aligned} R^d_{A,t+1} &= \psi^d_t \lambda_A R_{A,t+1} \\ R^m_{t+1} &= \left(\frac{(1-\omega\lambda_A)\psi^d_t + \omega - 1}{1-\lambda_A} \right) \lambda_A R_{A,t+1} \end{aligned}$$

Where

$$\psi^d_t := \frac{\left(\frac{n_t}{d_{A,t}} + 1 + \frac{1-\omega\lambda_A}{1-\lambda_A} \frac{m_t}{d_{A,t}} \right)}{\left(1 + \frac{1-\omega\lambda_A}{1-\lambda_A} \frac{m_t}{d_{A,t}} \right)}$$

To show this, first not that from the binding borrowing constraint

$$R^d_{A,t+1} = \lambda_A R_{A,t+1} \left(\frac{n_t}{d_{A,t}} + 1 \right) - \left(R^m_{t+1} - \omega\lambda_A R_{A,t+1} \right) \frac{m_t}{d_{A,t}}$$

Rearranging (3.53) gives

$$R^m_{t+1} = \frac{(1-\omega\lambda_A)}{1-\lambda_A} R^d_{A,t+1} + \frac{(\omega-1)\lambda_A R_{A,t+1}}{1-\lambda_A}$$

Thus, the deposit rate satisfies

$$R^d_{A,t+1} = \lambda_A R_{A,t+1} \left(\frac{n_t}{d_{A,t}} + 1 + \omega \frac{m_t}{d_{A,t}} \right) - \left(\frac{(1-\omega\lambda_A)}{1-\lambda_A} R^d_{A,t+1} + \frac{(\omega-1)\lambda_A R_{A,t+1}}{1-\lambda_A} \right) \frac{m_t}{d_{A,t}}$$

Solving for $R^d_{A,t+1}$:

$$\begin{aligned} R^d_{A,t+1} &= \lambda_A R_{A,t+1} \left(\frac{n_t}{d_{A,t}} + 1 + \frac{(1-\omega\lambda_A)m_t}{(1-\lambda_A)d_{A,t}} \right) \left(1 - \frac{1-\omega\lambda_A}{1-\lambda_A} \frac{m_t}{d_{A,t}} \right)^{-1} \\ &= \psi^d_t \lambda_A R_{A,t+1} \end{aligned}$$

Further,

$$\begin{aligned} R^m_{t+1} &= \frac{(1-\omega\lambda_A)}{1-\lambda_A} \psi^d_t \lambda_A R_{A,t+1} + \frac{(\omega-1)\lambda_A R_{A,t+1}}{1-\lambda_A} \\ &= \lambda_A R_{A,t+1} \left(\frac{(1-\omega\lambda_A)\psi^d_t + \omega - 1}{1-\lambda_A} \right) \end{aligned}$$

We now use these two results to establish the lemma:

$$\begin{aligned}
\frac{R_{A,t+1}-R^m_{t+1}}{R_{A,t+1}-R^d_{A,t+1}} &= \frac{1-\omega\lambda_A}{1-\lambda_A} \text{ iff} \\
\frac{\left[1-\frac{\lambda_A}{1-\lambda_A}\left((1-\omega\lambda_A)\psi^d_t+\omega-1\right)\right]}{\left[1-\psi^d_t\lambda_A\right]} &= \frac{1-\omega\lambda_A}{1-\lambda_A}\text{iff} \\
(1-\lambda_A)-\lambda_A\left((1-\omega\lambda_A)\psi^d_t+\omega-1\right) &= \left[1-\psi^d_t\lambda_A\right](1-\omega\lambda_A) \text{ iff} \\
1-\lambda_A\left[(1-\omega\lambda_A)\psi^d_t+\omega\right] &= \left[1-\psi^d_t\lambda_A\right](1-\omega\lambda_A)
\end{aligned}$$

But the LHS can be written

$$\begin{aligned}
& 1-\lambda_A\left[(1-\omega\lambda_A)\psi^d_t+\omega\right] \\
=\ & -(1-\omega\lambda_A)\lambda_A\psi^d_t+(1-\omega\lambda_A) \\
=\ & (1-\omega\lambda_A)(1-\lambda_A\psi^d_t) \\
=\ & RHS
\end{aligned}$$

This completes the proof of the lemma. ■

Given this (3.54) becomes

$$d^*_{A,t}=\frac{\lambda_A\beta}{1+\lambda_A\beta}(1-\pi)w_t-\frac{\lambda_A(1+\beta)}{1+\lambda_A\beta}d_{g,t}-\frac{(1-\omega\lambda_A)\,m_t}{1+\lambda_A\beta}$$

Now,

$$k_{t+1}=x_A(n_t+d_{A,t})+x_Am_t$$

Thus we can write the law of motion for k_{t+1} as (noting $m_t=\frac{d_{g,t}}{1+\tau_m}$)

$$k_{t+1}=x_A\left[n_t+\frac{\lambda_A\beta}{1+\lambda_A\beta}(1-\pi)w_t\right]+x_Ad_{g,t}\left(\frac{\left[1-\frac{(1-\omega\lambda_A)}{1+\lambda_A\beta}\right]}{1+\tau_m}-\frac{\lambda_A(1+\beta)}{1+\lambda_A\beta}\right)$$

The first term simplifies to k_{t+1} absent policy, in the usual way.
Further, given that $\omega=1+\frac{\omega_g(1-\lambda_A)}{\lambda_A}$we can write

$$\begin{aligned}
1-\omega\lambda_A &= 1-\lambda_A\left(1+\frac{\omega_g(1-\lambda_A)}{\lambda_A}\right) \\
&= 1-\lambda_A-\omega_g(1-\lambda_A) \\
&= (1-\lambda_A)(1-\omega_g)
\end{aligned}$$

Substituting this in results in the expression for k_{t+1} in the text.

3.D.3 Equity Injection

Derivation of Law of Motion

When the government obtains x_g fraction of bank equity, optimal household saving is then given by

$$d_{i,t} = \frac{\beta}{1+\beta}\left[(1-\pi)w_t\right] - \frac{(1-x_g)}{(1+\beta)}\frac{V_{i,t+1}}{R^d_{i,t+1}} - d_{g,t} \tag{3.55}$$

To derive the equilibrium law of motion for k_{t+1} we follow the usual steps, first determining equilibrium in the banking sector.

With the banks' leverage constraints binding they demand deposits,[38]

$$d_{i,t} = \frac{\lambda_i R_{i,t+1}(n_t + n_{g,t})}{R^d_{i,t+1} - \lambda_i R_{i,t+1}}$$

Bank profits are given by[39]

$$V_{i,t+1} = \left(R_{i,t+1} - R^d_{i,t+1}\right) d_{i,t} + R_{i,t+1}(n_t + n_{g,t})$$

Following the usual steps, with the binding constraint

$$V_{i,t+1} = \frac{R_{i,t+1}(n_t + n_{g,t})}{R^d_{i,t+1} - \lambda_i R_{i,t+1}} R^d_{i,t+1}(1-\lambda_i)$$

Then, from (3.55) deposit supply is given by

$$d_{i,t} = \frac{\beta}{1+\beta}\left[(1-\pi)w_t\right] - \frac{(1-x_g)(1-\lambda_i)}{(1+\beta)}\frac{R_{i,t+1}(n_t + n_{g,t})}{R^d_{i,t+1} - \lambda_i R_{i,t+1}} - d_{g,t}$$

In deposit market equilibrium the supply and demand for deposits are equal

$$\frac{\lambda_i R_{i,t+1}(n_t + n_{g,t})}{R^d_{i,t+1} - \lambda_i R_{i,t+1}} = \frac{\beta}{1+\beta}\left[(1-\pi)w_t\right] - \frac{(1-x_g)(1-\lambda_i)}{(1+\beta)}\frac{R_{i,t+1}(n_t + n_{g,t})}{R^d_{i,t+1} - \lambda_i R_{i,t+1}} - d_{g,t}$$

Rearranging

$$\begin{aligned}
\frac{R_{i,t+1}(n_t + n_{g,t})}{R^d_{i,t+1} - \lambda_i R_{i,t+1}}\left[\lambda_i + \frac{(1-x_g)(1-\lambda_i)}{(1+\beta)}\right] &= \frac{\beta}{1+\beta}\left[(1-\pi)w_t\right] - d_{g,t} \\
\frac{R_{i,t+1}(n_t + n_{g,t})}{R^d_{i,t+1} - \lambda_i R_{i,t+1}}\left[(1+\beta)\lambda_i + (1-x_g)(1-\lambda_i)\right] &= \beta\left[(1-\pi)w_t\right] - (1+\beta)d_{g,t} \\
\frac{R_{i,t+1}(n_t + n_{g,t})}{R^d_{i,t+1} - \lambda_i R_{i,t+1}} &= \frac{\beta\left[(1-\pi)w_t\right] - (1+\beta)d_{g,t}}{(1+\beta)\lambda_i + (1-x_g)(1-\lambda_i)}
\end{aligned}$$

[38] Note the addition of $n_{g,t}$ which is absent with no equity injection.

[39] The formula (save for the $n_{g,t}$ term) for bank profits has not changed here. What changes is *who* gets them once they're realised, i,e, the split between households and the government.

Then from the banks' deposit demand equation, equilibrium deposits are given by

$$d_{i,t}^{*}=\frac{\lambda_i\beta(1-\pi)w_t-(1+\beta)d_{g,t}\lambda_i}{(1+\beta)\lambda_i+(1-x_g)(1-\lambda_i)}$$

This reduces to the no-policy equilibrium level of deposits when $d_{g,t}=0$ and $x_g=0$.

Finally, $k_{t+1}=x_i\left(n_t+n_{g,t}+d_{i,t}\right)$, so using $\frac{d_{g,t}}{(1+\tau_{gn})}=n_{g,t}$ we have

$$k_{t+1}=x_i\left(n_t+\frac{\lambda_i\beta(1-\pi)w_t}{(1+\beta)\lambda_i+(1-\lambda_i)(1-x_g)}\right)+d_{g,t}x_i\left(\frac{1}{1+\tau_{gn}}-\frac{(1+\beta)\lambda_i}{(1+\beta)\lambda_i+(1-\lambda_i)(1-x_g)}\right) \tag{3.56}$$

The presence of the policy term x_g on the denominator makes this expression harder to compare to the other two policy cases, so we re-write it to put it into a comparable form.

Note that

$$\frac{1}{(1+\beta)\lambda_i+(1-\lambda_i)(1-x_g)}=\frac{1}{1+\beta\lambda_i}+\left[\frac{(1-\lambda_i)x_g}{\left((1+\beta)\lambda_i+(1-\lambda_i)(1-x_g)\right)(1+\beta\lambda_i)}\right]$$

Thus, we can write

$$\begin{aligned}k_{t+1} &= x_i\left(n_t+\frac{\lambda_i\beta(1-\pi)w_t}{(1+\beta\lambda_i)}\right)+\frac{x_i(1-\lambda_i)x_g\lambda_i\beta(1-\pi)w_t}{\left((1+\beta)\lambda_i+(1-\lambda_i)(1-x_g)\right)(1+\beta\lambda_i)}\\ &\quad +d_{g,t}x_i\left(\frac{1}{1+\tau_{gn}}-\frac{(1+\beta)\lambda_i}{(1+\beta\lambda_i)}\right)-\frac{d_{g,t}x_i(1-\lambda_i)x_g(1+\beta)\lambda_i}{\left((1+\beta)\lambda_i+(1-\lambda_i)(1-x_g)\right)(1+\beta\lambda_i)}\end{aligned}$$

After simplifications, this can be written as

$$\begin{aligned}k_{t+1} &= x_i\left(\left(\frac{\pi+\lambda_i\beta}{1+\lambda_i\beta}\right)(1-\alpha)k_t^{\alpha}\right)+x_id_{g,t}\left[\frac{1}{(1+\tau_{gn})}-\frac{\lambda_i(1+\beta)}{1+\lambda_i\beta}\right]\\ &\quad +x_g\frac{x_i\lambda_i(1-\lambda_i)\left[\beta(1-\pi)w_t-(1+\beta)d_{g,t}\right]}{\left[(1+\beta)\lambda_i+(1-\lambda_i)(1-x_g)\right](1+\beta\lambda_i)}\end{aligned} \tag{3.57}$$

An additional effect of equity is directly raising λ_A, it being a function of $n_t+n_{g,t}$:

$$\lambda_A(n_t+n_{g,t})$$

Then, the impact of an equity injection (with investment in sector A) can be written as

$$\begin{aligned}k_{t+1} &= x_A\left(\left(\frac{\pi+\lambda_A(n_t+n_{g,t})\beta}{1+\lambda_A(n_t+n_{g,t})\beta}\right)(1-\alpha)k_t^{\alpha}\right)\\ &\quad +x_Ad_{g,t}\left[\frac{1}{(1+\tau_{gn})}-\frac{\lambda_A(n_t+n_{g,t})(1+\beta)}{1+\lambda_A(n_t+n_{g,t})\beta}\right]\\ &\quad +\frac{x_Ad_{g,t}\lambda_A(n_t+n_{g,t})(1-\lambda_A(n_t+n_{g,t}))\left[\beta(1-\pi)w_t-(1+\beta)d_{g,t}\right]}{(1+\tau_{gn})(1+\beta\lambda_A(n_t+n_{g,t}))\left[\frac{d_{g,t}}{(1+\tau_{gn})}(1+\beta)\lambda_A(n_t+n_{g,t})+(1+\beta\lambda_A(n_t+n_{g,t}))\gamma n_t\right]}\end{aligned}$$

We note that policy directly affects the first term, "crowding in" depositors. We re-write the expression to make it comparable to the baseline case.

After some algebra, we can show that:

$$\frac{\pi + \lambda_A(n_t + n_{g,t})\beta}{1 + \lambda_A(n_t + n_{g,t})\beta} = \frac{\pi + \lambda_A(n_t)\beta}{1 + \lambda_A(n_t)\beta} + \frac{\beta(1-\pi)\left[\lambda_A(n_t + n_{g,t}) - \lambda_A(n_t)\right]}{\left(1 + \lambda_A(n_t + n_{g,t})\beta\right)\left(1 + \lambda_A(n_t)\beta\right)}$$

Further

$$\frac{\lambda_A(n_t + n_{g,t})}{1 + \lambda_A(n_t + n_{g,t})\beta} = \frac{\lambda_A(n_t)}{1 + \lambda_A(n_t)\beta} + \frac{\lambda_A(n_t + n_{g,t}) - \lambda_A(n_t)}{\left[1 + \lambda_A(n_t + n_{g,t})\beta\right]\left[1 + \lambda_A(n_t)\beta\right]}$$

Thus, we can write

$$\begin{aligned}
& x_A\left(\frac{\pi + \lambda_A(n_t + n_{g,t})\beta}{1 + \lambda_A(n_t + n_{g,t})\beta}\right) w_t - x_A d_{g,t}\frac{\lambda_A(n_t + n_{g,t})(1+\beta)}{1 + \lambda_A(n_t + n_{g,t})\beta} \\
= \; & x_A \frac{\pi + \lambda_A(n_t)\beta}{1 + \lambda_A(n_t)\beta} w_t - x_A d_{g,t}\frac{\lambda_A(n_t)(1+\beta)}{1 + \lambda_A(n_t)\beta} \\
& + x_A \frac{\left[\lambda_A(n_t + n_{g,t}) - \lambda_A(n_t)\right]}{\left(1 + \lambda_A(n_t + n_{g,t})\beta\right)\left(1 + \lambda_A(n_t)\beta\right)}\left(w_t\beta(1-\pi) - d_{g,t}(1+\beta)\right)
\end{aligned}$$

Thus, in full we can write

$$\begin{aligned}
k_{t+1} \; = \; & x_A\left(\frac{\pi + \lambda_A(n_t)\beta}{1 + \lambda_A(n_t)\beta} w_t\right) + x_A d_{g,t}\left[\frac{1}{(1+\tau_{gn})} - \frac{\lambda_A(n_t)(1+\beta)}{1 + \lambda_A(n_t)\beta}\right] \\
& + x_A \frac{\left[\lambda_A\left(n_t + \frac{d_{g,t}}{1+\tau_{gn}}\right) - \lambda_A(n_t)\right]}{\left(1 + \lambda_A\left(n_t + \frac{d_{g,t}}{1+\tau_{gn}}\right)\beta\right)\left(1 + \lambda_A\left(n_t\right)\beta\right)}\left[w_t\beta(1-\pi) - d_{g,t}(1+\beta)\right] \\
& + x_A d_{g,t} \frac{\lambda_A\left(n_t + \frac{d_{g,t}}{1+\tau_{gn}}\right)\left(1 - \lambda_A\left(n_t + \frac{d_{g,t}}{1+\tau_{gn}}\right)\right)}{(1+\tau_{gn})\left(1 + \beta\lambda_A\left(n_t + \frac{d_{g,t}}{1+\tau_{gn}}\right)\right)} \\
& \cdot \frac{\left[\beta(1-\pi)w_t - (1+\beta)d_{g,t}\right]}{\left[\frac{d_{g,t}}{(1+\tau_{gn})}(1+\beta)\lambda_A\left(n_t + \frac{d_{g,t}}{1+\tau_{gn}}\right) + \left(1 + \beta\lambda_A\left(n_t + \frac{d_{g,t}}{1+\tau_{gn}}\right)\right)\gamma n_t\right]}
\end{aligned}$$

This gives expression (3.41) in the text.

Other Results

We first establish an expression for the impact of policy:

$$\frac{dk_{t+1}}{d(d_{g,t})}$$

We go through the various components of (3.41) step by step.

The first term is straightforward with derivative

$$x_A \left[\frac{1}{(1+\tau_{gn})} - \frac{\lambda_A(n_t)(1+\beta)}{1+\lambda_A(n_t)\beta} \right]$$

The derivative for the second term is given by

$$x_A \frac{\lambda_A' \left(n_t + \frac{d_{g,t}}{1+\tau_{gn}} \right) [w_t \beta(1-\pi) - d_{g,t}(1+\beta)]}{(1+\tau_{gn}) \left(1 + \lambda_A \left(n_t + \frac{d_{g,t}}{1+\tau_{gn}} \right) \beta \right)^2} - x_A \frac{\left[\lambda_A \left(n_t + \frac{d_{g,t}}{1+\tau_{gn}} \right) - \lambda_A(n_t) \right] (1+\beta)}{\left(1 + \lambda_A \left(n_t + \frac{d_{g,t}}{1+\tau_{gn}} \right) \beta \right) (1+\lambda_A(n_t)\beta)}$$

To ease notation, let

$$\begin{aligned} f(d_{g,t}) \quad : \quad = & \frac{\lambda_A \left(n_t + \frac{d_{g,t}}{1+\tau_{gn}} \right) \left(1 - \lambda_A \left(n_t + \frac{d_{g,t}}{1+\tau_{gn}} \right) \right)}{(1+\tau_{gn}) \left(1 + \beta \lambda_A \left(n_t + \frac{d_{g,t}}{1+\tau_{gn}} \right) \right)} \\ & \cdot \frac{[\beta(1-\pi)w_t - (1+\beta)d_{g,t}]}{\left[\frac{d_{g,t}}{(1+\tau_{gn})} (1+\beta) \lambda_A \left(n_t + \frac{d_{g,t}}{1+\tau_{gn}} \right) + \left(1 + \beta \lambda_A \left(n_t + \frac{d_{g,t}}{1+\tau_{gn}} \right) \right) \gamma n_t \right]} \end{aligned}$$

Then the third term can be written as $x_A d_{g,t} f(d_{g,t})$.
It has derivative

$$x_A f(d_{g,t}) + x_A d_{g,t} f'(d_{g,t})$$

Thus, we have

$$\begin{aligned} \frac{dk_{t+1}}{d(d_{g,t})} = & x_A \left[\frac{1}{(1+\tau_{gn})} - \frac{\lambda_A(n_t)(1+\beta)}{1+\lambda_A(n_t)\beta} \right] + x_A \frac{\lambda_A' \left(n_t + \frac{d_{g,t}}{1+\tau_{gn}} \right) [w_t \beta(1-\pi) - d_{g,t}(1+\beta)]}{(1+\tau_{gn}) \left(1 + \lambda_A \left(n_t + \frac{d_{g,t}}{1+\tau_{gn}} \right) \beta \right)^2} \\ & - x_A \frac{\left[\lambda_A \left(n_t + \frac{d_{g,t}}{1+\tau_{gn}} \right) - \lambda_A(n_t) \right] (1+\beta)}{\left(1 + \lambda_A \left(n_t + \frac{d_{g,t}}{1+\tau_{gn}} \right) \beta \right) (1+\lambda_A(n_t)\beta)} + x_A f(d_{g,t}) + x_A d_{g,t} f'(d_{g,t}) \end{aligned} \tag{3.58}$$

Corollary 93

$$\begin{aligned} \left\{ \frac{dk_{t+1}}{d(d_{g,t})} \right\}_{d_{g,t=0}} = & x_A \left[\frac{1}{(1+\tau_{gn})} - \frac{\lambda_A(n_t)(1+\beta)}{1+\lambda_A(n_t)\beta} \right] + x_A \frac{\lambda_A'(n_t) w_t \beta(1-\pi)}{(1+\tau_{gn})(1+\lambda_A(n_t)\beta)^2} \\ & + x_A \frac{\lambda_A(n_t)(1-\lambda_A(n_t)) \beta(1-\pi) w_t}{(1+\tau_{gn})(1+\beta\lambda_A(n_t)) [(1+\beta\lambda_A(n_t)) \gamma n_t]} \end{aligned}$$

We now prove Lemma (64).

Proof. From the preceding line,

$$\begin{aligned}\left\{\frac{dk_{t+1}}{d\left(d_{g,t}\right)}\right\}_{d_{g,t}=0} &< 0 \\ &\text{iff} \frac{1}{(1+\tau_{gn})}-\frac{\lambda_A(n_t)(1+\beta)}{1+\lambda_A(n_t)\beta}+\frac{\lambda_A'\left(n_t\right)\left[w_t\beta(1-\pi)\right]}{(1+\tau_{gn})\left(1+\lambda_A\left(n_t\right)\beta\right)^2} \\ +\frac{\lambda_A\left(n_t\right)\left(1-\lambda_A(n_t)\right)\beta(1-\pi)w_t}{(1+\tau_{gn})\left(1+\beta\lambda_A\left(n_t\right)\right)^2\gamma n_t} &< 0\end{aligned}$$

This holds iff

$$\begin{aligned}\frac{1}{(1+\tau_{gn})}\left[1+\frac{\lambda_A'\left(n_t\right)\left[w_t\beta(1-\pi)\right]}{\left(1+\lambda_A\left(n_t\right)\beta\right)^2}+\frac{\lambda_A\left(n_t\right)\left(1-\lambda_A(n_t)\right)\beta(1-\pi)w_t}{\left(1+\beta\lambda_A\left(n_t\right)\right)^2\gamma n_t}\right] &< \frac{\lambda_A(n_t)(1+\beta)}{1+\lambda_A(n_t)\beta}\text{iff} \\ \frac{1+\lambda_A(n_t)\beta}{\lambda_A(n_t)(1+\beta)}\left[1+\frac{\left[\gamma n_t\lambda_A'\left(n_t\right)+\lambda_A\left(n_t\right)\left(1-\lambda_A(n_t)\right)\right]\left[w_t\beta(1-\pi)\right]}{\left(1+\lambda_A\left(n_t\right)\beta\right)^2\gamma n_t}\right]-1 &< \tau_{gn}\text{ iff} \\ \frac{1-\lambda_A(n_t)}{\lambda_A(n_t)(1+\beta)}+\frac{\left[\gamma n_t\lambda_A'\left(n_t\right)+\lambda_A\left(n_t\right)\left(1-\lambda_A(n_t)\right)\right]\left[\beta(1-\pi)\right]}{\lambda_A(n_t)(1+\beta)\left(1+\lambda_A\left(n_t\right)\beta\right)\gamma\pi} &< \tau_{gn}\end{aligned}$$

This condition can be written:

$$\tau_{gn}>\frac{1-\lambda_A(n_t)}{\lambda_A(n_t)(1+\beta)}\left[1+\frac{\left[\gamma n_t\lambda_A'\left(n_t\right)+\lambda_A\left(n_t\right)\left(1-\lambda_A(n_t)\right)\right]\left[\beta(1-\pi)\right]}{(1-\lambda_A(n_t))(1+\lambda_A(n_t)\beta)\gamma\pi}\right]$$

This completes the proof of the Lemma (64) ■

We now establish the sufficient conditions for the maximum marginal impact of an equity injection to be at $d_{g,t}=0$, first establishing a useful lemma.

Lemma 94 *Suppose* $\lambda_A(n_t)>\frac{-1+\sqrt{1+(\beta(2+\beta))}}{\beta(2+\beta)}$
Then

$$f'(d_{g,t})<0$$

Proof. It is clear that, treating λ_A as a constant, increasing $d_{g,t}$ decreases $f(d_{g,t})$. Now $d_{g,t}$ increases λ_A, so it's enough to show that $f(d_{g,t})$ is decreasing in λ_A. We write the relevant part as

$$\frac{\lambda(1-\lambda)}{(1+\beta\lambda)\left[(\alpha\lambda+(1+\beta\lambda)\gamma n_t\right]}=\frac{\lambda-\lambda^2}{(1+\beta\lambda)\left[\lambda\left[\alpha+\beta\gamma n_t\right]+\gamma n_t\right]}$$

where $\alpha:=\frac{d_{g,t}}{(1+\tau_{gn})}(1+\beta)$

Then, taking the derivative wrt λ :

$$\frac{(1-2\lambda)(1+\beta\lambda)[\lambda[\alpha+\beta\gamma n_t]+\gamma n_t]-\lambda(1-\lambda)[\beta[\lambda[\alpha+\beta\gamma n_t]+\gamma n_t]+[\alpha+\beta\gamma n_t](1+\beta\lambda)]}{(1+\beta\lambda)^2[\lambda[\alpha+\beta\gamma n_t]+\gamma n_t]^2}$$

$$= \quad -\left[\frac{\lambda^2(\alpha(1+\beta)+\beta n_t\gamma(2+\beta))+2\lambda\gamma n_t-\gamma n_t}{(1+\beta\lambda)^2[\lambda[\alpha+\beta\gamma n_t]+\gamma n_t]^2}\right]$$

This expression is then negative iff $\lambda > \frac{-2\gamma n_t+\sqrt{4\gamma^2 n_t^2+4\gamma n_t(\alpha(1+\beta)+\beta n_t\gamma(2+\beta))}}{2(\alpha(1+\beta)+\beta\gamma n_t(2+\beta))}$ iff

$$\lambda > \frac{-2\gamma n_t+\sqrt{4\gamma^2 n_t^2+4\gamma n_t\left(\frac{d_{g,t}}{(1+\tau_{gn})}(1+\beta)^2+\beta\gamma n_t(2+\beta)\right)}}{2(\frac{d_{g,t}}{(1+\tau_{gn})}(1+\beta)^2+\beta\gamma n_t(2+\beta))}$$

Note that the RHS is decreasing in $d_{g,t}$ hence it's sufficient that λ is greater than the expression when $d_{g,t}=0$

Evaluated at $d_{g,t}=0$, we require

$$\begin{aligned}\lambda &> \frac{-2\gamma n_t+\sqrt{4\gamma^2 n_t^2+4\gamma n_t(\beta\gamma n_t(2+\beta))}}{2\beta\gamma n_t(2+\beta)}\\ &= \frac{-1+\sqrt{1+(\beta(2+\beta))}}{\beta(2+\beta)}\end{aligned}$$

Note the required $\lambda < \frac{1}{2}$ ■

Proposition 95 *Suppose.*

$$\lambda_A''\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right) < \frac{2\beta\left[\lambda_A'\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\right]^2}{\left(1+\lambda_A\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\beta\right)}$$

and

$$\lambda_A(n_t) > \frac{-1+\sqrt{1+\beta(2+\beta)}}{\beta(2+\beta)}$$

then

$$\frac{dk_{t+1}}{d(d_{g,t})} \text{ is maximised at } d_{g,t}=0$$

Further, if

$$\tau_{gn} > \frac{1-\lambda_A(n_t)}{\lambda_A(n_t)(1+\beta)}\left[1+\frac{[\gamma n_t\lambda_A'(n_t)+\lambda_A(n_t)(1-\lambda_A(n_t))][w_t\beta(1-\pi)]}{(1-\lambda_A(n_t))(1+\lambda_A(n_t)\beta)\gamma n_t}\right]$$

Then an equity injection lowers k_{t+1} *for all* $d_{g,t}>0$.

Proof. [40]First consider the following term:

$$\frac{\lambda_A'\left(n_t + \frac{d_{g,t}}{1+\tau_{gn}}\right)}{\left(1+\lambda_A\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\beta\right)^2}$$

Its derivative is negative iff

$$\begin{aligned} & \lambda_A''\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\frac{1}{1+\tau_{gn}}\left(1+\lambda_A\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\beta\right)^2 \\ < \quad & 2\lambda_A'\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\left(1+\lambda_A\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\beta\right)\lambda_A'\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\frac{\beta}{1+\tau_{gn}} \end{aligned}$$

iff

$$\lambda_A''\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right) < \frac{2\beta\left[\lambda_A'\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\right]^2}{\left(1+\lambda_A\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\beta\right)}$$

We now show the following term is increasing in $d_{g,t}$:

$$\frac{\left[\lambda_A\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)-\lambda_A(n_t)\right]}{\left(1+\lambda_A\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\beta\right)}$$

It's derivative is positive iff

$$\begin{aligned} & \lambda_A'\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\frac{\left(1+\lambda_A\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\beta\right)}{1+\tau_{gn}} \\ > \quad & \left[\lambda_A\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)-\lambda_A(n_t)\right]\lambda_A'\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\frac{\beta}{1+\tau_{gn}} \end{aligned}$$

iff

$$1+\lambda_A(n_t)\beta > 0$$

Thus, it follows that the following term is decreasing in $d_{g,t}$.

$$-x_A\frac{\left[\lambda_A\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)-\lambda_A(n_t)\right](1+\beta)}{\left(1+\lambda_A\left(n_t+\frac{d_{g,t}}{1+\tau_{gn}}\right)\beta\right)(1+\lambda_A(n_t)\beta)}$$

Consider (3.58). Under the given conditions the first three terms are all decreasing in $d_{g,t}$. This leaves $x_A f(d_{g,t}) + x_A d_{g,t} f'(d_{g,t})$. As $f'(d_{g,t}) < 0$ under the given conditions, the first term is also decreasing in $d_{g,t}$. Finally, as $f'(d_{g,t}) < 0$, $x_A d_{g,t} f'(d_{g,t})$ takes it's maximum value for non-negative $d_{g,t}$ at $d_{g,t} = 0$.

[40]This is the proof of Lemma 65

From Lemma (64) given

$$\tau_{gn} > \frac{1-\lambda_A(n_t)}{\lambda_A(n_t)(1+\beta)} \left[1 + \frac{\left[\gamma n_t \lambda_A'(n_t) + \lambda_A(n_t)(1-\lambda_A(n_t))\right][w_t\beta(1-\pi)]}{(1-\lambda_A(n_t))(1+\lambda_A(n_t)\beta)\gamma n_t}\right] \tag{3.59}$$

$\left\{\frac{dk_{t+1}}{d(d_{g,t})}\right\}_{d_{g,t}=0} < 0$ which implies that $\left\{\frac{dk_{t+1}}{d(d_{g,t})}\right\}_{d_{g,t}=0} < 0$ for all $d_{g,t} > 0$ under the conditions given here.

This completes the proof. ■

3.D.4 Comparison of Policies

Here we prove Proposition 71, comparing the efficacy of an equity injection and discount window lending.

Proposition 96 *Suppose $\omega_g < \frac{1+\beta}{2+\beta}$ and we have the second equity pricing rule. Consider two credit crunches with associated banking system net worth n_1, n_2 with $n_1 > n_2$, so n_2 is the more severe crunch. Suppose (3.42) and (3.43) hold for n_1 and further that*

$$\begin{aligned}
\frac{(1+\tau_{gn})\left[(1+\beta\lambda_A(n_1)) - (1-\omega_g)(1-\lambda_A(n_1))\right]}{\left[1+(1+\beta)\lambda_A(n_1) + \frac{\lambda_A'(n_1)w_t\beta(1-\pi)}{(1+\lambda_A(n_1)\beta)}\right]} - 1 &> \tau_m \\
\frac{(1+\tau_{gn})\left[(1+\beta\lambda_A(n_2)) - (1-\omega_g)(1-\lambda_A(n_2))\right]}{\left[1+(1+\beta)\lambda_A(n_2) + \frac{\lambda_A'(n_2)w_t\beta(1-\pi)}{(1+\lambda_A(n_2)\beta)}\right]} - 1 &< \tau_m
\end{aligned}$$

Then an equity injection is more effective in raising k_{t+1} in the more severe credit crunch (n_2), for a range of $d_{g,t} > 0$, whilst discount window lending is more effective in the milder credit event (n_1) for all $d_{g,t} > 0$.

Proof. Under the given conditions, the marginal impact of an equity injection on k_{t+1} is greatest at $d_{g,t} = 0$. As the impact of discount window lending is linear in $d_{g,t}$ it is more effective in raising k_{t+1} for all $d_{g,t}$ than an equity injection if the marginal impact is greater at $d_{g,t} = 0$:

$$\begin{aligned}
\frac{1-(1-\omega_g)\left(\frac{(1-\lambda_A(n_t))}{(1+\lambda_A(n_t)\beta)}\right)}{(1+\tau_m)} \quad > \quad & \frac{1}{(1+\tau_{gn})} + \frac{\lambda_A'(n_t)\,w_t\beta(1-\pi)}{(1+\tau_{gn})(1+\lambda_A(n_t)\beta)^2} \\
& + \frac{\lambda_A(n_t)(1-\lambda_A(n_t))\beta(1-\pi)w_t}{(1+\tau_{gn})(1+\beta\lambda_A(n_t))^2\gamma n_t}
\end{aligned}$$

With the second equity pricing rule this reduces to

$$\frac{1-(1-\omega_g)\left(\frac{(1-\lambda_A(n_t))}{(1+\lambda_A(n_t)\beta)}\right)}{(1+\tau_m)} > \frac{1}{(1+\tau_{gn})}\left[\frac{1+(1+\beta)\lambda_A(n_t)}{(1+\beta\lambda_A(n_t))} + \frac{\lambda_A'(n_t)\,w_t\beta(1-\pi)}{(1+\lambda_A(n_t)\beta)^2}\right]$$

Rearranging this condition gives

$$\begin{aligned}
& (1+\tau_{gn})\left[1-(1-\omega_g)\left(\frac{(1-\lambda_A(n_t))}{(1+\lambda_A(n_t)\beta)}\right)\right] \\
> \quad & (1+\tau_m)\left[\frac{1+(1+\beta)\lambda_A(n_t)}{(1+\beta\lambda_A(n_t))}+\frac{\lambda_A'(n_t)\,w_t\beta(1-\pi)}{(1+\lambda_A(n_t)\,\beta)^2}\right] \text{ iff} \\
& \frac{(1+\tau_{gn})\left[(1+\beta\lambda_A(n_t))-(1-\omega_g)(1-\lambda_A(n_t))\right]}{\left[1+(1+\beta)\lambda_A(n_t)+\frac{\lambda_A'(n_t)w_t\beta(1-\pi)}{(1+\lambda_A(n_t)\beta)}\right]} \\
> \quad & (1+\tau_m)
\end{aligned}$$

We look for conditions under which the RHS is increasing in λ_A, so whether this holds or not can vary with the state of the economy. It will be increasing iff

$$\begin{aligned}
& [\beta+1-\omega_g]\left[1+(1+\beta)\lambda_A(n_t)+\frac{\lambda_A'(n_t)\,w_t\beta(1-\pi)}{(1+\lambda_A(n_t)\,\beta)}\right] \\
> \quad & [(1+\beta\lambda_A(n_t))-(1-\omega_g)(1-\lambda_A(n_t))]\left[(1+\beta)+\frac{d}{d\lambda_A}\left(\frac{\lambda_A'(n_t)\,w_t\beta(1-\pi)}{(1+\lambda_A(n_t)\,\beta)}\right)\right] \\
= \quad & [1+\lambda_A(n_t)(\beta+(1-\omega_g))-(1-\omega_g)]\left[(1+\beta)+\frac{d}{d\lambda_A}\left(\frac{\lambda_A'(n_t)\,w_t\beta(1-\pi)}{(1+\lambda_A(n_t)\,\beta)}\right)\right]
\end{aligned}$$

Given $\omega_g < \frac{1+\beta}{2+\beta}$, $\omega_g < 1+\beta$. Suppose n_t is sufficiently large that $\lambda_A''(n_t) < 0$, then a sufficient condition for the RHS increasing in λ_A is

$$[\beta+1-\omega_g]\,(1+(1+\beta)\lambda_A(n_t)) > [1+\lambda_A(n_t)(\beta+(1-\omega_g))-(1-\omega_g)]\,(1+\beta)$$

This reduces to the condition we assume:

$$\omega_g < \frac{1+\beta}{2+\beta}$$

This completes the proof of the proposition. ■

Bibliography

[1] Adelino, Manuel, Antoinette Schoar and Felipe Severino (2012) "Credit Supply and House Prices: Evidence From Mortgage Market Segmentation", *National Bureau of Economic Research* Working Paper 17832

[2] Adrian, Tobias, Paolo Colla and Hyun Song Shin (2012) "Which financial frictions? Parsing Evidence from the Financial Crisis of 2007-09", *Federal Reserve Bank of New York,* Staff Report No. 528

[3] Angelini, Paolo, Stefano Neri and Fabio Panetta (2012). "Monetary and Macroprudential Policies", *European Central Bank* Working Paper, No. 1449.

[4] Bank of England (2013) "The Financial Policy Committee's Powers to Supplement Capital Requirements"

[5] Benmelech, Efraim and Nittai Bergman (2012). "Credit Traps", *American Economic Review*, vol. 102(6), 3004-32

[6] Brueckner, Jan, Paul Calem and Leonard Nakamura (2012) "Subprime Mortgages and the Housing Bubble", *Journal of Urban Economics*, 71, 230-243

[7] Calhoun, Charles (2005) "The Hidden Risks of Piggyback Lending", *Calhoun Consulting Report.*

[8] Case, Karl and Robert Shiller (1994) "A Decade of Boom and Bust in the Prices of Single-Family Homes: Boston and Los Angeles, 1983 to 1993", *New England Economic Review* 41, March/April 1994: 40-51

[9] Case, Karl and Robert Shiller (2003) "Is there a Bubble in the Housing Market?", *Cowles Foundation Paper* No. 1089

[10] Case, Karl, Robert Shiller and Anne Thompson (2012) "What Have They Been Thinking? Homebuyer Behaviour in Hot and Cold Markets", *Brookings Papers on Economic Activity,* Fall 2012

[11] Christensen, Ian, Cesaire Meh, and Kevin Moran (2011). "Bank Leverage Regulation and Macroeconomic Dynamics", *Bank of Canada* Working Paper, No. 2011-32.

[12] Claessens, Stijn, Ayhan Kose and Marco Terrones (2008) "What Happens During Recessions, Crunches and Busts?", *International Monetary Fund* Working Paper 08/274

[13] Coleman IV, Major, Michael LaCour-Litte and Kerry Vandell (2008), "Subprime Lending and the Housing Bubble: Tail Wags Dog?", *Working Paper*

[14] Credit Suisse (2007) "Mortgage Liquidity du Jour: Underestimated no More", *Company Report*

[15] Davis, Morris, Francois Ortalo-Magne, and Peter Rupert, "What's Really Going on in Housing Markets?", *Federal Reserve Bank of Cleveland,* Economic Commentary 2007

[16] Dell'Ariccia, Giovanni, Deniz Igan and Luc Laeven (2008) "Credit Booms and Lending Standards: Evidence from the Subprime Mortgage Market", *International Monetary Fund* Working Paper 08/106

[17] Engelhardt, Gary (1996) "Consumption, Down Payments, and Liquidity Constraints", *Journal of Money, Credit and Banking* Vol.28, No.2, 255-271

[18] Favara, Giovanni and Jean Imbs (2011) "Credit Supply and the Price of Housing", *Working Paper*

[19] Gertler, Mark and Nobu Kiyotaki (2011) "Financial Intermediation and Credit Policy in Business Cycle Analysis", *Handbook of Monetary Economics,* Volume 3A

[20] Glaeser, Edward, Joshua Gottlieb and Joseph Gyourko (2010) "Can Cheap Credit Explain the Housing Boom?", *National Bureau of Economic Research* Working Paper 16230

[21] Glaeser, Edward, Joseph Gyourko and Albert Saiz (2008) "Housing Supply and Housing Bubbles", *Journal of Urban Economics,* vol. 64(2), 198-217

[22] Guerrieri, Veronica, Daniel Hartley and Erik Hurst (2011), "Endogenous Gentrification and Housing Price Dynamics", *Working Paper*

[23] Ferreira, Fernando and Joseph Gyourko (2011), "Anatomy of the Beginning of the Housing Boom: U.S. Neighbourhoods and Metropolitan Areas, 1993-2009", *National Bureau of Economic Research* Working Paper 17374

[24] Haughwout, Andrew, Donghoon Lee, Joseph Tracy and Wilbert van der Klaauw (2011), "Real Estate Investors, the Leverage Cycle, and the Housing Market Crisis", *Federal Reserve Bank of New York,* Staff Report No. 514

[25] Haughwout, Andrew, Richard Peach, John Sporn and Joseph Tracy (2012). "The Supply Side of the Housing Boom and Bust of the 2000s", *Federal Reserve Bank of New York,* Staff Report No. 556

[26] Kiyotaki, Nobu, Alexander Michaelides and Kalin Nikolov (2007) "Winners and Losers in Housing Markets", *Working Paper*

[27] La Cava, Gianni (2013) "Credit Supply Shocks in the US Housing Market", *PhD, London School of Economics, 2013*

[28] Iacoviello, Matteo (2005) "House Prices, Borrowing Constraints, and Monetary Policy in the Business Cycle", *American Economic Review*, vol. 95, No. 3, 739-764.

[29] Landvoigt, Tim, Monika Piazzesi and Martin Schneider (2012) "The Housing Market(s) of San Diego", *National Bureau of Economic Research* Working Paper 17723

[30] Matsuyama, Kiminori (2007), "Credit Traps and Credit Cycles". *American Economic Review*, Vol. 97, No. 1, 503-516.

[31] Mayer, Christopher (1993) "Taxes, Income Distribution, and the Real Estate Cycle: Why All Houses do not Appreciate at the Same Rate", *New England Economic Review* May/June 1993

[32] Mayer, Christopher and Gary Engelhardt (1996) "Gifts, Down Payments and Housing Affordability", *Journal of Housing Research*, Vol.7, Issue 1, 59-77

[33] Mian, Atif and Amir Sufi (2009) "The Consequences of Mortgage Credit Expansion: Evidence From the U.S. Mortgage Default Crisis", *Quarterly Journal of Economics,* November 2009, 1449-1496

[34] Ortalo-Magne, Francois and Sven Rady (1999) "Why are Housing Prices so Volatile? Income Shocks in a Stochastic Heterogeneous-Agents Model", *Working Paper*

[35] Poterba, James (1984) "Tax Subsidies to Owner-Occupied Housing: An Asset-Market Approach", *Quarterly Journal of Economics*, November 1984, 729-752

[36] Poterba, James (1991) "House Price Dynamics: The Role of Tax Policy and Demography", *Brookings Papers on Economic Activity*, Vol. 1991, No. 2, 143-203

[37] Shleifer, Andrei and Robert Vishny (1992). "Liquidation Values and Debt Capacity: a Market Equilibrium Approach", *Journal of Finance*, 47(4), 1343-66.

[38] Siaz, Albert (2008) "On Local Housing Supply Elasticity" *Working Paper*, University of Pennsylvania, The Wharton School

[39] Smith, Barton and William Tesarek, (1991) "House Prices and Regional Real Estate Cycles: Market Adjustments in Houston", *AREUEA Journal*, vol. 19, No.3, 396-416

[40] Standard and Poors (2009) "S&P/Case-Shiller Home Price Indices, Index Methodology"

[41] Stein, Jeremy (1995) "Prices and Trading Volume in the Housing Market: A Model with Down-Payment Effects", *Quarterly Journal of Economics*, Vol. 110, No.2, 379-406

[42] Van Nieuwerburgh, Stijn and Pierre-Olivier Weill, "Why Has House Price Dispersion Gone Up?", *The Review of Economic Studies*, 2010, 77 (4): 1567-1606.

[43] Wheaton, William and Gleb Nechayev (2007) "The 1998-2005 Housing "Bubble" and the Current "Correction": What's Different This Time?", *Working Paper*